More on coupling coefficients for the most degenerate representations of $SO(n)$

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Abstract. We present explicit closed-form expressions for the $6j$-symbols of the most degenerate representations of $SO(n)$. The general group-theoretical factor appearing in the α-topology of a high-temperature expansion of $SO(n)$-symmetric lattice models, which is closely related to these $6j$-symbols, is discussed in detail.

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1. Introduction

In this paper we extent our previous studies [1] on coupling coefficients for the so-called most degenerate (also named class-one) representations of $SO(n)$. Here we will present new explicit results for the corresponding $6j$-symbols. These coupling coefficients are important in many fields of theoretical physics such as atomic and nuclear physics. For example, in connection with the Jahn-Teller effect an extensive study of particular $6j$-symbols is due to Judd and coworkers [2]. The $6j$-symbols, in particular those of the most degenerate representations of $SO(n)$, also appear as group-theoretical factors in the high-temperature expansion of $SO(n)$-symmetric classical lattice models [3, 4] such as the XY-model ($n = 2$) and the Heisenberg model ($n = 3$). For example, the so-called α-graph contributes with the following group-theoretical factor to the high-temperature expansion of the free energy of such models [3, 4, 1].

$$\begin{aligned} I_n &\equiv I_n(\ell_1,\ell_2,\ell_3|\ell_4,\ell_5,\ell_6) \\ &:= \int\limits_{SO(n)} \mathrm{d}g_1 \int\limits_{SO(n)} \mathrm{d}g_2 \int\limits_{SO(n)} \mathrm{d}g_3\, \mathcal{D}^{\ell_1}_{00}(g_1)\mathcal{D}^{\ell_2}_{00}(g_2)\mathcal{D}^{\ell_3}_{00}(g_3)\mathcal{D}^{\ell_4}_{00}(g_2^{-1}g_3)\mathcal{D}^{\ell_5}_{00}(g_3^{-1}g_1)\mathcal{D}^{\ell_6}_{00}(g_1^{-1}g_2) \end{aligned} \qquad (1)$$

Here $\mathcal{D}^{\ell}_{00}(g)$ denotes a particular matrix element (the zonal spherical function) of the ℓth unitary irreducible class-one representation of $SO(n)$, $\ell \in \mathbb{N}_0$, and $\mathrm{d}g$ is the normalised

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invariant Haar measure on $SO(n)$. For details we refer to our earlier work [1]. Here we only note the relation of the above integral with the $6j$-symbols of $SO(n)$:

$$\begin{aligned} I_n = \;& (-1)^{\ell_4+\ell_5+\ell_6} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}_{(n)} \begin{pmatrix} \ell_1 & \ell_5 & \ell_6 \\ 0 & 0 & 0 \end{pmatrix}_{(n)} \begin{pmatrix} \ell_4 & \ell_2 & \ell_6 \\ 0 & 0 & 0 \end{pmatrix}_{(n)} \\ & \times \begin{pmatrix} \ell_3 & \ell_4 & \ell_5 \\ 0 & 0 & 0 \end{pmatrix}_{(n)} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \end{Bmatrix}_{(n)} , \end{aligned} \tag{2}$$

where

$$\begin{aligned} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2_{(n)} &:= \int\limits_{SO(n)} \mathrm{d}g\, \mathcal{D}^{\ell_1}_{00}(g)\mathcal{D}^{\ell_2}_{00}(g)\mathcal{D}^{\ell_3}_{00}(g) \\ &= \frac{\Gamma(J+n-2)}{\Gamma^2(n/2)\Gamma(n-2)\Gamma(J+n/2)} \prod_{i=1}^{3} \left[\frac{\Gamma(n-1)\Gamma(\ell_i+1)\Gamma(J-\ell_i+\frac{n-2}{2})}{2\,\Gamma(\ell_i+n-2)\Gamma(J-\ell_i+1)}\right] \end{aligned} \tag{3}$$

denotes the square of a $3j$-symbol which vanishes unless $J := (\ell_1 + \ell_2 + \ell_3)/2$ is a non-negative integer, $J \in \mathbb{N}_0$, and the ℓ's obey the triangular relation well-known from the case $n = 3$. This result, in essence, goes back to an earlier one of Vilenkin, ref [6] equation (6) on page 490. For the derivation of (3) see equations (21)-(24) in [1] where also a phase convention for the $3j$-symbol is given. This together with an explicit expression for I_n then also leads to a closed-form expression for the $6j$-symbol which is denoted with curly brackets in (2).

The purpose of this paper is to present an explicit closed-form expression for the above group integral (1) for arbitrary values of the ℓ's. So far only particular results have been known. For example, for arbitrary n and $(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) = (1, 1, 2, 1, 1, 2)$ an explicit expression has been given by Domb [4], the case $\ell_4 = 0$ can be found in [1] and, rather recently, some results have been given for the cases where one of the ℓ's equals one or two [5].

The remaining part of this paper deals with the derivation of an explicit closed-form expression for $I_n(\ell_1, \ell_2, \ell_3|\ell_4, \ell_5, \ell_6)$, which is given below in (10) in combination with (6), (14) and (15). Together with the above expression for the $3j$-symbol we have thus also obtained a closed-form expression for the corresponding $6j$-symbol. We finally present some explicit results for arbitrary n and $\ell_i \in \{1, 2, 3, 4\}$ after briefly discussing the symmetry properties of $I_n(\ell_1, \ell_2, \ell_3|\ell_4, \ell_5, \ell_6)$.

2. Explicit integration of (1)

In this section we will make extensive use of our previous results [1]. In the following if we refer to equations of ref [1] we will add the subscript 1 to the equation number. For example, $(18)^1$ refers to eq (18) of [1] which shows that the zonal spherical functions can be expressed in terms of Gegenbauer polynomials. In fact, using this relation the integral (1) may be rewritten as follows:

$$\begin{aligned} I_n = & \left[\prod_{i=1}^{6} \frac{\Gamma(\ell_i+1)\Gamma(n-2)}{\Gamma(\ell_i+n-2)}\right] \int\limits_{S^{n-1}} \frac{\mathrm{d}^{n-1}\boldsymbol{e}_1}{|S^{n-1}|} \int\limits_{S^{n-1}} \frac{\mathrm{d}^{n-1}\boldsymbol{e}_2}{|S^{n-1}|} \int\limits_{S^{n-1}} \frac{\mathrm{d}^{n-1}\boldsymbol{e}_3}{|S^{n-1}|} \\ & \times C_{\ell_1}^{(n-2)/2}(\boldsymbol{a}\cdot\boldsymbol{e}_1)\, C_{\ell_2}^{(n-2)/2}(\boldsymbol{a}\cdot\boldsymbol{e}_2)\, C_{\ell_3}^{(n-2)/2}(\boldsymbol{a}\cdot\boldsymbol{e}_3) \\ & \times C_{\ell_4}^{(n-2)/2}(\boldsymbol{e}_2\cdot\boldsymbol{e}_3)\, C_{\ell_5}^{(n-2)/2}(\boldsymbol{e}_3\cdot\boldsymbol{e}_1)\, C_{\ell_6}^{(n-2)/2}(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)\ . \end{aligned} \tag{4}$$

Here and in the following we will use the same notation as in [1]. Denoting with θ_i the polar angle of the unit vector $\boldsymbol{e}_i \in S^{n-1}$ we have $\boldsymbol{e}_i = (\sin\theta_i \boldsymbol{f}_i, \cos\theta_i)$ with $\boldsymbol{f}_i \in S^{n-2}$. Using $\boldsymbol{a}\cdot\boldsymbol{e}_i = \cos\theta_i$ and the addition theorem for Gegenbauer polynomials [7]

$$\begin{aligned} C_\ell^{n/2-1}(\boldsymbol{e}_i\cdot\boldsymbol{e}_j) = & \sum_{m=0}^{\ell} a(n,\ell,m)\, \sin^m\theta_i\, C_{\ell-m}^{m+n/2-1}(\cos\theta_i)\, \sin^m\theta_j\, C_{\ell-m}^{m+n/2-1}(\cos\theta_j) \\ & \times C_m^{(n-3)/2}(\boldsymbol{f}_i\cdot\boldsymbol{f}_j)\ , \end{aligned} \tag{5}$$

where we have set

$$a(n,\ell,m) := \frac{2^{2m}\Gamma(n-3)\Gamma(\ell-m+1)\Gamma^2(m+n/2-1)}{\Gamma^2(n/2-1)\Gamma(\ell+m+n-2)}\,(2m+n-3)\ , \tag{6}$$

the above integrations can be factorised into those over the polar angles and the remaining integrals over S^{n-2}. For this we have also to make use of $(39)^1$ in the form

$$\int\limits_{S^{n-1}} \frac{\mathrm{d}^{n-1}\boldsymbol{e}}{|S^{n-1}|}\,(\cdot) = \frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)} \int\limits_0^\pi \mathrm{d}\theta \sin^{n-2}\theta \int\limits_{S^{n-2}} \frac{\mathrm{d}^{n-2}\boldsymbol{f}}{|S^{n-2}|}\,(\cdot)\ . \tag{7}$$

The part of (4) which involves the $\boldsymbol{f}$-integrations reads $(m_i = 0,1,\ldots,\ell_{3+i})$

$$\begin{aligned} F_n := & \int\limits_{S^{n-2}} \frac{\mathrm{d}^{n-2}\boldsymbol{f}_1}{|S^{n-2}|} \int\limits_{S^{n-2}} \frac{\mathrm{d}^{n-2}\boldsymbol{f}_2}{|S^{n-2}|} \int\limits_{S^{n-2}} \frac{\mathrm{d}^{n-2}\boldsymbol{f}_3}{|S^{n-2}|} \\ & \times C_{m_1}^{(n-3)/2}(\boldsymbol{f}_2\cdot\boldsymbol{f}_3)\ C_{m_2}^{(n-3)/2}(\boldsymbol{f}_3\cdot\boldsymbol{f}_1)\ C_{m_3}^{(n-3)/2}(\boldsymbol{f}_1\cdot\boldsymbol{f}_2) \\ = & \prod_{i=1}^{3}\left[\frac{\Gamma(m_i+n-3)}{\Gamma(m_i+1)\Gamma(n-3)}\right] \\ & \times \int\limits_{SO(n-1)} \mathrm{d}h_1 \int\limits_{SO(n-1)} \mathrm{d}h_2 \int\limits_{SO(n-1)} \mathrm{d}h_3\ D_{00}^{m_1}(h_2^{-1}h_3)\ D_{00}^{m_2}(h_3^{-1}h_1)\ D_{00}^{m_3}(h_1^{-1}h_2) \end{aligned} \tag{8}$$

where D_{00}^m denotes zonal spherical functions of the subgroup $SO(n-1)$. These group integrations are easily performed via the orthogonality relation for the $SO(n-1)$ matrix elements D_{00}^m, cf $(12)^1$. As a consequence all m_i's have to be equal, $m \equiv m_1 = m_2 = m_3 = 0,1,2,\ldots,\min\{\ell_4,\ell_5,\ell_6\}$, and the result reads

$$F_n = \sum_{m=0}^{\min\{\ell_4,\ell_5,\ell_6\}} \delta_{mm_1}\delta_{mm_2}\delta_{mm_3}\, \frac{\Gamma(m+n-3)}{\Gamma(m+1)\Gamma(n-3)} \left(\frac{n-3}{2m+n-3}\right)^2\ . \tag{9}$$

With the help of this result we are now able to put the quantity of our interest into the form

$$\begin{aligned} I_n = & \left[\prod_{i=1}^{6} \frac{\Gamma(\ell_i+1)\Gamma(n-2)}{\Gamma(\ell_i+n-2)}\right] \left(\frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)}\right)^3 \\ & \times \sum_{m=0}^{\min\{\ell_4,\ell_5,\ell_6\}} \frac{\Gamma(m+n-3)}{\Gamma(m+1)\Gamma(n-3)} \left(\frac{n-3}{2m+n-3}\right)^2 \left[\prod_{i=4}^{6} a(n,\ell_i,m)\right] \\ & \times G_n(\ell_1,\ell_5,\ell_6,m)\, G_n(\ell_2,\ell_4,\ell_6,m)\, G_n(\ell_3,\ell_4,\ell_5,m) \end{aligned} \tag{10}$$

and thus have reduced it to three elementary integrals of the form

$$\begin{aligned} G_n(j_1,j_2,j_3,m) := & \int_0^{\pi} \mathrm{d}\theta\, \sin^{2m+n-2}\theta \\ & \times C_{j_1}^{n/2-1}(\cos\theta)\, C_{j_2-m}^{m+n/2-1}(\cos\theta)\, C_{j_3-m}^{m+n/2-1}(\cos\theta)\,. \end{aligned} \tag{11}$$

This integral is a special case of a class of integrals already studied in [1], cf $(41)^1$, where we have been able to represent such integrals by three finite sums. However, because of its special form we have decided to evaluate (11) in a different way. In doing so we first recall the recurrence relation [8] for the Gegenbauer polynomials

$$C_j^{\lambda}(x) = \frac{\lambda}{j+\lambda}\left(C_j^{\lambda+1}(x) - C_{j-2}^{\lambda+1}(x)\right)\,, \tag{12}$$

which is also valid for $j=0,1$ if we use the convention that Gegenbauer polynomials with a "negative degree" (the lower index) vanish identically. Iterating this recurrence relation m times we find

$$C_j^{n/2-1}(\cos\theta) = \sum_{k=0}^{\min\{m,[j/2]\}} (-1)^k\, b(n,j,k,m)\, C_{j-2k}^{m+n/2-1}(\cos\theta) \tag{13}$$

where we have introduced

$$b(n,j,k,m) := \frac{\Gamma(m+1)\,\Gamma(m+\frac{n-2}{2})\,\Gamma(j-k+\frac{n-2}{2})}{\Gamma(k+1)\,\Gamma(m-k+1)\,\Gamma(\frac{n-2}{2})\,\Gamma(j+m-k+\frac{n}{2})}\,(j+m-2k+\tfrac{n-2}{2})\,. \tag{14}$$

Replacing now the first Gegenbauer polynomial in (11) with the help of (13) we realize that the integral (11) represents in essence a $3j$-symbol of the group $SO(2m+n)$, cf $(21)^1$:

$$\begin{aligned} G_n(j_1,j_2,j_3,m) = & \frac{\sqrt{\pi}\,\Gamma(m+\frac{n-1}{2})}{\Gamma(m+\frac{n}{2})\,\Gamma^3(2m+n-2)} \sum_{k=0}^{\min\{m,[j_1/2]\}} (-1)^k\, b(n,j_1,k,m) \\ & \times \frac{\Gamma(j_1-2k+2m+n-2)\,\Gamma(j_2+m+n-2)\,\Gamma(j_3+m+n-2)}{\Gamma(j_1-2k+1)\,\Gamma(j_2-m+1)\,\Gamma(j_3-m+1)} \\ & \times \begin{pmatrix} j_1-2k & j_2-m & j_3-m \\ 0 & 0 & 0 \end{pmatrix}^2_{(n+2m)}\,. \end{aligned} \tag{15}$$

Thus we have succeeded in expressing the integral (11) by a single finite sum and in turn found a closed-form expression for the integral (1) in terms of four finite sums and $3j$-symbols of $SO(2m+n)$ with $m=0,1,\ldots,\min\{\ell_4,\ell_5,\ell_6\}$. Expression (10) together with (6), (14) and (15) thus provides us with an explicit formula for $I_n(\ell_1,\ell_2,\ell_3|\ell_4,\ell_5,\ell_6)$, which is fairly simple and can easily be evaluated using, for example, some computer-algebra program like *Mathematica*.§

3. Discussion

In this section we will briefly discuss the symmetry properties of $I_n(\ell_1,\ell_2,\ell_3|\ell_4,\ell_5,\ell_6)$ and the corresponding $6j$-symbol. First we note that the $3j$-symbol (3) is obviously invariant under any permutation of the ℓ's. In addition we note that because of the first $3j$-symbol appearing on the right-hand side of (2) the phase factor in front of it may be replaced by $(-1)^{\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6}$ as $\ell_1+\ell_2+\ell_3$ is required to be an even integer. As a consequence $I_n(\ell_1,\ell_2,\ell_3|\ell_4,\ell_5,\ell_6)$ and the $6j$-symbol have identical symmetry properties. Using the invariance property of the Haar measure in (1) one easily verifies that

$$\begin{aligned}\left\{\begin{matrix}\ell_1 & \ell_2 & \ell_3\\ \ell_4 & \ell_5 & \ell_6\end{matrix}\right\}_{(n)} &= \left\{\begin{matrix}\ell_2 & \ell_3 & \ell_1\\ \ell_5 & \ell_6 & \ell_4\end{matrix}\right\}_{(n)} = \left\{\begin{matrix}\ell_3 & \ell_1 & \ell_2\\ \ell_6 & \ell_4 & \ell_5\end{matrix}\right\}_{(n)} = \left\{\begin{matrix}\ell_2 & \ell_1 & \ell_3\\ \ell_5 & \ell_4 & \ell_6\end{matrix}\right\}_{(n)} \\ &= \left\{\begin{matrix}\ell_1 & \ell_3 & \ell_2\\ \ell_4 & \ell_6 & \ell_5\end{matrix}\right\}_{(n)} = \left\{\begin{matrix}\ell_3 & \ell_2 & \ell_1\\ \ell_6 & \ell_5 & \ell_4\end{matrix}\right\}_{(n)} = \left\{\begin{matrix}\ell_1 & \ell_5 & \ell_6\\ \ell_4 & \ell_2 & \ell_3\end{matrix}\right\}_{(n)} .\end{aligned} \tag{16}$$

These are indeed the well-known [9] symmetries of the $6j$-symbols for the group $SO(3)$ which are thus shown to be valid for all $n \geq 3$ if class-one representations are considered only. The additional Regge symmetry [10] known for the case $n=3$ cannot be verified by these methods and thus it is not clear whether it holds for arbitrary $n>3$. In combination with these symmetry properties Table 1 presents for all $\ell_i \in \{1,2,3,4\}$ explicit values for $I_n(\ell_1,\ell_2,\ell_3|\ell_4,\ell_5,\ell_6)$ and

$$c^{(\alpha)}_{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6} := d_{\ell_1}\, d_{\ell_2}\, d_{\ell_3}\, d_{\ell_4}\, d_{\ell_5}\, d_{\ell_6}\, I_n(\ell_1,\ell_2,\ell_3|\ell_4,\ell_5,\ell_6)\ , \tag{17}$$

where d_ℓ denotes the dimension of the ℓth representation and is given in $(8)^1$. Note that the quantity (17) is the actual contribution of the α-topology to the high-temperature expansion of $SO(n)$-symmetric lattice models [4]. Thus with the tabulated quantities (17) one can derive high-temperature expansions for $SO(n)$-symmetric lattice models to rather high order in the inverse temperature. For example, with only a few of the tabulated values of (17) one can find all expansion coefficients up to order ten for the free energy [11] of a mixed isovector-isotensor model which recently has attracted much attention [5, 12].

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§ A *Mathematica* package which implements the results of this paper can be obtained from the authors upon request.

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Table 1. Explicit expressions for the non-vanishing integrals $I_n(\ell_1,\ell_2,\ell_3|\ell_4,\ell_5,\ell_6)$ defined in (1) and the corresponding quantity $c^{(\alpha)}_{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6}$ defined in (17).

$(\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6)$	$I_n(\ell_1,\ell_2,\ell_3\|\ell_4,\ell_5,\ell_6)$	$c^{(\alpha)}_{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6}$
(1 1 2 1 1 2)	$\frac{4\ (n-2)}{(n-1)\ n^3\ (n+2)^3}$	$\frac{(n-2)\ (n-1)\ n}{n+2}$
(1 1 2 1 3 2)	$\frac{24}{(n-1)^2\ n\ (n+2)^3\ (n+4)}$	$\frac{(n-1)\ n^3}{n+2}$
(1 1 2 2 2 1)	$\frac{8\ (n-2)}{(n-1)^2\ n^2\ (n+2)^3}$	$(n-2)\ (n-1)\ n$
(1 1 2 2 2 3)	$\frac{48\ (n-2)}{(n-1)^3\ n\ (n+2)^3\ (n+4)^2}$	$\frac{(n-2)\ (n-1)\ n^2}{n+4}$
(1 1 2 2 4 3)	$\frac{288}{(n-1)^3\ n\ (n+2)^2\ (n+4)^2\ (n+6)}$	$\frac{(n-1)\ n^3\ (n+1)}{2\ (n+4)}$
(1 1 2 3 3 2)	$\frac{72\ (n-2)\ (n+1)}{(n-1)^3\ n^2\ (n+2)^3\ (n+4)^2}$	$\frac{(n-2)\ (n-1)\ n^2\ (n+1)}{2\ (n+2)}$
(1 1 2 3 3 4)	$\frac{864\ (n-2)}{(n-1)^3\ n^3\ (n+2)\ (n+4)^2\ (n+6)^2}$	$\frac{(n-2)\ (n-1)\ n^2\ (n+1)}{2\ (n+6)}$
(1 1 2 4 4 3)	$\frac{1152\ (n-2)}{(n-1)^3\ n^3\ (n+1)\ (n+4)^2\ (n+6)^2}$	$\frac{(n-2)\ (n-1)\ n^2\ (n+1)\ (n+2)}{6\ (n+4)}$
(1 2 3 1 2 3)	$\frac{72\ (n-2)}{(n-1)^3\ n\ (n+2)^3\ (n+4)^3}$	$\frac{(n-2)\ (n-1)\ n^3}{2\ (n+2)\ (n+4)}$
(1 2 3 1 4 3)	$\frac{864}{(n-1)^3\ n^2\ (n+2)\ (n+4)^3\ (n+6)}$	$\frac{(n-1)\ n^3\ (n+1)}{2\ (n+4)}$
(1 2 3 2 3 2)	$\frac{288\ (n-2)\ (n+1)}{(n-1)^4\ n\ (n+2)^3\ (n+4)^3}$	$\frac{(n-2)\ (n-1)\ n^2\ (n+1)}{n+4}$
(1 2 3 2 3 4)	$\frac{1728\ (n-2)}{(n-1)^4\ n\ (n+2)^2\ (n+4)^3\ (n+6)^2}$	$\frac{(n-2)\ (n-1)\ n^3\ (n+1)}{2\ (n+4)\ (n+6)}$
(1 2 3 3 2 3)	$\frac{864\ (n-2)\ (n+1)}{(n-1)^4\ n\ (n+2)^3\ (n+4)^3\ (n+6)}$	$\frac{(n-2)\ (n-1)\ n^3\ (n+1)}{(n+2)\ (n+6)}$
(1 2 3 3 4 3)	$\frac{5184\ (n-2)}{(n-1)^4\ n^2\ (n+2)\ (n+4)^3\ (n+6)^2}$	$\frac{(n-2)\ (n-1)\ n^3\ (n+1)}{2\ (n+6)}$
(1 2 3 4 3 2)	$\frac{864\ (n-2)}{(n-1)^4\ n\ (n+2)^2\ (n+4)^3\ (n+6)}$	$\frac{(n-2)\ (n-1)\ n^3\ (n+1)}{4\ (n+4)}$
(1 2 3 4 3 4)	$\frac{20736\ (n-2)}{(n-1)^4\ n^2\ (n+1)\ (n+4)^3\ (n+6)^2\ (n+8)}$	$\frac{(n-2)\ (n-1)\ n^3\ (n+1)\ (n+2)}{2\ (n+4)\ (n+8)}$
(1 3 4 1 3 4)	$\frac{3456\ (n-2)}{(n-1)^3\ n^3\ (n+1)\ (n+4)^3\ (n+6)^3}$	$\frac{(n-2)\ (n-1)\ n^3\ (n+1)}{6\ (n+4)\ (n+6)}$
(1 3 4 2 4 3)	$\frac{20736\ (n-2)\ (n+2)}{(n-1)^4\ n^3\ (n+1)\ (n+4)^3\ (n+6)^3}$	$\frac{(n-2)\ (n-1)\ n^2\ (n+1)\ (n+2)^2}{2\ (n+4)\ (n+6)}$
(1 3 4 3 3 4)	$\frac{62208\ (n-2)}{(n-1)^4\ n^2\ (n+1)\ (n+4)^3\ (n+6)^3\ (n+8)}$	$\frac{(n-2)\ (n-1)\ n^4\ (n+1)}{2\ (n+6)\ (n+8)}$
(1 3 4 4 4 3)	$\frac{124416\ (n-2)\ (n+3)}{(n-1)^4\ n^2\ (n+1)^2\ (n+4)^3\ (n+6)^3\ (n+8)}$	$\frac{(n-2)\ (n-1)\ n^4\ (n+1)\ (n+3)}{4\ (n+4)\ (n+8)}$
(2 2 2 2 2 2)	$\frac{64\ (n-2)\ (n^2+4n-24)}{(n-1)^5\ (n+2)^3\ (n+4)^3}$	$\frac{(n-2)\ (n-1)\ (n+2)^3\ (n^2+4n-24)}{(n+4)^3}$
(2 2 2 2 2 4)	$\frac{768\ (n-2)\ n}{(n-1)^5\ (n+2)^3\ (n+4)^3\ (n+6)}$	$\frac{(n-2)\ (n-1)\ n^2\ (n+1)\ (n+2)^2}{(n+4)^3}$
(2 2 2 2 4 4)	$\frac{4608\ (n-2)}{(n-1)^5\ (n+1)\ (n+2)\ (n+4)^3\ (n+6)^2}$	$\frac{(n-2)\ (n-1)\ n^2\ (n+1)\ (n+2)^3}{2\ (n+4)^3}$

Table 1. *cont.*

$(\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6)$	$I_n(\ell_1,\ell_2,\ell_3\vert\ell_4,\ell_5,\ell_6)$
(2 2 2 3 3 3)	$\frac{864\ (n-2)\ (n+1)\ (n^3+8n^2-28n-48)}{(n-1)^5\ n^2\ (n+2)^3\ (n+4)^3\ (n+6)^2}$
(2 2 2 4 4 4)	$\frac{18432\ (n-2)\ (n^3+12n^2-24n-128)}{(n-1)^5\ n^2\ (n+1)^2\ (n+4)^3\ (n+6)^2\ (n+8)^2}$
(2 2 4 2 2 4)	$\frac{2304\ (n-2)\ n^2}{(n-1)^5\ (n+1)\ (n+2)^3\ (n+4)^3\ (n+6)^3}$
(2 2 4 2 4 4)	$\frac{55296\ (n-2)}{(n-1)^5\ (n+1)^2\ (n+4)^3\ (n+6)^3\ (n+8)}$
(2 2 4 3 3 3)	$\frac{20736\ (n-2)}{(n-1)^5\ (n+2)^2\ (n+4)^3\ (n+6)^3}$
(2 2 4 4 4 2)	$\frac{13824\ (n-2)\ n\ (n+3)}{(n-1)^5\ (n+1)^2\ (n+2)^2\ (n+4)^3\ (n+6)^3}$
(2 2 4 4 4 4)	$\frac{663552\ (n-2)\ (n+3)}{(n-1)^5\ (n+1)^3\ (n+4)^3\ (n+6)^3\ (n+8)^2}$
(2 3 3 2 3 3)	$\frac{2592\ (n-2)\ (n+1)\ (2n^4+17n^3-14n^2-84n-72)}{(n-1)^5\ n^3\ (n+2)^3\ (n+4)^3\ (n+6)^3}$
(2 3 3 3 4 4)	$\frac{124416\ (n-2)\ (n^3+10n^2-20n-48)}{(n-1)^5\ n^3\ (n+1)\ (n+4)^3\ (n+6)^3\ (n+8)^2}$
(2 3 3 4 3 3)	$\frac{15552\ (n-2)\ (n^3+11n^2-48n-36)}{(n-1)^5\ n^3\ (n+2)\ (n+4)^3\ (n+6)^3\ (n+8)}$
(2 4 4 2 4 4)	$\frac{221184\ (n-2)\ (n+2)\ (3n^4+40n^3+72n^2-192n-512)}{(n-1)^5\ n^3\ (n+1)^3\ (n+4)^3\ (n+6)^3\ (n+8)^3}$
(2 4 4 4 4 4)	$\frac{3981312\ (n-2)\ (n+2)\ (n+3)\ (n^3+14n^2-16n-128)}{(n-1)^5\ n^2\ (n+1)^4\ (n+4)^3\ (n+6)^3\ (n+8)^3\ (n+10)}$
(3 3 4 3 3 4)	$\frac{186624\ (n-2)\ (4n^2+37n-50)}{(n-1)^5\ n^2\ (n+1)\ (n+4)^3\ (n+6)^3\ (n+8)^3}$
(3 3 4 4 4 3)	$\frac{373248\ (n-4)\ (n-2)\ (n+3)\ (n+20)}{(n-1)^5\ n^2\ (n+1)^2\ (n+4)^3\ (n+6)^3\ (n+8)^3}$
(4 4 4 4 4 4)	$\frac{11943936\ (n-2)\ (n+3)\ (n^6+43n^5+400n^4-212n^3-6752n^2-5888n+15360)}{(n-1)^5\ n^2\ (n+1)^5\ (n+4)^3\ (n+6)^3\ (n+8)^3\ (n+10)^3}$

Table 1. *cont.*

$(\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6)$	$c^{(\alpha)}_{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6}$
(2 2 2 3 3 3)	$\frac{(n-2)\ (n-1)\ n\ (n+1)\ (n^3+8n^2-28n-48)}{2\ (n+6)^2}$
(2 2 2 4 4 4)	$\frac{(n-2)\ (n-1)\ n\ (n+1)\ (n+2)^3\ (n+6)\ (n^3+12n^2-24n-128)}{6\ (n+4)^3\ (n+8)^2}$
(2 2 4 2 2 4)	$\frac{(n-2)\ (n-1)\ n^4\ (n+1)\ (n+2)}{4\ (n+4)^3\ (n+6)}$
(2 2 4 2 4 4)	$\frac{(n-2)\ (n-1)\ n^3\ (n+1)\ (n+2)^3}{2\ (n+4)^3\ (n+8)}$
(2 2 4 3 3 3)	$\frac{(n-2)\ (n-1)\ n^4\ (n+1)}{(n+6)^2}$
(2 2 4 4 4 2)	$\frac{(n-2)\ (n-1)\ n^4\ (n+1)\ (n+2)\ (n+3)}{8\ (n+4)^3}$
(2 2 4 4 4 4)	$\frac{(n-2)\ (n-1)\ n^4\ (n+1)\ (n+2)^2\ (n+3)\ (n+6)}{2\ (n+4)^3\ (n+8)^2}$
(2 3 3 2 3 3)	$\frac{(n-2)\ (n-1)\ n\ (n+1)\ (n+4)\ (2n^4+17n^3-14n^2-84n-72)}{2\ (n+2)\ (n+6)^3}$
(2 3 3 3 4 4)	$\frac{(n-2)\ (n-1)\ n^2\ (n+1)\ (n+2)\ (n^3+10n^2-20n-48)}{2\ (n+6)\ (n+8)^2}$
(2 3 3 4 3 3)	$\frac{(n-2)\ (n-1)\ n^2\ (n+1)\ (n+4)\ (n^3+11n^2-48n-36)}{4\ (n+6)^2\ (n+8)}$
(2 4 4 2 4 4)	$\frac{(n-2)\ (n-1)\ n\ (n+1)\ (n+2)^3\ (n+6)\ (3n^4+40n^3+72n^2-192n-512)}{6\ (n+4)^3\ (n+8)^3}$
(2 4 4 4 4 4)	$\frac{(n-2)\ (n-1)\ n^3\ (n+1)\ (n+2)^2\ (n+3)\ (n+6)^2\ (n^3+14n^2-16n-128)}{4\ (n+4)^3\ (n+8)^3\ (n+10)}$
(3 3 4 3 3 4)	$\frac{(n-2)\ (n-1)\ n^4\ (n+1)\ (n+4)\ (4n^2+37n-50)}{4\ (n+6)\ (n+8)^3}$
(3 3 4 4 4 3)	$\frac{(n-4)\ (n-2)\ (n-1)\ n^4\ (n+1)\ (n+3)\ (n+20)}{8\ (n+8)^3}$
(4 4 4 4 4 4)	$\frac{(n-2)\ (n-1)\ n^4\ (n+1)\ (n+3)\ (n+6)^3\ (n^6+43n^5+400n^4-212n^3-6752n^2-5888n+15360)}{16\ (n+4)^3\ (n+8)^3\ (n+10)^3}$