MICROLOCAL ANALYSIS AND INTERACTING QUANTUM FIELD THEORIES: RENORMALIZATION ON PHYSICAL BACKGROUNDS

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Dedicated to the memory of Professor Roberto Stroffolini

Abstract. We present a perturbative construction of interacting quantum field theories on smooth globally hyperbolic (curved) space-times. We develop a purely local version of the Stückelberg-Bogoliubov-Epstein-Glaser method of renormalization by using techniques from microlocal analysis. Relying on recent results of Radzikowski, Köhler and the authors about a formulation of a local spectrum condition in terms of wave front sets of correlation functions of quantum fields on curved space-times, we construct time-ordered operator-valued products of Wick polynomials of free fields. They serve as building blocks for a local (perturbative) definition of interacting fields. Renormalization in this framework amounts to extensions of expectation values of time-ordered products to all points of space-time. The extensions are classified according to a microlocal generalization of Steinmann scaling degree corresponding to the degree of divergence in other renormalization schemes. As a result, we prove that the usual perturbative classification of interacting quantum field theories holds also on curved space-times. Finite renormalizations are deferred to a subsequent paper.

As byproducts, we describe a perturbative construction of local algebras of observables, present a new definition of Wick polynomials as operator-valued distributions on a natural domain, and we find a general method for the extension of distributions which were defined on the complement of some surface.

Contents

1. Introduction

The quest of the existence of a non trivial quantum field theory in four space-time dimensions is still without any conclusive result. Nonetheless, physicists are working daily, with success, on concrete models which describe very efficiently physics at wide energy scales. This description is based on expansions of physical quantities like amplitudes of a scattering process in terms of power series of "physical" parameters, as coupling constants, masses, charges. The higher order terms of these power series are usually ill defined, in a naive approach, but physicists have learned soon how to make sense out of them through the procedure now known as *renormalization* [51]. The rigorous extension of this procedure to *curved Lorentzian* space-times will be the main topic of this paper. The question whether the power series approximate the corresponding quantities in a full quantum field theory goes beyond the scope of this paper and will not be touched.

The renormalization procedure on Minkowski space-time led to impressive results in the case of quantum electrodynamics [38, 62], where observable quantities were calculated and agree with high precision with the experimental values [42]. Based on this example, a general method of renormalization of interacting fields was found and successfully applied to the standard models of elementary particles.

There is another approach to quantum field theory ("axiomatic quantum field theory") which assumes the existence of a class of models satisfying certain first principles. Under this assumption several structural properties could be derived which therefore hold for every model in this class. To name some, CPT and Spin-Statistics Theorems are among the main successes of this line of thought, and nowadays the application of these methods to specific kind of theories, like conformally covariant theories on low dimensional spaces, is expected to give new insights. Nevertheless, all these schemes (see for instance [12] for a recent survey), either analytic [55] or algebraic [29], by now seem to have missed the challenge for the concreteness needed by, say, particle physicists.

A notable exception is the rigorous formulation of perturbation theory [34, 20, 53, 64] which may be considered as interpolating between the world of phenomenological physics and the mathematical schemes mentioned above. This point of view has been pioneered, in particular, by K. Hepp [34], who gave solid foundations to perturbation theory for quantum field theories on Minkowski space-time. This philosophy proved to be correct for instance in constructive quantum field theory [26] where rigorous renormalization ideas were used as fundamental inputs.

One of the aims of the authors is to put forward a formulation of perturbation theory which satisfies the needs of axiomatic field theory, much in the sense of [53], and is at the same time applicable to phenomenology. In distinction to earlier approaches we give a purely local formulation which is meaningful also on curved space-times.

Our principal result is:

Main Theorem . *All polynomially interacting (scalar) quantum field theories on smooth curved globally hyperbolic space-times of dimension four follow the same perturbative classification as on Minkowski space-time.*

Before starting the description of our claim we continue the description of the interplay between perturbation theory and rigorous methods.

One of the most puzzling things in physics is that all the attempts to include gravity in the renormalization program failed: More recent proposals look for theories of a different kind like string theory [28] or its generalizations which are hoped to describe all known forces, or Ashtekar program [1].

Because of the large difference between the Planck scale ($\approx 10^{-33}$ cm), where "quantum gravity" effects are expected to become important, and the scale relevant for the Standard Model [31] ($\approx 10^{-17}$ cm), a reasonable approximation should be to consider gravity as a *classical* background field and therefore investigate quantum field theory in curved space-time. This Ansatz already led to interesting results, the most famous being the Hawking radiation of black-holes [32]. A look through the literature (see, e.g., [60, 3, 25]), however, shows that predominantly *free* field theories were treated on curved physical backgrounds. In fact, most of the papers on interacting quantum field theories on curved space-times deal with the (locally) Euclidean case and discuss renormalization only for particular Feynman diagrams. We are aware of only one attempt to a complete proof of renormalizability, that given by Bunch [13] for the case of $\lambda\varphi^4$-model. His attempt was however confined to the rather special case of real analytic space-times which allow an analytic continuation to the (locally) Euclidean situation. It is interesting to note that the main technical tool of his paper is a kind of local Fourier transformation and that some of his mathematical claims can be justified in the framework of Hadamard "parametrices," both of which belong to the powerful techniques of microlocal analysis that we use in this paper.

The situation is then uncertain for general smooth space-times with the Lorentzian signature for the metric. Here, to our knowledge, more or less nothing has been done.

Why is the problem of renormalization so difficult on curved space-times?

Again the precise perspective gained from the rigorous approach is helpful. The main problem is the absence of translation invariance which in the rigorous schemes plays a decisive rôle. In general, no global (time-like) Killing vector field exists (no energy-momentum operator), so there is no canonical notion of a vacuum state, which is a central object in most formulations of quantum field theory; the spectrum condition (positivity of the energy-momentum operator) can not be formulated. There is no general connection between quantum field theories on Riemannian and Lorentzian space-times corresponding to the Osterwalder-Schrader Theorem [47], and the meaning of path integrals for quantum field theories on curved space-times is unclear. As a result, the rigorous frameworks described before cannot simply be generalized, and the more formal description based on Euclidean ideas and path integrals does not help much.

On the other hand, physically motivated by Einstein Equivalence Principle [59], a quick look at the possible ultraviolet (short distance) divergences indicates that they are of the same nature as on Minkowski space, so no real obstruction for renormalization on curved space-times is visible. Despite of the interest in its own right, renormalization on curved space-time might also trigger a conceptual revisitation of renormalization theory on Minkowski space in the light of the principle of locality [18].

To develop perturbation theory in a form which is suitable to extensions to curved Lorentzian space-times, we mainly rely on a construction given by Epstein and Glaser [20] at the beginning of the seventies and on some improvements suggested later by Stora [54]. This construction makes precise older ideas of Stückelberg [56] and Bogoliubov [5]. In spite of its elegance it was widely ignored (compare, e.g., its neglection in several books on quantum field theory, with the exception of [38]). Recently, it was further developed and applied to gauge field theories by Scharf and his collaborators [50] [1] (after earlier work by [4]).

We offer some intuitive explanations of the ideas behind the approach of Stückelberg-Bogoliubov-Epstein-Glaser (we refrain from using an acronym for this and simply call it the Epstein and Glaser approach). For simplicity we discuss the case of flat Minkowskian space-time.

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[1] Note that the term "finite" in Scharf's book refers to the fact that in the Epstein and Glaser approach (as in the similar BPHZ method) no regularization is necessary. It does not mean that the indeterminacy connected with the divergence of naive perturbation theory disappears.

The basic idea is that, in the asymptotic past and future, the interacting quantum fields approach, in a sense to be specified, *free* fields, i.e. fields satisfying linear hyperbolic equations of motion. For free fields there exists a precise construction which can be used for a perturbative description of interacting fields. Now, in a translationally invariant theory, the interacting fields approach the asymptotic free fields only in a rather weak sense (LSZ-asymptotic condition [34]). Moreover, Haag Theorem [29] forbids the construction of interacting fields in the vacuum Hilbert space of the time-0 free fields. In the Epstein and Glaser scheme, these problems are, in a first step, circumvented by choosing interactions which take place only in a bounded region of space-time. Then the scattering operator can be defined in the interaction picture as the time evolution operator from the past, before the interaction was switched on, to the future, after the interaction was switched off.

A localized interaction here is thought to be a smooth function of time, t, with compact support with values in the local operators associated with the free field. In the simplest case it is $H_{\mathrm{int}}(t) = \varphi(f_t)$ where φ is a free field, i.e. an operator-valued distribution on a Hilbert space, and where $f_t(x) = \delta(x^0 - t)f(x)$ for some test function f. The S-matrix is then an operator-valued functional $S(f)$ on the test function space. The functional equation for the evolution operator implies a factorization property for the S-matrix if the support of the interaction (as a function of time) consists of disjoint intervals. In the case above with the interaction being a linear function of the free field we even find the factorization,

$$S(f_1 + f_2) = S(f_1)S(f_2) \ , \tag{1}$$

whenever there exists some $t \in \mathbb{R}$ such that $\mathrm{supp}(f_1) \subset \{x|x^0 \geq t\}$ and $\mathrm{supp}(f_2) \subset \{x|x^0 \leq t\}$, and where f_1 and f_2 need not be smooth at the hypersurface $x^0 = t$. This stronger factorization property is not expected to hold for more singular interactions. Instead we require the following consequence of (1),

$$S(f_1 + f_2 + f_3) = S(f_1 + f_2)S(f_2)^{-1}S(f_2 + f_3) \ , \tag{2}$$

to hold for test functions f_1, f_2, f_3, whenever the supports of f_1 and f_3 can be separated by a Cauchy surface such that $\mathrm{supp}(f_1)$ lies in the future and $\mathrm{supp}(f_3)$ in the past of this surface. Together with the normalization condition $S(0) = \mathbf{1}$ (identity operator on Hilbert space) it implies the first mentioned weaker factorization condition in the case $f_2 = 0$.

The functional equation (2) has an interesting property; if S is a functional solving it we get other solutions S_f by defining $S_f(g) = S(f)^{-1}S(f + g)$ (the *relative* S-matrices) where f is an arbitrary test function. In particular we get commutativity in case $\mathrm{supp}(g_1)$ and $\mathrm{supp}(g_2)$ are space-like separated,

$$S_f(g_1 + g_2) = S_f(g_2)S_f(g_1) = S_f(g_1)S_f(g_2) \ . \tag{3}$$

Thus the relative S-matrices satisfy the locality condition required for local observables. They serve as generating functionals for the interacting fields.

Unfortunately, a construction of $S(f)$ in four dimensions is known only in the case of interaction Hamiltonians which are linear or quadratic in the free field, but in two dimensions Wrezinski [63] proved that, at least in the particular case of factorizable f, $f(t, x) = g(t)h(x)$, such a construction is possible for φ^4. One therefore mainly relies on the “infinitesimal” description of the local S-matrix $S(g)$

by studying its formal power series [6] expansion in terms of the "coupling constant" g. The connection with the usual formulation may be done via the *adiabatic limit*, i.e. the limit for $S(g)$ when $g \to 1$ all over space-time is the S-matrix, or in cases where the limit for $S(g)$ does not exist due to infrared divergences, the limit for the vacuum expectation values of $S_g(f)$, $g \to 1$, is the generating functional for the time-ordered correlation functions.

The description given so far emphasizes the fact that the Epstein and Glaser method is local in spirit, and it might be a favorite candidate for developing a renormalization theory on curved space-times. A closer inspection, however, shows that also in this method translation invariance plays an important rôle, both conceptually and technically, and it will require a lot of work to replace it by other structures. A similar problem was studied by Dosch and Müller [16]. These authors developed the Epstein and Glaser method on Minkowski space for quantum electrodynamics with external time independent electromagnetic fields. Their use of the Hadamard parametrices for the Dirac operator is already much in the spirit of a local formulation of perturbation theory; by the assumption of time independence of the external fields, however, time translation invariance still plays a crucial rôle in their approach. As a matter of fact, it will turn out that *microlocal analysis* [35] is ideally suited to carry through the program where in particular the concept of the wave front set proves to be extremely useful. We note, *en passant*, that other reseachers [15, 36, 37] had previously used these tools in quantum field theory and that more recently Verch [58] has developped a generalization of the concept of wave front sets which can be applied in algebraic quantum field theory.

This paper is an extended version of a previous one [9] where we sketched the main ideas. Here on we give all the necessary mathematical details.

The paper is organized as follows: After this introduction, Section 2 provides some useful grounding concepts and fixes the notations. Moreover, we present a new construction of Wick polynomials which may be of independent interest. In Section 3 we state the first principles by which we build up the perturbative method on smooth curved globally hyperbolic space-times. The most important change w.r.t. the Epstein and Glaser method is a characterization of the singularity structure of the time-ordered numerical distributions replacing translation covariance. In the course of this part we show a local version of the so called "Theorem 0" of Epstein and Glaser which provides the necessary mathematical properties of the building blocks of the construction. In Section 4 we start the inductive procedure which aims at constructing the time-ordered functions up to the small diagonal of the product manifold M^n, where all "dangerous" singularities are located. Sections 5 and 6, have a more mathematical flavour; we introduce the concept of scaling degree at a point, following essentially Steinmann [53], and its generalizations in terms of microlocal analysis. The main aim of this section is the description of the extension to all space of distributions defined on the complement of a submanifold. These tools are needed for the classification and implementation of renormalizability. The next, Section 7, contains the end of the inductive procedure by which we prove the theories with polynomial interactions to follow the same perturbative classification as on Minkowski space-time.

We emphasize that the method of defining the local S-matrix joins perturbation theory with the more abstract algebraic formulation of quantum field theory. In fact, we are able to define a unique family (net, precosheaf etc.) of $*$-algebras of observables on globally hyperbolic space-times via the idea of the local relative S-matrices. Section 8 describes this construction which seems to be widely unknown, in spite of the fact that it may already be found, in a preliminary form, in [52]. This Section partly justify the rather abstract starting point of Section 2.

An outlook, Section 9, concludes the paper. Finally, we stress that the procedure works for general field theories but for simplicity we stick to the notationally easiest case of a single scalar (massive) field theory with self interactions without derivatives.

2. General Theory of Quantized Fields and Microlocal Analysis

In order to fix our notations we recall some basic geometrical concepts. Further details may be found in some books on general relativity and Lorentzian geometry (see, for instance, [59] and [2]). We shall work on a *space-time* (M, g), where by this we mean that M is a connected, Hausdorff, boundaryless topological space of pure dimension $d \geq 2$ which (i) is paracompact, (ii) is equipped with a smooth structure, (iii) is endowed with a Lorentzian metric g_{ab}, i.e. a smooth 2-cotensor of signature $(1, d-1)$, i.e. $(+, -, \cdots, -)$ and (iv) is oriented and time oriented. Given the metric we have a canonically associated derivative, namely the Levi-Civita derivative denoted by ∇ and an associated curvature tensor R^a_{bcd}, with R the scalar curvature. The notion of totally geodesic submanifold, i.e., that one for which all tangential geodesics stay on the submanifold, is used in Section 6.

Some words on notations. Sometimes we write a zero section of a vector bundle $\mathcal{B}$ as $\{0\}$ some other times to precise that it belongs to that bundle we shall write $Z(\mathcal{B})$. However, in order to avoid any abuse, we use the notation $\dot{\mathcal{B}}$ to denote the bundle deprived from its zero section, i.e., $\mathcal{B} \backslash Z(\mathcal{B})$. We shall also use the notation M^n, whenever we treat the n-th order cartesian product of a manifold M, and by Δ_K, where $K \subseteq \{1, \ldots, n\}$, we mean the (smooth, closed) submanifold of M^n for which any of its points $(p_1, \dots, p_n)$ are such that $p_{k_1} = p_{k_2}$ for any pair $k_1 \neq k_2$ in K.

The causality principle plays a crucial rôle in our construction. Therefore we restrict our space-times to be *globally hyperbolic*. This means that M is homeomorphic to $\mathbb{R} \times \Sigma$ where Σ is a $(d-1)$-dimensional topological submanifold of M and for each $t \in \mathbb{R}$, $\{t\} \times \Sigma$ is a (spacelike) Cauchy surface. A Cauchy surface is a subset of M which every inextendible non spacelike curve intersects exactly once. Given a subset S of M we define the causal future/past sets $J^{\pm}(S)$ as the subsets of M which consist of all points $p \in M$ for which there exists some point $s \in S$ connected to p by a non space-like future/past directed curve. If M is globally hyperbolic, the set $J^+(p) \cap J^-(q)$ is compact for any pair $p, q \in M$. Finally, if $p \in M$ then the induced metric on tangent space T_pM and cotangent space T_p^*M are Minkowskian, and we define the future/past light-cones $V_\pm$ over these spaces (based on p) in the usual way.

Quantum field theories on more general spaces pose consistency problems (see, e.g., Hawking's "Chronology Protection Conjecture" [33] or the divergence of the energy momentum tensor at the Cauchy horizon observed in [40]). We remark, however, that since our constructions will be purely local, one can as well consider a globally hyperbolic submanifold of *any* Lorentzian space-time.

In many concrete cases, exact solutions of the Einstein equation, like Minkowski, de Sitter, Schwartzschild are real analytic. In these cases some of our results might be sharpened by working with the analytic version of microlocal analysis [44]. In this respect, we should mention some recent results of Bros, Epstein and Moschella for a Gårding-Wightman-like description of quantum field theories on de Sitter space-time [7] where analytic function techniques play a major rôle.

2.1. **Wave front sets and Hadamard states for free fields.** For the (massive) free field φ satisfying the (generalized) Klein-Gordon equation of motion

$$(\Box_g + m^2 + \kappa R)\varphi = 0 \ , \tag{4}$$

where $\Box_g$ is the d'Alembertian (or Laplace-Beltrami) operator w.r.t. the Lorentzian metric g, $m \geq 0$ and $\kappa \in \mathbb{R}$, one may associate an algebra of observables defined in the following way: Let E_{ret} resp. E_{adv} be the retarded resp. advanced Green functions of the Klein-Gordon operator which are *uniquely* defined on globally hyperbolic space-times, and let $E = E_{ret} - E_{adv}$. Then we consider the unital $*$-algebra $\mathfrak{A}$ which is generated by the symbols $\varphi(f)$, $f \in \mathcal{D}(M)$ (space of complex-valued smooth and compactly supported functions), with the following relations:

1. The map $f \mapsto \varphi(f)$ is linear,
2. $\varphi(f)^* = \varphi(\bar{f})$,
3. $[\varphi(f), \varphi(g)] = iE(f \otimes g)\mathbf{1}, \qquad \forall f, g \in \mathcal{D}(M)$,
4. $\varphi((\Box_g + m^2 + \kappa R)f) = 0, \qquad \forall f \in \mathcal{D}(M)$,

where the symbol $[\varphi(f), \varphi(g)]$ stands for $\varphi(f)\varphi(g) - \varphi(g)\varphi(f)$ and $\bar{f}$ means complex conjugation. A state is, by definition, a linear functional ω on $\mathfrak{A}$ (the *expectation value*) which is positive (i.e. $\omega(a^*a) \geq 0$) and normalized ($\omega(\mathbf{1}) = 1$). It is uniquely determined by a sequence of multilinear functionals ω_n, $n = 0, 1, \ldots$ (the *n-point functions*) on the test function space $\mathcal{D}(M)$,

$$\omega_n(f_1, \ldots, f_n) = \omega(\varphi(f_1) \cdots \varphi(f_n)) \ . \tag{5}$$

We only consider states whose n-point functions are distributions and restrict furthermore our attention to the states called *quasi-free*, namely, those states whose only non trivial n-point functions have n even and are generated in terms of the 2-point functions (see, e.g. [29]). Among them a distinguished class is formed by the so called Hadamard states (see, e.g. [14, 41]). They are thought to be the appropriate analogue of the concept of the *vacuum* which has no direct counterpart on generic space times. In fact, they are quasifree states whose 2-point functions have a prescribed short-distance behaviour which is partially motivated by the fact that it allows the definition of the expectation value of the energy-momentum tensor (see, e.g. [60]). As first observed by Radzikowski [49] the 2-point functions of Hadamard states can be characterized in terms of their wave front set.

To discuss this characterization we need to enter into the realm of microlocal analysis. We give some motivations to the basic notions of wave front sets and present those basic results which are used throughout the paper. We leave the reader the task to look further into the large existing literature [35]. Physicists might, for concreteness, start from the well-written short exposition of Junker in [39], where they can find definitions and results about pseudodifferential operators, which we hold as known.

We shall denote by $\mathcal{E}(\mathbb{R}^n)$ the space of complex-valued smooth functions and by $\mathcal{E}'(\mathbb{R}^n)$ its dual space, i.e., the space of compactly supported distributions.

It is a standard result in distribution theory that $u \in \mathcal{E}'(\mathbb{R}^n)$ is a smooth function iff its Fourier transform $\widehat{u}$ decays rapidly in Fourier dual space $\mathbb{R}_n$, i.e. for any integer N there exists a constant C_N such that $|\widehat{u}(k)| \leq C_N(1+|k|)^{-N}$ for all $k \in \dot{\mathbb{R}}_n$, where $\mathbb{R}_n \backslash \{0\} \doteq \dot{\mathbb{R}}_n$. In case u is not smooth the Fourier transform may still rapidly decay in certain directions. We may describe this set of directions by an open cone in $\dot{\mathbb{R}}_n$ and define $\Sigma(u)$ as its complement in $\dot{\mathbb{R}}_n$. It is easy to see that $\Sigma(\phi u) \subset \Sigma(u)$ when $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$. This property suggests a strategy for the general case in which u is not of compact support. So, considering $u \in \mathcal{D}'(\mathbb{R}^n)$ and a point $x \in \mathbb{R}^n$ in the support of u, $\mathrm{supp}(u) \subset \mathcal{O}$, $\mathcal{O}$ open subset of $\mathbb{R}^n$, we first localize u via multiplication with some $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that $\phi(x) \neq 0$ and then consider the Fourier transform of ϕu, now a distribution of compact support. We then define the set $\Sigma_x(u) \doteq \cap_\phi \Sigma(\phi u)$ where the intersection is taken w.r.t. all smooth functions of compact support ϕ such that $\phi(x) \neq 0$. This may be called the set of *singular directions* of u over x. It is empty whenever $x \notin \mathrm{singsupp}(u)$.

Hence, finally, we define the (*smooth*) *wave front set* for $u \in \mathcal{D}'(\mathbb{R}^n)$ as $\mathrm{WF}(u) \doteq \{(x,k) \in \mathbb{R}^n \times \dot{\mathbb{R}}_n \mid k \in \Sigma_x(u)\}$. This set is readily seen to be closed and conic, where the last means that if $k \in \Sigma_x(u)$ so do any λk for all $\lambda > 0$.

It is now crucial that the notion of the wave front set can be lifted to any smooth manifold $\mathcal{M}$ where it is invariantly defined as a subset of the cotangent bundle $\dot{T}^*\mathcal{M}$. This covariance under coordinate transformations is what gives to the definition its real technical power.

Among the results which will be important for us we mention that derivatives do not enlarge the wave front set of a distribution, i.e. $\mathrm{WF}(\partial u) \subseteq \mathrm{WF}(u)$, and the following criterion called Hörmander criterion for multiplication of distributions:

M1. Product. Picking two distributions $u_1, u_2 \in \mathcal{D}'(\mathcal{M})$, the *pointwise* product $u_1 u_2$ exists as a *bona-fide* distribution whenever $\mathrm{WF}(u_1) + \mathrm{WF}(u_2)$ does not intersect the zero section $Z(T^*\mathcal{M})$, i.e., if for all covectors $k_i \in \mathrm{WF}(u_i)$, $i = 1, 2$, based over the same point one finds that $k_1 + k_2 \neq 0$. Moreover, if $\mathrm{WF}(u_i) \subset \Gamma_i$, $i = 1, 2$, then $\mathrm{WF}(u_1 u_2) \subset \Gamma_1 \cup \Gamma_2 \cup (\Gamma_1 + \Gamma_2)$.

We shall also refer frequently to a certain continuity property in microlocal analysis which in the body of the paper is sometimes called "Hörmander (pseudo) topology." It has to do with the notion of convergent sequences which respect also wave front set properties:

M2. Continuity. Let $\mathcal{D}'_\Gamma(\mathcal{M}) \doteq \{v \in \mathcal{D}'(\mathcal{M}) \mid \mathrm{WF}(v) \subset \Gamma\}$, where Γ is a closed conic set in $\dot{T}^*\mathcal{M}$. A sequence $\{u_i\}_{i\in\mathbb{N}} \subset \mathcal{D}'_\Gamma(\mathcal{M})$ converges to $u \in \mathcal{D}'_\Gamma(\mathcal{M})$ in the sense of the Hörmander (pseudo) topology whenever the following two properties hold true:

(a) $u_i \to u$ weakly* (i.e. in $\mathcal{D}'(\mathcal{M})$),

(b) for any properly supported pseudodifferential operator A such that $\mu\mathrm{supp}(A) \cap \Gamma = \emptyset$, we have that $Au_i \to Au$ in the sense of $\mathcal{E}(\mathcal{M})$. ($\mu\mathrm{supp}(A)$ is the projection onto the second component of the wave front set of the Schwartz kernel of A.)

A last property is connected with the sequential continuity, in the sense of **M2**, of the operation of restriction of a distribution to a submanifold:

M3. Trace. Let $\mathcal{N} \subset \mathcal{M}$ denote a submanifold, and let $u \in \mathcal{D}'(\mathcal{M})$. Then u can be restricted to the submanifold $\mathcal{N}$ whenever $\mathrm{WF}(u)$ does not intersect the conormal bundle $N^*\mathcal{N}$ of $\mathcal{N}$. Moreover, if $\mathrm{WF}(u) \subset \Gamma$, with Γ a closed conic set such that $\Gamma \cap N^*\mathcal{N} = \emptyset$, then the operator of restriction (trace) γ can be lifted as a sequentially continuous operator, in the sense of **M2**, from $\mathcal{D}'_\Gamma(\mathcal{M})$ to $\mathcal{D}'(\mathcal{N})$.

For later purpose, it is convenient to have a coordinate dependent formulation of **M2**(b) by using Fourier transforms. Namely, let $x_0 \in \mathcal{M}$ and let V be an open conical neighbourhood of Γ_{x_0}, where the last denotes a set of covectors associated to the point x_0. Choose a chart (φ, U) at x_0 such that $\Gamma_x \subset V$ for all $x \in U$.

Let $\chi \in \mathcal{D}(U)$ with $\chi(x_0) \neq 0$. Then the Fourier transform of χu, $u \in \mathcal{D}'_\Gamma(\mathcal{M})$, is strongly decreasing in the complement of V, and

$$\sup_{k \notin V} |(\widehat{\chi u_i} - \widehat{\chi u})(k)|(1+|k|)^N \to 0 \ , \tag{6}$$

for all $N \in \mathbb{N}$ if $u_i \to u$ in $\mathcal{D}'_\Gamma(\mathcal{M})$. If, on the contrary, the above convergence holds true for all choices of x_0, V, (φ, U) and χ, we obtain **M2**(b).

After this disgression into microlocal analysis we briefly describe Radzikowski's characterization of Hadamard states [49]. The idea is to use wave front sets for a formulation of a spectral condition. The antisymmetric part of the 2-point function is the commutator function E. Its wave front set is

$$\mathrm{WF}(E) = \{(x,k;x',-k') \in \dot{T}^*M^2 \mid (x,k) \sim (x',k')\} \ . \tag{7}$$

Here the equivalence relation $\sim$ means that there exists a null geodesic from x to x' such that k is coparallel to the tangent vector of the geodesic and k' is its parallel transport from x_1 to x_2. For coinciding points, the relation is defined as consisting of the degenerate (i.e., only one point) geodesic at $x = x'$ which has covector k still along the boundary of the light-cone and $k' \equiv k$. We remark, for later purpose, that since only light-like covectors are present, one can restrict E, and, whenever local coordinates are chosen, its derivative w.r.t. time $\dot{E}$, to any spacelike Cauchy hypersurface.

As a result of [49, 43], the 2-point function of a Hadamard state has a wave front set which is just the positive frequency part of $\mathrm{WF}(E)$,

$$\mathrm{WF}(\omega_2) = \{(x,k;x',-k') \in \mathrm{WF}(E) \mid \ k \in \overline{V}_+\} \ . \tag{8}$$

Since (8) restricts the singular support of $\omega_2(x_1, x_2)$ to points x_1 and x_2 which are null related, ω_2 is smooth for all other points. The smoothness for *space-like* related points is known to be true for quantum field theories on Minkowski space satisfying the spectrum condition by the Bargmann-Hall-Wightman Theorem [55]. For time-like related points, however, a similar general prediction on the smoothness does not exist.

Another deep result from Radzikowski [49] shows that the Duistermaat-Hörmander [17] distinguished parametrices for the Klein-Gordon equation are nothing else than the (Stückelberg-)Feynman–anti-Feynman propagators (up to C^∞) for quasi-free Hadamard states. We recall that the time-ordered 2-point function E_F arising from ω_2 is given by

$$iE_F(x_1, x_2) = \omega_2(x_1, x_2) + E_{ret}(x_1, x_2) \ .$$

Its wave front set [49] is

$$\mathrm{WF}(E_F) = O \cup D \ ,$$

where the off-diagonal piece is given by,

$$O = \{(x, k; x', -k') \in \dot{T}^*M^2 \mid (x, k) \sim (x', k'), x \neq x', k \in \overline{V}_\pm \text{ if } x \in J^\pm(x')\} \ ,$$

and the diagonal one by,

$$D = \{(x, k; x, -k) \in \dot{T}^*M^2 \mid x \in M, k \in \dot{T}^*_x M\} \ .$$

Now, one can see why in naive perturbation theory we may find divergences. Indeed, the perturbative expansion in terms of Feynman graphs in position space leads to pointwise products of Feynman propagators. But these products do not satisfy Hörmander criterion for multiplication of distributions since covectors based on the diagonal piece D can add up to zero.

2.2. **A new construction of Wick polynomials.** In a previous paper [8] we constructed Wick polynomials as operator-valued distributions. We considered a fixed Hadamard state ω and the induced GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ for the $*$-algebra $\mathfrak{A}$ and found the Wick polynomials as operator-valued distributions on the dense cyclic domain generated by Ω_ω. We recall that by a GNS triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ we mean a complex Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathfrak{A}$ by unbounded operators on $\mathcal{H}_\omega$, and finally by $\Omega_\omega \in \mathcal{H}_\omega$ the cyclic vector representing the state ω for which one has the connection equation

$$\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega) \ , \qquad \forall A \in \mathfrak{A} \ .$$

The dependence of the construction of Wick polynomials on the choice of the Hadamard state led to two problems: The first one is due to the convention that the expectation value of a Wick polynomial vanishes in the chosen Hadamard state. Other choices lead to a finite redefinition, a problem well known from the definition of the expectation value of the energy momentum tensor [61]. Since we shall not discuss finite renormalizations in this paper, we do not treat this problem at the moment. The other problem is of a more technical nature: The smeared Wick polynomials are unbounded operators. We know from the work of Verch [57] that, locally, i.e. in bounded regions of space time, different Hadamard states lead to equivalent representations. But this theorem does not guarantee that the domains of definition for different choices of the cyclic vector coincide. We therefore give here a new definition which depends only on the representation but not on the cyclic vector. Its restriction to the cyclic subspaces coincides with the previous definition.

It is well known that the operators $\varphi(f)$ (now, representatives under π_ω of the abstract elements of the $*$-algebra $\mathfrak{A}$ in subSection 2.1) for a real valued test function are essentially self adjoint on the cyclic domain generated by Ω_ω, and that the Weyl operators $W(f) = \exp(i\varphi(f)^{**})$ (where now the $*$-operation denotes the Hilbert space adjoint) satisfy the Weyl relation $W(f)W(g) = \exp(-\frac{i}{2}E(f \otimes g))W(f+g)$. The expectation value in the given Hadamard state is

$$\omega(W(f)) = \exp(-\frac{1}{2}\omega_2(f,f)) \ . \tag{9}$$

Let $:W(f): \doteq \exp(\frac{1}{2}\omega_2(f,f))W(f)$, and define for $\Psi \in \mathcal{H}_\omega$ the vector-valued function $\Psi(f) = \,:W(f):\,\Psi$.

Definition 2.1. *We say that* $\Psi(f)$ *is* infinitely often differentiable *at* $f = 0$ *if there exists for every integer* $n \geq 0$ *a symmetrical vector-valued distribution* $\delta^n\Psi/\delta f^n$ *on* $\mathcal{D}(M^n)$*, and continuous seminorms* p_n *on the test function space* $\mathcal{D}(M)$ *with* $p_{n+1} \geq p_n$*, such that*

(a) *if* $p_0(h) = 0$ *then* $\Psi(h) = \Psi(0)$,

(b) *if* $h \to 0$ *with* $p_n(h) \neq 0$ *then*

$$\left\| \Psi(h) - \sum_{l=0}^{n} \frac{1}{l!} \frac{\delta^l \Psi}{\delta f^l}(h^{\otimes l}) \right\| \ p_n(h)^{-n} \longrightarrow 0 \ ,$$

where $\| \cdot \|$ *stands for the Hilbert space norm in* $\mathcal{H}_\omega$.

The kernel of the functional derivative can be written,

$$\frac{\delta^n \Psi}{\delta f(x_1) \cdots \delta f(x_n)} = i^n :\varphi(x_1) \cdots \varphi(x_n): \Psi \ . \tag{10}$$

The right hand side of Eqn. (10) defines what is called a Wick monomial.

We want to find those vectors on which the Wick monomials can be restricted to partial diagonals. In view of the criterion **M3** for the restriction of distributions, we define as the *microlocal domain of smoothness* the following set:

$$\begin{aligned}\mathcal{D} = \Big\{ &\Psi \in \mathcal{H}_\omega \mid \Psi(f) \text{ is infinitely often differentiable at } f = 0\text{, and for every} \\ &n \in \mathbb{N} \text{ the wavefront set of } \frac{\delta^n \Psi}{\delta f^n} \text{ is contained in the set} \\ &\{(x_1, k_1; \ldots ; x_n, k_n) \in \dot{T}^* M^n \mid k_i \in \overline{V}_-,\ i = 1, \ldots , n\} \Big\} \ .\end{aligned} \tag{11}$$

The vector-valued distributions (10) with $\Psi \in \mathcal{D}$ can be restricted to all partial diagonals, and give all possible Wick polynomials. Moreover, according to **M1**, they may also be multiplied by distributions whose wavefront sets do not contain elements $(x_1, k_1; \ldots ; x_n, k_n) \in \dot{T}^* M^n,\ k_i \in \overline{V}_+,\ i = 1, \ldots , n$. The domain $\mathcal{D}$ is invariant under application of Weyl operators and smeared Wick polynomials.

We state a crucial property:

Lemma 2.2. *Let $\Phi \in \mathcal{H}_\omega$ induce some quasi-free Hadamard state ω'. Then $\Phi \in \mathcal{D}$.*

Proof. The main point rests on the validity of Leibniz rule. Indeed, we can write,

$$\Phi(f) = \exp(\frac{1}{2}(\omega_2(f, f) - \omega_2'(f, f)) \exp(\frac{1}{2}\omega_2'(f, f) + i\varphi(f)^{**})\Phi \ ,$$

and differentiate w.r.t. f. The general n-th order derivative gives

$$\frac{\delta^n \Phi}{\delta f^n}(h^{\otimes n}) = \sum_{I \subseteq \{1,\ldots,n\}, |I| \text{ even}} \chi(h,h)^{|I|/2} \, \frac{\delta^{|I^c|}\Phi'}{\delta f^{|I^c|}}(h^{\otimes |I^c|}) \ , \tag{12}$$

where, $\Phi'(f) = \exp(\frac{1}{2}\omega_2'(f,f) + i\varphi(f)^{**})\Phi$, $\chi = \omega_2 - \omega_2'$ is a smooth function on M^2 and a solution of the Klein-Gordon equation in both entries.

Now, $\Phi'(h)$ satisfies the estimate in Definition 2.1 with the seminorms $p_n'(h) = \sqrt{\omega_2'(h,h)}$ for all n, and the numerical prefactor with the seminorms $p_n(h) = \sqrt{(\omega_2 + \omega_2')(h,h)}$, hence for the whole expression we may also use the seminorms p_n. Actually, as was shown by Verch in [57], there exist two positive constants A and B, such that

$$A\omega_2(f,g) \leq \omega_2'(f,g) \leq B\omega_2(f,g) \ , \qquad f,g \in \mathcal{D}(M) \ ,$$

hence all these seminorms are equivalent. We conclude that $\Phi(f)$ is infinitely differentiable at $f = 0$.

The wave front sets of the functional derivatives of $\Phi(f)$ and $\Phi'(f)$ coincide, since χ is smooth. Using the formula

$$\left\| \frac{\delta^n \Phi'}{\delta f^n}(h^{\otimes n}) \right\|^2 = n! \ \omega_2'(h,h)^n \ , \tag{13}$$

and the information on the wave front set of Hadamard states, we find

$$\mathrm{WF}\left(\frac{\delta^n \Phi}{\delta f^n}\right) = \{(x_1,k_1;\ldots;x_n,k_n) \in \dot{T}^* M^n \mid k_i \in \partial V_-, \ i = 1,\ldots,n\} \ , \tag{14}$$

so $\Phi \in \mathcal{D}$. ■

We use the so called (generalized) Wick expansion formula which is the basic combinatorial formula for perturbation theory. Let us denote by $\mathfrak{L}$ any Wick polynomial on $\mathcal{D} \subset \mathcal{H}_\omega$, and we define the "derivative" of a Wick polynomial with respect to φ to be $\partial\mathfrak{L}/\partial\varphi$. We can characterize it by the following result:

Lemma 2.3. *There is a unique Wick polynomial $\partial\mathfrak{L}/\partial\varphi$ which satisfies the equation,*

$$[\mathfrak{L}(x), \varphi(y)] = \frac{\partial\mathfrak{L}}{\partial\varphi}(x)E(x,y) \ , \tag{15}$$

in the sense of operator-valued distributions on $\mathcal{D}$.

Proof. By linearity it is sufficient to prove it for any Wick monomial e.g. $:\varphi^n(x):$. It is obvious, that $n{:}\varphi^{n-1}(x){:}$ satisfies (15), hence we only need to proof that if A is any Wick polynomial for which $A(x)E(x,y) = 0$ this means that $A \equiv 0$. But this follows from the fact that for every $x \in M$ we can find some test function f such that the (smooth) solution $E(x,f)$ of the Klein-Gordon equation does not vanish at x. ■

Now, let us consider, for any Wick polynomial $\mathfrak{L}$, the fields $\mathfrak{L}^{(j)} = \partial^j\mathfrak{L}/\partial\varphi^j$, $j \in \mathbb{N}$, which, by induction, are uniquely defined according to the previous Lemma.

Theorem 2.4 (Generalized Wick expansion Theorem)**.** *Let $\mathfrak{L}_k$, $k = 1, \dots, n$, be Wick polynomials. The following relation holds:*

$$\mathfrak{L}_1(x_1)\cdots\mathfrak{L}_n(x_n) = \sum_{j_1,\dots,j_n} (\Omega_\omega, \mathfrak{L}_1^{(j_1)}(x_1)\cdots\mathfrak{L}_n^{(j_n)}(x_n)\Omega_\omega) \times \\ \times \frac{:\varphi^{j_1}(x_1)\cdots\varphi^{j_n}(x_n):}{j_1!\cdots j_n!} \ ,\tag{16}$$

where the summations over the j_k's go from 0 to the order of the corresponding $\mathcal{L}_k$.

For a proof see [34] or just use the previous notion of differentiability and apply induction. Note that the products in the Theorem above exist because the wave front sets of their expectation values satisfy Hörmander criterion **M1** due to the convexity of the forward light cone.

The wave front sets for the Wightman distributions of Wick polynomials may be larger than those of the corresponding distributions for the free field φ. Consider as an example the 2-point Wightman function for the Wick monomial $:\varphi^2(x):$, i.e. $(\Omega_\omega, :\varphi^2(x_1)::\varphi^2(x_2):\Omega_\omega)$. According to the Theorem, this is equal to $2\omega_2(x_1, x_2)^2$. This product exists according to the Hörmander criterion, and its wave front set is contained in $(\mathrm{WF}(\omega_2) + \mathrm{WF}(\omega_2)) \cup \mathrm{WF}(\omega_2) \doteq \mathcal{F}_2$. The set $\mathcal{F}_2$ will be instrumental for some results below. Now, $\mathrm{WF}(\omega_2) + \mathrm{WF}(\omega_2)$ contains directions which lie *inside* the light-cone as it is clear by adding up two covectors $k_1 + k_2$ for points on the diagonal. One thus sees how already the smallest possible non-linearity may give rise to additional singular directions w.r.t. those already present in the wave front sets of the Wightman functions for the original field φ. Another important remark is that $\mathcal{F}_2$ is an involutive closed cone, i.e. is a closed cone which is stable under sums, and, as a straightforward result, it gives that $(\omega_2(x, y))^n$ still has $\mathrm{WF}(\omega_2^n) \subset \mathcal{F}_2$.

The general structure for multi-point expectation values of Wick polynomials can be found as follows. For a more compact notation some definitions from graph theory are used: Let $\mathcal{G}_n$ denote the set of all finite nonoriented graphs with vertices $V = \{1, \dots, n\}$ and let E^G denote the set of edges of a given graph G. Moreover, for any vertex $i \in V$ we denote by E_i^G the subset of edges which belong to the vertex i, possibly empty, and by $|E_i^G|$ their number and similarly by E_{ij}^G the subset of edges connecting points i and j, with the obvious relation $|E_i^G| = \sum_j |E_{ij}^G|$. For any edge $e \in E$ connecting points i and j we use the "source and range" notation, i.e. $i = s(e)$ and $j = r(e)$, whenever $i < j$.

It is sufficient, by linearity, to restrict ourselves to the treatment of products of Wick monomials. Indeed, let us denote by $\omega_n^{m_1,\dots,m_n}$ the expectational value, w.r.t. the GNS-vector Ω_ω for a quasi-free Hadamard state ω, of the product of Wick monomials $:\varphi^{m_1}(x_1):\cdots:\varphi^{m_n}(x_n):$, and define as $\widehat{\mathcal{G}}_n(m_1,\dots,m_n)$ the set of all graphs G for which all vertices j with m_j edges are saturated, i.e. $|E_j^G| = m_j$. Moreover, following [8], we call a triple (x,γ,k) an *immersion* of any graph $G \subset \mathcal{G}_n$ into the manifold M whenever, (a) $x : V \to M$ is a map from all vertices i of G to points x_i of M; (b) γ maps edges $e \in E^G$ to null geodesics γ_e connecting points $x_{s(e)}$ and $x_{r(e)}$; (c) k maps edges $e \in E^G$ to future directed covector fields $k_{\gamma(e)} \equiv k_e$ which are coparallel to the tangent vector $\dot\gamma_e$ of the null geodesic. Hence, generalizing the set Γ_2 above by,

$$\Gamma_n \doteq \Big\{(x_1,k_1;\dots;x_n,k_n) \in \dot T^*M^n | \quad \exists\, G \in \widehat{\mathcal{G}}_n(m_1,\dots,m_n) \text{ and an immersion } (x,\gamma,k) \text{ of } G \text{ such that } k_i = \sum_{\substack{e\in E^G\\ s(e)=i}} k_e(x_i) - \sum_{\substack{e\in E^G\\ r(e)=i}} k_e(x_i)\Big\}\ , \tag{17}$$

we have:

Proposition 2.5. *The wave front set of $\omega_n^{m_1,\dots,m_n}$ is geometrically bounded as*

$$\mathrm{WF}(\omega_n^{m_1,\dots,m_n}) \subset \Gamma_n\ . \tag{18}$$

Proof. Let us define for notational purpose $\widehat{\mathcal{G}}_n \doteq \widehat{\mathcal{G}}_n(m_1, \ldots, m_n)$. As follows from Theorem 2.4 the expectation value is,

$$\omega_n^{m_1,\ldots,m_n}(x_1, \ldots, x_n) \doteq \sum_{G \in \widehat{\mathcal{G}}_n} \omega_n^G(x_1, \ldots, x_n) = \sum_{G \in \widehat{\mathcal{G}}_n} \prod_{e \in G} \omega_2(x_{s(e)}, x_{r(e)}) \ . \tag{19}$$

Considering one graph G in this sum we see that, in explicit form,

$$\begin{aligned}\omega_n^G(x_1, \ldots, x_n) &\doteq \prod_{e \in G} \omega_2(x_{s(e)}, x_{r(e)}) \\ &= \omega_2(x_1, x_2)^{|E^G_{1,2}|} \omega_2(x_1, x_3)^{|E^G_{1,3}|} \cdots \omega_2(x_{n-1}, x_n)^{|E^G_{n-1,n}|}.\end{aligned}$$

In the last equality the 2-point distributions should be understood as distributions on the product manifold M^n. Hence their wave front set is given, when $i < j$, by,

$$\mathrm{WF}(\omega_2^{|E^G_{i,j}|}) \subset \{(x_1, 0; \ldots; x_i, k_{i,j}; \ldots; x_j, -k_{i,j}; \ldots; x_n, 0) | (x_i, k_{i,j}; x_j, -k_{i,j}) \in F_2\}.$$

It is straightforward to see from the last expression that the claim of the Proposition is correct. Indeed a general covector k_i will have the following expression,

$$k_i = -k_{1,i} - \cdots - k_{i-1,i} + k_{i,i+1} + \cdots + k_{i,n} \ , \tag{20}$$

where some of the $k_{l,m}$ may be zero. Now, as follows from Eqn. (8), to any edge e which consists of pair of joined vertices i, j in the graph G there exist points on the manifold x_i, x_j and a null geodesic γ_e connecting them together with a future directed covector field k_e which is coparallel to the tangent vector of the geodesic and is such that, in agreement with Eqn. (20),

$$k_i = \sum_{\substack{e\in E^G \\ s(e)=i}} k_e(x_i) - \sum_{\substack{e\in E^G \\ r(e)=i}} k_e(x_i) \ .$$

Since this applies equally well to all graphs, and since the wave front set of sums of distributions is bounded by the union of the wave front sets, we get the thesis. ■

3. On a Local Formulation of Perturbation Theory

We have recalled in the introduction the main ideas of the Epstein and Glaser formulation of perturbation theory. Here we give the details of our generalization.

We start from the Gell-Mann and Low formula for the S-matrix for quantum theories on four dimensional Minkowski space-time $\mathbb{M}$; this means adopting the following formal expression,

$$S_\lambda = T(e^{i\lambda \int_{\mathbb{M}} \mathcal{L}_{int}(x) d^4x}) \ ,$$

where T denotes the notion of time ordering, $\mathcal{L}_{int}$, the interaction Lagrangian, is some local field and λ is the strength of the interaction. Developing in Taylor series w.r.t. λ gives

$$S_\lambda = \sum_{k=0}^{\infty} \frac{(i\lambda)^k}{k!} \int_{\mathbb{M}} \cdots \int_{\mathbb{M}} T(\mathcal{L}_{int}(x_1)\mathcal{L}_{int}(x_2)\cdots\mathcal{L}_{int}(x_k)) \prod_{i=1}^{k} d^4x_i \ .$$

Hence, the perturbative solution to scattering theory is reduced to quadratures once one finds the general solution of the time ordering operation inside the integral. For noncoinciding points the solution is given by the following expression,

$$T(\mathcal{L}_{int}(x_1)\cdots\mathcal{L}_{int}(x_n)) =$$

$$\sum_{\pi\in\mathcal{P}_{1,\dots,n}} \theta(x_{\pi(1)}-x_{\pi(2)})\cdots\theta(x_{\pi(n-1)}-x_{\pi(n)})\mathcal{L}_{int}(x_{\pi(1)})\cdots\mathcal{L}_{int}(x_{\pi(n)}) \ , \tag{21}$$

where $\mathcal{P}_{1,\dots,n}$ is the set of all permutations of the index set $\{1,\dots,n\}$ and θ is the Heaviside step function,

$$\theta(x) = \begin{cases} 1, & \text{if } x^0 > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where x^0 denotes the time component of the points in $\mathbb{M}$. As well known, this expression leads to the description of scattering processes by Feynman graphs [19]. Due to local commutativity of the Lagrangian the singularities of the Heaviside step function at coinciding times are harmless, as long as all points are different. Unfortunately, this is no longer true for coincident points, since $\mathcal{L}_{int}$ is an operator-valued distribution which cannot, in general, be multiplied by a discontinuous function; if one tries to define the products by convolutions in momentum space, this leads, in a naive approach, to the occurrence of ultraviolet divergences.

Several procedures have been found to cope with these singularities. But typically they are nonlocal and are therefore not immediately generalizable to the case of Lorentzian curved backgrounds. Better is the situation in Euclidean field theory (see e.g. [45] for a generalization of dimensional renormalization to the curved case).

Let us consider, as possible interaction Lagrangians $\mathcal{L}$, Wick polynomials of the free field φ. From Eqns. (21) and (16) we find

$$T(\mathcal{L}_1(x_1)\cdots\mathcal{L}(x_n)) = \sum_{j_1,\ldots,j_n} (\Omega_\omega, T(\mathcal{L}_1^{(j_1)}(x_1)\cdots\mathcal{L}_n^{(j_n)}(x_n))\Omega_\omega) \times \\ \times \frac{:\varphi^{j_1}(x_1)\cdots\varphi^{j_n}(x_n):}{j_1!\cdots j_n!}\ , \qquad (22)$$

where, however, the expectation value in the right-hand side is *a priori* not defined all over $\mathbb{M}^n$, but only over $\mathbb{M}^n\backslash\Sigma$ where Σ is the union of all diagonals in $\mathbb{M}^n$. So, the main problem is to give a mathematical meaning to this formula on all points.

3.1. **Formulation of the local S-matrix.** The general starting idea, due to Bogoliubov, is to consider the usual Gell-Mann and Low formulation of the S-matrix but supplemented by the hypothesis necessary to the implementation of the causality principle. In one stroke one finds also the solution to the problem of the correct treatment of the operator-valued distributions. Choosing as the interaction Lagrangian $\mathcal{L}_{int}(x) = \mathcal{L}(x)\eta(x)$, a Wick polynomial $\mathcal{L}$ multiplied by a space-time function of compact support $\eta \in \mathcal{D}(M)$ (considered as a generalized "coupling constant"), we *define* the local S-matrix $S_\lambda(\eta)$ as a formal power series (see, for instance, [6]) in the coupling strength λ,

$$S_\lambda(\eta) \doteq \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\lambda)^n}{n!} \int_{M^n} T(\mathcal{L}_{int}(x_1)\cdots\mathcal{L}_{int}(x_n))d\mu_1\cdots d\mu_n\ , \qquad (23)$$

where $d\mu$ is the natural invariant volume measure on the globally hyperbolic space-time (M, g), and $\mathbf{1}$ is the Hilbert space identity operator. One can enlarge the definition e.g. by using a more general Lagrangian,

$$\lambda \mathcal{L}_{int} = \sum_{k=1}^{l} \eta_k \mathcal{L}_k \ , \tag{24}$$

with Wick polynomials $\mathcal{L}_k$ and associating to each a different "coupling constant" $\eta_k \in \mathcal{D}(M)$, where the additional "Lagrangians" $\mathcal{L}_k$ are defined as terms like currents, external fields etc., including in particular all derivatives of the basic interaction Lagrangian. Using this extended Lagrangian we may replace the time-ordered operator in Eqn. (23) by,

$$T(\mathcal{L}_{int}(x_1) \cdots \mathcal{L}_{int}(x_n)) = \sum_{k_1,\ldots,k_n} T(\mathcal{L}_{k_1}(x_1) \cdots \mathcal{L}_{k_n}(x_n)) \eta_{k_1}(x_1) \cdots \eta_{k_n}(x_n) \ ,$$

where the summation over the k's go from 1 to l, the number of the terms in the extended Lagrangian.

We remark that, eventually, the test function(s) η should be sent to a fixed value over all space-time. This procedure, known as adiabatic limit, amounts to treat the infrared nature of the theory. Some studies of this limit in the case of Minkowski space-time have been performed by Epstein and Glaser themselves [21]. It is not clear how to generalize their study to curved spaces. It is therefore gratifying that all local properties of the theory are already obtained via the construction of the local S-matrices, and this point of view might also be useful in cases (like non abelian gauge theories) where due to infrared problems the S-matrix in the adiabatic limit does not properly exist.

3.2. **Defining properties.** Our main goal is the inductive construction of the time-ordered products of Wick polynomials,

$$T(\mathcal{L}_1(x_1)\cdots\mathcal{L}_n(x_n))\ .$$

Following Epstein and Glaser we require the following properties:

P1. Well-posedness. The symbols $T(\mathcal{L}_1(x_1)\cdots\mathcal{L}_n(x_n))$ are well defined operator-valued distributions on the GNS-Hilbert space $\mathcal{H}_\omega$ i.e. (multilinear, strongly continuous) maps $\mathcal{D}(M^n) \to \mathtt{End}(\mathcal{D})$ where $\mathcal{D} \subset \mathcal{H}_\omega$ is the dense subspace (microlocal domain of smoothness) defined in (11).

P2. Symmetry. Any time-ordered product $T(\mathcal{L}_1(x_1)\cdots\mathcal{L}_n(x_n))$ is symmetric under permutations of indices, i.e. the action of the permutation group $\mathcal{P}_{\{1,\dots,n\}}$ of the index set $\{1,\dots,n\}$ gives,

$$T(\mathcal{L}_{\pi(1)}(x_{\pi(1)})\cdots\mathcal{L}_{\pi(n)}(x_{\pi(n)})) \equiv T(\mathcal{L}_1(x_1)\cdots\mathcal{L}_n(x_n))\ ,$$

for any $\pi \in \mathcal{P}_{\{1,\dots,n\}}$, in the sense of distributions.

This symmetry property corresponds to the fact that the time-ordered products are functional derivatives of the local S-matrix. More crucial is the following causality property, which follows from the Eqn. (2);

P3. Causality. Consider any set of points $(x_1,\dots,x_n) \in M^n$ and any full partition of the set $\{1,\dots,n\}$ into two non empty subsets I and I^c such that no point x_i with $i \in I$ is in the past of the points x_j with $j \in I^c$, i.e. $x_i \notin J^-(x_j)$ for any $i \in I$ and $j \in I^c$. Then the time-ordered distributions are required to satisfy the following factorization property

$$T(\mathcal{L}_1(x_1)\cdots\mathcal{L}_n(x_n)) = T(\prod_{i\in I}\mathcal{L}_i(x_i))\; T(\prod_{j\in I^c}\mathcal{L}_j(x_j)) \ .$$

In the Epstein and Glaser scheme on Minkowski space-time one requires, in addition, translation covariance of the time-ordered products. If the free field is among the possible terms in the Lagrangian one can show that the time-ordered products are sums of pointwise products of Wick polynomials with translation invariant numerical distributions. (Such products exist due to Theorem 0 of Epstein and Glaser, an easy proof of which follows from our microlocal characterization of the domain of definition of Wick products.) Moreover, the condition,

$$[T(\mathcal{L}_1(x_1)\cdots\mathcal{L}_n(x_n)),\varphi(y)] = \sum_{i=1}^{n} T(\mathcal{L}_1(x_1)\cdots\frac{\partial\mathcal{L}_i}{\partial\varphi}(x_i)\cdots\mathcal{L}_n(x_n))E(x_i,y) \ , \tag{25}$$

fixes the coefficients to be vacuum expectation values of time-ordered products of those Wick polynomials which are of lower order w.r.t. the chosen interacting Lagrangian (from now on, we shall call them *sub*-Wick polynomials), hence the problem is reduced to a problem for numerical distributions. Unfortunately, in the case of a curved space-time, we have not yet determined the class of fields which are relatively local to the scalar free field, i.e., what is known in the literature as the Borchers' class [22]. One also needs a replacement of the condition of translation covariance. Our idea is to impose a condition on the time-ordered distributions which in a sense employs both ideas of invariance and spectrality crucial in the Minkowskian case. Since, as emphasized in the previous section, spectrality for us means wave front sets properties we now look for a condition which fixes the properties of the singularities of the time-ordered distributions. We use the graph theoretic definitions of Section 2.

P4. Spectrality. For the expectation value $t_n \in \mathcal{D}'(M^n)$, $n \geq 2$, of any time-ordered product it holds,

$$\mathrm{WF}(t_n) \subset \Gamma_n^{\mathrm{to}} \ ,$$

where

$$\begin{aligned}\Gamma_n^{\mathrm{to}} = \Big\{ & (x_1, k_1; \ldots ; x_n, k_n) \in \dot{T}^* M^n \quad | \quad \exists \quad \text{a graph } G \in \mathcal{G}_n \text{ and} \\ & \text{an immersion } (x, \gamma, k) \text{ of } G \text{ in which } k_e \text{ is future directed} \\ & \text{whenever } x_{s(e)} \notin J^-(x_{r(e)}) \text{ and such that,} \\ & k_i = \sum_{m:s(m)=i} k_m(x_i) - \sum_{n:r(n)=i} k_n(x_i) \Big\} \ .\end{aligned}$$

This may be motivated by the fact that, for non coinciding points, t_n can be expressed in terms of the usual Feynman graphs and for the set of coinciding points we have an infinitesimal remnant of translation invariance, since all covectors at coinciding points sum up to zero.

We can now formulate a microlocal version of Theorem 0 of Epstein and Glaser.

Theorem 3.1 (Microlocal Theorem 0)**.** *If* **P4** *holds for* t_n *then*

$$t_n(x_1, \ldots , x_n) {:} \varphi^{l_1}(x_1) \cdots \varphi^{l_n}(x_n) {:} \ ,$$

is a well defined operator-valued distribution, for any n *and any choice of indices* $l_1, \ldots , l_n$*, on the dense invariant domain* $\mathcal{D}$ *in the Hilbert space* $\mathcal{H}_\omega$*.*

Proof. Let $\Psi \in \mathcal{D}$. The vector-valued distribution $:\varphi^{l_1}(x_1)\cdots\varphi^{l_n}(x_n):\Psi$ is a restriction of $\delta^l\Psi/\delta f^l$, $l = \sum_{i=1}^n l_i$, to a partial diagonal with wave front set contained in $\bigcup_{x\in M^n}\{x\} \times \overline{V}_-^{\times n}$. Since Γ_n^{to} does not contain elements of the form $(x_1, k_1, \ldots, x_n, k_n)$ with $k_i \in \overline{V}_+$, $i = 1, \ldots, n$, the product is a well defined vector-valued distribution, and after smearing with some test function one obtains again a vector in $\mathcal{D}$. ■

In particular, formula (22) makes sense everywhere, provided the expectation values of all time-ordered products of sub-Wick polynomials satisfy **P4**. Moreover, every expansion into a sum of products of Wick polynomials by numerical distributions which satisfies (25) is of this form:

P5. Causal Wick Expansion.

$$T(\mathcal{L}_1(x_1)\cdots\mathcal{L}(x_n)) = \sum_{j_1,\ldots,j_n} (\Omega_\omega, T(\mathcal{L}_1^{(j_1)}(x_1)\cdots\mathcal{L}_n^{(j_n)}(x_n))\Omega_\omega) \times \\ \times \frac{:\varphi^{j_1}(x_1)\cdots\varphi^{j_n}(x_n):}{j_1!\cdots j_n!} \,. \tag{26}$$

4. Inductive Construction up to the Small Diagonal

The properties defined in the previous section allow us to set up an inductive procedure in the spirit of Epstein and Glaser. We rely on a variation of their construction proposed by Stora [54]. We start with a linear space $\mathcal{W}$ of Wick polynomials which contains all respective sub-Wick polynomials and want to define the time-ordered products $T(\mathcal{L}_1(x_1)\cdots\mathcal{L}(x_n))$ as a family of operator-valued distributions which are multilinear in the entries $\mathcal{L}_i \in \mathcal{W}$ and satisfy the properties **P1-5**. We start the induction by setting $T(\mathbf{1}) = \mathbf{1}$ and $T(\mathcal{L}) = \mathcal{L}$ and assume that the time-ordered products for $1 < l \leq n-1$ factors have been constructed and satisfy all the defining properties. In a first step, we aim at constructing time-ordered products of n factors on $M^n \backslash \Delta_n$ where Δ_n is the small diagonal submanifold of M^n, i.e. the set of points $(x_1, \dots, x_n)$ with the property $x_1 = x_2 = \cdots = x_n$.

We use the space-time notion of causality in order to define a certain partition of unity for $M^n \backslash \Delta_n$: Let us denote by $\mathcal{J}$ the family of all non-empty proper subsets I of the index set $\{1, \dots, n\}$ and define, accordingly, the sets $\mathcal{C}_I = \{(x_1, \dots, x_n) \in M^n \mid x_i \notin J^-(x_j), i \in I, j \in I^c\}$ for any $I \in \mathcal{J}$. Note that the defining relation for the $\mathcal{C}_I$'s is related to causality on M and *not* on M^n. It is fairly easy to show that

Lemma 4.1. *Let M be a globally hyperbolic space-time, then it holds*

$$\bigcup_{I \in \mathcal{J}} \mathcal{C}_I = M^n \backslash \Delta_n \ .$$

Proof. The inclusion $\cup_I \mathcal{C}_I \subset M^n \backslash \Delta_n$ is obvious. The opposite inclusion is proved as follows. Consider any set of points $(x_1, \dots, x_n)$ such that $x_i \neq x_j$ for some $i \neq j$, then the points x_i and x_j can be separated by a Cauchy surface Σ as follows from the global hyperbolicity assumption. One may choose it as to contain none of the points x_k, $k = 1, \dots, n$. Hence, defining $I = \{k \mid x_k \in J^+(\Sigma)\}$ and noting that $I \in \mathcal{J}$ we find $(x_1, \dots, x_n) \in \mathcal{C}_I$. ■

We use the short hand notations

$$T^I(x_I) = T(\prod_{i \in I} \mathcal{L}_i(x_i)), \qquad x_I = (x_i, i \in I) \ . \tag{27}$$

The first step now is to set on any $\mathcal{C}_I$

$$T_I(x) \doteq T^I(x_I) \ T^{I^c}(x_{I^c}) \ , \tag{28}$$

as an operator-valued distribution since according to the induction hypothesis and the fact that I is proper this is a well defined operation on $\mathcal{D}(\mathcal{C}_I)$.

We now glue together all operators previously defined on different elements of the cover. For this we need to prove a sheaf consistency condition. Indeed, different $\mathcal{C}_I$'s overlap but due to the causality hypothesis **P3** and the causal Wick expansion **P5** valid for the lower order terms, the following property holds:

Proposition 4.2. *For any choice of $I_1, I_2 \in \mathcal{J}$ such that $\mathcal{C}_{I_1} \cap \mathcal{C}_{I_2} \neq \emptyset$ we have*

$$T_{I_1} \restriction_{\mathcal{C}_{I_1} \cap \mathcal{C}_{I_2}} = T_{I_2} \restriction_{\mathcal{C}_{I_1} \cap \mathcal{C}_{I_2}} \ ,$$

in the sense of operator-valued distributions over $M^n \backslash \Delta_n$.

Proof. Let $I_1, I_2 \in \mathcal{J}$ and $x = (x_1, \ldots, x_n) \in \mathcal{C}_{I_1} \cap \mathcal{C}_{I_2}$. Using the causality property **P3** which is by assumption valid for time-ordered products of less than n factors we find,

$$\begin{aligned} T^{I_1}(x_{I_1}) &= T^{I_1 \cap I_2}(x_{I_1 \cap I_2}) \, T^{I_1 \cap I_2^c}(x_{I_1 \cap I_2^c}) \ , \\ T^{I_1^c}(x_{I_1^c}) &= T^{I_1^c \cap I_2}(x_{I_1^c \cap I_2}) \, T^{I_1^c \cap I_2^c}(x_{I_1^c \cap I_2^c}) \ , \end{aligned} \tag{29}$$

and similarly for T^{I_2} and $T^{I_2^c}$. Now the terms $T^{I_1 \cap I_2^c}$ and $T^{I_1^c \cap I_2}$ commute. Namely, they are based on mutually space-like points, thus using the Wick expansion for these terms this follows from local commutativity for the Wick polynomials of the free scalar field φ. Hence from definition (28) we get on $\mathcal{C}_{I_1} \cap \mathcal{C}_{I_2}$

$$\begin{aligned} T_{I_1} &= T^{I_1\cap I_2} T^{I_1^c\cap I_2} T^{I_1\cap I_2^c} T^{I_1^c\cap I_2^c} \ , && (\text{Eqn.}(28) + \mathbf{P3}), \\ &= T^{I_2} T^{I_2^c} \ , && (\text{Eqn.}(29)), \\ &= T_{I_2} \ , && (\text{Eqn.}(28)). \end{aligned}$$

■

Let now $\{f_I\}_{I\in\mathcal{J}}$ be a locally finite smooth partition of unity of $M^n \backslash \Delta_n$ subordinate to $\{\mathcal{C}_I\}_{I\in\mathcal{J}}$. We formally define, following (27) and (28),

$$^0T(\mathcal{L}_1(x_1)\ldots\mathcal{L}_n(x_n)) \doteq \sum_{I\in\mathcal{J}} f_I \ T_I \ . \tag{30}$$

Hence, we get our first crucial result, namely,

Theorem 4.3. *The symbols ^0T are well defined operator-valued distributions over $M^n \backslash \Delta_n$ which satisfy the defining properties on $\mathcal{D} \subset \mathcal{H}_\omega$.*

Proof. We first prove that the definition does not depend on the choice of the partition of unity. Indeed, let $\{f'_I\}_{I\in\mathcal{J}}$ be another such partition. Consider $x \in M^n \backslash \Delta_n$, and let $\mathcal{K} = \{I \in \mathcal{J} \mid x \in \mathcal{C}_I\}$. Then there exists a neighbourhood V of x such that $V \subset \cap_{I\in\mathcal{K}}\mathcal{C}_I$, and supp$(f_I)$ and supp(f'_I) do not meet V for all $I \notin \mathcal{K}$. In this case

$$\sum_{I\in\mathcal{J}} (f_I - f'_I) \ T_I \restriction_V = \sum_{I\in\mathcal{K}} (f_I - f'_I) \ T_I \restriction_V \ .$$

However, on V T_I is independent of the choice of $I \in \mathcal{K}$. Since $\sum_{I\in\mathcal{K}} f_I = \sum_{I\in\mathcal{K}} f'_I = 1$ on V, we arrive at the conclusion. Furthermore, an inspection of the formula readily gives that the operator ${}^0T$ is defined on the domain $\mathcal{D}$, the microlocal domain of definition of the Wick monomials, because of induction. Hence property **P1**.

As far as the symmetry property **P2** is concerned we just observe that the permuted distribution ${}^0T^\pi(x_1, \dots, x_n) = {}^0T(\mathcal{L}_{\pi(1)}(x_{\pi(1)}) \dots \mathcal{L}_{\pi(n)}(x_{\pi(n)}))$, has the expansion,

$$ {}^0T^\pi = \sum_{I \in J} f_I^\pi T_I^\pi = \sum_{I \in J} f_{\pi(I)}^\pi \; T_{\pi(I)}^\pi \ , $$

where we used the fact that the set J is invariant under permutations, but $T_{\pi(I)}^\pi = T_I$ and $\{f_{\pi(I)}^\pi\}_{I \in J}$ is a partition of unity subordinate to $\{\mathcal{C}_I\}_{I \in J}$, so symmetry follows from the result of the previous paragraph about the independence of ${}^0T$ on the choice of the partition of unity.

Causality **P3** follows from an argument similar to the one used for the independence from the partition of unity. Indeed, taken any point $x \in M^n \setminus \Delta_n$, as before $x \in V \subset \cap_{I \in \mathcal{K}} \mathcal{C}_I$. From (30),

$$ {}^0T(x) = \sum_{I \in \mathcal{K}} f_I(x) T_I(x) \ . $$

Since $T_I \restriction_V$ does not depend anymore on $I \in \mathcal{K}$ and $\sum_{I \in \mathcal{K}} f_I = 1$ over V hence ${}^0T(x) \equiv T_I(x)$ which from (28) satisfies causality by definition.

Now, we want to show that property **P4** holds on $M^n \setminus \Delta_n$. It is sufficient to check that this property is satisfied for each T_I on $\mathcal{C}_I$. We apply Wick Theorem to the components of the product in the definition (28). It can be easily checked that the distributions t_I in the Wick expansion of T_I are sums of terms of the following form

$$f_I(x)t^I(x_I)t^{I^c}(x_{I^c}) \cdot \prod_{(i,j)\in I\times I^c} \omega_2(x_i,x_j)^{a_{i,j}} \ , \tag{31}$$

with $a_{i,j} \in \mathbb{N}_0$, and where t^I, t^{I^c} are expectation values of lower order time-ordered products.

The wave front set of (31) is contained in the convex combination of the wave front sets of its factors. Hence it is given in terms of immersions of graphs with vertex sets I, I^c, resp., and of $a_{i,j}$ graphs with vertex sets $\{i,j\}$. All these immersions satisfy the condition in **P4**, the first two by assumption, and the last ones because of the definition of $\mathcal{C}_I$ and the properties of the wave front set of a Hadamard state. The union of these graphs is a graph with vertices $\{1,\dots,n\}$, and any convex combination of the components is given by an admissible immersion of this graph.

Finally, property **P5** follows from expression (28) by a straightforward application of the (generalized) Wick Theorem. ■

5. STEINMANN SCALING DEGREE AND THE EXTENSION OF DISTRIBUTIONS

We now want to extend ${}^0T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n))$ to the whole M^n. As discussed before the problem can be reduced to the extension of the numerical time-ordered distributions ${}^0t(x_1,\dots,x_n) \doteq (\Omega_\omega, {}^0T(\mathcal{L}_1(x_1)\dots\mathcal{L}_n(x_n))\Omega_\omega)$.

The extension can be performed in two steps. First ${}^0t$ is extended by continuity to the subspace of test-functions which vanish on Δ_n up to a certain order, and then it is arbitrarily defined on a complementary subspace. It is this last step which corresponds to the method of counterterms in the classical procedure of perturbative renormalization. The extension of ${}^0t$ by continuity requires some topology on test-function space. The seminorms used by Epstein and Glaser in their paper are quite complicated, and their generalization to curved space-times appears to be rather involved. We found it preferable therefore to apply a different method already introduced by Steinmann [53], namely the concept of *scaling degree* at a point of a distribution (see also [16]). Its generalization to curved space-time is very similar to the concept of the scaling limit as introduced by Haag, Narnhofer and Stein [30] and further developed by Fredenhagen and Haag [24]. A similar technique is used in [50].

On Minkowski space, by translation invariance, the distribution is in terms of relative coordinates everywhere defined up to the origin, and there the concept of the *scaling degree* at a point leads to a rather smooth and economic method of renormalization, see e.g. [48] where the relation to differential renormalization is elaborated. On a curved space-time one needs the corresponding notion for a scaling degree with respect to the submanifold Δ_n, and one also needs some uniformity of the singularity along Δ_n as well as control of the wave front sets during the extension process.

Our strategy will be that, at first we introduce this improvement in the case of $\mathbb{R}^n$, then we discuss the case of manifolds. There we try to set up a procedure which allows to restrict the discussion to the pointwise case.

5.1. **The scaling degree.** For simplicity, we work at first on $\mathbb{R}^d$. Hence, consider a distribution $t \in \mathcal{D}'(\mathbb{R}^d)$. Let the action of the positive reals (dilations) be defined via the map

$$\begin{aligned} \Lambda : \mathbb{R}_+ \times \mathcal{D}(\mathbb{R}^d) &\longrightarrow \mathcal{D}(\mathbb{R}^d) \\ (\lambda, \phi) \qquad &\longmapsto \phi^\lambda \doteq \lambda^{-d}\phi(\lambda^{-1}\,\cdot\,)\ , \end{aligned}$$

and obtain, by pull-back, the map over distributions $t \in \mathcal{D}'(\mathbb{R}^d)$ as,

$$(\Lambda^* t)(\phi) \doteq t_\lambda(\phi) \doteq t(\phi^\lambda)\ ,$$

where this operation in case $t \in L^1_{\text{loc}}(\mathbb{R}^d)$ is given by the explicit formula,

$$t_\lambda(\phi) = \int t(\lambda x)\phi(x) d^d x\ , \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^d)\ . \tag{32}$$

The map Λ is clearly continuous w.r.t. the topology of $\mathcal{D}(\mathbb{R}^d)$ and we shall sometimes use the previous formula (32), by the usual abuse of notation, also in the general case.

We say that t has *scaling degree* $\text{sd}(t) = \omega$ w.r.t. the origin in $\mathbb{R}^d$, if ω is the infimum of all $\omega' \in \mathbb{R}$ for which,

$$\lim_{\lambda \downarrow 0} \lambda^{\omega'} t_\lambda = 0\ , \tag{33}$$

holds in the sense of $\mathcal{D}'(\mathbb{R}^d)$.

It should be clear from the definition that every distribution $t \in \mathcal{D}'(\mathbb{R}^d)$ has a scaling degree $\omega \in [-\infty, +\infty[$. If the distribution is not defined at the point we want to check, then the scaling degree might also be equal to $+\infty$. We give some examples.

Examples .

1. Trivial example. Every $\phi \in \mathcal{E}(\mathbb{R}^d)$ has $\mathrm{sd}(\phi) \leq 0$.
2. Dirac measure. Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$ with $\mu(\phi) = \phi(0)$, $\phi \in \mathcal{E}(\mathbb{R}^d)$, then $\mathrm{sd}(\mu) = d$.
3. Feynman propagator. In the case of a free massive scalar field which is covariant under translation on Minkowski space-time, the Feynman propagator can be written as

   $$E_F(x) = (2\pi)^{-d} \int \frac{e^{ip\cdot x}}{p^2 - m^2 + i\epsilon} d^d p \ ,$$

   from which it is readily seen that $\mathrm{sd}(E_F) = d - 2$.
4. Homogeneous distributions. If $t \in \mathcal{D}'(\mathbb{R}^d)$ is homogeneous of order α at the origin, i.e. $t_\lambda = \lambda^\alpha t$, then $\mathrm{sd}(t) = -\alpha$.
5. Infinite degree. The smooth function $x \to \exp(1/x)$, $x \in \mathbb{R}_+$, is not defined at the origin and its scaling degree w.r.t. the origin is clearly infinite.

As inferred from the 4th example, the scaling degree may be seen as a generalization of the notion of the degree of homogeneity. Actually, our extension method is similar to the extension to all space of a homogeneous distribution as discussed in Hörmander's book [35] which on the other hand is also quite similar to the Epstein and Glaser procedure of distribution splitting [20]. The fact that homogenous extensions not always exist is the mathematical origin of the logarithmic corrections to scaling found in renormalization. Here, a discussion about space-time symmetries and their implementation after renormalization is absent. It will be presented in [11].

Lemma 5.1. *The scaling degree obeys the following properties:*

(*a*) *Let* $t \in \mathcal{D}'(\mathbb{R}^d)$ *have* $\mathrm{sd}(t) = \omega$ *at* 0 , *then*

1. *Let* $\alpha \in \mathbb{N}^d$ *be any multiindex, then* $\mathrm{sd}(\partial^\alpha t) \le \omega + |\alpha|$.
2. *Let* $\alpha \in \mathbb{N}^d$ *be any multiindex, then* $\mathrm{sd}(x^\alpha t) \le \omega - |\alpha|$.
3. *Let* $f \in \mathcal{E}(\mathbb{R}^d)$, *then* $\mathrm{sd}(ft) \le \mathrm{sd}(t)$.

(*b*) *For* $t_i \in \mathcal{D}'(\mathbb{R}^{d_i}), i = 1, 2$ *we have* $\mathrm{sd}(t_1 \otimes t_2) = \mathrm{sd}(t_1) + \mathrm{sd}(t_2)$.

Proof. The first two cases in (a) as well as (b) are straightforward. The third case in (a) follows from the fact that, by the Banach-Steinhaus principle, a convergent sequence of distributions is uniformly bounded. Hence, for every $\omega' > \mathrm{sd}(t)$ and every compact set $K \subset \mathbb{R}^d$ there is some polynomial P such that

$$|\lambda^{\omega'} t_\lambda(\phi)| \le \sup_{x \in K} |P(\partial)\phi(x)| \equiv ||\phi||_{\infty,P} \ . \tag{34}$$

Hence, for $f \in \mathcal{E}(\mathbb{R}^d)$, we have

$$|(ft)_\lambda(\phi)| = |t_\lambda(f_\lambda \phi)| \le \lambda^{\omega'} ||f_\lambda \phi||_{\infty,P} \ . \tag{35}$$

The statement follows now from the boundedness of the sequence $||f_\lambda \phi||_{\infty,P}$ as $\lambda \to 0$. ■

5.2. **Extensions of distributions to a point.** We now want to show how to extend a distribution $t \in \mathcal{D}'(\mathbb{R}^d \backslash \{0\})$ to all space by using the concept of the scaling degree. The scaling degree can easily be defined for such distributions by restricting the test functions appropriately. Equivalently, we can also, for each $\chi \in \mathcal{E}(\mathbb{R}^d)$ with $0 \notin \mathrm{supp}(\chi)$ look at the behaviour of the sequences χt_λ, now considered as sequences in $\mathcal{D}'(\mathbb{R}^d)$.

There are three possible cases; when the scaling degree is $+\infty$; in this case no extension to a distribution on $\mathbb{R}$ exists; when the scaling degree ω is finite, but $\omega \geq d$; then a finite dimensional set of extensions exists; or otherwise $\omega < d$. We first study the third case.

Theorem 5.2. *Let* $t_0 \in \mathcal{D}'(\mathbb{R}^d \backslash \{0\})$ *have scaling degree* $\omega < d$ *w.r.t. the origin. There exists a unique* $t \in \mathcal{D}'(\mathbb{R}^d)$ *with scaling degree* ω *such that* $t(\phi) = t_0(\phi)$, $\phi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$.

Proof. The uniqueness is easy. Indeed, the difference among two possible extensions would be a distribution with support at $\{0\}$. By a well known structural Theorem of distribution theory this last is given by $P(\partial)\delta$ where P is a polynomial of degree $\deg(P)$ and δ is Dirac measure at the origin. But this distribution has scaling degree equal to $d + \deg(P)$, hence a contradiction.

Let us now consider a smooth function of compact support ϑ such that $\vartheta = 1$ in a neighbourhood of the origin. Set $\vartheta_\lambda(x) \doteq \vartheta(\lambda x), \lambda \in \mathbb{R}$ and

$$t^{(n)} \doteq (1 - \vartheta_{2^n}) t_0 \ , \qquad n \in \mathbb{N} \ ,$$

where now $t^{(n)}$ is a sequence of distributions defined on the whole $\mathbb{R}^d$. We wish to show that the sequence converges in the weak* topology of $\mathcal{D}'(\mathbb{R}^d)$. Because of the sequential completeness of $\mathcal{D}'(\mathbb{R}^d)$ it is sufficient to prove that it is a Cauchy sequence. Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ and look at

$$\begin{aligned} (t^{(n+1)} - t^{(n)})(\phi) &= (\phi t_0)(\vartheta_{2^n} - \vartheta_{2^{n+1}}) \\ &= 2^{-nd} (\phi t_0)_{2^{-n}} (\vartheta - \vartheta_2) \ . \end{aligned} \tag{36}$$

According to Lemma 5.1, (a)3., this sequence is majorized, for every $\omega' \in \,]\omega, d[$ by const. $2^{n(\omega'-d)}$, hence it is summable as required. The limit

$$t(\phi) \doteq \lim_{n\to\infty} t^{(n)}(\phi) \ , \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^d) \ ,$$

then defines an extension of t_0. It is obvious that the scaling degree of t is not smaller than ω. It remains to proof that it is not bigger than ω.

Pick $\phi \in \mathcal{D}(\mathbb{R}^d)$ and consider the following expression:

$$t_\lambda(\phi) = \lim_{n\to\infty} \lambda^{-d} t_0((1-\vartheta_{2^n})\phi_{\lambda^{-1}}) \ .$$

Let $R, \epsilon > 0$ be such that $\mathrm{supp}(\phi) \subset \{x, |x| < R\}$ and $\vartheta(x) = 1$ for $|x| < \epsilon$. Then,

$$(1 - \vartheta(2^n x))\phi(\lambda^{-1}x) = 0 \ ,$$

whenever $2^{-n}\epsilon > \lambda R$. Let us choose $n_\lambda \in \mathbb{N}$ such that $2^{-n_\lambda}\epsilon > \lambda R > 2^{-(n_\lambda+1)}\epsilon$.

We have,

$$\begin{aligned} t_\lambda(\phi) &= \sum_{n=n_\lambda}^{\infty} \lambda^{-d} t_0((\vartheta_{2^n} - \vartheta_{2^{n+1}})\phi_{\lambda^{-1}}) \\ &= \sum_{n=n_\lambda}^{\infty} (2^n\lambda)^{-d} (t_0)_{2^{-n}}((\vartheta - \vartheta_2)\phi_{2^{-n}\lambda^{-1}}) \ . \end{aligned} \tag{37}$$

The set $\{(\vartheta - \vartheta_2)\phi_\mu, \mu < \text{const.}\}$ is bounded in $\mathcal{D}(\mathbb{R}^d\setminus\{0\})$. Hence for every $\omega' > \omega$ we find a constant $c > 0$ such that

$$|(t_0)_{2^{-n}}((\vartheta - \vartheta_2)\phi_{2^{-n}\lambda^{-1}})| \leq \ c\ 2^{n\omega'}, \qquad n \geq n_\lambda \ .$$

Inserting this estimate back to Eqn. (37) we have

$$
\begin{aligned}
|t_\lambda(\phi)| <\ & c\,\lambda^{-d} \sum_{n=n_\lambda}^{\infty} 2^{-n(d-\omega')} = \ c\,\lambda^{-d} \frac{2^{-n_\lambda(d-\omega')}}{1-2^{-(d-\omega')}} \\
& \leq \frac{c}{1-2^{-(d-\omega')}} \lambda^{-d} \left(\frac{2R}{\epsilon}\right)^{d-\omega'} \lambda^{d-\omega'} \leq c' \lambda^{-\omega'}
\end{aligned}
$$

for some constant $c' > 0$. This proves the assertion. ■

We now deal with the extension procedure in case a distribution has a finite scaling degree $\omega \geq d$. This extension procedure corresponds to renormalization in other schemes. To adhere more to the standard notation we introduce the degree of singularity $\rho \doteq \omega - d$. This is the analog of the degree of divergence of a Feynman diagram.

Let $\mathcal{D}_\rho(\mathbb{R}^d)$ be the set of all smooth functions of compact support which vanish of order ρ at the origin and let W be a projection from $\mathcal{D}(\mathbb{R}^d)$ onto $\mathcal{D}_\rho(\mathbb{R}^d)$. Since the orthogonal complement of $\mathcal{D}_\rho(\mathbb{R}^d)$ consists of the derivatives of the δ-function up to order ρ, W is of the form

$$
W\phi = \phi - \sum_{|\alpha| \leq \rho} \mathfrak{w}_\alpha \partial^\alpha \phi(0) \ , \tag{38}
$$

with $\mathfrak{w}_\alpha$ being smooth functions of compact support such that $\partial^\alpha \mathfrak{w}_\beta(0) = \delta^\alpha_\beta$.

Theorem 5.3. *Let $t_0 \in \mathcal{D}'(\mathbb{R}^d \backslash \{0\})$ have a finite scaling degree $\omega \geq d$. Then there exist extensions $t \in \mathcal{D}'(\mathbb{R}^d)$ of t_0 with the same scaling degree, and, given W, they are uniquely determined by their values on the test functions $\mathfrak{w}_\alpha$.*

Proof. Any $\phi \in \mathcal{D}(\mathbb{R}^d)$ can be uniquely decomposed as $\phi = \phi_1 + \phi_2$ where $\phi_1 = \sum_{|\alpha|\leq\rho} \mathfrak{w}_\alpha \partial^\alpha \phi(0)$ and $\phi_2 \in \mathcal{D}_\rho(\mathbb{R}^d)$. ϕ_2 has the form,

$$\phi_2(x) = \sum_{|\alpha|=[\rho]+1} x^\alpha \psi_\alpha(x) \ ,$$

with $\psi_\alpha \in \mathcal{D}(\mathbb{R}^d)$. We set

$$\langle t, \phi \rangle = \sum_{|\alpha|=[\rho]+1} \langle x^\alpha t_0, \psi_\alpha \rangle + \langle t, \phi_1 \rangle \ . \tag{39}$$

Since, by Lemma 5.1, $x^\alpha t_0$ has scaling degree equal to $\rho - [\rho] - 1 + d$ which is strictly smaller than d this term has a unique extension by Theorem 5.2.

We now prove that t has the same scaling degree as t_0. We write,

$$t_\lambda(\phi) = (t_0 \circ W)_\lambda(\phi) - \sum_{|\alpha|\leq\rho} t(\mathfrak{w}_\alpha) \left(\partial^\alpha \phi\right)(0) \lambda^{-d-|\alpha|} \ .$$

The second term clearly has scaling degree less or equal to $\rho + d = \omega$. The first term can be written in the form

$$(t_0)_\lambda(W\phi) + (t_0)_\lambda((W\phi_{\lambda^{-1}})_\lambda - W\phi) \ .$$

By assumption, the first term has scaling degree ω. To analyze the second term we write

$$((W\phi_{\lambda^{-1}})_\lambda - W\phi)(x) = \sum_{|\alpha|\leq\rho} \Big(\mathfrak{w}_\alpha(x) - \lambda^{-|\alpha|}\mathfrak{w}_\alpha(\lambda x)\Big) \left(\partial^\alpha \phi\right)(0) \ .$$

(Note that $(\mathfrak{w}_\alpha(\cdot) - \lambda^{-|\alpha|}\mathfrak{w}_\alpha(\lambda\ \cdot)) \in \mathcal{D}_\rho(\mathbb{R}^d)$). Using the identity,

$$\begin{aligned}(\mathfrak{w}_\alpha(x) - \lambda^{-|\alpha|}\mathfrak{w}_\alpha(\lambda x)) &= \int_\lambda^1 \frac{d}{d\mu}\mu^{-|\alpha|}\mathfrak{w}_\alpha(\mu x)\ d\mu \\ &= \int_\lambda^1 (\mu x_k(\partial_k \mathfrak{w}_\alpha)(\mu x) - |\alpha|\mathfrak{w}_\alpha(\mu x))\,\mu^{-|\alpha|-1}\ d\mu\ , \end{aligned} \tag{40}$$

we get, after a moment of reflection for the exchange of the order between integration and duality, that,

$$(t_0)_\lambda((W\phi_{\lambda^{-1}})_\lambda - W\phi) =$$

$$\sum_{|\alpha|\leq\rho} (\partial^\alpha\phi)\,(0) \int_\lambda^1 \mu^{-d-|\alpha|-1}(t_0)_{\lambda\mu^{-1}}((x_k\partial_k - |\alpha|)\mathfrak{w}_\alpha)d\mu\ .$$

The integrand can be estimated according to Lemma 5.1. Indeed, for any $\omega' > \omega$ we have,

$$\left|(t_0)_{\lambda\mu^{-1}}\left((x_k\partial_k - |\alpha|)\mathfrak{w}_\alpha\right)\ \right| \leq \text{const.}\left(\lambda^{-1}\mu\right)^{\omega'}\ ,$$

and therefore

$$\left|\int_\lambda^1 \mu^{-d-|\alpha|-1}(t_0)_{\lambda\mu^{-1}}\left((x_k\partial_k - |\alpha|)\mathfrak{w}_\alpha\right)d\mu\right| \leq \text{const.}\ \lambda^{-\omega'}\ \frac{1-\lambda^{\omega'-d-|\alpha|}}{\omega'-d-|\alpha|}\ ,$$

which proves the assertion. ■

The expert reader can now proceed from this point to study the renormalizability of any theory which admits space-time translation covariance. The ambiguity of the extension is given by terms localized over the origin. The coefficients of these terms can be fixed by additional requirements, as customary in perturbative quantum field theory. We refer the reader to [11] for more details.

During this process, one needs estimates on the scaling degrees of the arising distributions, corresponding to the power counting rules. In addition to Lemma 5.1 estimates on scaling degrees of products of distributions (provided they exist) are required. These can be obtained by explicit calculations (see e.g. the analogous estimates in [20]). Much more elegant is a general method which exploits a microlocal version of the scaling degree. This technique is actually necessary if one wants to generalize the methods above to generic manifolds. We shall describe it in the next section.

6. SURFACES OF UNIFORM SINGULARITY AND THE MICROLOCAL SCALING DEGREE

The generalization of the previous procedure to the case of submanifolds is what we really need in the treatment of perturbation theory on curved spaces. Indeed, the description given in Section 4 led to the notion of a scaling degree w.r.t. the small diagonal Δ_n of the topological product M^n. Here we classify the behaviour of distributions near some surface by a microlocal version of the scaling degree. We introduce two different notions. The first one, the (microlocal) scaling degree at a surface, involves only the surface under consideration, the second one, the transversal scaling degree, involves a fibration of the surface by transversal surfaces. The first notion behaves very nicely under tensor products and restrictions, whereas the second one admits an easy generalization of the extension procedure. As a matter of fact, the notions can be shown to be equivalent.

6.1. **Scaling degrees at submanifolds.** Let $\mathcal{M}$ be a smooth paracompact manifold of dimension d and t be a distribution in $\mathcal{D}'(\mathcal{M})$. Let $\mathcal{N} \subset \mathcal{M}$ be a submanifold such that the wave front set of t is orthogonal to the tangent bundle $T\mathcal{N}$ of $\mathcal{N}$, i.e.

for $(x,k) \in \mathrm{WF}(t)$ with $k \in T^*_x\mathcal{M}, x \in \mathcal{N}$,

$$\langle k, \xi \rangle = 0 \ , \qquad \forall \xi \in T_x\mathcal{N} \ . \tag{41}$$

Under these circumstances, t can be restricted to a sufficiently small submanifold $\mathcal{C} \subset \mathcal{M}$ which intersects $\mathcal{N}$ in a single point x_0, such that the intersection of their tangent spaces at x_0 is trivial and their sum spans the whole tangent space (the submanifolds $\mathcal{C}$ and $\mathcal{N}$ are *transversal*, see e.g. [27], symbolically $\mathcal{C} \pitchfork \mathcal{N}$). This is due to the fact that $\mathrm{WF}(t)$ does not intersect the conormal bundle $N^*\mathcal{C} = \{(x,k) \in T^*\mathcal{M} | \langle k, \xi \rangle = 0, \forall \xi \in T_x\mathcal{C}\}$ of $\mathcal{C}$. Namely, for $k \in T^*_{x_0}\mathcal{M}$, $(x_0, k) \in \mathrm{WF}(t)$ we have $\langle k, \xi \rangle = 0$ for $\xi \in T_{x_0}\mathcal{N}$, hence $\langle k, \xi \rangle \neq 0$ for some $\xi \in T_{x_0}\mathcal{C}$, thus $(x_0, k) \notin N^*\mathcal{C}$. But $\mathrm{WF}(t) \cap N^*\mathcal{C}$ is a closed conical set in $\dot{T}^*_{\mathcal{C}}\mathcal{M}$, hence its complement is an open conical neighbourhood of $\dot{T}^*_{x_0}\mathcal{M}$, in particular it contains a set $\dot{T}^*_{U_0}\mathcal{M}$ where U_0 is an open neighbourhood of x_0 in $\mathcal{C}$. By choosing $\mathcal{C} = U_0$ we arrive at the conclusion. So we proved,

Lemma 6.1. *Let $t \in \mathcal{D}'(\mathcal{M})$ be a distribution on a smooth manifold $\mathcal{M}$ and let $\mathcal{N}$ be a submanifold such that* $\mathrm{WF}(t) \perp T\mathcal{N}$*. Then t can be restricted to every sufficiently small submanifold $\mathcal{C}$ such that $\mathcal{N} \pitchfork \mathcal{C}$.*

The singularity of $t\!\restriction_{\mathcal{C}}$ at x_0 may be classified by a covariant extension of the notion of the scaling degree, or better by a slight extension which uses microlocal analysis. For the economy of the presentation we first look at the concept of scaling degree at some surface $\mathcal{N}$ which reduces for each transversal surface $\mathcal{C}$ to the scaling degree at the intersection point. This last will just be a pointwise reduction of the general case we proceed to discuss right now.

Let U be a star-shaped neighbourhood of the zero section $Z(T_{\mathcal{N}}\mathcal{M})$ and consider a map $\alpha : U \to \alpha(U) \subset \mathcal{N} \times \mathcal{M}$ which is a diffeomorphism onto its range and such that the following properties hold true

(*i*) $\alpha(x, 0) = (x, x)$, $x \in \mathcal{N}$;

(*ii*) $\alpha(T\mathcal{N} \cap U) \subset \mathcal{N} \times \mathcal{N}$;

(*iii*) $\alpha(x, \xi) \in \{x\} \times \mathcal{M}$, $x \in \mathcal{N}, \xi \in T_x\mathcal{M}$;

(*iv*) $d_\xi \alpha(x, \cdot)\restriction_{\xi=0} = \mathrm{id}_{T_x\mathcal{M}}$.

A concrete example of such a map α can be defined, whenever we consider the manifold $\mathcal{M}$ endowed with a (semi-)Riemannian metric, in terms of the exponential map, namely, $\alpha(x, \xi) \doteq (x, \exp_x \xi)$, provided the submanifold $\mathcal{N}$ is totally geodesic, as will be the case in our applications. In the general case, we shall call the set of all such maps by $\mathcal{Z}$.

Let $\alpha \in \mathcal{Z}$ and set $t^\alpha = (\mathbf{1} \otimes t) \circ \alpha$ on $\mathcal{D}'(U)$ and $t^\alpha_\lambda(x, \xi) \doteq t^\alpha(x, \lambda\xi)$, $0 < \lambda \leq 1$. Here, $\langle \mathbf{1} \otimes t, \phi \otimes \psi \rangle = \int \phi \cdot \langle t, \psi \rangle$ for test-densities $\phi \in \mathcal{D}_1(\mathcal{N})$ and $\psi \in \mathcal{D}_1(\mathcal{M})$. Since U is starshaped, $\lambda^{-1}U \supset U$ for $0 < \lambda \leq 1$, hence t^α_λ can be considered as a distribution on $\mathcal{D}_1(U)$.

As a preliminary step we have the following

Proposition 6.2. *For any* $t \in \mathcal{D}'(\mathcal{M})$ *which satisfies the hypothesis of Lemma 6.1, there exists a closed conic set* $\Gamma \subset \dot{T}^*U$ *such that*

(*i*) $\Gamma \perp T(T\mathcal{N} \cap U)$*;*

(*ii*) $\mathrm{WF}(t^\alpha_\lambda) \subset \Gamma$.

Proof. Since α maps $T\mathcal{N}\cap U$ into $\mathcal{N}\times\mathcal{N}$, its derivative $\alpha_* : TU \to T(\mathcal{N}\times\mathcal{M})$ maps $T(T\mathcal{N}\cap U)$ into $T(\mathcal{N}\times\mathcal{N})$. But $\mathrm{WF}(t) \perp T\mathcal{N}$ implies $\mathrm{WF}(\mathbf{1}\otimes t) \perp T(\mathcal{N}\times\mathcal{N})$, hence

$$\mathrm{WF}(t^\alpha) = \alpha^*\mathrm{WF}((\mathbf{1}\otimes t)\restriction_{\alpha(U)}) \perp \alpha_*^{-1}(T_{\alpha(U)}(\mathcal{N}\times\mathcal{N})) = T(T\mathcal{N}\cap U)\ .$$

Now,

$$\mathrm{WF}(t^\alpha_\lambda) = \{(x,\xi;k) \in T^*_{(x,\xi)}(U) | (x,\lambda\xi;k) \in \mathrm{WF}(t^\alpha)\}\ .$$

Here, we identified the cotangent spaces at the points (x,ξ) and $(x,\lambda\xi)$ by the isomorphism induced by the diffeomorphism $U \to \lambda U$, $(x,\xi') \to (x,\lambda\xi')$, $\xi' \in T_x\mathcal{M}$. Now, let $\xi \in T_x\mathcal{N}$ and $\eta \in T_{(x,\xi)}\mathcal{N}$. We may identify η with a vector in $T_{(x,\lambda\xi)}\mathcal{N}$ and observe that it is orthogonal to $\mathrm{WF}(t^\alpha)$ and hence also to $\mathrm{WF}(t^\alpha_\lambda)$. We now set $\Gamma = \overline{\cup_{0<\lambda\leq 1}\mathrm{WF}(t^\alpha_\lambda)}$, where the closure is performed within $\dot{T}^*U$. It remains to prove (i).

Let $(x,\xi_n;k_n) \in \mathrm{WF}(t^\alpha_{\lambda_n})$ be a convergent sequence in T^*U with limit $(x,\xi;k)$, $k \neq 0$ and $\xi \in T_x\mathcal{N}$. There is a corresponding sequence $(x,\lambda_n\xi_n;k_n) \in \mathrm{WF}(t^\alpha)$. Let $\lambda \in [0,1]$ be a limit point of the bounded sequence $\{\lambda_n\}_{n\in\mathbb{N}}$. Then, there is a subsequence converging to $(x,\lambda\xi;k) \in \mathrm{WF}(t^\alpha)$. But $\lambda\xi \in T_x\mathcal{N}$, hence we have $k \perp T_{(x,\lambda\xi)}(T\mathcal{N})$. If we again identify the tangent spaces at $(x,\lambda\xi)$ and (x,ξ) we obtain the desired result. ■

Choosing first any map $\alpha \in \mathcal{Z}$ we are ready for the following:

Definition 6.3. *A distribution $t \in \mathcal{D}'(\mathcal{M})$ has the microlocal scaling degree ω at a submanifold $\mathcal{N}$ w.r.t. a closed conical set $\Gamma_0 \subseteq \dot{N}^*\mathcal{N}$, symbolically $\omega = \mu\mathrm{sd}^{\Gamma_0}_{\mathcal{N}}(t,\alpha)$, if,*

- (i) *there exists a closed conic set* $\Gamma \subset \dot{T}^*(T_{\mathcal{N}}\mathcal{M})$ *with the properties stated in Proposition 6.2, with the first one replaced by* $\Gamma\restriction_{T\mathcal{N}} \subset \alpha^*(Z(T^*\mathcal{N}) \times \Gamma_0)$,
- (ii) ω *is the infimum of all those* ω' *for which,*

$$\lim_{\lambda \downarrow 0} \lambda^{\omega'} t^{\alpha}_{\lambda} = 0 \ , \tag{42}$$

in the sense of the Hörmander topology on $\mathcal{D}'_{\Gamma}(T_{\mathcal{N}}\mathcal{M})$.

Now, depending on the position of Γ_0 one can give different refined versions of the microlocal scaling degree. Indeed, when the inclusion in $\dot{N}^*\mathcal{N}$ is proper we speak of the *strict* microlocal scaling degree. When $\Gamma_0 \equiv \dot{N}^*\mathcal{N}$, we call it simply the scaling degree at the submanifold, symbolically $\mathrm{sd}_{\mathcal{N}}(t)$. Moreover, when $\Gamma_0 = \emptyset$ we speak of the *smooth* scaling degree.

The definition seems to depend on the choice of the map $\alpha \in \mathcal{Z}$. In our concrete case we could make use of the metric to choose a canonical diffeomorphism α in terms of the exponential map, but for reasons which will become clear in the following it is helpful to proof its independence.

Before coming to that point we show an example for the computation of the scaling degree which is relevant for the physical discussion, namely, the generalization of the example (3) in subSection 5.1, the Feynman propagator E_F, to curved space-time. The Feynman propagator is considered as a distribution on $M \times M$, and we are interested in the microlocal scaling degree at the diagonal $\Delta_2 \subset M \times M$. Indeed, the wave front set of E_F is orthogonal to the tangent bundle of the diagonal. We choose $\alpha : T_{\Delta_2}M^2 \to \Delta_2 \times M^2 \simeq M \times M^2$ as $\alpha(x, \xi_1, \xi_2) = (x, \exp_x \xi_1, \exp_x \xi_2)$ and obtain as on Minkowski space $\mathrm{sd}_{\Delta_2}(E_F) = d - 2$. A similar result holds for the 2-point Wightman distribution ω_2. So,

Lemma 6.4. *The microlocal scaling degrees of the 2-point Wightman distribution* ω_2 *at the diagonal* Δ_2 *w.r.t.* $\Gamma_0 = \{(x, k; x, -k) | x \in M, k \in \partial V_+, k \neq 0\}$ *is given by* $\mu\mathrm{sd}^{\Gamma_0}_{\Delta_2}(\omega_2) = d - 2$*, where* d *is the dimension of the space-time.*

6.2. **Invariance and properties for the scaling degrees.**

Let us then choose two maps $\alpha_1, \alpha_2 \in \mathcal{Z}$ and state the following:

Proposition 6.5. *Let* $t \in \mathcal{D}'(\mathcal{M})$*. Let* $\omega_i = \mu\mathrm{sd}^{\Gamma_0}_{\mathcal{N}}(t, \alpha_i)$*,* $i = 1, 2$ *be the microlocal scaling degrees w.r.t.* $\mathcal{N}$ *and* Γ_0 *resp. for the two arbitrarily chosen maps. Then* $\omega_1 = \omega_2$*.*

Proof. It is simple to check that

$$t^{\alpha_2}_\lambda(\phi) = t^{\alpha_1}_\lambda(\phi \circ \beta^{-1}_\lambda) \ , \qquad \forall \phi \in \mathcal{D}_1(T_\mathcal{N}\mathcal{M}) \ , \tag{43}$$

where $\beta_\lambda(x, \xi) = \lambda^{-1}\beta(x, \lambda\xi)$ and $\beta = \alpha^{-1}_1 \circ \alpha_2$. Now, assume ω_1 is the scaling degree for t w.r.t. $\mathcal{N}$ and Γ_0 associated with α_1. We should proof that Eqn. (42) for α_2 converges in the sense of $\mathcal{D}'(T_\mathcal{N}\mathcal{M})$ as well, at the same rate as $\lambda \downarrow 0$.

The convergence in the sense of distributions is simple. Indeed, if $\mathrm{supp}(\phi) \subset K$, K a compact subset, then there exists a λ_0 such that $\beta_\lambda(\mathrm{supp}(\phi)) \subset K$ for all $\lambda \leq \lambda_0$. Hence it suffices by Banach-Steinhaus principle to proof that the family $\{\phi \circ \beta^{-1}_\lambda | 0 < \lambda \leq \lambda_0\}$ is bounded, uniformly in λ, in $\mathcal{D}_1(K)$ w.r.t. the family of continuous seminorms which gives the appropriate Fréchet topology. This check proceeds easily from the chain rule and the verifiable fact that the only contribution comes from the 0-th and 1-st order derivatives w.r.t. β^{-1}_λ. In the limit they are the only terms which survive giving resp. the identity map on the bundle and the derivative of the identity map. Hence, we get the same rate of convergence as far as plain distribution convergence is concerned.

A little bit trickier is the convergence in the sense of seminorms for **M2** (*b*). Starting again from Eqn. (43), via the multiplication of a smooth test function of compact support ψ such that $\psi \equiv 1$ on a small neighbourhood of supp(ϕ), we have that ψt is of compact support and then, by a partition of unity with functions with support on charts, that we are working on $\mathbb{R}^\delta \times \mathbb{R}^d$, where δ is the dimension of the submanifold $\mathcal{N}$. Now, let us multiply the test function ϕ with the term $\exp(i\langle k,\ \cdot\ \rangle)$, and use inverse Fourier transform to get,

$$\begin{aligned}\widehat{\phi t_\lambda^{\alpha_2}}(k) &= {\beta_\lambda}^* t_\lambda^{\alpha_1}(\phi \exp(i\langle k,\ \cdot\ \rangle)) \\ &= \int \widehat{\psi t_\lambda^{\alpha_1}}(p) I_\phi(p,k;\beta_\lambda) d^{d+\delta}p\ ,\end{aligned}$$

where,

$$I_\phi(p,k;\beta_\lambda) = \int e^{-i(\langle \beta_\lambda(\xi),p\rangle - \langle \xi,k\rangle)} \phi(\xi)\ d^{d+\delta}\xi\ ,$$

where in all these expressions the coordinates ξ are the local coordinates of $T_\mathcal{N}\mathcal{M}$ and k and p are their dual coordinates.

We use the idea of the proof for the stationary phase Theorem, see for instance Theorem 7.7.1 in Hörmander books [35]. Because of $\beta_\lambda \to \text{id}$ for $\lambda \to 0$ the oscillatory integral I_ϕ falls off rapidly outside of any conical neighbourhood of the diagonal $p = k$ in $\mathbb{R}^{d+\delta} \times \mathbb{R}^{d+\delta}$, uniformly for λ sufficiently small, i.e. for every $\epsilon > 0$ there exists a $\lambda_0 > 0$ such that for every $N \in \mathbb{N}$

$$\sup_{0<\lambda<\lambda_0} \sup_{|p-k|>\epsilon|k|} (1+|p|+|k|)^N |I_\phi(p,k,\beta_\lambda)| < \infty\ . \tag{44}$$

Now let $\Gamma \subset \mathbb{R}^{d+\delta}$ be a closed cone such that

$$\sup_C (1+|p|)^N |\widehat{\psi t_\lambda^{\alpha_1}}(p)| \lambda^\omega \to 0 \ , \ \lambda \to 0 \ , \tag{45}$$

for all closed cones C with $C \cap \Gamma = \emptyset$ and all $N \in \mathbb{N}$.

We now want to show that the same property holds for $\widehat{\phi t_\lambda^{\alpha_2}}$. So let C be a closed cone such that the closed cone

$$C' = \{p \in \mathbb{R}^{d+\delta}, |p-k| \le \epsilon |k| \text{ for some } k \in C\} \tag{46}$$

does not intersect Γ. Then we split the region of integration over p into the parts $|p-k| \le \epsilon|k|$ and the rest. In the first region we can estimate k by p and use the fast decay of $(1+|p|)^N |\widehat{\psi t_\lambda^{\alpha_1}}(p)|$ within C' and the polynomial boundedness of I_ϕ; in the second region the polynomial boundedness of $(1+|p|)^N |\widehat{\psi t_\lambda^{\alpha_1}}(p)|$ and the fast decay of I_ϕ (44). This proves the desired estimate for $\widehat{\phi t_\lambda^{\alpha_2}}$. ■

The microlocal scaling degree μsd has similar properties as the scaling degree, as described in Lemma 5.1. In addition, one finds the following two properties:

Lemma 6.6. *Let $t_1, t_2 \in \mathcal{D}'(\mathcal{M})$ with μsd ω_1 resp. ω_2 at $\mathcal{N} \subset \mathcal{M}$, w.r.t. Γ_0^1 resp. Γ_0^2 and such that $Z(N^*\mathcal{N}) \notin (\Gamma_0^1 + \Gamma_0^2)$. Then the pointwise product $t_1 t_2$ exists in a small neighbourhood of $\mathcal{N}$ and has the microlocal scaling degree $\omega \le \omega_1 + \omega_2$ at $\mathcal{N}$ w.r.t. $\Gamma_0 = \Gamma_0^1 \cup \Gamma_0^2 \cup (\Gamma_0^1 + \Gamma_0^2)$.*

Proof. By assumption, the wave front sets of $t_{1,\lambda}^\alpha$ and $t_{2,\lambda}^\alpha$, some $\alpha \in \mathcal{Z}$, on a sufficiently small neighbourhood of $\mathcal{N}$ satisfy the condition $(\mathrm{WF}(t_{1,\lambda}^\alpha) + \mathrm{WF}(t_{2,\lambda}^\alpha)) \cap Z(T^*(T_\mathcal{N}\mathcal{M})) = \emptyset$, hence their product exists there by **M1**. Because of the sequential continuity of the products in the Hörmander topology **M2**, the microlocal scaling degree is given by the sum w.r.t. the stated conic region as follows from **M1** and does not depend on the choice of the map α. ■

The following nice property follows from the sequential continuity of the restriction operator to submanifolds **M3**:

Lemma 6.7. *Let* $\mathcal{N}_1$ *be a submanifold of* $\mathcal{N}$*, and let* $t \in \mathcal{D}'(\mathcal{M})$ *have the microlocal scaling degree* ω *at* $\mathcal{N}$ *w.r.t.* Γ_0*. Then the microlocal scaling degree of* t *at* $\mathcal{N}_1$ *w.r.t. the restriction* Γ_1 *of* Γ_0 *to* $\mathcal{N}_1$ *is less or equal to* ω*,* $\mu\mathrm{sd}^{\Gamma_1}_{\mathcal{N}_1}(t) \leq \mu\mathrm{sd}^{\Gamma_0}_{\mathcal{N}}(t)$*.*

A last word is devoted to a pointlike trivialization of the above procedure. This case can be derived straightforwardly by considering $\mathcal{N} \equiv \{p\}$, where $p \in \mathcal{M}$ is a generic point and thought of as (a rather singular case of) a submanifold.

The translation to this simpler case is done via the following correspondence between geometrical and analytical quantities:

$$U \in T_{\mathcal{N}}\mathcal{M} \qquad \longrightarrow \qquad U_p \in T_p\mathcal{M} \ ,$$

$$\alpha : U \to \mathcal{N} \times \mathcal{M} \qquad \longrightarrow \qquad \alpha : U_p \to \mathcal{M} \ , \quad (i) \text{ and } (iv) \text{ valid,}$$

$$t^\alpha = (\mathbf{1} \otimes t) \circ \alpha \qquad \longrightarrow \qquad t^\alpha = t \circ \alpha \ ,$$

$$\Gamma \subset \dot{T}^*U \ , \quad \Gamma \perp T(T\mathcal{N} \cap U) \quad \longrightarrow \qquad \Gamma \subset \dot{T}^*U_p \ , \quad \Gamma \perp TU_p \ ,$$

$$\Gamma_0 \subset \dot{N}^*\mathcal{N} \qquad \longrightarrow \qquad \Gamma_p \subset \dot{T}_p^*\mathcal{M} \ .$$

6.3. **Transversal scaling degree.** Instead of blowing up distributions on $\mathcal{M}$ to distributions on $\mathcal{N} \times \mathcal{M}$ in the definition of the scaling degree, one could also use a fixed fibration of a neighbourhood of $\mathcal{N}$ in $\mathcal{M}$ by transversal surfaces. For this purpose we decompose $T_{\mathcal{N}}\mathcal{M}$ into complementary subbundles, $T_{\mathcal{N}}\mathcal{M} = T\mathcal{N} + C$. The map $\alpha_C := \pi_2 \circ \alpha\restriction_{C \cap V}$, with the projection $\pi_2 : \mathcal{N} \times \mathcal{M} \to \mathcal{M}$ onto the second factor and with V being a sufficiently small neighbourhood of the zero section $Z(T_{\mathcal{N}}\mathcal{M})$, is then a diffeomorphism onto some neighbourhood of $\mathcal{N}$. The images of the fibers of C are transversal surfaces. The transversally (w.r.t. α and C) scaled distribution is then defined by

$$t_{\lambda,\perp}(x,\eta) = t \circ \alpha_C(x, \lambda\eta) \ , \ (x,\eta) \in C \ . \tag{47}$$

The *transversal* microlocal scaling degree may then be defined as the infimum of all $\omega \in \mathbb{R}$ such that the sequence $\lambda^{\omega} t_{\lambda,\perp}$ converges to zero within $\mathcal{D}'_{\Gamma_C}(C)$, with a closed conical set $\Gamma_C \subset \dot{T}^*C$ with $\Gamma_C\restriction_{Z(C)} = \alpha_C^*(\Gamma_0)$. Fortunately, it turns out that this new concept of a scaling degree at a surface coincides with the old one. Thus, in particular, the transversal scaling degree does not depend on the choice of the fibration.

Proposition 6.8. *The transversal microlocal scaling degree defined above coincides with the microlocal scaling degree defined in (42).*

Proof. We may restrict ourselves to a sufficiently small neighbourhood of a point at the surface $\mathcal{N}$. In suitable coordinates, $\mathcal{M}$ is a subset of $\mathbb{R}^{\delta} \times \mathbb{R}^{d-\delta}$ where the first factor corresponds to $\mathcal{N}$ and the second factor to the transversal surfaces. $T_{\mathcal{N}}\mathcal{M}$ is a subset of $\mathbb{R}^{\delta} \times \mathbb{R}^{\delta} \times \mathbb{R}^{d-\delta}$ with the first factor corresponding to $\mathcal{N}$, the second to the tangent spaces of $\mathcal{N}$ and the third one to the fibers of the transversal bundle C. The map α may be chosen as

$$\alpha(x, \xi, \eta) = (x, x + \xi, \eta) \ . \tag{48}$$

Then α_C becomes the identity. The distribution t may be replaced by a distribution with compact support, and the factor $\mathbf{1}$ in the blow up of t may be replaced by a test function $\chi \in \mathcal{D}(\mathcal{N})$. For the Fourier transforms we then obtain

$$\widehat{t_{\lambda}}(p, q, k) = \lambda^{-d} \widehat{\chi}(p - \lambda^{-1} q) \widehat{t}(\lambda^{-1} q, \lambda^{-1} k) \ , \tag{49}$$

and

$$\widehat{t_{\lambda,\perp}}(p,k) = \lambda^{d-\delta}\widehat{t}(p,\lambda^{-1}k) \ . \tag{50}$$

Furthermore, using a corresponding trivialization of the respective cotangent bundles, we may identify $\Gamma_0 = \Gamma_C$ with $\{0\} \times K$, where K is a closed cone in $\mathbb{R}^{d-\delta}$, considered as the transversal part of the cotangent space, and Γ with $\{0\}\times\{0\}\times K$. The convergence of t_λ may be discussed in terms of seminorms of the form $\int_V (1+|p|)^N |t_\lambda|$ with, respectively, conical neighbourhoods V of Γ and some $N \in (-\mathbb{N})$ and closed conical sets V in the complement of Γ and all $N \in \mathbb{N}$. Since χ is strongly decreasing, these seminorms of t_λ can be estimated in terms of the corresponding seminorms of $t_{\lambda,\perp}$. ■

6.4. **Extension of distributions to surfaces.** We now want to apply these concepts to the extension problem of distributions $t \in \mathcal{D}'(\mathcal{M}\backslash\mathcal{N})$. The wavefront set of the extension shall be orthogonal to $T\mathcal{N}$, hence a necessary condition is that this holds true for the closure of $\mathrm{WF}(t)$ within $T^*\mathcal{M}$. We extend the notion of the microlocal scaling degree to such distributions in an analogous way as in the extension problem to a single point. Namely, for an arbitrary function $\chi \in \mathcal{E}(\mathcal{M})$ with $\mathrm{supp}\chi \cap \mathcal{N} = \emptyset$, $(1\otimes\chi)\circ\alpha\cdot t_\lambda^\alpha$ can be considered as a distribution on $U \subset T_{\mathcal{N}}\mathcal{M}$. The microlocal scaling degree of t at $\mathcal{N}$ is then defined in terms of all sequences so obtained.

We choose a fibration of a neighbourhood of $\mathcal{N}$ by transversal surfaces. It is easy to see that if t has a scaling degree ω at $\mathcal{N}$, than its restriction to a transversal surface has a scaling degree at the point of intersection with $\mathcal{N}$ which is less or equal to ω. We therefore obtain the corresponding extension theorem.

Theorem 6.9. *Let* $\mathcal{N}$ *be a submanifold of the manifold* $\mathcal{M}$*, and let* $t_0 \in \mathcal{D}'(\mathcal{M}\backslash\mathcal{N})$*.*

- *(i) If* $\mathrm{sd}_{\mathcal{N}}(t_0) < \mathrm{codim}(\mathcal{N})$ *there exists a unique distribution* $t \in \mathcal{D}'(\mathcal{M})$ *extending* t_0 *with* $\mathrm{sd}_{\mathcal{N}}(t) = \mathrm{sd}_{\mathcal{N}}(t_0)$
- *(ii) If* $\mathrm{codim}(\mathcal{N}) \le \mathrm{sd}_{\mathcal{N}}(t_0) < \infty$ *there exist extensions* $t \in \mathcal{D}'(\mathcal{M})$ *with* $\mathrm{sd}_{\mathcal{N}}(t) = \mathrm{sd}_{\mathcal{N}}(t_0)$*. They are uniquely characterized by their values on some closed subspace of* $\mathcal{D}(\mathcal{M})$ *which is complementary to the space of all test functions which vanish on* $\mathcal{N}$ *up to order* $\mathrm{sd}_{\mathcal{N}}(t_0) - \mathrm{codim}(\mathcal{N})$*.*

Proof. According to Theorems 5.2 and 5.3 there exist extensions of the restrictions of t_0 to every transversal surface with the same scaling degree. We have to show that they are restrictions of a unique distribution t on $\mathcal{M}$ with the same microlocal scaling degree. We first fix some normal fibration as described above and consider t_0 as a distribution on $U\backslash\{0\}$ with a neighbourhood U of the zero section of C. We perform the construction of t at all fibers, by choosing a smooth function $\vartheta \in \mathcal{E}(C)$ which is equal to 1 in a neighbourhood of the zero section and whose restrictions to every fiber have compact support. Moreover, we choose smooth functions $\mathfrak{w}_\beta \in \mathcal{E}(U)$, also with compact support on each fiber, which satisfy the condition $\partial_\xi^\gamma \mathfrak{w}_\beta(x,0) = \delta_\beta^\gamma$ where ξ denotes the variable in the fiber over $x \in \mathcal{N}$. We may take $\mathfrak{w}_\beta = \mathfrak{w}\xi^\beta/\beta!$ with some function $\mathfrak{w}$ which is identical to 1 in a neighbourhood of the zero section.

We set $\rho = \mathrm{sd}_{\mathcal{N}}(t_0) - \mathrm{codim}(\mathcal{N})$, $\vartheta_{\lambda^{-1}}(x,\xi) = \vartheta(x,\lambda^{-1}\xi)$ and $W\phi(x,\xi) = \phi(x,\xi) - \sum_{|\beta|\le\rho} \mathfrak{w}_\beta(x,\xi)\partial_\xi^\beta \phi(x,0)$. Let $\Gamma_C = \overline{\mathrm{WF}(t_0)} \cup N^*\mathcal{N}$. We already know that the sequence $t_n = t_0(1-\vartheta_{2^n}) \circ W$ converges on every fiber, and it is easy to see that it converges weakly in $\mathcal{D}'(U)$. We now want to show that the wave front set of t is perpendicular to $T\mathcal{N}$. For this purpose we show that the above sequence converges even in $\mathcal{D}'_{\Gamma_C}(U)$, i.e. that for every pseudodifferential operator A whose wave front set does not intersect Γ_C (hence At_n is smooth) the sequence At_n converges in the sense of smooth functions. For a pseudodifferential operator with smooth kernel the argument is essentially the same as for the weak convergence, hence we may restrict ourselves to pseudodifferential operators whose kernels have support in a sufficiently small neighbourhood of the diagonal of $U \times U$ (only here singularities may occur).

According to the discussion of condition **M2**(b), we may equivalently look at the Fourier transform of $\chi t_0(\vartheta_{2^m} - \vartheta_{2^{m+1}})$ where χ is a test function with sufficiently small support which does not vanish at some point $x_0 \in \mathcal{N}$. Introducing suitable local coordinates in a neighbourhood of x_0, we find

$$[\chi t_0(\vartheta_{2^m} - \vartheta_{2^{m+1}})]^\wedge(p,k) = 2^{-m(d-\delta)} \left[(\chi t_0)_{2^{-m},\perp}(\vartheta - \vartheta_2)\right]^\wedge(p, 2^{-m}k) \ ,$$

where the symbol $[\,\cdot\,]^\wedge$ means Fourier transform, the p's are the dual coordinates w.r.t. points $x \in \mathcal{N}$ and similarly the k's are dual coordinates w.r.t. the points $\xi \in C_x$ of the fibers C_x of C and finally δ is the dimension of $\mathcal{N}$.

By the assumption on the scaling degree of t_0 at $\mathcal{N}$ we know that for all $\varepsilon > 0$, $N \in \mathbb{N}$ and $\omega > \mathrm{sd}_{\mathcal{N}}(t_0)$ there exists some $c > 0$ such that

$$|[(\chi t_0)_{2^{-m},\perp}(\vartheta - \vartheta_2)]^\wedge(p,k)| \leq c\; 2^{m\omega}(1 + |p| + |k|)^N \ , \tag{51}$$

for all (p,k) with $|p| > \varepsilon |k|$ (i.e. outside of a certain conical neighbourhood of the normal bundle $\{(p,k), p = 0\}$). Therefore, the sequence

$$\sup_{|p|>\varepsilon|k|} |[\chi t_0(\vartheta_{2^m} - \vartheta_{2^{m+1}})]^\wedge(p,k)|(1 + |p| + |k|)^N \ , \tag{52}$$

is summable for $\omega < (d-\delta)$. If $\mathrm{sd}_{\mathcal{N}}(t_0) < (d-\delta)$ such an ω exists, and we conclude, that the sequence $t_0(1-\vartheta_{2^m})$ converges in $\mathcal{D}'_{\Gamma_C}(M)$.

In case of $\mathrm{sd}_{\mathcal{N}}(t_0) \geq (d-\delta)$ we have to apply the W-operation defined above, which, for χ with sufficiently small support, reduces to a subtraction of the Taylor series up to order $[\rho]$. We obtain from the integral formula for the remainder in the Taylor expansion

$$[\chi t_0(\vartheta_{2^m}-\vartheta_{2^{m+1}})\circ W]^\wedge(p,k) =$$
$$\int_0^1 \sum_{|\beta|=[\rho]+1} [\chi t_0(\vartheta_{2^m}-\vartheta_{2^{m+1}})\xi^\beta]^\wedge(p,\mu k)k^\beta \frac{(1-\mu)^{[\rho]}([\rho]+1)}{\beta!} d\mu \ .$$

Since $\chi t_0 \xi^\beta$ has scaling degree $\mathrm{sd}_{\mathcal{N}}(t_0) - [\rho] - 1 < (d-\delta)$, we may use the same estimate as before and find that the sequence $t_0(1-\vartheta_{2^m})\circ W$ converges in $\mathcal{D}'_{\Gamma_C}(M)$.

It remains to prove the stability of the scaling degree under the extension procedure. The argument is a straightforward combination of the techniques in the corresponding proofs in Section 5 and the arguments above and is therefore omitted.

■

7. Extension to the Diagonal and Renormalization

We come to the main point, namely to prove that the inductive analysis of Section 4 closes when supplemented by the information about the scaling degrees and gives well defined operator-valued distributions T_n all over the space M^n. This process is what it is usually called “renormalization.”

Toward this goal we must show that the scaling degree for the distributions of the nth order of the induction can be estimated in terms of those of lower orders.

As explained before it is sufficient to do it for the numerical distributions

$$^0t(x_1,\ldots,x_n)=\omega(^0T(x_1,\ldots,x_n))\ , \tag{53}$$

where ω is our reference Hadamard state. According to (31) ^0t is a finite sum of terms

$$f_I(x)t^I(x_I)t^{I^c}(x_{I^c})\prod_{(i,j)\in I\times I^c}\omega_2(x_i,x_j)^{a_{ij}}\ , \tag{54}$$

with nonnegative integers a_{ij} and a smooth partition of unity of $M^n\backslash\Delta_n$ $f_I\in\mathcal{E}(M^n\backslash\Delta_n)$ with $\mathrm{supp}f_I\subset\mathcal{C}_I$. M^n inherits a natural metric from M, and all partial diagonals Δ_I are totally geodesic submanifolds. The map $\alpha: TM^n\to M^n\times M^n$ may therefore be defined by $\alpha(x,\xi)=(x,\exp_x\xi)$. Then all restrictions of α to partial subdiagonals, $\alpha_I=\alpha\restriction_{T_{\Delta_I}M^n}$ satisfy the conditions before Proposition 6.4. We can choose the functions f_I with smooth scaling degree 0 at the small diagonal. We then may consider all factors in (54) as distributions on M^n. According to Lemma 6.7, their microlocal scaling degrees with respect to Δ_n are bounded from above by their microlocal scaling degrees with respect to the respective partial diagonals. Moreover the convex combinations of the respective conical subsets of the conormal bundle of the small diagonal do not meet the zero section. Hence, by Lemma 6.6, the scaling degree of the distribution in (54) is bounded by

$$\omega=\mathrm{sd}_{\Delta_I}(t_I)+\mathrm{sd}_{\Delta_{I^c}}(t_{I^c})+\sum_{ij}a_{ij}(d-2)\ . \tag{55}$$

From this we get the formula

$$\mathrm{sd}_{\Delta_n}(\omega(T(:\prod_{i=1}^{n}\varphi^{l_i}(x_i):))) \leq \sum_{i=1}^{n} l_i \frac{d-2}{2} \ . \tag{56}$$

We thus obtain

Theorem 7.1 (Main Theorem)**.** *All polynomially interacting quantum field theories based on the the scalar field on $d \geq 2$ dimensional globally hyperbolic spacetimes follow the same short-distance perturbative classification as on their respective Minkowskian cases.*

We recall that for simplicity we have considered only pure monomials as interacting terms, i.e. without derivatives, multiple interacting fields and so forth. The general case however can be derived straightforwardly from our construction and we leave the task to the reader.

We close this section by specifying the choices which have to be made in the abstract geometrical setting of Section 6. First we choose the normal fibration of M^n in a neighbourhood of Δ_n which was used in the proof of the Theorem 6.9 for the construction of the extension. Let $N\Delta_n$ be the orthogonal complement of $T\Delta_n$ in $T_{\Delta_n}M^n$ w.r.t. the metric in M^n,

$$N\Delta_n = \{(x, \xi_1, \dots, \xi_n) \in T_{\Delta_n}M^n, \sum \xi_i = 0\} \ . \tag{57}$$

Then $\pi_2 \circ \alpha\restriction_{N\Delta_n}$ describes the desired fibration,

$$\pi_2 \circ \alpha(x, \xi_1, \dots, \xi_n) = (\exp_x \xi_1, \dots, \exp_x \xi_n) \ . \tag{58}$$

x may be considered as the center of mass of the points $x_i = \exp_x \xi_i$, and the tangent vectors ξ_i with the constraint $\sum \xi_i = 0$ play the rôle of relative coordinates. We further choose a smooth function $\mathfrak{w}$ on TM which is equal to 1 on a neighbourhood of the zero section and has compact support on each fiber and use the function

$$\mathfrak{w}_n(x, \xi_1, \ldots, \xi_n) = \prod \mathfrak{w}(x, \xi_i) \ , \tag{59}$$

in the extension to the diagonal Δ_n.

By these conventions we get a reference definition of time-ordered products for all Wick products which involves only the chosen Hadamard state and the function $\mathfrak{w}$. The algebra of interacting fields can then be defined by choosing the coefficients in the Lagrangian.

8. On the Definition of the Net of Local Algebras of Observables

In the preceding Section we finished the construction of time-ordered products of Wick polynomials of the free fields. We now want to show that this already gives the full net of local algebras of observables (within perturbation theory). An "adiabatic limit," whatever this might mean on a curved space-time, is not required. Actually, these observations are not completely new. In [52] it was already observed that the local S-matrices of the Stückelberg-Bogoliubov-Epstein-Glaser approach give a local net of observables.

Let $\mathcal{W}$ be the set of Wick polynomials with coefficients from $\mathcal{D}(M)$. So every $A \in \mathcal{W}$ is an operator valued distribution with compact support which is relatively local to the free field. Our starting relation is the causal factorization (2)

$$S(A + B + C) = S(A + B)S(B)^{-1}S(B + C) \ , \tag{60}$$

for $A, B, C \in \mathcal{W}$, whenever $\operatorname{supp}(A)$ is later than $\operatorname{supp}(C)$.

Now let $\mathcal{L}$ be our interaction Lagrangian. Then $g\mathcal{L} \in \mathcal{W}$ for $g \in \mathcal{D}(M)$. We may define observables with respect to the interaction $g\mathcal{L}$ by Bogoliubov formula

$$S_{g\mathcal{L}}(A) = S(g\mathcal{L})^{-1} S(g\mathcal{L} + A) \ . \tag{61}$$

We now show that the interacting observables depend only locally on the interaction. More precisely, we have the following

Proposition 8.1. *Let* $\mathcal{O}$ *be a causally closed region, and let the test functions* g *and* g' *coincide on some neighbourhood of* $\mathcal{O}$*. Then there exists a unitary* V *such that for all* $A \in \mathcal{W}$ *with* $\operatorname{supp}(A) \subset \mathcal{O}$

$$V S_{g\mathcal{L}}(A) V^{-1} = S_{g'\mathcal{L}}(A) \ . \tag{62}$$

Proof. We may split $g' - g = a + b$ where a does not intersect the past of $\mathcal{O}$ and b not the future. Then $\operatorname{supp}(a\mathcal{L})$ is later than $\operatorname{supp}(A)$, hence from (60) we find

$$S(g'\mathcal{L} + A) = S(g'\mathcal{L}) S((g+b)\mathcal{L})^{-1} S((g+b)\mathcal{L} + A) \ , \tag{63}$$

thus $S_{g'\mathcal{L}}(A) = S_{(g+b)\mathcal{L}}(A)$. Moreover, $\operatorname{supp}(A)$ is later than $\operatorname{supp}(b\mathcal{L})$, hence

$$S((g+b)\mathcal{L} + A) = S(g\mathcal{L} + A) S(g\mathcal{L})^{-1} S((g+b)\mathcal{L}) \ . \tag{64}$$

Hence we obtain (62) with $V = S_{g\mathcal{L}}(b\mathcal{L})^{-1}$. ■

We conclude that the algebra $\mathfrak{A}_{g\mathcal{L}}(\mathcal{O})$ which is generated by $S_{g\mathcal{L}}(A)$, $\mathrm{supp}(A) \subset \mathcal{O}$ with $g \equiv 1$ in a neighbourhood of $\mathcal{O}$, is up to unitary equivalence uniquely fixed by $\mathcal{L}$.

We may formalize the construction in the following way. Let $\Theta(\mathcal{O})$ for a causally closed compact region $\mathcal{O}$ be the set of test functions which equal unity on a neighbourhood of $\mathcal{O}$. Consider for $g, g' \in \Theta(\mathcal{O})$ the set $\mathcal{V}_{gg'}(\mathcal{O})$ of unitaries V which satisfy the intertwining relation

$$VS_{g\mathcal{L}}(A) = S_{g'\mathcal{L}}(A)V \ , \ \mathrm{supp}(A) \subset \mathcal{O} \ . \tag{65}$$

The algebra of observables $\mathfrak{A}_{\mathcal{L}}(\mathcal{O})$ can now be defined as the algebra of covariantly constant sections of the bundle

$$\bigcup_{g \in \Theta(\mathcal{O})} \{g\} \times \mathfrak{A}_{g\mathcal{L}}(\mathcal{O}) \ . \tag{66}$$

Here, a section $A = (A_g)_{g\in\Theta(\mathcal{O})}$ is called covariantly constant if

$$VA_g = A_{g'}V \ , \qquad \forall V \in \mathcal{V}_{gg'}(\mathcal{O}) \ . \tag{67}$$

$\mathfrak{A}_{\mathcal{L}}(\mathcal{O})$ contains for example the elements $S_{\mathcal{L}}(A) = (S_{g\mathcal{L}}(A))_{g\in\Theta(\mathcal{O})}$. To complete the construction of the net of local algebras of observables we have to fix the imbeddings $i_{\mathcal{O}_2\mathcal{O}_1} : \mathfrak{A}_{\mathcal{L}}(\mathcal{O}_1) \to \mathfrak{A}_{\mathcal{L}}(\mathcal{O}_2)$ for $\mathcal{O}_1 \subset \mathcal{O}_2$. But this structure is inherited from the fibers and may be defined by the restriction of the section from $\Theta(\mathcal{O}_1)$ to $\Theta(\mathcal{O}_2)$.

9. SUMMARY AND OUTLOOK

We have proven that renormalization on curved backgrounds can be done in close analogy to renormalization on Minkowski space, and that the removal of singularities follows the well known power counting rules. This result was expected since the ultraviolet behaviour on smooth manifolds should be essentially identical to that on Minkowski space. But we had to overcome two major obstacles: On a generic space-time there is no reason to expect a decent infrared behaviour, hence we had to use a method which decouples completely the short distance from the long distance problem; the other source of problem was the absence of translation invariance which, on the technical side makes obsolete the usual momentum space methods, and on the side of physics, forbids to base the construction on a distinguished vacuum state and on the notion of particles.

We solved these problems by basing the construction on the local S-matrices of Stückelberg and Bogoliubov, by invoking the general ideas of algebraic quantum field theory and by replacing translation invariance by smoothness properties, making extensive use of techniques and concepts from microlocal analysis.

Besides the solution of the problem to which this paper is addressed, we solved several other problems which might be of independent interest. First we have found a new construction of Wick polynomials on a domain which depends only on the representation associated to a fixed Hadamard state but not on the state itself. Since according to Verch [57], all representations induced by Hadamard states are locally quasiequivalent this amounts to an algebraic construction of Wick polynomials.

Second, we gave a perturbative construction of algebraic quantum field theory. In particular, we proved that the theory (in the algebraic sense), is completely fixed if it is known locally. Actually this holds independently of perturbation theory and might be a hint for a construction (in the sense of constructive quantum field theory) of asymptotically free theories. There the construction in small volumes seems to be possible [46] but the infrared problem poses, at present, unsormountable difficulties. The message of this paper is that the construction of the algebra of observables is nevertheless possible once the algebra of observables for small space-time regions have been constructed. The long distance behaviour of such a theory would still require an extra investigation, but it would be the behaviour of an existing theory, quite similar to the computation of spectra of Hamiltonians which have been shown to be self-adjoint operators.

On the technical side we had to study the extension problem for distributions which are defined on the complement of some submanifold. This seems to be a natural mathematical question, and in a simple case it was treated in [23] by similar methods. What seems to be new is our concept of the (microlocal) scaling degree at a surface which combines the condition of smoothness along a surface with a classification of the singularity in a transversal direction.

The main open point in this paper is the fixing of the finite renormalizations. One expects that they can be chosen in terms of local functions of the metric, but a precise formulation meets a lot of problems. A similar problem was studied (and partially solved) in the definition of the expectation value of a renormalized energy momentum tensor of free fields by R. Wald [61]. We hope to return to this problem in a future publication [10].

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