Dimensional interpolation and addition formulae for Jacobi polynomial and hypergeometric function

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1. Introduction

In the early seventies several authors called their attention to "unfortunate gaps"[1] in the theory of Jacobi polynomials. While ultraspherical (Gegenbauer) polynomials "inherited" practically all the formulae known in the theory of Legendre polynomials, generalization of some similar formulae for Jacobi polynomials was not available. The addition and product formulae and the Laplace type integral representation are provided for examples of such lacunae. The gap was filled by T. Koornwinder [4-6] who suggested the addition formula

$$P_n^{\varkappa,\lambda}(\cos 2\alpha') = \sum_{k=0}^{n}\sum_{l=0}^{n} a_{n,k,l} F_n^{k,l}(\alpha) F_n^{k,l}(\tau) p_{k,l}(\beta,\varphi) \tag{1.1}$$

where

$$\cos^2\alpha' = cos^2\alpha\cos^2\tau + \sin^2\alpha\sin^2\tau\cos^2\beta + \frac{1}{2}\sin 2\alpha\sin 2\tau\cos\beta\cos\varphi, \tag{1.2}$$

$$p_{k,l}(\beta,\varphi) = \cos^{k-l}\beta P_l^{\varkappa-\lambda-1,\lambda+k-l}(\cos 2\beta) C_{k-l}^{\lambda}(\cos\varphi), \tag{1.3}$$

$$F_n^{k,l}(\alpha) = \sin^{k+l}\alpha\cos^{k-l}\alpha P_{n-k}^{\varkappa+k+l,\lambda+k-l}(\cos 2\alpha), \tag{1.4}$$

$$a_{n,k,l} = \frac{(\varkappa+k+l)\Gamma(\varkappa+\lambda+k+n+1)\Gamma(\varkappa+k)\Gamma(\lambda+1)\Gamma(\varkappa+n+1)\Gamma(n-k+1)}{\Gamma(\varkappa+\lambda+n+1)\Gamma(\varkappa+n+l+1)\Gamma(\lambda+k+1)\Gamma(\lambda+n-l+1)}, \tag{1.5}$$

when $\lambda = 0$, then

$$\frac{\lambda+k-l}{\lambda} C_{k-l}^{\lambda}(\cos\varphi) \tag{1.6}$$

is replaced by $2\cos(k-l)\varphi$ if $k-l \neq 0$ or by 1 if $k-l=0$.

In (1.1)-(1-6.) $P_n^{\varkappa,\lambda}(x)$ is Jacobi polynomial and $C_l^{\lambda}(x)$ is Gegenbauer polynomial.

Derivation of the addition formula (1.1) with its unexpected structure, namely, with the double summation in the right-hand side of (1.1) instead of simple summation in the analogous formulae known before, attracted considerable interest of not only the author of [4]-[6] but also other mathematicians. The Koornwinder's proof was based both on the theory of group representations and on some theorems of the integral geometry. Later R. Askey [1] gave a direct analytical proof of (1.1) for those who "might use the Koornwinder results but have no background for understanding the proof".

Another pecularity of the structure of (1.1), on the contrary, common both for (1.1) and for the addition formulae known before 1972 is separation of variables. The following method of deriving addition formulae is widely used in practice. It is based on a singular solution of a certain second order partial differential equation (PDE) which allows for separation of variables. If the PDE is invariant under a group of transformations. then there exist two representations of the same solution in two coordinate systems. Equating the two forms of the solution yields the addition formula connected with the group of transformation. Whittaker and Watson [12] characterized this kind of proof as based on physical reasons, though it is not less rigorous than the analytical proof which was of fundamental importance in the twenties. After the developing the theory of distributions the role of the fundamental and complementary methods in the field might be changed.

The aim of the present paper is to interprete the addition formula (1.1) in terms of PDE with separable variables. From the above discussion it follows an alternative proof of (1.1), which, on the one hand, retains the geometrical spirit of the representation theory and, on the other hand, provides a good deal of elementarity. It should be noted that the interaction between the symmetry and separation of variables is currently the subject of intensive investigations [6].

Let us take into consideration another feature of deriving the addition formulae from the symmetry considerations. Usually (Koornwinder's paper [4] is not an exception) special functions appear to be spherical functions in some n-dimensional space. Therefore formula of (1.1) type can be derived under some restrictions imposed on the indices (which are most commonly half- integer). Using analytical continuation makes the formula valid for arbitrary indices, i.e. a sort of interpolation in fractional dimensions takes place. The object of the present paper consists in the realisation of the interpolation from the outset. We would also like to receive PDE with separable variables possessed the needed symmetry for arbitrary indices. Nevertheless the relation with symmetries of n-dimensional space is preserved. Multidimensional Laplace equation averaged over all except a few angular variables plays the role of the interpolational PDE. Such approach is used in (mono)spherical coordinates in Section 2. Rotation of singular solution yields the known addition theorem for ultraspherical polynomials.It appears reasonable to seek the addition formulae for Jacobi polynomials in polyspherical coordinate systems. The Laplace equation in bispherical coordinates is considered in Section 3. Separation of variables in these coordinates [11, p. 475] leads to Jacobi polynomials. It is convenient to introduce the adjoint Jacobi function close connected to coefficients $F_n^{k,l}$(1.4). The adjoint functions are specified by two indices(in some sense this is the reason of the occurrence of double summation in (1.1)). However, the symmetry of the averaged Laplace equation is insufficient to obtain (1.1). Instead of it the alternative addition formula arises, which is asymmetric in variables α (coordinate) and τ (parameter of rotation group). Therefore the seeking is continued in trispherical coordinate system in Section 4. We succeeded in construction of the particular case of (1.1) for polynomials $P_n^{\varkappa,0}(x)$ under the specific restriction on the parameters. It might be noted that the same consequence occured in deriving (1.1) using the representation theory. At first, R.L. Šapiro [10] obtained (1.1) in the case λ=0, then T. Koornwinder repeated the result independently and arrived at the general case using the analitical continuation. Finally, the proof of general formula was obtained. The approach connected with separation of variables is rewarded in Section 4 by obtaining a new generating function for polynomials $P_n^{\varkappa,0}(x)$ by summarizing the Fourer series for the singular solution due to translation symmetry of the interpolational PDE.

The further ascent along the staircase of polyspherical coordinates or extending the chain of the angular variables do not result in the symmetric addition formula. The required PDE is obtained in Section 5 by explicit use of the scalar product $\cos\omega = (\xi, I\xi)$ (see [4]) in the Laplace equation. This variable enters in the set of invariants of the particular rotation (so called pratactical displacement[8,9]) used by T. Koornwinder for the proof of (1.1). As it is pointed out in [4], the choise of the rotation is determined by Cartan's decompositon of $SO(p+q)$ group. The Laplace equation in which the dependence from angular variables in terms of "paratactical invariant" $\cos\omega$ is used instead of simple averaging appears to be thedesired interpolational PDE. The rotation of singular solution yields the symmetric addition formula (1.1). Finally, the transition from Euclidean space to pseudoeuclidean one allows to construct the addition formula for hypergeometric function of general form.

2. Laplace equation in monospherical coordinates. The addition formula for Gegenbauer polynomials.

The theory of Gegenbauer polynomials is known to be closely connected with the presentation of the group $SO(n)$ of rotations of the n-dimensional Euclidean space [11]. Specifically, the addition theorem for ultraspherical polynomials follows from the definition of matrix elements of the presentation of SO(n) in the canonical basis. However, the general formula appears to be "inconvenient for application because of complexity of the expression of matrix elements in terms of Gegenbauer polynomials" [11, p. 450]. Vilenkin considered a particular case in which the vector being rotated has only two non-zero angular coordinates θ_{n-1} and θ_{n-2}; he used rotation in (x_{n-1},x_n)-plane to obtain a simpler formula. The resulted formula for Gegenbauer polynomials

$$\begin{aligned} &C_l^p(\cos\theta\cos\tau+\sin\theta\sin\tau\cos\varphi) = \frac{\Gamma(2p-1)}{\Gamma^{2(}(p)}\times \\ &\sum_{m=0}^{l}\frac{2^{2m}\Gamma^2(p+m)(l-m)!(2m+2p-1)}{\Gamma(l+m+2p)}\times \\ &C_{l-m}^{p+m}(\cos\theta)\sin^m\theta C_{l-m}^{p+m}(\cos\tau)\sin^m\tau C_m^{p-\frac{1}{2}}(\cos\varphi), \end{aligned} \tag{2.1}$$

where $\theta = \theta_{n-1}$, $\varphi = \theta_{n-2}$, $p = (n-2)/2$, has the special three-dimensional form, and is valid for arbitrary p (or n). The proof of (2.1) given below will be a pattern of "interpolation in dimension". The n-dimensional Laplace equation averaged over all but two angular spherical coordinates plays the role of the interpolated PDE that allows the required one-parametrical group of transformations.

Let us start from developing the Laplace equation in monospherical coordinates. In 3-dimensional case it is convenient to use an intermediate cylindrical coordinate system for transition from the cartesian coordinates to spherical ones. Generalysing the technique we introduce the monospherical coordinates $r, \theta_1, \theta_2, ..., \theta_{n-1}$ in $n-$dimensional Euclidean space E_n by the chain of identities

$$\begin{gathered}\rho_0 = r, \quad x_k = \rho_{k-1}\cos\theta_k, \quad \rho_k = \rho_{k-1}\sin\theta_k, \quad k = 1, 2, ...n-1; \\ x_n = \rho_{n-1}; \quad 0 \le \theta_k \le \pi, \quad k = 1, 2, ..., n-2; \quad 0 \le \theta_{n-1} \le 2\pi. \end{gathered} \tag{2.2}$$

Differentiation of (2.2) yields

$$\begin{aligned} dx_k &= \cos\theta_k d\rho_{k-1} - \rho_{k-1}\sin\theta_k d\theta_k, \\ d\rho_k &= \sin\theta_k d\rho_{k-1} + \rho_{k-1}\cos\theta_k d\theta_k \end{aligned}$$

and, consequently,

$$dx_k^2 + d\rho_k^2 = d\rho_{k-1}^2 + \rho_{k-1}^2 d\theta_k^2. \tag{2.3}$$

By summing the relations (2.3) over k from 1 to $n-1$ we obtain the familiar expression for the squared linear element

$$ds^2 = \sum_{k=1}^{n} dx_k^2 = dr^2 + r^2 d\theta_1^2 + \sum_{k=2}^{n-1} \rho_{k-1}^2 d\theta_k^2 = dr^2 + r^2 \sum_{k=1}^{n-1} q_k^2 d\theta_k^2, \tag{2.4}$$

where, for brevity, we put (cf. [8])

$$q_1 = 1, \quad q_j^2 = \prod_{k=1}^{j-1} \sin^2\theta_k, j \ge 2. \tag{2.5}$$

Using (2.4) and (2.5) we can express Lame coefficients as

$$H_1 = 1, \quad H_k = rq_k, \quad k = 2, ..., n-1. \tag{2.6}$$

The Laplace equation in orthogonal curvilinear coordinates $z_k = z_k(x_1, ..., x_n), \quad k = 1, ..., n-1$ is written in the form [8]

$$\Delta u = \frac{1}{J}\sum_{k=1}^{n} \frac{\partial}{\partial z_k}(\frac{J}{H_k^2}\frac{\partial u}{\partial z_k}), \tag{2.7}$$

where

$$J = H_1 H_2 ... H_n. \tag{2.8}$$

Substituting (2.6) in (2.7), (2.8) we obtain the Laplace equation in (mono)spherical coordinates ($z_1 = r, z_k = \theta_{k-1}, k = 2, ..., n$) as

$$\begin{aligned}&\frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^{n-2}\theta_1}\frac{\partial}{\partial\theta_1}\sin^{n-2}\theta_1\frac{\partial u}{\partial\theta_1} + \\ &\frac{1}{r^2\sin^2\theta_1\sin^{n-3}\theta_2}\frac{\partial}{\partial\theta_2}\sin^{n-3}\theta_2\frac{\partial u}{\partial\theta_2} + ... + \\ &\frac{1}{r^2\sin^2\theta_1\sin^2\theta_2 ... \sin^2\theta_{n-3}\sin\theta_{n-2}} \times \frac{\partial}{\partial\theta_{n-2}}\sin\theta_{n-2}\frac{\partial u}{\partial\theta_{n-2}} + \\ &\frac{1}{r^2\sin^2\theta_1\sin^2\theta_2 ... \sin^2\theta_{n-2}}\frac{\partial^2 u}{\partial u^2} = 0.\end{aligned} \tag{2.9}$$

The averaging of (2.9) over $\theta_3, ..., \theta_{n-1}$ is equivalent to crossing out the derivatives with respect to these arguments :

$$\begin{aligned}Lu \equiv &\frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^{n-2}\theta_1}\frac{\partial}{\partial\theta_1}\sin^{n-2}\theta_1\frac{\partial u}{\partial\theta_1} + \\ &\frac{1}{r^2\sin^2\theta_1\sin^{n-3}\theta_2}\frac{\partial}{\partial\theta_2}\sin^{n-3}\theta_2\frac{\partial u}{\partial\theta_2} = 0.\end{aligned} \tag{2.10}$$

Equation (2.10) is the interpolated Laplace equation that we have been seeking for. Its transformational properties are convenient to analyze in Cartesian coordinates

$$x_1 = r\cos\theta_1, \quad x_2 = r\sin\theta_1\cos\theta_2, \quad x_3 = r\sin\theta_1\sin\theta_2. \tag{2.11}$$

Thus we get

$$Lu = \sum_{k=1}^{3}\frac{\partial^2 u}{\partial x_k^2} + (m-3)\frac{1}{x_3}\frac{\partial u}{\partial x_3}. \tag{2.12}$$

It is seen from (2.12) that the equation $Lu = 0$ is invariant both under rotations in $(x_1, x_2)-$plane:

$$x_1' = x_1\cos\tau + x_2\sin\tau, \quad x_2' = -x_1\sin\tau + x_2\cos\tau \tag{2.13}$$

and parallel transfers in the plane

$$x_1' = x_1 + x_{10}, \quad x_2' = x_2 + x_{20}. \tag{2.14}$$

Equation (2.10) allows separation of variables. The substitution $u = r^l v(\theta_1, \theta_2)$ yields the equation for spherical functions in the form

$$\begin{aligned}&\frac{1}{r^2\sin^{n-2}\theta_1}\frac{\partial}{\partial\theta_1}\sin^{n-2}\theta_1\frac{\partial v}{\partial\theta_1} + \\ &\frac{1}{r^2\sin^2\theta_1\sin^{n-3}\theta_2}\frac{\partial}{\partial\theta_2}\sin^{n-3}\theta_2\frac{\partial v}{\partial\theta_2} + l(l+n-2)v = 0.\end{aligned} \tag{2.15}$$

For solutions depending on only θ_1, equation (2.15) reduces to

$$\frac{1}{\sin^{n-2}\theta_1}\frac{d}{d\theta_1}\sin^{n-2}\theta_1\frac{dv_0}{d\theta_1}+l(l+n-2)v_0=0, \quad 0<\theta_1<\pi. \tag{2.16}$$

The solution of (2.16) which is bounded at both $\theta_1=0$ and $\theta_1=\pi$ exists if l is integer and may be expressed in terms of Gegenbauer polynomial

$$v_{0,l}=C_l^{(n-2)/2}(\cos\theta_1).$$

By separating the variable θ_2 we obtain equation (2.16) in which n is replaced by $n-1$ with the eigenfunction

$$y_q(\theta_2)=C_q^{(n-3)/2}(\cos\theta_2), \quad q=0,1,2,...$$

Then the spherical function (the solution of (2.15) bounded on the sphere) may be expressed in the form

$$v_{q,l}(\theta_1,\theta_2)=v_{q,l}(\theta_1)y(\theta_2).$$

Substitution of this expression in (2.15) yields the equation for "adjoint Gegenbauer functions" $v_{q,l}(\theta_1)$:

$$\frac{1}{\sin^{n-2}\theta_1}\frac{d}{d\theta_1}\sin^{n-2}\theta_1\frac{dv_{q,l}}{d\theta_1}+[l(l=n-2)-\frac{q(q+n-3)}{\sin^2\theta_1}]v_{q,l}=0$$

with a bounded solution of the form

$$v_{q,l}(\theta_1)=\sin^q\theta_1 C_{l-q}^{(n-2)/2+q}(\cos\theta_1).$$

Hence the particular solution of (2.10) can be expressed in the form

$$\begin{aligned} u^*_{q,l}(r,\theta_1,\theta_2) &= \sin^q\theta_1 C_{l-q}^{(n-2)/2+q}(\cos\theta_1)C_q^{(n-3)/2}(\cos\theta_2)\times \\ &\times (A_{q,l}r^l+B_{q,l}r^{2-n-l}). \end{aligned} \tag{2.17}$$

Equation (2.10) possesses a centrally symmetric solution of the form $const\times r^{2-n}$. Because of the invariance of (2.10) under transformations (2.14), there exists a singular solution which has the form

$$const(r_0^2-2rr_0\cos\theta_1+r^2)^{2-n/2}.$$

If we construct the same solution with the aid of particular solutions (2.17), we receive the well-known generating function for Gegenbauer polynomials

$$\sum_{l=0}^{\infty}C_l^{\lambda}(x)z^n=(1-2xz+z^2)^{-\lambda}, \quad |z|<1.$$

Transfer of the singularity by the use of rotation (2.14) gives the addition formula (2.1).

3. Bispherical coordinates. Adjoint Jacobi functions. Asymmetric addition formula.

Let us introduce bispherical coordinates by the use of the chain of relations

$$r_1 = r\cos\alpha, \quad r_2 = r\sin\alpha, \quad 0 \le \alpha \le \pi/2. \tag{3.1}$$

$$\rho_{1,0} = r_1; \quad x_{1,k} = \rho_{1,k-1}\cos\theta_{1,k}, \quad \rho_{1,k} = \rho_{1,k-1}\sin\theta_{1,k}, \quad k = 1, ..., m-1;$$

$$x_{1,m} = \rho_{1,m-1}; \quad 0 \le \theta_{1,k} \le \pi, \quad k = 1, ..., m-2; \quad 0 \le \theta_{1,m-1} \le 2\pi. \tag{3.2}$$

$$\rho_{2,0} = r_2; \quad x_{2,k} = \rho_{2,k-1}\cos\theta_{2,k}, \quad \rho_{2,k} = \rho_{2,k-1}\sin\theta_{2,k}, \quad k = 1, ..., n-1;$$

$$x_{2,n} = \rho_{2,n-1}; \quad 0 \le \theta_{1,k} \le \pi, \quad k = 1, ..., n-2; \quad 0 \le \theta_{1nm-1} \le 2\pi. \tag{3.3}$$

Using the same procedure as in Sec. 2, we obtain the following expression for a square of a linear element

$$\begin{aligned} ds^2 &= \sum_{k=1}^{m} dx_{1,k}^2 + \sum_{k=1}^{n} dx_{2,k}^2 = \\ &= dr^2 + r^2 d\alpha^2 + r^2\cos^2\alpha \sum_{k=1}^{m-1} q_{1,k}^2 d\theta_{1,k}^2 + r^2\sin^2\alpha \sum_{k=1}^{n-1} q_{2,k}^2 d\theta_{2,k}^2, \end{aligned} \tag{3.4}$$

where $q_{i,1} = 1, \quad q_{i,k}^2 = \prod_{j=1}^{k-1} \sin^2\theta_{i,j}, \quad k \ge 2, \quad i = 1, 2.$

It follows from (3.4) and (2.7) that the Laplace equation in bispherical coordinates can be written in the form

$$\begin{aligned} &\frac{1}{r^{m+n-1}}\frac{\partial}{\partial r}r^{m+n-1}\frac{\partial u}{\partial r} + \\ &+\frac{1}{r^2\cos^{m-1}\alpha\sin^{n-1}\alpha}\frac{\partial}{\partial\alpha}\cos^{m-1}\alpha\sin^{n-1}\alpha\frac{\partial u}{\partial\alpha} + \\ &+\frac{1}{r^2\cos^2\alpha}[\frac{1}{\sin^{m-2}\theta_{1,1}}\frac{\partial}{\partial\theta_{1,1}}\sin^{m-2}\theta_{1,1}\frac{\partial u}{\partial\theta_{1,1}} + \\ &+\frac{1}{\sin^2\theta_{1,1}\sin^{m-3}\theta_{1,2}}\frac{\partial}{\partial\theta_{1,2}}\sin^{m-3}\theta_{1,2}\frac{\partial u}{\partial\theta_{1,2}} + ... + \\ &+\frac{1}{\sin^2\theta_{1,1}\sin^2\theta_{1,2}...\sin^2\theta_{1,m-2}}\frac{\partial^2 u}{\partial\theta_{m-1}^2}] + \\ &+\frac{1}{r^2\sin^2\alpha}[\frac{1}{\sin^{n-2}\theta_{2,1}}\frac{\partial}{\partial\theta_{2,1}}\sin^{n-2}\theta_{2,1}\frac{\partial u}{\partial\theta_{2,1}} + \\ &+\frac{1}{\sin^2\theta_{2,1}\sin^{n-3}\theta_{2,2}}\frac{\partial}{\partial\theta_{2,2}}\sin^{n-3}\theta_{2,2}\frac{\partial u}{\partial\theta_{2,2}} + ... + \\ &+\frac{1}{\sin^2\theta_{2,1}\sin^2\theta_{2,2}...\sin^2\theta_{2,n-2}}\frac{\partial^2 u}{\partial\theta_{n-1}^2}] = 0 \end{aligned} \tag{3.5}$$

Equation (3.5) averaged over the angles $\theta_{1,2},...,\theta_{1,m-1}, \quad \theta_{2,2},...,\theta_{2,n-1}$ reduces to

$$\begin{aligned}
Lu \equiv & \frac{1}{r^{m+n-1}}\frac{\partial}{\partial r}r^{m+n-1}\frac{\partial u}{\partial r}+\\
+&\frac{1}{r^2\cos^{m-1}\alpha\sin^{n-1}\alpha}\frac{\partial}{\partial\alpha}\cos^{m-1}\alpha\sin^{n-1}\alpha\frac{\partial u}{\partial\alpha}+\\
+&\frac{1}{r^2\cos^2\alpha}\frac{1}{\sin^{m-2}\theta}\frac{\partial}{\partial\theta}\sin^{m-2}\theta\frac{\partial u}{\partial\theta}+\\
+&\frac{1}{r^2\sin^2\alpha}\frac{1}{\sin^{n-2}\psi}\frac{\partial}{\partial\psi}\sin^{n-2}\psi\frac{\partial u}{\partial\psi}=0,
\end{aligned} \tag{3.6}$$

where we put $\theta=\theta_{1,1}, \quad \psi=\theta_{2,1}$.
In Cartesian coordinates

$$x_1=\cos\alpha\cos\theta, \quad x_2=\cos\alpha\sin\theta, \quad x_3=\sin\alpha\cos\psi, \qquad x_4=\sin\alpha\sin\psi$$

Equation (3.6) rearranges to the form

$$Lu=\sum_{i=1}^{4}\frac{\partial^2 u}{\partial x_i^2}+(m-2)\frac{1}{x_2}\frac{\partial u}{\partial x_2}+(n-2)\frac{1}{x_4}\frac{\partial u}{\partial x_4}=0. \tag{3.7}$$

Equation (3.6) allows separation of variables. Substitution $r^l v(\alpha,\theta,\psi)$ yields the equation for (bi)spherical functions which has the form

$$\begin{aligned}
&\frac{1}{\cos^{m-1}\alpha\sin^{n-1}\alpha}\frac{\partial}{\partial\alpha}\cos^{m-1}\alpha\sin^{n-1}\alpha\frac{\partial v}{\partial\alpha}+\\
+&\frac{1}{\cos^2\alpha}\frac{1}{\sin^{m-2}\theta}\frac{\partial}{\partial\theta}\sin^{m-2}\theta\frac{\partial v}{\partial\theta}+\\
+&\frac{1}{\sin^2\alpha}\frac{1}{\sin^{n-2}\psi}\frac{\partial}{\partial\psi}\sin^{n-2}\psi\frac{\partial v}{\partial\psi}+p(p+m+n-2)v=0.
\end{aligned} \tag{3.8}$$

For the solution depending on α alone, (3.8) reduces to

$$\begin{aligned}
&\frac{1}{\cos^{m-1}\alpha\sin^{n-1}\alpha}\frac{\partial}{\partial\alpha}\cos^{m-1}\alpha\sin^{n-1}\alpha\frac{\partial v_0}{\partial\alpha}+p(p+m+n-2)v_0=0,\\
&0<\alpha<\pi/2.
\end{aligned} \tag{3.9}$$

The solution of (3.9) bounded at $\alpha=0,\pi/2$ exists if $p=2q, \quad q$ is integer, and has the form

$$v_o=P_q^{\varkappa,\lambda}(\cos 2\alpha), \quad \varkappa=(n-2)/2, \quad \lambda=(m-2)/2.$$

By separating the variables θ and ψ we come to Sturm-Liouville problems similar to (2.18) with the following solutions

$$\begin{aligned}
&\Theta_l(\theta)=C_l^{(m-2)/2}(\cos\theta),\\
&\Psi_k(\psi)=C_k^{(n-2)/2}(\cos\psi), \quad l=0,1,..., \quad k=0,1...
\end{aligned}$$

The bispherical function, i.e. the bounded solution of (3.8) is expressed in the form

$$v(\alpha, \theta, \psi) = y(\alpha)\Theta_l(\theta)\Psi_k(\psi). \tag{3.10}$$

Substituting (3.10) in (3.8) we obtain (cf. [11, p. 475]) the equation

$$\begin{aligned}
&\frac{1}{\cos^{m-1}\alpha\sin^{n-1}\alpha}\frac{d}{d\alpha}\cos^{m-1}\alpha\sin^{n-1}\alpha\frac{dy}{d\alpha}+ \\
&+[p(p+m+n-2)-\frac{l(l+m-2)}{\cos^2\alpha}-\frac{k(k+n-2)}{\sin^2\alpha}]y=0, \\
&0<\alpha<\pi/2,
\end{aligned} \tag{3.11}$$

which is natural to call as the equation for Jacobi adjoint functions. These functions have the form

$$y_q^{k,l}(\alpha) = \sin^k\alpha\cos^l\alpha P_{q-s}^{\varkappa+k,\lambda+l}(\cos 2\alpha), \tag{3.12}$$

where $k+l=2s, s=0,1,...,q$ if $p=2q$ and $k+l=2s+1$ if $p=2q+1$.

From the formulae written above it follows that biaxially-symmetric solution of (3.6) with the singularity at $\alpha=0, \quad r=r_0$ can be written in the form

$$\sum_{q=0}^{\infty} A_q(\frac{r}{r_0})^{2q}P_q^{\varkappa,\lambda}(\cos 2\alpha), \quad r<r_0. \tag{3.13}$$

From the Cartesian form (3.8) it is seen that Eq.(3.6) is invariant under rotation

$$x_1' = x_1\cos\tau - x_3\sin\tau, \quad x_3' = x_1\sin\tau + x_3\cos\tau, \quad x_2' = x_2, \quad x_4' = x_4.$$

This fact is equivalent to invariance of (3.6) under transformation

$$\begin{aligned}
\cos^2\alpha' &= \cos^2\alpha(\cos^2\theta\cos^2\tau + \\
&+ \sin^2\theta) + \sin^2\alpha\sin^2\tau\cos^2\psi + (1/2)\sin 2\alpha\sin 2\tau\cos\theta\cos\psi.
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\tan\theta' &= \frac{\cos\alpha\sin\theta}{\cos\alpha\cos\theta\cos\tau - \sin\alpha\cos\psi\sin\tau}, \\
\tan\psi' &= \frac{\sin\alpha\sin\psi}{\cos\alpha\cos\theta\sin\tau + \sin\alpha\cos\psi\cos\tau}.
\end{aligned} \tag{3.15}$$

By transforming the singular solution (3.13) to variables (3.14), (3.15) we come to the addition formula

$$P_n^{\varkappa,\lambda}(\cos 2\alpha') = \sum_{k=0}^{n}\sum_{l=0}^{n} b_{n,k,l}F_n^{k,l}(\alpha)G_n^{k,l}(\tau)C_l^{\varkappa-1/2}(\cos\theta)C_k^{\lambda-1/2}(\cos\psi). \tag{3.16}$$

Formula (3.16) is asymmetric about variables α and τ (because of the analogous asymmetry of transformation (3.14)). The coefficients $G_n^{k,l}$ are expressed with the use of a quadrature along the circle of singularities

$$G_n^{k,l}(\tau) = \int_0^{2\pi} F_n^{k,l}(\alpha_1)C_l^{\varkappa-1/2}(\cos\psi_1)d\theta^*, \tag{3.17}$$

where $\cos\alpha_1 = \cos\theta^*\cos\tau, \quad \tan\psi_1 = \tan\theta^*/\cos\tau$.

To find the symmetric addition formula we had to consider trispherical coordinates.

4. Laplace equation in trispherical coordinates. The addition formula for $\mathbf{P}_n^{\varkappa,0}(x)$

Trispherical coordinates are introduced by the relations

$$r_1 = r\cos\alpha, \quad r_2 = r\sin\alpha, \quad 0 \le \alpha \le \pi/2; \tag{4.1}$$

$$r_{2,1} = r_2\cos\beta, \quad r_{2,2} = r_2\sin\beta, \quad 0 \le \beta \le \pi/2 \tag{4.2}$$

$$\rho_0 = r_1, \quad x_k = \rho_{k-1}\cos\theta_k, \quad \rho_k = \rho_{k-1}\sin\theta_k, \quad k = 1,2,...p-1; \quad x_p = \rho_{p-1};$$

$$\begin{aligned}
&0 \le \theta_k \le \pi, \quad k = 1,2,...,p-2; \quad 0 \le \theta_{p-1} \le 2\pi; && (4.3)\\
&\rho_p = r_1, \quad x_{p+k} = \rho_{p+k-1}\cos\theta_{p+k}, \quad \rho_{p+k} = \rho_{p+k-1}\sin\theta_{p+k}, &&\\
&k = 1,2,...q-1; \quad x_{p+q} = \rho_{p+q-1}; &&\\
&0 \le \theta_{p+k} \le \pi, \quad k = 1,2,...,q-2; \quad 0 \le \theta_{p+q-1} \le 2\pi; && (4.4)\\
&\rho_{p+q} = r_{2,2}, \quad x_{p+q=k} = \rho_{p+q=k-1}\cos\theta_{p+q+k}, &&\\
&\rho_{p+q+k} = \rho_{p+q+k-1}\sin\theta_{p+q+k}, \quad k = 1,2,...s-1; \quad x_{p+q+s} = \rho_{p+q+s-1}; &&\\
&0 \le \theta_{p+q+k} \le \pi, \quad k = 1,2,...,s-2; \quad 0 \le \theta_{p+q+s-1} \le 2\pi; && (4.5)
\end{aligned}$$

The square of a linear element can be written in the form

$$\begin{aligned}
ds^2 &= dr^2 + r^2 d\alpha^2 + r^2\sin^2\alpha d\beta^2 + \\
&+ r^2\cos^2\alpha \sum_{k=1}^{p-1}\prod_{j=1}^{k-1}\sin^2\theta_j d\theta_k^2 + \\
&+ r^2\sin^2\alpha\cos^2\beta \sum_{k=1}^{q-1}\prod_{j=1}^{k-1}\sin^2\theta_{p+j} d\theta_{p+k}^2 + \\
&+ r^2\sin^2\alpha\sin^2\beta \sum_{k=1}^{s-1}\prod_{j=1}^{k-1}\sin^2\theta_{p+q+j} d\theta_{p+q+k}^2.
\end{aligned} \tag{4.6}$$

The Laplace equation in trispherical coordinates is found from (4.6) and (2.7) in the form

$$\begin{aligned}
&\frac{1}{r^{p+q+s-1}}\frac{\partial}{\partial r}r^{p+q+s-1}\frac{\partial u}{\partial r} + \frac{1}{r^2\cos^{p-1}\alpha\sin^{q+s-1}\alpha}\frac{\partial}{\partial\alpha}\cos^{p-1}\alpha\sin^{q+s-1}\alpha\frac{\partial u}{\partial\alpha} + \\
&+\frac{1}{r^2\sin^2\alpha\cos^{q-i}\beta\sin^{s-1}\beta}\frac{\partial}{\partial\beta}\cos^{q-i}\beta\sin^{s-1}\beta\frac{\partial u}{\partial\beta} + \\
&+\frac{1}{\cos^2\alpha}\Big(\frac{1}{\sin^{p-2}\theta_1}\frac{\partial}{\partial\theta_1}\sin^{p-2}\theta_1\frac{\partial u}{\partial\theta_1} + \\
&+\frac{1}{\sin^2\theta_1\sin^{p-3}\theta_2}\frac{\partial}{\partial\theta_2}\sin^{p-3}\theta_2\frac{\partial u}{\partial\theta_2} + ... + \frac{1}{\sin^2\theta_1\sin^2\theta_2 ... \sin^2\theta_{p-2}}\frac{\partial^2 u}{\partial\theta_{p-1}^2}\Big) + \\
&+\frac{1}{r^2\sin^2\alpha\cos^2\beta}\Big(\frac{1}{\sin^{q-2}\theta_{p+1}}\frac{\partial}{\partial\theta_{p+1}}\sin^{q-2}\theta_{p+1}\frac{\partial u}{\partial\theta_{p+1}} + ... +
\end{aligned} \tag{4.7}$$

$$
\begin{aligned}
&+\frac{1}{\sin^2\theta_{p+1}\sin^{q-3}\theta_{p+2}}\frac{\partial}{\partial\theta_{p+2}}\sin^{p-3}\theta_{p+2}\frac{\partial u}{\partial\theta_{p+2}}+\\
&+\frac{1}{\sin^2\theta_{p+1}\sin^2\theta_{p+2}...\sin^2\theta_{p+q-2}}\frac{\partial^2 u}{\partial\theta^2_{p+q-1}})+\\
&+\frac{1}{r^2\sin^2\alpha\sin^2\beta}(\frac{1}{\sin^{s-2}\theta_{p+q1}}\frac{\partial}{\partial\theta_{p+q1}}\sin^{s-2}\theta_{p+q+1}\frac{\partial u}{\partial\theta_{p+q+1}}+...+\\
&+\frac{1}{\sin^2\theta_{p+q+1}\sin^{s-3}\theta_{p+q+2}}\frac{\partial}{\partial\theta_{p+q+2}}\sin^{s-3}\theta_{p+q+2}\frac{\partial u}{\partial\theta_{p+q+2}}+\\
&+\frac{1}{\sin^2\theta_{p+q+1}\sin^2\theta_{p+q+2}...\sin^2\theta_{p+q+s-2}}\frac{\partial^2 u}{\partial\theta^2_{p+q+s-1}})=0.
\end{aligned}
$$

As before we are interested in Eq.(4.7) averaged over arguments $\theta_2, ..., \theta_{p-1}$, $\theta_{p+2}, ...,$ $\theta_{p+q+1}, ..., \theta_{p+q+s-1}$. Denoting $\theta_1 = \theta$, $\theta_{p+1} = \psi$, we can write the interpolated equation in the form

$$
\begin{aligned}
&\frac{1}{r^{p+q+s-1}}\frac{\partial}{\partial r}r^{p+q+s-1}\frac{\partial u}{\partial r}+\\
&+\frac{1}{r^2\cos^{p-1}\alpha\sin^{q+s-1}\alpha}\frac{\partial}{\partial\alpha}\cos^{p-1}\alpha\sin^{q+s-1}\alpha\frac{\partial u}{\partial\alpha}+\\
&+\frac{1}{r^2\sin^2\alpha\cos^{q-i}\beta\sin^{s-1}\beta}\frac{\partial}{\partial\beta}\cos^{q-i}\beta\sin^{s-1}\beta\frac{\partial u}{\partial\beta}+\\
&+\frac{1}{r^2\cos^2\alpha}\frac{1}{\sin^{p-2}\theta}\frac{\partial}{\partial\theta}\sin^{p-2}\theta\frac{\partial u}{\partial\theta}+\\
&+\frac{1}{r^2\sin^2\alpha\cos^2\beta}\frac{1}{\sin^{q-2}\psi}\frac{\partial}{\partial\psi}\sin^{q-2}\psi\frac{\partial u}{\partial\psi}=0.
\end{aligned}
\tag{4.8}
$$

In Cartesian coordinates

$$
\begin{aligned}
x_1 &= r\cos\alpha\cos\theta, \quad x_2 = r\cos\alpha\sin\theta, \\
x_{p+1} &= r\sin\alpha\cos\beta\cos\psi, \quad x_{p+2} = r\sin\alpha\cos\beta\sin\psi, \quad x_{p+q+1} = r\sin\alpha\sin\beta
\end{aligned}
\tag{4.9}
$$

Eq. (4.8) rearranges to the form

$$
\begin{aligned}
&\sum_{k=1}^{2}(\frac{\partial^2 u}{\partial x_k^2}+\frac{\partial^2 u}{\partial x_{p+k}^2})+\frac{\partial^2 u}{\partial x_{p+q+1}^2}+\frac{(p-2)}{x_2}\frac{\partial u}{\partial x_2}+\\
&+\frac{(q-2)}{x_{p+2}}\frac{\partial u}{\partial x_{p+2}}+\frac{(s-1)}{x_{p+q+1}}\frac{\partial u}{\partial x_{p+q+1}}=0.
\end{aligned}
\tag{4.10}
$$

It is seen from (4.10) that Eq.(4.10) (and Eq.(4.8) respectively) permits only rotations and parallel transfers in (x_1, x_{p+1})-plane for arbitrary values of p, q and s. This rotation yields only an asymmetric addition formula as in the case of bispherical coordinates. However, we can obtain the symmetric formula in trispherical case using paratactical displacement in the subspace $(x_1, x_2, x_{p+1}, x_{p+2})$:

$$
x'_k = x_k\cos\tau - x_{p+k}\sin\tau, \quad x'_{p+k} = x_k\sin\tau + x_{p+k}\cos\tau, \quad k=1,2. \tag{4.11}
$$

The axis x_{p+q+1} remains motionless:

$$x'_{p+q+1} = x_{p+q+1}. \tag{4.12}$$

Let us introduce the trispherical coordinates (4.1)-(4.5) in the initial coordinate system $(x_1, ..., x_{p+q+1})$ and in the transformed system $(x'_1, ..., x'_{p+q+1})$. Then from the first two formulae (4.11) we obtain the following relation

$$\cos^2\alpha' = \cos^2\alpha\cos^2\tau + \sin^2\alpha\sin^2\tau\cos^2\beta - (1/2)\sin 2\alpha\sin 2\tau\cos\beta\cos(\theta-\psi) \tag{4.13}$$

i.e the symmetric transformation which was included in (1.1). The paratactical displacement (4.11) is permissible for Eqs. (4.8), (4.10) only if $p = q = 2$, and therefore the only parameter for interpolation is parameter s.

Let us construct the singular solution of (4.8) ($p = q = 2$) with a singularity at $r = r_0, \alpha = 0$. It is seen from (4.9) that the line of singularities is the circumference $x_{10} = r_0\cos\theta^*, x_{20} = r_0\sin\theta^*$ in the plane $x_{p+1} = x_{p+2} = x_{p+q+1} = 0$. This solution can be expressed in terms of the central-symmetric solution $const \times r^{-s-2}$. Indeed, Eq.(4.8) permits parallel transfers $x'_1 = x_1 + x_{10}$, $x'_{20} = x_2 + x_{20}$, hence there exists a singular solution of the form

$$\begin{aligned}&[(x_1-x_{10})^2 + (x_2-x_{20})^2 + x^2_{p+1} + x^2_{p+2} + x^2_{p+q+1}]^{-(s+2)/2} = \\ &= (r^2 + r_0^2 - 2rr_0\cos(\theta-\theta^*))^{-(s+2)/2}.\end{aligned} \tag{4.14}$$

Constructing the same solution in the separated trispherical coordinates gives the identity

$$\begin{aligned}&(1+t^2-2t\cos\alpha\cos\varphi)^{-(s+2)/2} = \\ &= \frac{1}{\Gamma(s/2+1)}[\sum_{q=0}^{\infty} t^{2q}\frac{\Gamma(s/2+1+q)}{q!}P_q^{s/2,0}(\cos 2\alpha) + \\ &+2\sum_{q=0}^{\infty} t^{2q}\sum_{j=0}^{q}\frac{\Gamma(s/2+q+1+j)}{(q+j)!}\cos^j\alpha P_{q-j}^{s/2,0}(\cos 2\alpha)\cos 2j\varphi + \\ &+2\sum_{q=o}^{\infty} t^{2q+1}\sum_{j=0}^{q}\frac{\Gamma(s/2+q+j+2)}{(q+j+1)!}\cos^{j+1/2}\alpha\, \times \\ &\times P_{q-j}^{s/2,0}(\cos 2\alpha)\cos(2j+1)\varphi].\end{aligned} \tag{4.15}$$

Integrating (4.15) along the circle of singularities, we receive the generating function for Jacobi polynomials $P_n^{\varkappa,0}(x)$:

$$\begin{aligned}&\int_0^{2\pi}(1+t^2-2t\cos\alpha\cos\varphi)^{-(s+2)/2}d\varphi = 2\pi t_2^{-s/2}F(\frac{s}{2},\frac{s}{2},1;\frac{t_1}{t_2}) = \\ &= \frac{2\pi}{\Gamma(s+1/2)}\sum_{q=0}^{\infty} t^{2q}\frac{\Gamma(s/2+1+q)}{q!}P_q^{s/2,0}(\cos 2\alpha),\end{aligned} \tag{4.16}$$

where $t_{1,2} = (1/2)(1 \pm \sqrt{(1+t^2)^2 - 4t^2 \cos\alpha})$ and $F(\alpha, \beta, \gamma; z)$ is the standard notation for the hypergeometric function.

The transformation of the circle of singularities by rotation (4.11),(4.12) in the circle $r' = r_0$, $\alpha' = \tau$, $\beta' = 0$, $\theta' = \theta^*$, $\psi' = \theta^* + \pi sign(\pi - \theta^*)$ and comparison between the singular solutions (4.16) in the initial coordinate system and in the rotated one result in the addition formula which is a particular case of (1.1) if $\lambda = 0$.

5. The addition formulae for Jacobi polynomials (general case) and hypergeometric functions

It is seen from (4.13) that the transformed variable α' depends on θ and ψ only in terms of difference $\theta - \psi$.So it is convenient to introduce the variable $\theta - \psi$ in Eq.(4.8) at $p = q = 2$. With the change of variables $\omega = \theta - \psi$, $\chi = \theta$ the last two terms of (4.8) ($p = q = 2$) may be rearranged to give

$$\frac{1}{r^2 \cos^2\alpha}\left(\frac{\partial^2 u}{\partial \omega^2} + \frac{\partial^2 u}{\partial \chi^2} + 2\frac{\partial^2 u}{\partial \chi \partial \omega}\right) + \frac{1}{r^2 \sin^2\alpha \cos^2\beta}\frac{\partial^2 u}{\partial \omega^2}.$$

Therefore, function $u(r, \alpha, \beta, \omega)$ is a solution of the equation

$$\begin{aligned}&\frac{1}{r^{3+s}}\frac{\partial}{\partial r} r^{3+s}\frac{\partial u}{\partial r} + \frac{1}{r^2 \cos\alpha \sin^{s+1}\alpha}\frac{\partial}{\partial \alpha}\sin^{s+1}\alpha \cos\alpha \frac{\partial u}{\partial \alpha} + \\ &+ \frac{1}{r^2 \sin^2\alpha \cos\beta \sin^{s-1}\beta}\frac{\partial}{\partial \beta}\cos\beta \sin^{s-1}\beta \frac{\partial u}{\partial \beta} + \frac{1}{r^2}\left(\frac{1}{\cos^2\alpha} + \right. \\ &\left. + \frac{1}{\sin^2\alpha \cos^2\beta}\right)\frac{\partial^2 u}{\partial \omega^2} = 0.\end{aligned} \tag{5.1}$$

Similarly we should replace the arguments θ, ψ by the new variable ω in the formulae of transformation of coordinates. After simple calculations, using (4.9) and (4.11), we have

$$\begin{aligned}&\cos^2\alpha' = \cos^2 \cos^2\tau + \sin^2\alpha \cos^2\beta \sin^2\tau - (1/2)\sin 2\alpha \sin 2\tau \cos\beta \cos\omega, \\ &\sin\alpha' \sin\beta' = \sin\alpha \sin\beta, \\ &\sin 2\alpha' \cos\beta' \cos\omega' = (\cos^2\alpha - \sin^2\alpha \cos^2\beta)\sin 2\tau + \sin 2\alpha \cos 2\tau \cos\omega.\end{aligned} \tag{5.2}$$

It is evident that Eq.(5.1) is invariant under the transformation of variables (5.2).

The left-hand side of Koornwinder's addition formula (1.1) is a solution of the more general equation than (5.1), namely,

$$\begin{aligned}&\frac{1}{r^{2p-1+s}}\frac{\partial}{\partial r} r^{2p-1+s}\frac{\partial u}{\partial r} + \frac{1}{r^2 \cos^{p-1}\alpha \sin^{p+s-1}\alpha}\frac{\partial}{\partial \alpha}\sin^{p+s-1}\alpha \cos^{p-1}\alpha \frac{\partial u}{\partial \alpha} + \\ &+ \frac{1}{r^2 \sin^2\alpha \cos^{p-1}\beta \sin^{s-1}\beta}\frac{\partial}{\partial \beta}\cos^{p-1}\beta \sin^{s-1}\beta \frac{\partial u}{\partial \beta} + \\ &+ \frac{1}{r^2}\left(\frac{1}{\cos^2\alpha} + \frac{1}{\sin^2\alpha \cos^2\beta}\right)\frac{1}{\sin^{p-2}\omega}\frac{\partial}{\partial \omega}\sin^{p-2}\omega \frac{\partial u}{\partial \omega} = 0.\end{aligned} \tag{5.3}$$

Equation (5.1) is a particular case of (5.3) at $p=2$. By direct but cumbersome calculation it is easy to verify that Eq.(5.3) is invariant under transformations (5.2) .However we make use of an alternative way of generalizing the transition from Eq.(4.8)($p=q=2$) to Eq.(5.1) to reveal the geometrical sense of (5.2) and (5.3).

Let us consider the Euclidean space of $2p+s$ dimensions E_{2p+s} referred to Cartesian coordinates $x_1,...,x_{2p}$, $x_{2p+1,...,}x_{2p+s}$ (i.e., the particular case of the Cartesian coordinates we used to introduce trispherical coordinates (4.1)-(4.5), at $p=q$). Let E_{2p+s} be transformed by rotation into the subspace E_{2p} (the paratactical displacement):

$$\begin{aligned} x_i' &= x_i\cos\tau - x_{p+i}\sin\tau, \quad x_{p+i}' = x_i\sin\tau + x_{p+i}, \\ i &= 1,...,p; \quad x_{2p+i}' = x_{2p+i}, i=1,...,s. \end{aligned} \tag{5.4}$$

We introduce the quadratic forms of variables $x_1,...,x_{2p}$ as

$$\left\{ \begin{array}{lll} r_1^2 &=& x_1^2+x_2^2+...+x_p^2, \\ r_{2,1}^2 &=& x_{p+1}^2+x_{p+2}^2+...+x_{2p}^2, \\ \Omega &=& x_1x_{p+1}+x_2x_{p+2}+...+x_px_{2p}.. \end{array} \right. \tag{5.5}$$

The first two forms (5.5) are the squares of trispherical coordinates introduced by (4.1)-(4.2). Under the rotation (5.4) the forms (5.5) are transformed through each other

$$\left\{ \begin{array}{lll} r_1'^2 &=& x_1'^2+...+x_p'^2 = r_1^2+\cos^2\tau + r_{2,1}^2\sin^2\tau - \Omega\sin 2\tau, \\ r_{2,1}'^2 &=& x_{p+1}'^2+...+x_{2p}'^2 = r_1^2\sin^2\tau + r_{2,1}^2\cos^2\tau + \Omega\sin 2\tau, \\ \Omega' &=& x_1'x_{p+1}'+...+x_p'x_{2p}' = (1/2)(r_1^2\sin 2\tau - r_{2,1}^2 + 2\Omega\cos 2\tau). \end{array} \right. \tag{5.6}$$

Using the notation of (4.1), (4.2) and denoting

$$\cos\omega = \Omega/r_1r_{2,1} \tag{5.7}$$

we rearrange the first and the third lines of (5.6) to the corresponding lines of (5.2).The second line of (5.6) transforms into

$$\sin^2\alpha'\cos^2\beta' = \cos^2\alpha\sin^2\tau.$$

Now we can add the latter to the the first line of (5.2) to obtain the second line of (5.2).Thus, if we have a function of the variables $r=\sqrt{x_{1.}^2+...x_{2p+s}^2}$, $r_1, r_{2,1}$ and Ω , then, after transformation (5.4), this function transforms into a function of variables r, r_1', $r_{2,1}', \Omega$:

$$u(r,\alpha,\beta,\omega) = v(r,r_1,r_{2,1},\Omega) = v'(r,r_1',r_{2,1}',\Omega) = u'(r,\alpha',\beta',\omega'). \tag{5.8}$$

The variables $r, r_{1,}r_{2,1}$ were introduced in trispherical coordinate system above. Now we should make certain that the variable ω defined by (5.7) coincides with the argument ω in (5.3). Then we ensure that Eq.(5.3) is invariant under transformation (5.2) due to invariance Laplace equation under rotation (5.4). It is convenient to carry out the change of variables $\omega=\omega(\theta_1,...,\theta_{p-1},\theta_{p+1},...,\theta_{2p-1})$ not in trispherical but in Cartesian coordinates. Substituting (5.5) in (5.7), we obtain

$$\cos\omega = \frac{x_1x_{p+1}+x_2x_{p+2}+...+x_px_{2p}}{\sqrt{(x_1^2+x_2^2+...+x_p^2)(x_{p+1}^2+x_{p+2}^2+...x_{2p}^2)}}. \tag{5.9}$$

Since Laplace equation in E_{2p+s} is written in the form $\sum_{i=1}^{2p+s}\frac{\partial^2 u}{\partial x_i^2}=0$ the differential operator with respect to ω (and only ω) may be expressed in the form

$$L_\omega u=\frac{\partial^2 u}{\partial\omega^2}\sum_{i=1}^{2p}(\frac{\partial\omega}{\partial x_i})^2+\frac{\partial u}{\partial\omega}\sum_{i=1}^{2p}\frac{\partial^2\omega}{\partial x_i^2}. \tag{5.10}$$

Differentiating (5.9) we obtain

$$\begin{aligned}-\sin\omega\frac{\partial\omega}{\partial x_i}&=\frac{x_{p+i}}{r_1 r_{2,1}}-\frac{x_i\cos\omega}{r_1^2},\\ -\sin\omega\frac{\partial\omega}{\partial x_{p+i}}&=\frac{x_i}{r_1 r_{2,1}}-\frac{x_{p+i}\cos\omega}{r_{2,1}^2},\quad i=1,...,p.\end{aligned} \tag{5.11}$$

which is substituted in (5.10) to give

$$\sum_{i=1}^{2p}(\frac{\partial\omega}{\partial x_i})^2=\frac{1}{r_1^2}+\frac{1}{r_{2,1}^2}=\frac{1}{r^2}(\frac{1}{\cos^2\alpha}+\frac{1}{\sin^2\alpha\cos^2\beta}). \tag{5.12}$$

Then, differentiating (5.11), using (5.11) and (5.12) and summing the results from $i=1$ to $i=p$ we have

$$\sin\omega\sum_{i=1}^{p}(\frac{\partial^2\omega}{\partial x_i^2}+\frac{\partial^2\omega}{\partial x_{p+i}^2})=(p-2)\cos\omega(\frac{1}{r_1^2}+\frac{1}{r_{2,1}^2}). \tag{5.13}$$

Substitution of (5.12) and (5.13) in (5.10) leads to the conclusion that the operator L_ω (5.10) coincides with the last term in the left-hand side of (5.3). The remaining terms in (5.3) do not change the form under the transformation $\omega=\omega(x_1,...,x_{2p})$. Hence, Eq.(5.3) is the Laplace equation in E_{2p+s} for functions $u(r,\alpha,\beta,\omega)$, and (5.3) is invariant under the transformation of one-parametric group (5.2).

Now we can repeat our general procedure, i.e., construct the solution of (5.3) with singularity at $r=r_0,\alpha=0$, subject this solution to transformation (5.2) and equate the solutions in variables (r,α,β,ω) and $(r,\alpha',\beta',\omega')$ to obtain the addition formula (1.1).

The same technique can be used in the case of hypergeometric function. Let us consider pseudo-Euclidean space $E_{2p+s-1,1}$ instead of the Euclidean space E_{2p+s}. Introducing the coordinates similar to the trispherical ones by the change of relations (4.1) to

$$r_1=r\cosh\alpha,\quad r_2=\sinh\alpha,\quad 0\le\alpha<\infty \tag{5.14}$$

and retaining the formulae (4.2)-(4.5) in the previous form we rearrange the first two terms in the Laplace equation (4.7) to

$$\begin{aligned}&-\frac{1}{r^{p+q+s-1}}\frac{\partial}{\partial r}r^{p+q+s-1}\frac{\partial u}{\partial r}+\\&+\frac{1}{r^2\cosh^{p-1}\alpha\sinh^{q+s-1}\alpha}\frac{\partial}{\partial\alpha}\cosh^{p-1}\alpha\sinh^{q+s-1}\alpha\frac{\partial u}{\partial\alpha}.\end{aligned}$$

Limiting ourselves to the case $p = q$, introducing the variable ω by the relations (5.5), (5.7) and separating coordinate r in the resulting equation we come to the equation for the spherical functions (on hyperboloid) in the form

$$\begin{aligned}
&\frac{1}{\cosh^{p-1}\alpha\sinh^{p+s-1}\alpha}\frac{\partial}{\partial\alpha}\cosh^{p-1}\alpha\sinh^{p+s-1}\alpha\frac{\partial u}{\partial\alpha}+\\
&+\frac{1}{\sinh^2\alpha\cos^{p-1}\beta\sin^{s-1}\beta}\frac{\partial}{\partial\beta}\cos^{p-1}\beta\sin^{s-1}\beta\frac{\partial u}{\partial\beta}+\\
&+(\frac{1}{\cosh^2\alpha}+\frac{1}{\sinh^2\alpha\cos^2\beta})\frac{1}{\sin^{p-2}}\frac{\partial}{\partial\omega}\sin^{p-2}\frac{\partial u}{\partial\omega}+\\
&+\nu(2p+s+\nu-2)u=0.
\end{aligned} \tag{5.15}$$

Equation (5.15) is invariant under the transformation

$$\begin{cases}
\begin{aligned}
\cosh^2\alpha' = &\quad \cosh^2\alpha\cosh^2\tau+\sinh^2\alpha\sinh^2\tau\cos^2\beta\\
- &\quad (1/2)\sinh 2\alpha\sinh 2\tau\cos\beta\cos\omega,\\
\sinh\alpha'\sin\beta' = &\quad \sinh\alpha\sin\beta,\\
\sinh 2\alpha'\cos\beta'\cos\omega' = &\quad (\cosh^2\alpha+\sinh^2\alpha\cos^2\beta)\sinh 2\tau+\\
+ &\quad \sinh 2\alpha\cosh 2\tau\cos\omega.
\end{aligned}
\end{cases} \tag{5.16}$$

connected with the hyperbolic rotation of $E_{2p+s-1,1}$ of the form

$$\begin{aligned}
&x'_i = x_i\cosh\tau + x_{p+i}\sinh\tau, \quad x'_{p+i} = x_i + x_{p+i}\cosh\tau,\\
&i = 1,...,p; \quad x'_{2p+i} = x_{2p+i}, \quad i = 1,...,s.
\end{aligned} \tag{5.17}$$

Singular solution of (5.15) dependent on α alone can be expressed by the Jacobi function $P_\nu^{\varkappa,\lambda}(\cosh 2\alpha)$, where

$$P_\nu^{\varkappa,\lambda}(x) = \frac{\Gamma(\varkappa+1)}{\Gamma(\varkappa-\nu+1)\Gamma(\nu+1)}F(-\nu,\nu+\varkappa+\lambda+1,\varkappa+1;\frac{1-x}{2}). \tag{5.18}$$

The transformation of the singular solution by rotation (5.16) results in serial addition theorem for Jacobi function (5.18) (i.e., for the general hypergeometric function) which has the form

$$P_\nu^{\varkappa,\lambda}(\cosh 2\alpha') = \sum_{k=0}^{\infty}\sum_{l=0}^{n} a_{\nu,k,l}F_\nu^{k,l}(\alpha)F_\nu^{k,l}(\tau)p_{k,l}(\beta,\omega), \tag{5.19}$$

where

$$F_\nu^{k,l}(\alpha) = \sinh^{k+l}\alpha\cosh^{k-l}\alpha P_{\nu-k}^{\varkappa+k-l,\lambda+k-l}(\cosh 2\alpha)$$

and coefficients $a_{\nu,k,l}$ are received from the coefficients $a_{n,k,l}$(1.1) by replacing n for ν. A similar formula can be written for Jacobi function of second type, i.e., for the second particular solution of the equation

$$\frac{1}{\cosh^{p-1}\alpha\sinh^{p+s-1}\alpha}\frac{d}{d\alpha}\cosh^{p-1}\alpha\sinh^{p+s-1}\alpha\frac{du}{d\alpha}+\nu(2p+s+\nu-2)u=0.$$

The alternative way of deriving the addition formula for Jacobi function $P_\nu^{\varkappa,\lambda}(x)$ consists in using Mellin integral transform in r for presenting the singular solution of Eq.(5.3) and transforming of the modified singular solution by rotation (5.2).

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References

[1] Askey R. Jacobi polynomials. I. New proof of Koornwinder's Laplace type integral representation and Bateman's bilinear sum.-SIAM J. Math. Anal., 1974, v. 5, 117-124.

[2] Bateman H., Erdélyi A. Higher transcendental function, v. 2, N.Y., Toronto,L., Mc Graw-Hill Book Comp. Inc., 1953.

[3] Koornwinder T.H. The addition formula for Jacobi polynomials. I. Summary of results.-Indag. Math., 1972, v. 34, 188–191.

[4] Koornwinder T.H. The addition formula for Jacobi polynomials and spherical harmonics.-SIAM J. Appl.Math., 1973, v. 25, 236-246.

[5] Koornwinder T. Jacobi polynomials. II. An analytic proof of the product formula.-SIAM J. Math.Anal., 1974, v. 5, 125-137.

[6] Miller W., Jr. Symmetry and separation of variables. L.-Amsterdam-Don Mills--Ontario-Sydney-Tokyo. Addison-Wesley Publ. Comp., 1977.

[7] Michlin S.G. Linear partial differential equations [in Russain], Moscow, Vysshaya shkola, 1977.

[8] Rosenfeld B.A. Multidimensional spaces [in Russain], Moscow, Nauka, 1966

[9] Rosenfeld B.A. Noneuclidean spaces [in Russain], Moscow, Nauka, 1969.

[10] Šapiro R.L. Special functions connected with the representations of group $SU(n)$, class I, relative to $SU(n-1)(n \geq 3)$ [in Russain], Izv.Vuz.Math., 1968, v. 4(71), 97-107.

[11] Vilenkin N.Ja. Special functions and group representation theory [in Russain], Moscow, Nauka, 1991.

[12] Whittaker E.T., Watson G.N. A course of modern analysis. An introduction to the general theory. of infinite processes; with an account of the principal transcendental functions. Cambridge, Univ.Press, 1927.