C^*-Crossed Products and a Generalized Quantum Mechanical N-Body Problem

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To Eyvind H. Wichmann, an "hommage" for his 70^{th} *birthday.*

Abstract

For each finite dimensional real vector space X we construct a C^*-algebra $\mathfrak{C}^X$ graded by the lattice of all subspaces of X and we compute its quotient with respect to the algebra of compact operators. This allows one to describe the essential spectrum and to prove the Mourre estimate for the self-adjoint operators affiliated to $\mathfrak{C}^X$.

1 Introduction

Let $\mathfrak{C}$ be a C^*-algebra of bounded operators on a Hilbert space $\mathcal{H}$ and let H be a self-adjoint operator on $\mathcal{H}$. One says that H is *affiliated* to $\mathfrak{C}$ if $(H-z)^{-1} \in \mathfrak{C}$ for some $z \in \mathbb{C} \setminus \sigma(H)$. In this case $\varphi(H) \in \mathfrak{C}$ for all $\varphi \in C_0(\mathbb{R})$ (space of continuous functions which tend to zero at infinity).

Assume that the algebra of compact operators $K(\mathcal{H})$ on $\mathcal{H}$ is included in $\mathfrak{C}$. Then one can construct the quotient C^*-algebra $\widehat{\mathfrak{C}} = \mathfrak{C}/K(\mathcal{H})$ and one can consider the map $\widehat{H} : C_0(\mathbb{R}) \to \widehat{\mathfrak{C}}$ defined by $\widehat{H}(\varphi) = \widehat{\varphi(H)}$ = class of $\varphi(H)$ in the quotient. This is a morphism and $\widehat{H}$ should be thought as an "abstract" self-adjoint operator affiliated to the "abstract" C^*-algebra $\widehat{\mathfrak{C}}$. The notion of spectrum of $\widehat{H}$ has an obvious meaning and it is easy to show that the essential spectrum of H is given by $\sigma_{\text{ess}}(H) = \sigma(\widehat{H})$.

This point of view is not practically useful unless the algebra $\widehat{\mathfrak{C}}$ has a special structure. In any case $\widehat{\mathfrak{C}}$ should be, in some sense, simpler than $\mathfrak{C}$. A systematic treatment of a rather large class of examples in which $\widehat{\mathfrak{C}}$ and $\widehat{H}$ are explicitly computed can be found in [GI]. In particular, the relevance, in this context, of the C^*-algebras obtained as crossed products of algebras equipped with group actions is pointed out.

Our purpose in this note is to consider the quantum mechanical N-body problem from this point of view. By taking into account the kind of potentials involved in the N-body problem it is natural (at least, a posteriori) to consider in this case the algebra $\mathfrak{C} = \mathfrak{C}^X$ constructed as follows. Let X be a real finite dimensional vector space (the configuration space of the system of N particles) and $\mathcal{A}$ the C^*-algebra generated by functions on X of the form $\varphi \circ \pi_Y$, where Y is a vector subspace of X, $\varphi \in C_0(X/Y)$ and $\pi_Y : X \to X/Y$ is the natural map. The additive group X acts continuously by translations on $\mathcal{A}$. Then we take $\mathfrak{C}$ equal to the crossed product of $\mathcal{A}$ by the action of X.

The definition of crossed products used in this paper is adapted to our needs and we do not make any explicit reference to the general theory (whose usefulness in more complicated situations is, however, shown in [GI]). Let $\mathcal{H}(X)$ be the space of square integrable functions on X and denote by Q, P the position and momentum observables. If $\mathcal{A}$ is a $*$-algebra of bounded uniformly continuous functions on X and if $\mathcal{A}$ is stable under translations, then the (norm) closed linear subspace of $B(\mathcal{H}(X))$ generated by operators of the form $\varphi(Q)\psi(P)$, with $\varphi \in \mathcal{A}, \psi \in C_0(X^*)$, is a C^*-algebra isomorphic to the "abstract" crossed product $\mathcal{A} \rtimes X$ of $\mathcal{A}$ by the action of the additive group X. Here we simply take this as the definition of $\mathcal{A} \rtimes X$. Thus the algebra $\mathfrak{C}^X$ is obtained by choosing $\mathcal{A}$ equal to the closed linear space generated by functions of the form $\varphi \circ \pi_Y$ with $\varphi \in C_0(X/Y)$.

In section 2 we show that such crossed products appear quite naturally in the spectral analysis of quantum hamiltonians. Assume, for example, that one is interested in self-adjoint operators of the form $H = H_0 + V$, where H_0, the kinetic energy, is an elliptic differential operator of order m with constant coefficients and V is a symmetric differential operator of order $< m$ with coefficients in $\mathcal{A}$. Assume also that $\mathcal{A}$ contains the constant functions. Then the smallest C^*-algebra of operators on $\mathcal{H}(X)$ to which all these operators are affiliated (H_0 being fixed) is equal to $\mathfrak{C} = \mathcal{A} \rtimes X$. The main point of our approach is that this algebra often has a remarkable structure (determined by certain properties of $\mathcal{A}$) and this fact alone gives important information on the spectral properties of H. Note that we construct $\mathfrak{C}$ starting with a rather restricted class of perturbations V. However, the class of operators H affiliated to $\mathfrak{C}$ is very large and it allows quite singular perturbations (this point is studied in [DG] and will not be further discussed here).

Above X^* is the space dual to X and one may assume without loss of generality that $X = X^* = \mathbb{R}^n$. However, our approach is explicitly independent of a choice of an euclidean structure on the configuration space. This not only simplifies the presentation but opens the way to generalizations which allow one to study many channel hamiltonians quite different from those of N-body type. Indeed, one can replace X by a locally compact abelian group and the subspaces Y by subgroups. The case of non abelian groups is also interesting, for example the "symplectic algebra" associated to a symplectic space (to which N-body hamiltonians with magnetic fields are affiliated) corresponds to the Heisenberg group.

Technically speaking, the main result of this paper is Theorem 5.2. We shall state now several consequences of this theorem.

Theorem 1.1 *Let H be a self-adjoint operator on $\mathcal{H}(X)$ affiliated to $\mathfrak{C}^X$. Then for each $\omega \in X$, $\omega \neq 0$, the limit $\lim_{|\lambda|\to\infty} \tau_{\lambda\omega}[H] := H_\omega$ exists in strong-resolvent sense and*

$$\sigma_{\text{ess}}(H) = \bigcup_{\omega \in X \setminus \{0\}} \sigma(H_\omega). \tag{1.1}$$

Here $\tau_x[H] = \mathrm{e}^{i\langle x,P\rangle} H \mathrm{e}^{-i\langle x,P\rangle}$ and $(\mathrm{e}^{i\langle x,P\rangle} f)(y) = f(x+y)$. Note that H_ω depends only on the one dimensional space generated by ω. The proof of the fact that $\sigma_{\text{ess}}(H)$ is equal to the closure of the right-hand side in (1.1) is quite easy (see section 8.4.3 in [ABG] and note that the spectrum of a direct sum, not necessarily finite, of self-adjoint operators is equal to the closure of the union of the spectra). The fact that the right-hand side in (1.1) is closed is more difficult and the proof can be found in [DG].

In order to get a version of Theorem 1.1 which resembles more to the standard N-body version, we shall consider only a rather particular class of H. Take $X = X^* = \mathbb{R}^n$ and let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $c^{-1}\langle x\rangle^{2s} \leq h(x) \leq c\langle x\rangle^{2s}$ if $|x| > r$, where s, r, c are strictly positive constants and $\langle x\rangle = (1 + |x|^2)^{1/2}$. Then $H_0 = h(P)$ is a self-adjoint operator whose form domain is the Sobolev space $\mathcal{H}^s(X)$. For each linear subspace $Y \subset X$ let V_Y be a continuous symmetric sesquilinear form on $\mathcal{H}^s(X)$, identified with an operator $\mathcal{H}^s(X) \to \mathcal{H}^{-s}(X)$, such that:

(i) $[\mathrm{e}^{i\langle y,P\rangle}, V_Y] = 0 \quad \forall\, y \in Y$;
(ii) $\big\|[\mathrm{e}^{i\langle Q,k\rangle}, V_Y]\big\|_{\mathcal{H}^s\to\mathcal{H}^{-s}} \to 0$ if $k \in Y$, $k \to 0$;
(iii) $\big\|[\mathrm{e}^{i\langle Q,k\rangle} - 1]V_Y\big\|_{\mathcal{H}^s\to\mathcal{H}^{-s}} \to 0$ if $k \in Y^\perp$, $k \to 0$.

We have denoted $\mathrm{e}^{i\langle Q,k\rangle}$ the operator of multiplication by the function $\mathrm{e}^{i\langle x,k\rangle}$. For $Y = O = \{0\}$ we take $V_Y = 0$. Note that in (ii) (but not in (iii)) the euclidean structure of $X = \mathbb{R}^n$ is used. Furthermore, we assume

$$\sum_{Y\subset X} \|V_Y\|_{\mathcal{H}^s\to\mathcal{H}^{-s}} < \infty,$$

in particular $V_Y \neq 0$ only for a countable number of subspaces Y. Finally, we ask that $V_Y \geq -\mu_Y H_0 - \delta_Y$ as forms on $\mathcal{H}^s(X)$, where $\mu_Y, \delta_Y \geq 0$ are numbers such that $\sum_Y \mu_Y < 1$ and $\sum_Y \delta_Y < \infty$. Then $H = H_0 + \sum_Y V_Y$ is a self-adjoint, bounded from below operator on $\mathcal{H}(X)$, H is affiliated to $\mathfrak{C}^X$, and

$$\sigma_{\mathrm{ess}}(H) = \bigcup_{Y\in\mathcal{M}} \sigma(H_Y).$$

Here $H_Y = H_0 + \sum_{Z\supset Y} V_Z$ and $\mathcal{M}$ is the set of minimal elements (for the inclusion relation) of the class of subspaces of the form $Y_1 \cap \cdots \cap Y_k \neq 0$ with $Y_i \in \{Y \mid V_Y \neq 0\}$.

Theorem 5.2 can also be used in order to prove the Mourre estimate for operators affiliated to $\mathfrak{C}^X$. We shall present here only the simplest case when H_0 is the (positive) laplacian and the V_Y do not depend on the projection on Y of the momentum, so $V_Y = 1 \otimes V^Y$ if $\mathcal{H}(X) = \mathcal{H}(Y) \otimes \mathcal{H}(Y^\perp)$. More precisely, it suffices to replace (ii) by the stronger condition:

(ii*) $[\mathrm{e}^{i\langle Q,k\rangle}, V_Y] = 0$ if $k \in Y$.

Now H looks exactly as in the non-relativistic N-body problem, the only difference is that we allow an infinite lattice of subspaces. Thus we can define operators H^Y acting in $\mathcal{H}(Y^\perp)$ for each $Y \subset X$ and these operators have a structure similar to that of H. Note that in the N-body case the subspaces Y are usually denoted X_a; moreover, one must interchange the rôles of Y and $Y^\perp$ in order to agree with the conventions from [ABG].

Theorem 1.2 *Let H be of the form described above and let D be the generator of the dilation group in $\mathcal{H}(X)$. Assume that the quadratic form $[D, (H+i)^{-1}]$, with domain equal to the domain of D, extends to a bounded operator which belongs to $\mathfrak{C}^X$. Let $\mathcal{L}$ be the family of subspaces of the form $Y_1 \cap \cdots \cap Y_k$ where Y_i are subspaces such that $V_{Y_i} \neq 0$. Define the set of* *thresholds of* H *by*

$$\tau(H) = \bigcup_{Y \in \mathcal{L} \setminus \{O\}} \sigma_p(H^Y).$$

Then the Mourre estimate for H *with respect to* D *holds outside* $\tau(H)$.

As in case of Theorem 1.1, the proof of the fact that $\tau(H)$ is the closure of the right-hand side in the above formula is a straightforward application of Theorem 5.2 and of Theorem 8.4.3 from [ABG] (infinite version), see section 9.4.1 from [ABG]. The fact that the right-hand side is actually closed is not so easy, see [DG] where the Mourre estimate is proved for more general classes of hamiltonians.

2 Crossed Products and Quantum Hamiltonians

We begin with some notations and conventions adopted in this paper. If X is a locally compact topological space then $C_b(X)$ is the C^*-algebra of continuous bounded complex functions on it, $C_0(X)$ the C^*-subalgebra of functions convergent to zero at infinity, and $C_c(X)$ the subalgebra of functions with compact support. If $\mathcal{H}$ is a Hilbert space then $B(\mathcal{H})$ and $K(\mathcal{H})$ are the C^*-algebras of all bounded and compact operators on $\mathcal{H}$ respectively. By *morphism* between two $*$-algebras we mean $*$-morphism. If $\mathfrak{A}, \mathfrak{B}$ are subspaces of an algebra $\mathfrak{C}$ then we denote by $\mathfrak{A} \cdot \mathfrak{B}$ the linear subspace of $\mathfrak{C}$ generated by the elements of the form AB with $A \in \mathfrak{A}, B \in \mathfrak{B}$; if $\mathfrak{C}$ is a C^*-algebra then $[\![\mathfrak{A} \cdot \mathfrak{B}]\!]$ is the norm closure of $\mathfrak{A} \cdot \mathfrak{B}$ in $\mathfrak{C}$. A family $\{\mathfrak{C}_i\}_{i \in I}$ of subalgebras of $\mathfrak{C}$ is *linearly independent* if for each family $\{S_i\}_{i \in I}$ such that $S_i \in \mathfrak{C}_i \ \forall i, S_i \neq 0$ for at most a finite number of i and $\sum_{i \in I} S_i = 0$, one has $S_i = 0$ for all $i \in I$.

Let X be a finite dimensional real vector space. For each $x \in X$ we denote by τ_x the automorphism of $C_b(X)$ associated to the translation by x, so $(\tau_x \varphi)(y) = \varphi(y + x)$. Then the set $C_{bu}(X)$ of functions $\varphi \in C_b(X)$ such that $x \mapsto \tau_x \varphi$ is norm continuous is the C^*-algebra of bounded uniformly continuous functions on X. If $X = O = \{0\}$ is the zero dimensional vector space then $C_b(X) = C_{bu}(X) = C_0(X) = \mathbb{C}$.

We denote by $\mathcal{H}(X)$ the space of (equivalence classes of) functions on X square integrable with respect to a Haar measure (i.e. a translation invariant positive Radon measure on X). There is no canonical norm on $\mathcal{H}(X)$, the Haar measure being determined only modulo a positive constant factor. However, it is clear that the norms in the spaces $B(\mathcal{H}(X))$ and $K(\mathcal{H}(X))$ are independent of the choice of the measure. We set $\mathfrak{B}(X) = B(\mathcal{H}(X))$

and $\mathfrak{K}(X) = K(\mathcal{H}(X))$; these are C^*-algebras uniquely determined by X. If $X = O = \{0\}$ is the zero dimensional vector space we set $\mathcal{H}(X) = \mathbb{C}$ and $\mathfrak{B}(O) = \mathfrak{K}(O) = \mathbb{C}$.

We embed $C_b(X)$ in $\mathfrak{B}(X)$ by identifying a function $u \in C_b(X)$ with the operator of multiplication by u in $\mathcal{H}(X)$. If we denote $u(Q)$, or $u(Q^X)$, this operator, then the map $u \mapsto u(Q)$ is an isometric morphism of $C_b(X)$ into $\mathfrak{B}(X)$. So we realize $C_b(X)$ as a C^*-subalgebra of $\mathfrak{B}(X)$.

Let X^* be the dual space of X. For $x \in X, x^* \in X^*$ let $\langle x, x^* \rangle = x^*(x)$ and identify $X^{**} = X$ by setting $\langle x^*, x \rangle = \langle x, x^* \rangle$. We shall embed the algebra $C_b(X^*)$ of bounded continuous functions on X^* into $\mathfrak{B}(X)$ by the following procedure, explicitly independent of a Haar measure on X. It will be convenient here to use the notation $\mathrm{e}^{i\langle x,P\rangle}$ for the operator of translation by $x \in X$, more precisely $(\mathrm{e}^{i\langle x,P\rangle} f)(y) = f(x+y)$ (this is identical to the action of τ_x on functions). For each $v \in \mathcal{S}(X^*)$ (space of Schwartz test functions on X^*) there is a unique measure $\widehat{v}$ on X such that $v(x^*) = \int_X \mathrm{e}^{i\langle x,x^*\rangle} \widehat{v}(dx)$. We set $v(P) = \int_X \mathrm{e}^{i\langle x,P\rangle} \widehat{v}(dx)$ (if X has to be specified we set $v(P) = v(P^X)$). The map $v \mapsto v(P)$ extends to an isometric morphism of $C_0(X^*)$ into $\mathfrak{B}(X)$. This allows us to realize $C_0(X^*)$ as a C^*-subalgebra of $\mathfrak{B}(X)$ and it is easy to check that the action of $C_0(X^*)$ on $\mathcal{H}(X)$ is nondegenerate. This, in turn, will give us an embedding $C_b(X^*) \subset \mathfrak{B}(X)$, uniquely determined by the property $u(P)v(P) = (uv)(P)$ if $u \in C_b, v \in C_0$. Observe that $C_0(X^*)$ is just the group C^*-algebra of the additive group X.

One may define the Fourier transformation as a bijective map $F : \mathcal{S}(X) \to \mathcal{S}(X^*)$ such that $F \exp(i\langle x, P^X \rangle) = \exp(i\langle x, Q^{X^*} \rangle) F$ for all $x \in X$. This defines F modulo a complex factor and $\Phi_X[S] = FSF^{-1}$ gives a canonical isomorphism $\Phi_X : \mathfrak{B}(X) \to \mathfrak{B}(X^*)$ such that $\Phi_X[v(P^X)] = v(Q^{X^*})$ for all $v \in C_b(X^*)$.

We now prove a lemma which plays an important rôle in our arguments.

Lemma 2.1 *If $u \in C_{bu}(X)$ and $v \in \mathcal{S}(X^*)$ then for each number $\varepsilon > 0$ there are points $x_1, ..., x_N \in X$ and functions $v_1, ..., v_N \in \mathcal{S}(X^*)$ such that:*

$$\left\| v(P)u(Q) - \sum_{k=1}^{N} u(Q + x_k) v_k(P) \right\| < \varepsilon.$$

Proof. Set $S = u(Q)$ and $S(x) = e^{i\langle x,P\rangle} S e^{-i\langle x,P\rangle}$. Then $S(x) = u(Q + x)$, S is an element of $\mathfrak{B}(X)$, and the map $x \longmapsto S(x)$ is bounded by $\|S\|$ and is norm continuous. Choose a Haar measure dx on X. The measure $\widehat{v}$

is absolutely continuous, so we can identify it with a function in $\mathcal{S}(X)$ by $\widehat{v}(dx) \equiv \widehat{v}(x)dx$. Then

$$v(P)S = \int_X \mathrm{e}^{i\langle x,P\rangle} S\widehat{v}(x)dx = \int_X S(x)\mathrm{e}^{i\langle x,P\rangle}\widehat{v}(x)dx.$$

Let K be a compact subset of X and V a compact neighbourhood of the origin in X. One can find functions J_0, J_1, ..., J_N of class C^∞ on X such that: (i) $0 \leq J_k \leq 1$ if $0 \leq k \leq N$ and $\sum_{k=0}^N J_k(x) = 1$ for all $x \in X$; (ii) $K \cap \operatorname{supp} J_0 = \emptyset$; (iii) there are points $x_1, ..., x_N \in X$ and such that $\operatorname{supp} J_k \subset x_k + V$ if $1 \leq k \leq N$. Then we write:

$$\begin{aligned} v(P)S &= \sum_{k=1}^N S(x_k) \int_X J_k(x)\mathrm{e}^{i\langle x,P\rangle}\widehat{v}(x)dx \\ &+ \sum_{k=1}^N \int_X [S(x) - S(x_k)] J_k(x)\mathrm{e}^{i\langle x,P\rangle}\widehat{v}(x)dx \\ &+ \int_X S(x)J_0(x)\mathrm{e}^{i\langle x,P\rangle}\widehat{v}(x)dx. \end{aligned}$$

Let $v_k \in \mathcal{S}(X^*)$ be defined by $\widehat{v}_k(x) = J_k(x)\widehat{v}(x)$; then the first sum above is $\sum_{k=1}^N S(x_k)v_k(P)$. Denote $K_k = \operatorname{supp} J_k$ and let L be the integral of $|\,\widehat{v}\,|$. We estimate the last two terms as follows :

$$\left\| \sum_{k=1}^N \int_X [S(x) - S(x_k)] J_k(x)\mathrm{e}^{i\langle x,P\rangle}\widehat{v}(x)dx \right\| \leq \sup_{1\leq k\leq N} \sup_{x\in K_k} \|S(x) - S(x_k)\|\, L$$

because $\sum_{k=1}^N J_k(x) \leq 1$, and

$$\left\| \int_X S(x)J_0(x)\mathrm{e}^{i\langle x,P\rangle}\widehat{v}(x)dx \right\| \leq \|S\| \int_{X\setminus K} |\,\widehat{v}(x)\,|\, dx.$$

If K is large enough the second member in the last inequality can be made $< \varepsilon/2$. Finally, we have

$$\sup_{x\in K_k} \|S(x) - S(x_k)\| \leq \sup_{y\in V} \|S(y) - S\|$$

and this can be made $< \varepsilon/(2L)$ because the function $S(\cdot)$ is continuous. ■

Corollary 2.2 *If* $\mathcal{A} \subset C_{bu}(X)$ *is a translation invariant subspace, then* $[\![\mathcal{A} \cdot C_0(X^*)]\!] = [\![C_0(X^*) \cdot \mathcal{A}]\!]$. *In particular, if* $\mathcal{A}$ *is a translation invariant* $*$*-subalgebra of* $C_{bu}(X)$ *then* $[\![\mathcal{A} \cdot C_0(X^*)]\!]$ *is a* C^**-subalgebra of* $\mathfrak{B}(X)$.

If $\mathcal{A}$ is a C^*-subalgebra of $C_{bu}(X)$ stable under translations then $\mathcal{A}$ is equipped with a continuous action of the additive group X, so the "abstract" C^*-crossed product $\mathcal{A} \rtimes X$ is well defined. It can be shown (see [GI]) that $\mathcal{A} \rtimes X$ is canonically isomorphic to the C^*-subalgebra $[\![\mathcal{A} \cdot C_0(X^*)]\!]$ of $\mathfrak{B}(X)$. For this reason we shall call this subalgebra *crossed product of $\mathcal{A}$ by X* and we use the notation

$$\mathcal{A} \rtimes X = [\![\mathcal{A} \cdot C_0(X^*)]\!]. \tag{2.1}$$

Our next purpose is to explain the relevance of this notion in the setting of "algebras of energy observables" considered in [GI]. More precisely, we show that $\mathcal{A} \rtimes X$ coincides with the C^*-algebra generated by the hamiltonians of the form $H = H_0 + V$ where H_0, the kinetic energy, is fixed, while the interaction is given by a potential V of "type $\mathcal{A}$" in a sense that we shall specify below. So $\mathcal{A} \rtimes X$ can be thought as the algebra of all hamiltonians describing the motion of a system subject to a certain type ($\mathcal{A}$) of interactions.

Let us fix a C^*-algebra stable under translations and containing the constants $\mathcal{A} \subset C_{bu}(X)$. We set

$$\mathcal{A}^\infty = \{\varphi \in \mathcal{A} \cap C^\infty(X) \mid \text{all the derivatives of } \varphi \text{ belong to } \mathcal{A}\}.$$

We notice that $\mathcal{A}^\infty$ is a dense $*$-subalgebra of $\mathcal{A}$. Indeed, let $\theta \in C_c^\infty(X)$ with $\int \theta(x)dx = 1$ and let us set $\theta_\varepsilon(x) = \varepsilon^{-n}\theta(x/\varepsilon)$ if $\varepsilon > 0$. Then $\theta_\varepsilon * \varphi \to \varphi$ in sup norm as $\varepsilon \to 0$ if $\varphi \in \mathcal{A}$, because φ is uniformly continuous. So it suffices to prove that all the derivatives of $\theta_\varepsilon * \varphi$ belong to $\mathcal{A}$. Identifying $X = \mathbb{R}^n$ and taking $\varepsilon = 1$, we have for $\alpha \in \mathbb{N}^n$:

$$(\theta * \varphi)^{(\alpha)} = \theta^{(\alpha)} * \varphi = \int \theta^{(\alpha)}(-y)\tau_y\varphi \, dy.$$

The integral converges in norm in $C_{bu}(X)$ and $\mathcal{A}$ is a closed subspace, so $(\theta * \varphi)^{(\alpha)} \in \mathcal{A}$.

Theorem 2.3 *Let $\mathcal{A} \subset C_{bu}(X)$ be a translation invariant C^*-subalgebra containing the constant functions and let $h : X^* \to \mathbb{R}$ be a continuous function such that $|\, h(k) \,| \to \infty$ when $k \to \infty$ in X^*. Then the C^*-subalgebra of $\mathfrak{B}(X)$ generated by the self-adjoint operators of the form $h(P+k)+V(Q)$, where $k \in X^*$ and $V : X \to \mathbb{R}$ belongs to $\mathcal{A}^\infty$, coincides with $\mathcal{A} \rtimes X$.*

Proof. Let $\mathfrak{C}$ be the C^*-algebra generated by the operators $H = h(P + k) + V(Q) \equiv H_0 + V(Q)$, with $k \in X^*$ and $V \in \mathcal{A}^\infty$. By making a series expansion for large z

$$(z - H)^{-1} = \sum_{n \geq 0} (z - H_0)^{-1}[V(Q)(z - H_0)^{-1}]^n$$

we easily get $\mathfrak{C} \subset \mathcal{A} \rtimes X$. It remains to prove the opposite inclusion. Let $z \in \mathbb{C} \setminus h(X^*)$. Then for $\mu \in \mathbb{R}$ small enough we have $z \notin \sigma(H_\mu)$ if $H_\mu = h(P+k) + \mu V(Q)$. The function $\mu \mapsto (H_\mu - z)^{-1}$ is norm derivable at $\mu = 0$ with derivative $-(H_0 - z)^{-1} V(Q)(H_0 - z)^{-1}$. Hence $(H_0 - z)^{-1} V(Q)(H_0 - z)^{-1} \in \mathfrak{C}$. Let $\theta \in C_c(\mathbb{R})$ with $\theta(0) = 1$ and $\varepsilon > 0$. Then $\theta(\varepsilon H_0)(H_0 - z) = \varepsilon^{-1}\theta_1(\varepsilon H_0)$ if $\theta_1(t) = \theta(t)(t - z/\varepsilon)$. Since εH_0 is affiliated to $\mathfrak{C}$, we get $\theta(\varepsilon H_0)(H - z) \in \mathfrak{C}$. Thus:

$$\theta(\varepsilon H_0) V(Q)(H_0 - z)^{-1} = \theta(\varepsilon H_0)(H_0 - z) \cdot (H_0 - z)^{-1} V(Q)(H_0 - z)^{-1} \in \mathfrak{C}.$$

We can write $(H_0 - z)^{-1} = \lim_{n\to\infty} \psi_n(P)$ (norm limit) with $\psi_n \in C_c(X^*)$. Then, for each $s > 0$, $\psi_n(P)$ is a continuous map from $L^2(X)$ into the Sobolev space $\mathcal{H}^s(X)$ and $V(Q) \in B(\mathcal{H}^s(X))$. Clearly $\lim_{\varepsilon\to 0} \theta(\varepsilon H_0) = 1$ in norm in $B(\mathcal{H}^s(X), L^2(X))$. Thus we see that $\lim_{\varepsilon\to 0} \theta(\varepsilon H_0) V(Q)\psi_n(P) = V(Q)\psi_n(P)$ in norm in $B(L^2(X))$ for each n. On the other hand, we have $\left\| V(Q)\psi_n(P) - V(Q)(H_0 - z)^{-1} \right\| \to 0$ as $n \to \infty$. It follows then that $\lim_{\varepsilon\to 0} \theta(\varepsilon H_0) V(Q)(H_0 - z)^{-1} = V(Q)(H_0 - z)^{-1}$ in norm in $B(L^2(X))$.

This argument proves that

$$V(Q)(h(P+k) - z)^{-1} = V(Q)(H_0 - z)^{-1} \in \mathfrak{C}$$

for each $k \in X^*$. This clearly implies $V(Q)\xi(H_0) \in \mathfrak{C}$ for $\xi \in C_c(\mathbb{R})$.

The set of $\psi \in C_0(X^*)$ such that $V(Q)\psi(P) \in \mathfrak{C}$ is norm closed and contains all the functions of the form $\psi(P) = \xi(h(P+k))$ with $\xi \in C_c(\mathbb{R})$ and $k \in X^*$. The family consisting of such functions is a $*$-subalgebra of $C_0(X^*)$ which separates the points of X^* (because $|\, h(p+k) \,| \to \infty$ if $k \to \infty$). By the Stone-Weierstrass theorem, we see that this family is dense in $C_0(X^*)$. So we have $V(Q)\psi(P) \in \mathfrak{C}\ \forall \psi \in C_0(X^*)$. Here V is an arbitrary function in $\mathcal{A}^\infty$. Since $\mathcal{A}^\infty$ is dense in $\mathcal{A}$, we finally obtain $\varphi(Q)\psi(P) \in \mathfrak{C}$ for all $\varphi \in \mathcal{A}, \psi \in C_0(X^*)$. ■

Corollary 2.4 *Let $h : X^* \to \mathbb{R}$ be an elliptic polynomial of order m. Then the C^*-algebra of operators on $\mathcal{H}(X)$ generated by the self-adjoint operators $H = h(P) + W$, where W is a symmetric differential operator of order $< m$ with coefficients in $\mathcal{A}^\infty$, is equal to $\mathcal{A} \rtimes X$.*

Proof. If $V \in \mathcal{A}^\infty$, and if we identify $X^* = \mathbb{R}^n$, we have

$$h(P+k) + V(Q) = h(P) + \sum_{|\,\alpha\,| \geq 1} \frac{k^\alpha}{\alpha!} h^{(\alpha)}(P) + V(Q) \equiv h(P) + W$$

so we may use the preceding theorem. ■

3 Graded C^*-Algebras

In this paper we are interested in C^*-algebras which are graded by a semilattice $\mathcal{L}$. The case when $\mathcal{L}$ is finite is presented in Section 8.4 from [ABG]. Below we extend the formalism to the case of infinite $\mathcal{L}$. Note that in the present context it is convenient to interchange the rôles of the lower and upper bounds in the definition of the grading, which explains some differences in notations with respect to [ABG].

Let $\mathcal{L}$ be an arbitrary *semilattice*, i.e. $\mathcal{L}$ is a partially ordered set such that the lower bound $a \wedge b$ exists for all $a, b \in \mathcal{L}$. If $\mathcal{L}$ has a least or a greatest element, we denote it $\min \mathcal{L}$ or $\max \mathcal{L}$ respectively. We denote by $\mathbb{F}(\mathcal{L})$ the family of finite subsets $\mathcal{F} \subset \mathcal{L}$ such that $a \wedge b \in \mathcal{F}$ if $a, b \in \mathcal{F}$ (the empty set belongs to $\mathbb{F}(\mathcal{L})$). So each $\mathcal{F} \in \mathbb{F}(\mathcal{L})$ is a finite semilattice for the order relation induced by $\mathcal{L}$, in particular it has a least element $\min \mathcal{F}$. We equip $\mathbb{F}(\mathcal{L})$ with the order relation given by inclusion. Since $\mathcal{F}_1 \cap \mathcal{F}_2 \in \mathbb{F}(\mathcal{L})$ if $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{F}(\mathcal{L})$, the set $\mathbb{F}(\mathcal{L})$ becomes a semilattice. Note that for each finite part $F \subset \mathcal{L}$ the set of elements of the form $a_1 \wedge \cdots \wedge a_n$ with $a_1, \cdots, a_n \in F$ belongs to $\mathbb{F}(\mathcal{L})$ and contains F.

A *$\mathcal{L}$-graded C^*-algebra* is a C^*-algebra $\mathfrak{C}$ equipped with a linearly independent family $\{\mathfrak{C}(a)\}_{a\in\mathcal{L}}$ of C^*-subalgebras such that:
(i) $\mathfrak{C}(a) \cdot \mathfrak{C}(b) \subset \mathfrak{C}(a \wedge b)$ for all $a, b \in \mathcal{L}$;
(ii) if $\mathcal{F} \in \mathbb{F}(\mathcal{L})$ then $\mathfrak{C}(\mathcal{F}) := \sum_{a\in\mathcal{F}} \mathfrak{C}(a)$ is a closed subspace of $\mathfrak{C}$;
(iii) $\bigcup_{\mathcal{F}\in\mathbb{F}(\mathcal{L})} \mathfrak{C}(\mathcal{F}) \equiv \sum_{a\in\mathcal{L}} \mathfrak{C}(a)$ is dense in $\mathfrak{C}$.

It is clear that for each $\mathcal{F} \in \mathbb{F}(\mathcal{L})$ the space $\mathfrak{C}(\mathcal{F})$ is a C^*-subalgebra of $\mathfrak{C}$ and is equipped with a canonical structure of $\mathcal{F}$-graded C^*-algebra. The set of C^*-subalgebras of $\mathfrak{C}$ (ordered by inclusion) is a semilattice and $\mathcal{F} \longmapsto \mathfrak{C}(\mathcal{F})$ is an injective morphism of semilattices (i.e. $\mathfrak{C}(\mathcal{F}_1 \cap \mathcal{F}_2) = \mathfrak{C}(\mathcal{F}_1) \cap \mathfrak{C}(\mathcal{F}_2)$). In particular, we have $\mathfrak{C}(\mathcal{F}_1) \subset \mathfrak{C}(\mathcal{F}_2)$ if $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathfrak{C}$ coincides with the inductive limit of $\{\mathfrak{C}(\mathcal{F}) \mid \mathcal{F} \in \mathbb{F}(\mathcal{L})\}$, a directed system of C^*-algebras.

We use the notation $\mathfrak{C}(\mathcal{F}) = \sum_{a\in\mathcal{F}} \mathfrak{C}(a)$ for an arbitrary subset $\mathcal{F}$ of $\mathcal{L}$. If $\mathcal{F}$ is $\wedge$-stable then $\mathfrak{C}(\mathcal{F})$ is a $*$-algebra, but is not complete in general. In particular $\mathfrak{C}$ is just the closure of the $*$-algebra $\mathfrak{C}(\mathcal{L})$. For $a \in \mathcal{L}$ set $\mathcal{L}_a = \{b \mid a \leq b\}$, $\mathcal{L}'_a = \{b \mid a \not\leq b\}$ and let $\mathfrak{C}_a$, $\mathfrak{J}_a$ be the closure of $\mathfrak{C}(\mathcal{L}_a)$ and $\mathfrak{C}(\mathcal{L}'_a)$ respectively. Note that $\mathfrak{C}(\mathcal{L}_a)$ is a $*$-subalgebra of $\mathfrak{C}$ and $\mathfrak{C}(\mathcal{L}'_a)$ is a self-adjoint ideal in $\mathfrak{C}$. Hence $\mathfrak{C}_a, \mathfrak{J}_a$ are C^*-subalgebras of $\mathfrak{C}$ and $\mathfrak{J}_a$ is also a closed self-adjoint ideal in $\mathfrak{C}$.

Theorem 3.1 *For all $a \in \mathcal{L}$ one has $\mathfrak{C} = \mathfrak{C}_a + \mathfrak{J}_a$ and $\mathfrak{C}_a \cap \mathfrak{J}_a = \{0\}$. The projection $\mathcal{P}_a : \mathfrak{C} \to \mathfrak{C}_a$ determined by this linear direct sum decomposition is a morphism, in particular $\|\mathcal{P}_a\| = 1$.*

Proof. We clearly have $\mathfrak{C}(\mathcal{L}) = \mathfrak{C}(\mathcal{L}_a) + \mathfrak{C}(\mathcal{L}'_a)$ as a linear direct sum. Let $\mathcal{P}^\circ_a$ be the projection of $\mathfrak{C}(\mathcal{L})$ onto $\mathfrak{C}(\mathcal{L}_a)$ determined by this decomposition. Thus, if $T \in \mathfrak{C}(\mathcal{L})$ is given by $T = \sum_{b\in\mathcal{L}} T(b)$, where $T(b) \neq 0$ only for a finite number of b, we have $\mathcal{P}^\circ_a[T] = \sum_{a\leq b} T(b)$. Then $\mathcal{P}^\circ_a[T]^* = \mathcal{P}^\circ_a[T^*]$ and if $S = \sum_{b\in\mathcal{L}} S(b)$ with $S(b) \neq 0$ only for a finite number of b, then

$$ST = \sum_{b,c} S(b)T(c) = \sum_{d\in\mathcal{L}} \sum_{b\wedge c=d} S(b)T(c).$$

Thus

$$\begin{aligned}
\mathcal{P}^\circ_a[ST] &= \sum_{a\leq d} \sum_{b\wedge c=d} S(b)T(c) = \sum_{a\leq b\wedge c} S(b)T(c) \\
&= \sum_{a\leq b,\, a\leq c} S(b)T(c) = \mathcal{P}^\circ_a[S]\ \mathcal{P}^\circ_a[T].
\end{aligned}$$

Hence $\mathcal{P}^\circ_a$ is a morphism of the $*$-algebra $\mathfrak{C}(\mathcal{L})$ onto its $*$-subalgebra $\mathfrak{C}(\mathcal{L}_a)$.

Let $\mathcal{F} \in \mathbb{F}(\mathcal{L})$ such that $a \in \mathcal{F}$ and $T(b) = 0$ if $b \notin \mathcal{F}$ (we saw before that such a $\mathcal{F}$ exists). Then $\mathcal{P}^\circ_a|\mathfrak{C}(\mathcal{F})$ is a morphism of the C^*-algebra $\mathfrak{C}(\mathcal{F})$ onto the C^*-algebra $\mathfrak{C}(\mathcal{F}_a)$, with $\mathcal{F}_a = \mathcal{F}\cap\mathcal{L}_a$. Such a morphism always has norm ≤ 1. Hence we have $\|\mathcal{P}^\circ_a[T]\| \leq \|T\|$. Since this is valid for each $T \in \mathfrak{C}(\mathcal{L})$ and $\mathfrak{C}(\mathcal{L})$ is dense in $\mathfrak{C}$, we see that $\mathcal{P}^\circ_a$ extends to a morphism $\mathcal{P}_a : \mathfrak{C} \to \mathfrak{C}_a$ with $\mathcal{P}_a[T] = T$ if $T \in \mathfrak{C}_a$. In particular, $\mathcal{P}_a$ is also a linear projection of $\mathfrak{C}$ onto $\mathfrak{C}_a$ with $\|\mathcal{P}_a\| \leq 1$. Since $\mathfrak{C}_a \neq \{0\}$ (because $\mathfrak{C}(a) \neq \{0\}$) we have in fact $\|\mathcal{P}_a\| = 1$.

We have $\mathfrak{C}(\mathcal{L}'_a) = \ker\mathcal{P}^\circ_a \subset \ker\mathcal{P}_a$. Since $\mathfrak{C}(\mathcal{L})$ is dense in $\mathfrak{C}$, $\mathfrak{C}(\mathcal{L}'_a) = (1-\mathcal{P}_a)\mathfrak{C}(\mathcal{L})$, and $1-\mathcal{P}_a$ is a continuous surjective map of $\mathfrak{C}$ onto $\ker\mathcal{P}_a$, we get that $\mathfrak{C}(\mathcal{L}'_a)$ is dense in $\ker\mathcal{P}_a$. So $\ker\mathcal{P}_a = \mathfrak{J}_a$. ■

One can reformulate the preceding theorem in the following terms: *the map $\mathcal{P}^\circ_a : \mathfrak{C} \to \mathfrak{C}$ defined by $\mathcal{P}^\circ_a[\sum_b T(b)] = \sum_{a\leq b} T(b)$ extends to a norm 1 projection $\mathcal{P}_a$ of $\mathfrak{C}$ onto $\mathfrak{C}_a$ which is also a morphism.* Not also that the family of C^*-subalgebras $\{\mathfrak{C}_a\}_{a\in\mathcal{L}}$ is decreasing: if $a \leq b$ then $\mathfrak{C}_b \subset \mathfrak{C}_a$ and

$$\mathcal{P}_a\mathcal{P}_b = \mathcal{P}_b\mathcal{P}_a = \mathcal{P}_b \tag{3.1}$$

If $\mathcal{L}$ has a least element $\min\mathcal{L}$ then $\mathfrak{C}(\min\mathcal{L})$ is a closed self-adjoint ideal in $\mathfrak{C}$, hence one may construct the quotient C^*-algebra

$$\widehat{\mathfrak{C}} := \mathfrak{C}/\mathfrak{C}(\min\mathcal{L}). \tag{3.2}$$

We shall give a more explicit description of this object when $\mathcal{L}$ is an atomic semilattice. We recall that an *atom* of $\mathcal{L}$ is an element $a \neq \min \mathcal{L}$ such that: $b \leq a \Rightarrow b = \min \mathcal{L}$ or $b = a$. We denote by $\mathcal{M}$ the set of atoms of $\mathcal{L}$ and we say that $\mathcal{L}$ is *atomic* if each $b \neq \min \mathcal{L}$ is minorated by an atom. Then we can associate to $\mathfrak{C}$ a second C^*-algebra, namely

$$\widetilde{\mathfrak{C}} := \bigoplus_{a \in \mathcal{M}} \mathfrak{C}_a \tag{3.3}$$

where the direct sum is in the C^*-algebra sense. Observe that there is a natural morphism $\mathcal{P} : \mathfrak{C} \to \widetilde{\mathfrak{C}}$, namely

$$\mathcal{P}[T] = (\mathcal{P}_a[T])_{a \in \mathcal{M}}.$$

Theorem 3.2 *Assume that the semilattice $\mathcal{L}$ has a least element and is atomic. Then the kernel of the morphism $\mathcal{P}$ is equal to $\mathfrak{C}(\min \mathcal{L})$.*

Proof. First we note that the result is known (and easy to prove) if $\mathcal{L}$ is finite, see Theorem 8.4.1 in [ABG]; this particular case will be needed below. Clearly $\mathcal{P}_a[T] = 0$ if $T \in \mathfrak{C}(\min \mathcal{L})$ and $a \neq \min \mathcal{L}$, so $\mathfrak{C}(\min \mathcal{L}) \subset \operatorname{Ker} \mathcal{P}$. Reciprocally, let $T \in \mathfrak{C}$ such that $\mathcal{P}_a[T] = 0$ for all $a \in \mathcal{M}$. Then for each $\varepsilon > 0$ there is $\mathcal{F} \in \mathbb{F}(\mathcal{L})$ and there is $S \in \mathfrak{C}(\mathcal{F})$ such that $\|T - S\| \leq \varepsilon$. We assume, without loss of generality, that $\min \mathcal{L} \in \mathcal{F}$, hence $\mathcal{F}$ is a finite semilattice with $\min \mathcal{F} = \min \mathcal{L}$. If $b \neq \min \mathcal{L}$ is an element of $\mathcal{F}$ then there is $a \in \mathcal{M}$ such that $a \leq b$, hence $\mathcal{P}_b[T] = \mathcal{P}_b \mathcal{P}_a[T] = 0$. We get

$$\|\mathcal{P}_b[S]\| = \|\mathcal{P}_b[T - S]\| \leq \|T - S\| \leq \varepsilon.$$

Here $\mathcal{P}_b$ is the projection associated to the algebra $\mathfrak{C}$. However, it is clear (see the remark after the proof of Theorem 3.1) that the restriction of $\mathcal{P}_b$ to $\mathfrak{C}(\mathcal{F})$ coincides with the canonical projection of the $\mathcal{F}$-graded algebra $\mathfrak{C}(\mathcal{F})$ onto its subalgebra $\mathfrak{C}(\mathcal{F}_b)$, $\mathcal{F}_b = \{c \in \mathcal{F} \mid b \leq c\}$.

Let $\mathcal{N}$ be the set of atoms of $\mathcal{F}$. Then according to Theorem 8.4.1 from [ABG], the map $U \mapsto (\mathcal{P}_b[U])_{b \in \mathcal{N}}$, sending $\mathfrak{C}(\mathcal{F})$ into $\bigoplus_{b \in \mathcal{N}} \mathfrak{C}(\mathcal{F}_b)$, has $\mathfrak{C}(\min \mathcal{F}) = \mathfrak{C}(\min \mathcal{L})$ as its kernel. The map $\mathfrak{C}(\mathcal{F})/\mathfrak{C}(\min \mathcal{L}) \to \bigoplus_{b \in \mathcal{N}} \mathfrak{C}(\mathcal{F}_b)$ will be an isometry and since $\|\mathcal{P}_b[S]\| \leq \varepsilon$ for each $b \in \mathcal{N}$, the image of S in the quotient space $\mathfrak{C}(\mathcal{F})/\mathfrak{C}(\min \mathcal{L})$ has norm $\leq \varepsilon$. ¿From the definition of the quotient norm it follows that there is $S_0 \in \mathfrak{C}(\min \mathcal{L})$ such that $\|S - S_0\| \leq 2\varepsilon$ (in fact $\leq \varepsilon$).

Thus, we see that for each $\varepsilon > 0$ there is $S_0 \in \mathfrak{C}(\min \mathcal{L})$ such that

$$\|T - S_0\| = \|T - S + S - S_0\| \leq 3\varepsilon.$$

Since $\mathfrak{C}(\min \mathcal{L})$ is closed we get $T \in \mathfrak{C}(\min \mathcal{L})$. ■

The preceding theorem gives us a canonical embedding $\widehat{\mathfrak{C}} \subset \widetilde{\mathfrak{C}}$, more precisely

$$\mathfrak{C}/\mathfrak{C}(\min \mathcal{L}) \hookrightarrow \bigoplus_{a \in \mathcal{M}} \mathfrak{C}_a. \tag{3.4}$$

Although easy to prove, this result is important: it allows one to compute the essential spectrum and to prove the Mourre estimate under very general assumptions. The range of the map (3.4) can be explicitly described, but this is irrelevant for our purposes.

4 C^*-Algebras Associated to Subspaces

4.1. Let X be a finite dimensional real vector space and Y a linear subspace. We denote $\pi_Y = \pi_Y^X$ the canonical surjection of X onto the quotient vector space X/Y and $Y^\perp$ the set of $x^* \in X^*$ such that $\langle y, x^* \rangle = 0 \ \forall y \in Y$. We have canonical identifications $(X/Y)^* = Y^\perp$ and $X^*/Y^\perp = Y^*$.

We shall embed $C_0(X/Y) \subset C_b(X)$ with the help of the map $\varphi \longmapsto \varphi \circ \pi_Y$. Since $C_b(X) \subset \mathfrak{B}(X)$, we shall have

$$C_0(X/Y) \subset C_{bu}(X/Y) \subset C_{bu}(X) \subset \mathfrak{B}(X). \tag{4.1}$$

For $\varphi \in C_b(X/Y)$ we shall denote $\varphi(Q_Y) = (\varphi \circ \pi_Y)(Q)$ the operator in $\mathfrak{B}(X)$ associated to it. Sometimes it is important to specify in the notations the space X; then we set $\varphi(Q_Y) = \varphi(Q_Y^X)$.

The relation $Y^* = X^*/Y^\perp$ implies

$$C_b(Y^*) = C_b(X^*/Y^\perp) \subset C_b(X^*) \subset \mathfrak{B}(X). \tag{4.2}$$

For $\psi \in C_b(Y^*)$ we denote $\psi(P_Y)$ or $\psi(P_Y^X)$ the operator in $\mathfrak{B}(X)$ associated to it; we have $\psi(P_Y^X) = \Phi_{X^*}[\psi(Q_{Y^\perp}^{X^*})]$. Observe, in particular, that the group C^*-algebra $C_0(Y^*)$ of the additive group Y is embedded in $\mathfrak{B}(X)$.

Let $\mathbb{G}(X)$ be the grassmannian of X, i.e. the lattice of all vector subspaces of X with inclusion as order relation. Note that for $Y, Z \in \mathbb{G}(X)$ one has $Y \wedge Z = Y \cap Z$ and $Y \vee Z = Y + Z$. For each $Y \in \mathbb{G}(X)$ we have a C^*-subalgebra $C_0(X/Y)$ of $C_b(X)$ as explained above. In particular $C_0(X/O) = C_0(X)$ and $C_0(X/X) = \mathbb{C}$. Note that each $C_0(X/Y)$ is translation invariant, i.e. it is stable under all the automorphisms τ_x, $x \in X$.

If $\mathcal{F} \subset \mathbb{G}(X)$ is a family of vector subspaces of X then we set

$$C_0^X(\mathcal{F}) = \sum_{Y \in \mathcal{F}} C_0(X/Y). \tag{4.3}$$

This is the linear subspace of $C_{bu}(X)$ generated by $\bigcup_{Y\in\mathcal{F}} C_0(X/Y)$. Note that $C_0^X(\emptyset) = \{0\}$ and $C_0^X(Y) \equiv C_0^X(\{Y\}) = C_0(X/Y)$.

Lemma 4.1 (a) *The family* $\{C_0(X/Y) \,|\, Y \in \mathbb{G}(X)\}$ *of* C^**-subalgebras of* $C_b(X)$ *is linearly independent.*
(b) *If* $\mathcal{F} \subset \mathbb{G}(X)$ *is finite then* $C_0^X(\mathcal{F})$ *is a closed subspace of* $C_{bu}(X)$.
(c) *For each* $Y, Z \in \mathbb{G}(X)$ *the set* $C_0(X/Y) \cdot C_0(X/Z)$ *is a dense subalgebra of* $C_0(X/(Y \cap Z))$.

Proof. We give a detailed proof of this simple lemma because the same argument will be used later on in order to prove Theorem 4.5. Let $\mathcal{F} \subset \mathbb{G}(X)$ be finite and for each $Y \in \mathcal{F}$ let $\varphi_Y \in C_0(X/Y)$. Denote $\varphi^Y : X/Y \to \mathbb{C}$ the function such that $\varphi_Y = \varphi^Y \circ \pi_Y$. Then for each $\omega \in X$ one has $(\tau_\omega \varphi_Y)(x) = \varphi^Y(\pi_Y(x) + \pi_Y(\omega))$ for all $x \in X$. Hence, if we set $\mathcal{F}_\omega = \{Y \in \mathcal{F} \,|\, \omega \in Y\}$, then

$$\lim_{\lambda\to\infty} \tau_{\lambda\omega}[\sum_{Y\in\mathcal{F}} \varphi_Y] = \sum_{Y\in\mathcal{F}_\omega} \varphi_Y \tag{4.4}$$

pointwise on X. In particular

$$\left\| \sum_{Y\in\mathcal{F}_\omega} \varphi_Y \right\| \leq \left\| \sum_{Y\in\mathcal{F}} \varphi_Y \right\| \tag{4.5}$$

where $\|\cdot\|$ is the sup norm.

Let us prove that there is a number C such that for all $Z \in \mathcal{F}$ and all $\{\varphi_Y\}$ as above

$$\|\varphi_Z\| \leq C \left\| \sum_{Y\in\mathcal{F}} \varphi_Y \right\|. \tag{4.6}$$

This clearly implies (a) and (b). If the set

$$Z_0 = Z \setminus \bigcup_{Y\neq Z} Y = \bigcap_{Y\neq Z} [Z \setminus (Y \bigcap Z)]$$

is not empty then (4.5) with a choice $\omega \in Z_0$ gives (4.6) (with $C = 1$). Since $Y \cap Z$ are linear subspaces of Z one has $Z_0 = \emptyset$ if and only if there is $Y \in \mathcal{F}$ such that $Z \subset Y$ strictly. This cannot happen if Z is a maximal element in $\mathcal{F}$, hence (4.6) holds for such elements. Let $\mathcal{F}_1$ be the set of $Y \in \mathcal{F}$

which are not maximal elements in $\mathcal{F}$. Then we clearly get $\left\|\sum_{Y\in\mathcal{F}_1}\varphi_Y\right\| \leq C_1\left\|\sum_{Y\in\mathcal{F}}\varphi_Y\right\|$ for some constant C_1. By what we already proved we see then that (4.6) holds for the maximal elements Z of $\mathcal{F}_1$, etc.

We now prove (c). Let $E=(X/Y)\times(X/Z)$ equipped with the direct sum vector space structure. If $\varphi\in C_0(X/Y)$ and $\psi\in C_0(X/Z)$ then $\varphi\otimes\psi$ denotes the function $(s,t)\longmapsto\varphi(s)\psi(t)$, which belongs to $C_0(E)$. The subspace generated by the functions of the form $\varphi\otimes\psi$ is dense in $C_0(E)$ by the Stone-Weierstrass theorem. Let F be a linear subspace of E. Since each function in $C_0(F)$ extends to a function in $C_0(E)$ we see that the restrictions $(\varphi\otimes\psi)\mid F$ generate a dense linear subspace of $C_0(F)$.

Let us denote by π the map $x\longmapsto(\pi_Y(x),\pi_Z(x))$, so π is a linear map from X to E with kernel $V=Y\cap Z$. Let F be the range of π. Then there is a linear bijective map $\tilde{\pi}:X/V\to F$ such that $\pi=\tilde{\pi}\circ\pi_V$. So $\theta\longmapsto\theta\circ\tilde{\pi}$ is an isometric isomorphism of $C_0(F)$ onto $C_0(X/V)$. Hence for $\varphi\in C_0(X/Y)$ and $\psi\in C_0(X/Z)$ the function $\theta=(\varphi\otimes\psi)\circ\tilde{\pi}$ belongs to $C_0(X/V)$, it has the property $\theta\circ\pi_V=\varphi\circ\pi_Y\cdot\psi\circ\pi_Z$, and the functions of this form generate a dense linear subspace of $C_0(X/V)$. ■

We say that $\mathcal{F}\subset\mathbb{G}(X)$ is $\cap$-*stable* if $Y,Z\in\mathcal{F}\Rightarrow Y\cap Z\in\mathcal{F}$ (so $\mathcal{F}$ is a generalized flag of subspaces of X). Such a $\mathcal{F}$ is a semilattice when equipped with the order relation given by inclusion. We denote by $\mathbb{F}(X)=\mathbb{F}(\mathbb{G}(X))$ the set of finite $\cap$-stable subsets of $\mathbb{G}(X)$.

Corollary 4.2 *If $\mathcal{F}\in\mathbb{F}(X)$ then $C_0^X(\mathcal{F})$ is a C^*-subalgebra of $C_{bu}(X)$ equipped with a natural structure of $\mathcal{F}$-graded C^*-algebra. This algebra is unital if and only if $X\in\mathcal{F}$. For $\mathcal{F}_1,\mathcal{F}_2\in\mathbb{F}(X)$ one has*

$$C_0^X(\mathcal{F}_1)\bigcap C_0^X(\mathcal{F}_2)=C_0^X(\mathcal{F}_1\bigcap\mathcal{F}_2). \tag{4.7}$$

In particular, one has $C_0^X(\mathcal{F}_1)\subset C_0^X(\mathcal{F}_2)$ if and only if $\mathcal{F}_1\subset\mathcal{F}_2$.

4.2. We are now ready to define the noncommutative versions of the algebras $C_0(X/Y)$: they are crossed products of algebras of the form $C_0(X/Y)$ by the natural action τ of the additive group X. These algebras have been first introduced, in a rather different form, by Perry, Sigal and Simon in [PSS]. The connection between our formulation and theirs is clarified by Proposition 4.7 below. See also the introductions of chapters 8 and 9 in [ABG].

Definition 4.3 *If Y is a subspace of X then $\mathfrak{C}^X(Y)=C_0(X/Y)\rtimes X$ is the C^*-subalgebra of $\mathfrak{B}(X)$ obtained as norm closure of $C_0(X/Y)\cdot C_0(X^*)$. For* *each subset* $\mathcal{F} \subset \mathbb{G}(X)$ *let* $\mathfrak{C}^X(\mathcal{F})$ *be the linear subspace of* $\mathfrak{B}(X)$ *generated by the algebras* $\mathfrak{C}^X(Y)$ *with* $Y \in \mathcal{F}$*, so*

$$\mathfrak{C}^X(\mathcal{F}) = \sum_{Y \in \mathcal{F}} \mathfrak{C}^X(Y). \tag{4.8}$$

Observe that $\mathfrak{C}^X(\mathcal{F}) = [\![C_0^X(\mathcal{F}) \cdot C_0(X^*)]\!] = [\![C_0(X^*) \cdot C_0^X(\mathcal{F})]\!]$. So for $\cap$-stable $\mathcal{F}$ we have $\mathfrak{C}^X(\mathcal{F}) = C_0^X(\mathcal{F}) \rtimes X$.

To each vector subspace Y of X we have thus associated a C^*-subalgebra $\mathfrak{C}^X(Y)$ of $\mathfrak{B}(X)$. The only one which is abelian is

$$\mathfrak{C}^X(X) = C_0(X^*) = \{\varphi(P) \mid \varphi \in C_0(X^*)\}. \tag{4.9}$$

The algebra $\mathfrak{C}^X(O)$ is generated by $C_0(X) \cdot C_0(X^*)$ and, since the operators of the form $\varphi(Q)\psi(P))$ with $\varphi \in C_0(X), \psi \in C_0(X^*)$ are compact, we have

$$\mathfrak{C}^X(O) = \mathfrak{K}(X). \tag{4.10}$$

The algebras which play the main rôle in the N-body problem (as presented in ch. 9 of [ABG]) are of the form $\mathfrak{C}^X(\mathcal{F})$ with finite $\mathcal{F}$ and will be sudied in this section. The next one is devoted to the case $\mathcal{F} = \mathbb{G}(X)$.

We shall need an extension of the automorphism τ_x of $C_b(X)$ to an automorphism of $\mathfrak{B}(X)$: we set $\tau_x[S] = \mathrm{e}^{i\langle x,P\rangle} S \mathrm{e}^{-i\langle x,P\rangle}$ for each $x \in X$ and $S \in \mathfrak{B}(X)$. Observe that for $\varphi \in C_0(X/Y)$ and $\psi \in C_0(X^*)$ one has

$$\tau_x[\varphi(Q_Y)\psi(P)] = \varphi(Q_Y + \pi_Y(x))\psi(P).$$

This immediately gives the next lemma.

Lemma 4.4 *(i) If* $y \in Y$ *and* $S \in \mathfrak{C}^X(Y)$ *then* $\tau_y[S] = S$*;*
(ii) if $S \in \mathfrak{C}^X(Y)$ *and* $\pi_Y(x) \to \infty$ *then* $\tau_x[S] \to 0$ *in the strong operator topology.*

Theorem 4.5 (a) *The family* $\{\mathfrak{C}^X(Y) \,|\, Y \in \mathbb{G}(X)\}$ *of* C^**-subalgebras of* $\mathfrak{B}(X)$ *is linearly independent.*
(b) *If* $\mathcal{F} \subset \mathbb{G}(X)$ *is finite then* $\mathfrak{C}^X(\mathcal{F})$ *is* (*norm*) *closed in* $\mathfrak{B}(X)$*.*
(c) *If* $Y, Z \in \mathbb{G}(X)$ *then* $\mathfrak{C}^X(Y) \cdot \mathfrak{C}^X(Z)$ *is a dense linear subspace of the* C^**-algebra* $\mathfrak{C}^X(Y \cap Z)$*.*

Proof. Let $\mathcal{F} \subset \mathbb{G}(X)$ be finite and for each $Y \in \mathcal{F}$ let $T(Y) \in \mathfrak{C}^X(Y)$. Then

$$s-\lim_{\lambda\to\infty} \tau_{\lambda\omega}[\sum_{Y \in \mathcal{F}} T(Y)] = \sum_{Y \in \mathcal{F}_\omega} T(Y) \tag{4.11}$$

where the notations are as in the proof of Lemma 4.1. Indeed, this is an immediate consequence of (i) and (ii) above. Now (a) and (b) follow by the same argument as in Lemma 4.1.

We shall deduce (c) from the corresponding assertion of Lemma 4.1. By Corollary 2.2 $\mathfrak{C}^X(Y)$ is the norm closed linear space generated by the operators of the form $\psi_Y(P)\varphi_Y(Q_Y)$, with $\psi_Y \in C_0(X^*)$ and $\varphi_Y \in C_0(X/Y)$. On the other hand $\mathfrak{C}^X(Z)$ is, by definition, the norm closure of the linear space generated by the operators of the form $\varphi_Z(Q_Z)\psi_Z(P)$, with $\varphi_Z \in C_0(X/Z)$ and $\psi_Z \in C_0(X^*)$. By Lemma 4.1 one has $\varphi_Y(Q_Y)\varphi_Z(Q_Z) = \varphi_V(Q_V)$ for some $\varphi_V \in C_0(X/V)$, where $V = Y \cap Z$. So

$$\psi_Y(P)\varphi_Y(Q_Y) \cdot \varphi_Z(Q_Z)\psi_Z(P) = \psi_Y(P) \cdot \varphi_V(Q_V)\psi_Z(P)$$

which clearly belongs to $\mathfrak{C}^X(Y \cap Z)$. This proves that $\mathfrak{C}^X(Y) \cdot \mathfrak{C}^X(Z) \subset \mathfrak{C}^X(Y \cap Z)$.

Elements of the form $\psi_1(P)\varphi_V(Q_V)\psi_2(P)$, with ψ_1, $\psi_2 \in C_0(X^*)$ and $\varphi_V \in C_0(X/V)$, clearly generate $\mathfrak{C}^X(Y \cap Z)$ (because those of the form $\varphi_V(Q_V)\psi_1(P)\psi_2(P)$ do and we may use Lemma 2.1). Hence the density of $\mathfrak{C}^X(Y) \cdot \mathfrak{C}^X(Z)$ in $\mathfrak{C}^X(Y \cap Z)$ follows immediately from Lemma 4.1. ■

The following result is an immediate consequence of Theorem 4.5

Theorem 4.6 *If $\mathcal{F} \in \mathbb{F}(X)$ then $\mathfrak{C}^X(\mathcal{F})$ is a C^*-subalgebra of $\mathfrak{B}(X)$. If we equip $\mathcal{F}$ with the order relation given by inclusion then the family $\{\mathfrak{C}^X(Y)\}_{Y\in\mathcal{F}}$ of C^*-subalgebras of $\mathfrak{C}^X(\mathcal{F})$ provides $\mathfrak{C}^X(\mathcal{F})$ with a structure of $\mathcal{F}$-graded C^*-algebra.*

4.3. The choice of a supplementary subspace Z of Y in X will give us a canonical isomorphism between $\mathfrak{C}(X/Y)$ and the C^*-tensor product of the algebras $C_0(Y^*)$ and $\mathfrak{K}(Z)$:

$$\mathfrak{C}(X/Y) \cong C_0(Y^*) \otimes \mathfrak{K}(Z). \tag{4.12}$$

In order to define in a precise way this isomorphism let us introduce some notations. Since $X = Y + Z$ (direct sum) we have a canonical identification of X^* with $Y^* \oplus Z^*$. Let i_Y, i_Z be the inclusion maps of Y, Z into X respectively. By taking adjoints we get an isomorphism $(i_Y^*, i_Z^*) : X^* \to Y^* \oplus Z^*$ and we may define for any functions $u : Y^* \to \mathbb{C}$ and $v : Z^* \to \mathbb{C}$ the function $u \otimes v : X^* \to \mathbb{C}$ by $(u \otimes v)(x^*) = u(i_Y^* x^*)v(i_Z^* x^*)$. If $u \in C_0(Y^*), v \in C_0(Z^*)$ then clearly $u \otimes v \in C_0(X^*)$. Moreover, the projection $p_Z : X \to Z$ determined by the direct sum decomposition $X = Y + Z$ factorizes to an isomorphism $p_Z^\flat : X/Y \to Z$ and if for $w : Z \to \mathbb{C}$ we define $w^\flat : X/Y \to \mathbb{C}$ by $w^\flat = w \circ p_Z^\flat$, then $w^\flat \in C_0(X/Y)$ if $w \in C_0(Z)$.

Proposition 4.7 *There is a linear continuous map* $C_0(Y^*)\otimes\mathfrak{K}(Z) \to \mathfrak{C}^X(Y)$ *such that for each* $u \in C_0(Y^*), v \in C_0(Z^*)$ *and* $w \in C_0(Z)$ *the element* $u(P^Y) \otimes \left[w(Q^Z)v(P^Z)\right]$ *is sent into* $w^\flat(Q_Y)(u \otimes v)(P^X)$. *This map is uniquely defined and is an isomorphism.*

Proof. The uniqueness of the map and its surjectivity follow immediately from the fact that the elements of the form $u(P^Y) \otimes \left[w(Q^Z)v(P^Z)\right]$ and $w^\flat(Q_Y)(u \otimes v)(P^X)$ span dense linear subspaces in $C_0(Y^*) \otimes \mathfrak{K}(Z)$ and $\mathfrak{C}^X(Y)$ respectively. To prove the existence and the isomorphism properties observe that we get an isomorphism $\mathcal{J} : \mathcal{H}(X) \to \mathcal{H}(Y) \otimes \mathcal{H}(Z)$ by setting $(\mathcal{J}f)(y,z) = f(y+z)$. It remains to check that

$$u(P^Y) \otimes \left[w(Q^Z)v(P^Z)\right] \cdot \mathcal{J} = \mathcal{J} \cdot w^\flat(Q_Y)(u \otimes v)(P^X)$$

which is a straightforward consequence of the definitions. ■

The preceding tensor product decomposition of $\mathcal{H}(X)$ also gives canonical isomorphisms $C_0(X/Y) \cong 1 \otimes C_0(Z)$ and $C_0(X^*) \cong C_0(Y^*) \otimes C_0(Z^*)$. This induces a linear isomorphism of the vector spaces $C_0(X/Y) \cdot C_0(X^*)$ and $C_0(Y^*) \otimes [C_0(Z) \cdot C_0(Z^*)]$ which extends to the isomorphism between $\mathfrak{C}^X(Y)$ and $C_0(Y^*) \otimes \mathfrak{K}(Z)$ indicated above.

If X is equipped with a scalar product α and if $\mathcal{H}(X)$ is identified with $\mathcal{H}(Y)\otimes\mathcal{H}(Y_\alpha^\perp)$, where $Y_\alpha^\perp$ is the orthogonal space of Y in X, then $\mathfrak{C}^X(Y)$ will be equal to $C_0(Y^*)\otimes\mathfrak{K}(Y_\alpha^\perp)$. The algebras $C_0(Y^*)$ and $C_0(Y^*)\otimes\mathfrak{K}(Y_\alpha^\perp)$ are denoted $\mathbb{T}(Y)$ and $\mathcal{T}(Y_\alpha^\perp)$ in [ABG]. If α is replaced by a new scalar product β, so that Y has a different orthogonal subspace $Y_\beta^\perp$, then $C_0(Y^*) \otimes \mathfrak{K}(Y_\beta^\perp)$ gives (after the identification $\mathcal{H}(X) = \mathcal{H}(Y) \otimes \mathcal{H}(Y_\beta^\perp)$) the same algebra as $C_0(Y^*) \otimes \mathfrak{K}(Y_\alpha^\perp)$ (cf. Proposition 4.7). So the algebra $\mathcal{T}(Y_\alpha^\perp)$ is determined by Y, independently of any euclidean structure on X. Our present notation $\mathfrak{C}^X(Y)$ stresses this fact.

5 The Algebra $\mathfrak{C}^X$

5.1. In this section we shall study the C^*-algebra

$$\mathfrak{C}^X := \text{norm closure in } \mathfrak{B}(X) \text{ of } \mathfrak{C}^X(\mathbb{G}(X)). \tag{5.1}$$

For this we apply in the present context the general theory of Section 3: we take $\mathcal{L} = \mathbb{G}(X)$ and $\mathfrak{C}(Y) = \mathfrak{C}^X(Y)$ for $Y \in \mathbb{G}(X)$. So the algebra $\mathfrak{C}^X$ is $\mathbb{G}(X)$-graded and can be identified with the inductive limit of the family of

C^*-algebras $\{\mathfrak{C}^X(\mathcal{F}) \mid \mathcal{F} \in \mathbb{F}(X)\}$. Indeed, if we order $\mathbb{F}(X)$ by the inclusion relation then $\mathfrak{C}^X(\mathcal{F}_1) \subset \mathfrak{C}^X(\mathcal{F}_2)$ if and only if $\mathcal{F}_1 \subset \mathcal{F}_2$ and

$$\bigcup_{\mathcal{F}\in\mathbb{F}(X)} \mathfrak{C}^X(\mathcal{F}) = \mathfrak{C}^X(\mathbb{G}(X)) \tag{5.2}$$

is a dense $*$-subalgebra of $\mathfrak{C}^X$. Moreover, for all $\mathcal{F}_1, \mathcal{F}_2$ in $\mathbb{F}(X)$ we have

$$\mathfrak{C}^X(\mathcal{F}_1) \cap \mathfrak{C}^X(\mathcal{F}_2) = \mathfrak{C}^X(\mathcal{F}_1 \cap \mathcal{F}_2). \tag{5.3}$$

If $Y \in \mathbb{G}(X)$ then we denote by $\mathfrak{C}^X_Y$ and $\mathfrak{J}^X_Y$ the norm closures in $\mathfrak{B}(X)$ of the spaces $\sum_{Y\subset Z} \mathfrak{C}^X(Z)$ and $\sum_{Y\not\subset Z} \mathfrak{C}^X(Z)$ respectively. If $Y = O = \{0\}$ then $\mathfrak{C}^X_O = \mathfrak{C}^X$. In the next theorem, which is an immediate consequence of Theorem 3.1, we point out a canonical projection (in the sense of linear spaces) of $\mathfrak{C}^X$ onto its subspace $\mathfrak{C}^X_Y$.

Theorem 5.1 *$\mathfrak{C}^X_Y$ is a C^*-subalgebra of $\mathfrak{C}^X$, $\mathfrak{J}^X_Y$ is a closed self-adjoint ideal in $\mathfrak{C}^X$, and $\mathfrak{C}^X$ is equal to their direct sum: $\mathfrak{C}^X = \mathfrak{C}^X_Y + \mathfrak{J}^X_Y$ and $\mathfrak{C}^X_Y \bigcap \mathfrak{J}^X_Y = \{0\}$. The linear projection $\mathcal{P}_Y$ of the linear space $\mathfrak{C}^X$ onto its linear subspace $\mathfrak{C}^X_Y$ determined by the preceding direct sum decomposition is a morphism (in particular it is an operator of norm 1).*

The family $\{\mathfrak{C}^X_Y \mid Y \in \mathbb{G}(X)\}$ of C^*-subalgebras of $\mathfrak{C}^X$ is decreasing

$$Y \subset Z \Rightarrow \mathfrak{C}^X_Z \subset \mathfrak{C}^X_Y \tag{5.4}$$

and has a least element $\mathfrak{C}^X_X = C_0(X^*)$. Clearly (5.4) implies

$$\mathcal{P}_Z\mathcal{P}_Y = \mathcal{P}_Y\mathcal{P}_Z = \mathcal{P}_Z. \tag{5.5}$$

Our purpose now is to describe the quotient of the algebra $\mathfrak{C}^X$ with respect to the ideal $\mathfrak{C}^X(O) = \mathfrak{K}(X)$ of compact operators. The next result is an immediate consequence of Theorem 3.2.

Theorem 5.2 *Let $\mathbb{P}(X)$ be the projective space associated to X, i.e. the set of all one dimensional subspaces of X. Denote by $\widetilde{\mathfrak{C}}^X$ the C^*-direct sum of the algebras $\mathfrak{C}^X_Y$ with $Y \in \mathbb{P}(X)$:*

$$\widetilde{\mathfrak{C}}^X = \bigoplus_{Y\in\mathbb{P}(X)} \mathfrak{C}^X_Y. \tag{5.6}$$

Let $\mathcal{P} : \mathfrak{C}^X \to \widetilde{\mathfrak{C}}^X$ be defined by

$$\mathcal{P}[T] = \bigoplus_{Y\in\mathbb{P}(X)} \mathcal{P}_Y[T]. \tag{5.7}$$

Then $\mathcal{P}$ is a morphism and its kernel is equal to $\mathfrak{K}(X)$.

So $\mathcal{P}$ induces an embedding of the C^*-algebra $\mathfrak{C}^X/\mathfrak{K}(X)$ into $\widetilde{\mathfrak{C}}^X$. We shall identify $\mathfrak{C}^X/\mathfrak{K}(X)$ with a subalgebra of $\widetilde{\mathfrak{C}}^X$:

$$\mathfrak{C}^X/\mathfrak{K}(X) \subset \widetilde{\mathfrak{C}}^X. \tag{5.8}$$

5.2. We shall make here some final remarks concerning the algebra $\mathfrak{C}^X$. First we give another description of the maps $\mathcal{P}_Y$. Observe that by Theorem 4.5(a) each $T \in \mathfrak{C}^X(\mathbb{G}(X))$ can be written in a unique way as a sum $T = \sum_{Z\in\mathcal{F}} T(Z)$ with $\mathcal{F} \subset \mathbb{G}(X)$ finite and $T(Z) \in \mathfrak{C}^X(Z)$, $T(Z) \neq 0$. For such a T we have

$$\mathcal{P}_Y[T] = \sum_{Z\in\mathcal{F}, Z\supset Y} T(Z) \tag{5.9}$$

and this property uniquely characterizes $\mathcal{P}_Y$. If $\omega \in Y$ is such that $\omega \notin Z$ if $Z \in \mathcal{F}$ and $Y \not\subset Z$ (such a choice is possible because $\mathcal{F}$ is finite), then

$$s - \lim_{\lambda\to\infty} \tau_{\lambda\omega}[T] = \sum_{Z\supset Y} T(Z),$$

by Lemma 4.4. In other terms, for T as above we have

$$\mathcal{P}_Y[T] = s - \lim_{\lambda\to\infty} \tau_{\lambda\omega}[T]. \tag{5.10}$$

In particular we get:

Lemma 5.3 *If Y is a one dimensional subspace of X and $\omega \in Y \setminus \{0\}$ then one has for all $T \in \mathfrak{C}^X$:*

$$\mathcal{P}_Y[T] = s - \lim_{\lambda\to\infty} \tau_{\lambda\omega}[T]. \tag{5.11}$$

In particular, we see that the main assertion of Theorem 5.2, namely the relation $\operatorname{Ker}\mathcal{P} = \mathfrak{K}(X)$, is equivalent to the following one: *for $T \in \mathfrak{C}^X$ one has $T \in \mathfrak{K}(X)$ if and only if* $\mathrm{w} - \lim_{\lambda\to\infty} \tau_{\lambda\omega}[T] = 0$ *for each $\omega \in X \setminus \{0\}$.*

Notice that there is an abelian version of the algebra $\mathfrak{C}^X$, namely the closure C_0^X in $C_b(X)$ of $\sum_{Y\subset X} C_0(X/Y)$, and everything we have done applies to C_0^X too. In particular, for $f \in C_0^X$ we have: $f \in C_0(X)$ if and only if $\lim_{\lambda\to\infty} f(x+\lambda\omega) = 0$ for each $\omega \in X \setminus \{0\}$. A geometric proof of this not obvious fact (if $\dim X > 2$) has been shown to us by Radu-Alexandru Todor. We thank him for that.

Certain partitions of unity introduced by Froese and Herbst in [FH] have proved to be very useful in the usual treatment of N-body hamiltonians. We shall briefly present them and their relation with the algebras $\mathfrak{C}^X(\mathcal{F})$. Below we assume that an euclidean norm is given on X.

Let $\chi : X \to \mathbb{R}$ be a C^∞ function, homogeneous of degree zero outside the unit sphere. Since the algebra $\mathfrak{C}^X$ is generated by functions of the form $\varphi(Q)\psi(P)$ (or $\psi(P)\varphi(Q)$) with $\psi \in \mathcal{S}(X^*)$, it is easy to prove that $[\chi(Q), T] \in \mathfrak{K}(X)$ for all $T \in \mathfrak{C}^X$.

Now let $Z \subset X$ be a subspace and assume that $\chi(z) = 0$ if $z \in Z$, $|z| \geq 1$. Then for each $\varphi \in C_0(X/Z)$ the function $\chi \cdot \varphi \circ \pi_Z$ belongs to $C_0(X)$ (indeed, if $x \to \infty$ and $\pi_Z(x)$ is bounded, then $x/|x|$ approaches Z), hence $\chi(Q)\varphi(Q_Z)\psi(P)$ is a compact operator. It follows that $\chi(Q)T$ and $T\chi(Q)$ are compact operators if $T \in \mathfrak{C}^X(Z)$.

Let us fix $\mathcal{F} \in \mathbb{F}(X)$ with $O, X \in \mathcal{F}$ and let $Y \in \mathcal{F}$, $Y \neq X$. A C^∞ function $\chi_Y : X \to \mathbb{R}$ which is homogeneous of degree zero on $|x| \geq 1$ is called (according to Froese-Herbst) Y-*reducing* if: for each $Z \in \mathcal{F}$ with $Y \not\subset Z$ and each $z \in Z$, $|z| \geq 1$, one has $\chi_Y(z) = 1$. By what we said above, we see that
(i) $[\chi_Y(Q), T] \in \mathfrak{K}(X)\ \forall T \in \mathfrak{C}(\mathcal{F})$;
(ii) $\chi_Y(Q)T$ and $T\chi_Y(Q)$ belong to $\mathfrak{K}(X)$ if $T \in \mathfrak{C}(\mathcal{F}'_Y) = \sum_{Z \in \mathcal{F}, Y \not\subset Z} \mathfrak{C}^X(Z)$.

Let $\mathcal{N}$ be the set of atoms of $\mathcal{F}$. An $\mathcal{F}$-reducing partition of unity on X is a family $\{\chi_Y\}_{Y \in \mathcal{N}}$ such that χ_Y is Y-reducing and $\sum_{Y \in \mathcal{N}} \chi_Y^2 = 1$ on X. In [FH] such families are constructed. From (i) and (ii) above we then get: if $S \in \mathfrak{C}(\mathcal{F})$ and if we denote $S_Y = \mathcal{P}_Y[S]$ its canonical projection onto $\mathfrak{C}(\mathcal{F}_Y)$, then there is $K \in \mathfrak{K}(X)$ such that

$$S = K + \sum_{Y \in \mathcal{N}} \chi_Y(Q)\, S_Y\, \chi_Y(Q). \tag{5.12}$$

It is clear that $\mathcal{F}$ cannot be replaced by an infinite semilattice in the preceding construction. However, these partitions can be used to give an alternate and more elementary proof of the main assertion of Theorem 5.2, namely that T is compact if $T \in \mathfrak{C}^X$ and $\mathrm{w} - \lim_{\lambda \to \infty} \tau_{\lambda\omega}[T] = 0$ for each $\omega \in X \setminus \{0\}$. Indeed, for each $\varepsilon > 0$ we can find $\mathcal{F}$ as above and $S \in \mathfrak{C}(\mathcal{F})$ such that $\|T - S\| \leq \varepsilon$. Note that we can assume T and S self-adjoint. Write $S = \sum\{S(Z) \mid Z \in \mathcal{F}\}$ with $S(Z) \in \mathfrak{C}^X(Z)$, hence $S_Y = \sum\{S(Z) \mid Z \in \mathcal{F}, Z \supset Y\}$, and let $\mathcal{F}_\omega = \{Z \in \mathcal{F} \mid \omega \in Z\}$ for $\omega \in X \setminus \{0\}$. Then $\mathrm{s} - \lim_{\lambda \to \infty} \tau_{\lambda\omega}[S] = \sum_{Z \in \mathcal{F}_\omega} S(Z)$ and from the Fatou lemma we get $\|\sum_{Z \in \mathcal{F}_\omega} S(Z)\| \leq \varepsilon$. For each $Y \in \mathcal{N}$ we can find $\omega \in Y \setminus \{0\}$ such that $\omega \notin Y'$ if $Y' \in \mathcal{N} \setminus \{Y\}$. Hence we get $\|S_Y\| \leq \varepsilon$, or $-\varepsilon \leq S_Y \leq \varepsilon$, for each

$Y \in \mathcal{N}$. Then $-\varepsilon \leq \sum \chi_Y(Q)\, S_Y\, \chi_Y(Q) \leq \varepsilon$ because $\sum \chi_Y(Q)^2 = 1$. So from (5.12) we see that there is $K \in \mathfrak{K}(X)$ such that $\|S - K\| < \varepsilon$. This implies $\|T - K\| \leq 2\varepsilon$, which proves the assertion.

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