Fractional Fokker–Planck Equation for Nonlinear Stochastic Differential Equations Driven by Non-Gaussian Levy Stable Noises

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Abstract

The Fokker-Planck equation has been very useful for studying dynamic behavior of stochastic differential equations driven by Gaussian noises. In this paper, we derive a Fractional Fokker–Planck equation for the probability distribution of particles whose motion is governed by a *nonlinear* Langevin-type equation, which is driven by a non-Gaussian Levy-stable noise. We obtain in fact a more general result for Markovian processes generated by stochastic differential equations.

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1 Introduction and statement of the problem

The Fokker-Planck equation is one of the most celebrated equations in Physics, since it has been very useful for studying dynamic behavior of stochastic differential equations driven by Gaussian noises. There has recently been a mushrooming interest [1, 2, 3, 4, 5, 6] in the fact that the probability density of a (linear) Levy motion satisfies a generalized Fokker-Planck equation involving fractional orders of differentiation. This essentially corresponds to a re-interpretation of the characteristic function of a Levy motion and significant applications seem to require its nonlinear generalization.

We therefore consider the following nonlinear Langevin–like equation for a stochastic (real) quantity $X(t)$:

$$dX(t) = m(X(t), t)\ dt + \sigma(X(t), t)\ dL \tag{1}$$

where the driving source is a Levy stable motion L, i.e. a motion (e.g. [7]) whose increments δL are stationary and independent for any time lag δt and correspond to independent, identically distributed Levy stable variables [9, 10, 11, 12, 13]. Let us recall that a Levy stable motion is defined, as are its increments, by four parameters: its Levy stability index α $(0 < \alpha \leq 2)$, its skewness β $(-1 \leq \beta \leq 1)$, its center $\gamma\delta t$ and its scale parameter $D\ \delta t$ $(D \geq 0)$. Brownian motion corresponds to the limit case $\alpha = 2$, which also implies $\beta = 0$, and to the 'normal' diffusion law. The variance $Var[X(t) - X(t_0)]$ of the distance traveled by a brownian particle, is twice its scale parameter and therefore yields the classical Einstein relation: $Var[X(t) - X(t_0)] = 2D(t - t_0)$.

The linear case, which is the unique case studied until now, corresponds to:

$$m(X(t), t) \equiv m = Const., \quad \sigma(X(t), t) \equiv \sigma = Const. \tag{2}$$

$X(t) - X(t_0)$ is also a Levy motion which has the same Levy stability index α, but with a possible different center or trend (when $m \neq 0$) and scale or amplitude (when $\sigma \neq 1$).

In the nonlinear cases, $\sigma(X(t), t) \geq 0$ and $m(X(t), t)$ are nonlinear functions of $X(t)$ and t, which satisfy certain regularity constraints to be discussed later. We claim that:

Proposition 1 *The transition probability density:*

$$\forall t \geq t_0 : \quad p(x, t|x_0, t_0) = Pr(X(t) = x|X(t_0) = x_0) \tag{3}$$

corresponding to the nonlinear stochastic differential equation (Eq.1), with $\alpha \neq 1$ *or* $\beta = 0$, *is the solution of the following Fractional Fokker-Planck equation: .*

$$\frac{\partial}{\partial t}p(x,t|x_0,t_0) = -\frac{\partial}{\partial x}(\gamma\sigma(x,t)+m(x,t))p(x,t|x_0,t_0)$$
$$-D[(-\Delta)^{\alpha/2}+\beta\omega(\alpha)\frac{\partial}{\partial x}(-\Delta)^{(\alpha-1)/2}]\sigma(x,t)^{\alpha}p(x,t|x_0,t_0) \quad (4)$$

where $\omega(\alpha)$ is defined by:

$$\alpha \neq 1: \quad \omega(\alpha) = tan\frac{\pi\alpha}{2} \quad (5)$$

and where the fractional powers of the Laplacian Δ will be discussed in Sect.5. Proposition 1 and Eq.4 will be established for scalar processes (i.e. $\Delta \equiv \frac{\partial^2}{\partial x^2}$) and its extension to vector processes will be discussed and presented in Sect.7.

This Fractional Fokker-Planck equation will be established with the help of the much more general proposition:

Proposition 2 *The inverse Fourier transform of the second characteristic function or cumulant generating function of the increments of a Markov process $X(t)$ generates by convolution the Fokker-Planck equation of evolution of its transition probability $p(x,t|x_0,t_0)$.*

We will demonstrate this proposition in a straightforward, yet rigorous way. More precisely, we will establish the following:

$$\frac{\partial p}{\partial t}(x,t|x_0,t_0) = \int dy \frac{\partial \widetilde{K}}{\partial t}(x-y|y,t)p(y,t|x_0,t_0) \quad (6)$$

where $\widetilde{K}$ is the inverse Fourier transform of the cumulant generating function of the increments. Its arguments will become explicit in Sect.2.

This not only holds for processes with stationary and independent increments, as in the linear case (Eq.2) but for any Markov process, including those defined by the non-linear Langevin-like equation (Eq.1 with $m \neq Const.$, $\sigma \neq Const.$). As a consequence of Eq.6, we will demonstrate the following:

Proposition 3 *The Kramers-Moyal coefficients A_n of the Fokker-Planck equation of a Markov process $X(t)$:*

$$\frac{\partial p}{\partial t}(x,t|x_0,t_0) = \sum_{n\in J}\frac{\partial^n}{\partial x^n}[A_n(x,t)p(x,t|x_0,t_0)] \quad (7)$$

are directly related to the cumulants C_n of the increments:

$$A_n(x,t) = \frac{(-1)^n}{n!}C_n(x,t) \quad (8)$$

where the set J of the indices n is $\{1,2\}$ in the most classical case (e.g. [8], which is a particular case of $J \subseteq \mathbf{N}$ which corresponds to an analytic expansion of cumulants.

We will demonstrate this property (Prop.3), which at best is only mentioned in few standard text books on the (classical) Fokker-Plank equation, as well as its generalization for non analytic cumulant expansions, i.e. there are non integers indices $n \in J$. This latter property, discussed in Sect.5, will be exploited in Sect.6 in order to derive Prop.1 with $J = \{1, \alpha\}, 0 < \alpha \leq 2$.

2 The cumulant generating function of the increments

The first and second (conditional) characteristic functions are respectively the moment generating function $Z_X(k, t-t_0|x_0, t_0)$ and the cumulant generating function $K_X(k, t-t_0|x_0, t_0)$, associated with the transition probability $p(x, t|x_0, t_0)$ of a process $X(t)$. They are defined by the Fourier transform of the latter, with k being the conjugate variable of $x - x_0$:

$$\begin{aligned} F[p(x,t|x_0,t_0)] &\equiv Z_X(k, t-t_0|x_0,t_0) && (9)\\ &\equiv exp(K_X(k,t-t_0|x_0,t_0)) && (10)\\ &\equiv E[exp(ik(X(t)-X_0))|X(t_0)] = X_0] && (11) \end{aligned}$$

where $E[\cdot|\cdot]$ denote the conditional mathematical expectation, F and F^{-1} respectively the Fourier–transform and its inverse:

$$\begin{aligned} F[f] &= \hat{f}(k) = \int_{-\infty}^{\infty} dx \quad exp(ikx)f(x) && (12)\\ F^{-1}[\hat{f}] &= f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \quad exp(-ikx)\hat{f}(k) && (13) \end{aligned}$$

The corresponding quantities for increments $\delta X(\delta t) = X(t+\delta t) - X(t)$, corresponding to a given time lag $\delta t > 0$, are defined in a similar way:

$$\begin{aligned} F[p(x+\delta x, t+\delta t|x,t)] &= \delta Z_X(k, \delta t|x,t) && (14)\\ &\equiv exp(\delta K_X(k,\delta t|x,t)) && (15)\\ &= E[exp(ik(X(t+\delta t)-X))|X(t) = X] && (16) \end{aligned}$$

where k is the conjugate variable of δx. The cumulants of the increments C_n are the coefficients of the Taylor expansion of δK_X:

$$\delta K_X(k, \delta t|x,t) = \delta t \sum_{n \in J} \frac{(ik)^n}{n!} C_n(x,t) + o(\delta t) \qquad (17)$$

As already mentioned, the classical case corresponds to an analytic expansion of δK_X, i.e. $J \subseteq \mathbf{N}$, whereas we will be interested by a non-analytic expansion $J = \{1, \alpha\}$.

3 Processes with stationary and independent increments

Let us first consider the simple sub-case of a process with stationary and independent increments. It corresponds to $C_n(x,t) \equiv C_n = Const.$ in Eqs.7, 17 and as already discussed in Sect. 1, it includes the linear case (Eq.2) of the Langevin–like equation (Eq.1).

However we believe that the following derivation is not only somewhat pedagological on the role of the characteristic functions for the nonlinear case, but terser than derivations previously presented for the linear case.

The stationarity of the increments implies that the transition probability depends only on the time and space lags, i.e.:

$$p(x,t|x_0,t_0) = p(x-x_0, t-t_0) \tag{18}$$

and similarly, the characteristic functions of the increments are no longer conditioned, for instance:

$$\begin{aligned} Z_X(k,t-t_0|x_0,t_0) &\equiv Z_X(k,t-t_0) & (19)\\ K_X(k,t-t_0|x_0,t_0) &\equiv K_X(k,t-t_0) & (20) \end{aligned}$$

On the other hand, the independence of the increments implies that the transition probabilities satisfy a convolution (over any possible intermediate position y) for any given time lag δt:

$$\forall \delta t > 0 : p(x-x_0, t+\delta t - t_0) = \int dy \ \ p(x-y,\delta t) p(y-x_0, t-t_0) \tag{21}$$

and the corresponding characteristic functions merely factor (resp. add). Therefore, we have:

$$Z_X(k,t+\delta t-t_0) - Z_X(k,t-t_0) = Z_X(k,t-t_0)(\delta Z_X(k,\delta t)-1) \tag{22}$$

This in turn leads to:

$$Z_X(k,t+\delta t-t_0) - Z_X(k,t-t_0) = Z_X(k,t-t_0)\delta K_X(k,\delta t) + o(\delta t) \tag{23}$$

Its inverse Fourier transform yields:

$$p(x,t+\delta t|x_0,t_0) - p(x,t|x_0,t_0) = \int dy F^{-1}[\delta K_X(k,\delta t)] p(y-x_0,t-t_0) + o(\delta t) \tag{24}$$

This demonstrates (in the limit $\delta t \to 0$) Prop.2 and Eq.6, as well as Prop.3, since Eq.24 corresponds, with the help of Eq.17, to:

$$p(x,t+\delta t|x_0,t_0)-p(x,t|x_0,t_0)=\delta t\sum_n[C_n\frac{(-1)^n}{n!}\int dy\delta^{(n)}_{x-y}p(y,t|x_0,t_0)]+o(\delta t) \tag{25}$$

where δ^n_x denotes the n^{th} derivative of the Dirac function. Therefore, we obtain:

$$\frac{\partial}{\partial t}p(x,t|x_0,t_0)=\sum_{n\in J}A_n\frac{\partial^n}{\partial x^n}p(x,t|x_0,t_0) \tag{26}$$

which corresponds to the linear case of Eq.7.

4 More general Markov processes

In the case of a Markov process which does not have stationary and independent increments, there is no longer a simple convolution equation (Eq. 21) of the transition probabilities, nor a simple factorization of characteristic functions (Eq.22). However, the former satisfies a generalized convolution equation which corresponds to the Chapman-Kolmorogorov identity [12] valid for any Markov process $X(t)$:

$$\forall\delta t>0: p(x,t+\delta t|x_0,t_0))=\int dy\ p(x,t+\delta t|y,t)p(y,t|x_0,t_0) \tag{27}$$

which indeed reduces to a mere convolution (Eq. 21) in the case of processes with stationary and independent increments. This identity can be written under the equivalent form:

$$p(x,t+\delta t|x_0,t_0)=\int dy\int\frac{dk}{2\pi}e^{-iky+\delta K_X(k,\delta t|y,t)}p(y,t|x_0,t_0) \tag{28}$$

Noting that we have:

$$p(x,t|x_0,t_0)=\int dy\ p(y,t|x_0,t_0)\int\frac{dk}{2\pi}e^{-iky} \tag{29}$$

we obtain:

$$p(x,t+\delta t|x_0,t_0)-p(x,t|x_0,t_0)=\delta t\int dyF^{-1}[\delta K_X(k,\delta t|y,t)]p(y,t|x_0,t_0)+o(\delta t) \tag{30}$$

In the limit $\delta t\to 0$, this corresponds to Prop. 2 and Eq. 6. When $J\subseteq\mathbf{N}$, it yields with the help of Eq.17:

$$\delta p(x,t|x_0,t_0)=\delta t\sum_{n\in N}\int dy\delta^{(n)}_{x-y}[\frac{(-1)^n}{n!}C_n(y,t)p(y,t|x_0,t_0)]+o(\delta t) \tag{31}$$

The limit $\delta t\to 0$ corresponds to Eq.7 and demonstrates Prop. 3 for any (classical) Markow process.

5 Extension to fractional orders

In the two previous sections (Sects.3- 4), the fact that the indices $n \in J$ should be integers intervene at best only in the correspondence between (integer order) differentiation $\frac{\partial^n}{\partial x^n}$ (in Eq. 7) and powers of the conjugate variable k^n (in Eq. 17). However, by the very definition of fractional differentiation (e.g.[14]), this correspondence holds also for non integer orders. However, there is not a unique definition of fractional differentiation and therefore, as discussed in some details in [6]), we cannot expect to have a unique expression of the Fractional Fokker-Planck equation.

Since it will be sufficient for the following to consider an expansion of the characteristic function involving fractional powers of only the wave number $|k|$, it is interesting to consider Riesz's definition of a fractional differentiation. Indeed, the latter corresponds to consider fractional powers of the Laplacian:

$$-(-\Delta)^{\alpha/2} f(x) = F^{-1}[|k|^\alpha \hat{f}(k)] \tag{32}$$

which has furthermore the advantage of being valid for the vector cases. However, we will see in Sect. 7 that in general it does not apply in a straightforward manner for d-dimensional stable Lévy motions. Indeed the latter introduces rather (one-dimensional) directional Laplacians, i.e. (one-dimensional) Laplacians along a given direction $\underline{u}$ ($| \underline{u} |= 1$) :

$$-(-\Delta_{\underline{u}})^{\alpha/2} f(x) = F^{-1}[|(\underline{k}, \underline{u})|^\alpha \hat{f}(k)] \tag{33}$$

where(.,.) denotes the scalar product. On the other hand, it will be useful to consider the fractional power of the contraction of the Laplacian tensor $\underline{\underline{\underline{\Delta}}}$:

$$\Delta_{i,j} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \tag{34}$$

by a tensor $\underline{\underline{\sigma}}$, with the following definition:

$$-(-\underline{\underline{\underline{\Delta}}} : \underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\frac{\alpha}{2}} \equiv F^{-1}[| \, (\underline{k}, \underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*.\underline{k} \, |^{\frac{\alpha}{2}}] = F^{-1}[| \, \underline{\underline{\sigma}}^*.\underline{k} \, |^\alpha] \tag{35}$$

6 Levy case

The second characteristic function of the increments δL of the (scalar) Levy forcing is the following:

$$\delta K_L(k, \delta t) = \delta t[ik\gamma - D|k|^\alpha (1 - i\beta \frac{k}{|k|} \omega(k, \alpha)] + o(\delta t) \tag{36}$$

where $\omega(k, \alpha)$ is defined by:

$$\alpha \neq 1 : \quad \omega(k, \alpha) \equiv \omega(\alpha) = tan \frac{\pi \alpha}{2}; \qquad \alpha = 1 : \quad \omega(k, \alpha) = \frac{\pi}{2} log|k| \tag{37}$$

Considering an Ito-like forward integration of Eq.1, the increments δL generates the following (first) characteristic function for the increments δX of the motion $X(t)$:

$$\delta Z_X(k, \delta t|x-\delta x, t) = e^{ikm(X,t)}\delta Z_{\sigma L}(k, \delta t|x, t) + +o(\delta t) \tag{38}$$

which yields the following elementary cumulant generating function δK_X:

$$\delta K_X(k, \delta t|x, t) = \delta t[ikm(x,t) + ik\gamma\sigma(x,t) \tag{39}$$

$$-D|k|^\alpha(1 - i\beta\frac{k}{|k|}\omega(k,\alpha))\sigma(x,t)^\alpha] + o(\delta t) \tag{40}$$

and which is of the same type as Eq.17, with $J = \{1, \alpha\}$. Therefore, as discussed in Sect.5, we have fractional differentiations in the corresponding Eq.7, which will precisely correspond to Eq.4, and therefore establishes Prop. 1.

Let us discuss briefly the regularity constraints that should be satisfied by the nonlinear function $\sigma((X(t), t) \geq 0$ and $m((X(t), t)$. Obviously, they should be measurable. On the other hand, the uniqueness of the solution should require, as for the classical nonlinear Fokker-Planck equation (e;g. [18]), a Lipschitz condition for both $\sigma((X(t), t)$ and $m((X(t), t)$.

7 Extension to vector processes

With but one important exception, the extension of the previous results to higher dimensions is rather straightforward. The starting point of this extension is the following nonlinear stochastic equation ($\underline{X}(t) \in R^d$):

$$d\underline{X}(t) = \underline{m}(\underline{X}(t), t)dt + \underline{\underline{\sigma}}(\underline{X}(t), t).d\underline{L} \tag{41}$$

where $\underline{m}$ and $\underline{\underline{\sigma}}$ are the natural vector, respectively tensor, extensions of the deterministic-like trend, respectively modulation of the random driving force. $\underline{L}$ is a d-dimensional Levy stable motion and, as discussed below, the expression of its characteristic function corresponds to the source of the difficulty in extending the scalar results to high dimensions. On the contrary, it is straightforward to check that Props. 2, 3 are valid in the d-dimensional case, with the following extensions ($\underline{x} \in R^d$) for Eq. 6:

$$\frac{\partial p}{\partial t}(\underline{x}, t|x_0, t_0) = \int dy \frac{\partial \widetilde{K}}{\partial t}(\underline{x} - \underline{y}|\underline{y}, t)p(\underline{y}, t|x_0, t_0) \tag{42}$$

and for Eq. 7 ($\underline{n} \in J \subseteq \mathbf{N}^d, |\ \underline{n}\ | = \sum_{i=1}^d n_i$):

$$\frac{\partial p}{\partial t}(\underline{x}, t|\underline{x}_0, t_0) = \sum_{\underline{n} \in J} \frac{\partial^{|\underline{n}|}}{\partial x_1^{n_1} \partial x_2^{n_2} .. \partial x_d^{n_d}}[A_{\underline{n}}(\underline{x}, t)p(\underline{x}, t|\underline{x}_0, t_0)] \tag{43}$$

the relation to the cumulants $C_{\underline{n}}$ of the increments is now:

$$A_{\underline{n}}(\underline{x},t) = \frac{(-1)^{|\underline{n}|}}{(n_1)!(n_2)!..(n_d)!} C_{\underline{n}}(\underline{x},t) \tag{44}$$

On the other hand, Eq. 41 yields the following extension to Eq.38:

$$\delta Z_{\underline{X}}(\underline{k},\delta t|\underline{x},t) = e^{i\ \underline{k}.\underline{m}(\underline{x},t)} \delta Z_{\underline{\underline{\sigma}}.\underline{L}}(\underline{k},\delta t|\underline{x},t) \tag{45}$$

and therefore we have:

$$\delta K_{\underline{X}}(\underline{k},\delta t|\underline{x},t) = i\ \underline{k}.\underline{m}(\underline{x},t) + \delta K_{\underline{L}}(\underline{\underline{\sigma}}^*.\underline{k},\delta t|\underline{x},t) + o(\delta t) \tag{46}$$

Let us recall that a stable Lévy vector in the classical sense [9, 15, 16] (see [17] for a discussion and a generalization) corresponds to the limit of a sum of jumps, with a power-law distribution, along random directions $\underline{u} \in \partial B_1$, B_1 being the unit ball, distributed according to a (positive) measure $d\Sigma(\underline{u})$. The latter, which generalizes the scale parameter D of the scalar case, is the source of the difficulty since in general the probability distribution of a stable Lévy vector depends on this measure, and therefore is a non parametric distribution. However, as discussed below, there is at least a trivial exception: the case of isotropic stable Lévy vectors.

Corresponding to our previous remarks, a (classical) stable Lévy vector has the following (Fourier) cumulant generating function:

$$K_{\underline{L}}(\underline{k}) = \delta t[i(\underline{k},\underline{\gamma}) - \int_{\underline{u}\in\partial B_1} (i\underline{k},\underline{u})^{\alpha} d\Sigma(\underline{u})] + o(\delta t) \tag{47}$$

which yields with the help of the Eq.46:

$$\frac{\partial}{\partial t}\widetilde{K}_{\underline{X}}(\underline{k}) = -div(\underline{m} + \underline{\underline{\sigma}}.\underline{\gamma}) - F^{-1}[\int_{\underline{u}\in\partial B_1} (i\underline{\underline{\sigma}}^*(\underline{x},t).\underline{k},\underline{u})^{\alpha} d\Sigma(\underline{u})] \tag{48}$$

The scalar case (Eq.36) corresponds to:

$$0 \le p \le 1 : \beta = 2p-1,\ d\Sigma(u) = Dcos(\frac{\pi\alpha}{2})[p\delta_{(u-1)} + (1-p)\delta_{(u+1)}] \tag{49}$$

For any dimension d, the second term on the right hand side of Eq.48 corresponds to a fractional differentiation operator of order α. This operator can be slightly re-arranged. With the help of the odd $d\Sigma^-(\underline{u})$ and even $d\Sigma^+(\underline{u})$ parts of the measure $d\Sigma(\underline{u})$,

$$2\ d\Sigma^+(\underline{u}) = d\Sigma(\underline{u}) + d\Sigma(-\underline{u});\quad 2\ d\Sigma^-(\underline{u}) = d\Sigma(\underline{u}) - d\Sigma(-\underline{u}) \tag{50}$$

and the identity (θ being the Heaviside function):

$$(ik)^{\alpha} = |k|^{\alpha}[\theta(k)e^{i\frac{\alpha\pi}{2}} + \theta(-k)e^{-i\frac{\alpha\pi}{2}}] \tag{51}$$

one can write the extension of Eq.4 under the following form:

$$\frac{\partial}{\partial t} p(\underline{x}, t|\underline{x}_0, t_0) = -div[\underline{m}(\underline{x}, t) + \underline{\underline{\sigma}}(\underline{x}, t).\underline{\gamma})]p(\underline{x}, t|\underline{x}_0, t_0)]$$
$$-[< (-\underline{\underline{\Delta}} : \underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\frac{\alpha}{2}} >_{\Sigma+} - < (\underline{\nabla}.\underline{\underline{\sigma}}^*).(-\underline{\underline{\Delta}} : \underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\frac{\alpha-1}{2}} >_{\Sigma-}]p(\underline{x}, t|\underline{x}_0, t_0) \quad (52)$$

where the symmetric and antisymmetric operators are defined , similarly to Eq.35, in the following manner:

$$- < (-\underline{\underline{\Delta}} : \underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\frac{\alpha}{2}} >_{\Sigma+} \;=\; \int_{\underline{u}\in\partial B_1} d\Sigma^+(\underline{u})F^{-1}[|\; (\underline{\underline{\sigma}} * (\underline{x}, t).\underline{k}, \underline{u}) \;|^{\alpha}] \quad (53)$$
$$- < (\underline{\nabla}.\underline{\underline{\sigma}}^*).(-\underline{\underline{\Delta}} : \underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\frac{\alpha-1}{2}} >_{\Sigma-} \;=\; \int_{\underline{u}\in\partial B_1} d\Sigma^-(\underline{u})F^{-1}[(-i\underline{\underline{\sigma}}^*(\underline{x}, t).\underline{k}, \underline{u}) \;|\; (\underline{\underline{\sigma}}^*(\underline{x}, t).\underline{k}, \underline{u}) \;|^{\alpha-1}] \quad (54)$$

In general, each operator corresponds to a rather complex integration (which is indicated by the symbol $< . >_{\Sigma}$) of directional fractional Laplacians (Eq.33). However, the symmetric operator becomes simpler as soon as the even part $d\Sigma^+$ of the measure $d\Sigma$ is isotropic. Indeed, the integration over directions yields only a prefactor D:

$$< -(\underline{\underline{\Delta}} : \underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\frac{\alpha}{2}} >_{\Sigma+} = D\ (-\underline{\underline{\Delta}} : \underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)^{\frac{\alpha}{2}}$$
$$D = \int_{\underline{u}\in\partial B_1} d\Sigma^+(\underline{u}) \;|\; (\underline{u}_1, \underline{u}) \;|^{\alpha} \quad (55)$$

and for $\alpha = 2$ this corresponds to the classical term $(\underline{\underline{\Delta}} : \underline{\underline{\sigma}}.\underline{\underline{\sigma}}^*)$ of the standard d-dimensional Fokker-Planck equation. If $d\Sigma$ itself is rotation invariant, then the asymmetric operator vanishes, since $d\Sigma^- = 0$. If furthermore, $\underline{\underline{\sigma}}$ is rotation invariant, i.e. $\underline{\underline{\sigma}} = \sigma\underline{\underline{1}}$, then one obtains the following Fractional Fokker-Planck equation:

$$\frac{\partial}{\partial t} p(\underline{x}, t|\underline{x}_0, t_0) \;=\; -div[\underline{\sigma}.\underline{\gamma}(\underline{x}, t) + \underline{m}(\underline{x}, t)]p(\underline{x}, t|\underline{x}_0, t_0) \quad (56)$$
$$-\; D\ [(-\Delta)^{\alpha/2}]\sigma(x, t)^{\alpha} p(\underline{x}, t|\underline{x}_0, t_0) \quad (57)$$

Therefore, as one might expect it, due to the rotation symmetries, this corresponds to a rather trivial extension of the standard gaussian case: a fractional power α of the d-dimensional Laplacian, as in the pure scalar case (Eq.4). Obviously, the integration performed in Eq.52 is also greatly simplified as soon as $d\Sigma(\underline{u})$ is discrete, i.e. its support corresponds to a discrete set of directions $\underline{u}_i$.

On the other hand, let us note that the framework of generalized stable Lévy vectors [17], allows one to introduce a much stronger anisotropy than the the measure $d\Sigma$ does if for classical stable Lévy vectors. This therefore diminishes the importance of the asymmetry of the latter. Indeed, the components of a generalized stable Lévy vector do not have necessarily the same Lévy stability index, the latter being generalized into a second rank tensor. Similarly, the differential operators involved in the corresponding Fractional Fokker-Planck equation have no longer a unique order of differentiation. This is rather easy to check in case of a discrete measure $d\Sigma(\underline{u})$ and we will explore elsewhere the general case.

8 Conclusion

We have derived a Fractional Fokker-Planck equation, i.e. a kinetic equation which involves fractional derivatives, for the evolution of the probability distribution of nonlinear stochastic differential equations driven by non-Gaussian Levy stable noises. We first established this equation in the scalar case, where it has a rather compact expression with the help of fractional powers of the Laplacian, and then discussed its extension to the vector case. This Fractional Fokker-Planck equation generalizes broadly previous results obtained for a linear Langevin-like equation with a Lévy forcing, as well as the standard Fokker-Planck equation for a nonlinear Langevin equation with a Gaussian forcing.

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References

[1] Fogedby, H.C., *Phys.Rev.E, 50*, 2, 1657-1660, (1994)

[2] Chechkin, A.V., D. Schertzer, A.V. Tur, V.V. Yanovsky, *Ukr. J Phys., 40*, 5, 434-439 (1995).

[3] Compte, A *Phys.Rev.E, 53*, 4, 4191-4193, (1996)

[4] Marsan, D., D. Schertzer, S. Lovejoy, J. Geophys. Res., **31D**, 26333-26346 (1996).

[5] Chaves, A.S., Phys. Letters A, **239**, 13-16 (1998)

[6] Yanovsky,V. V., A. V. Chechkin, D. Schertzer and A. V. Tur, submitted to *Physica A*, 1999.

[7] Fristedt, B., L. Gray, A Modern Approach to Probability Theory, Birkhauser, Boston (1997).

[8] Gardiner, C.W., Handbok of Staochastic Methods for Physics, Chemistry and the Natural Sciences,. Springer-Verlag, Berlin (1985).

[9] Lévy P., Theorie de l”Addition des Variables Aléatoires. Gauthier–Villars, Paris (1937).

[10] Khintchine A.Y. and Lévy P., C. R. Acad. Sci.(Paris) **202**, 374 (1936).

[11] Gnedenko B.V. and Kolmogorov A.N., Limit Distributions for Sums of Independent Random Variables, Addison Wesley, Reading, MA, (1954).

[12] Feller W., An Introduction to Probability Theory and its Applications. John.Wiley & Sons, New.York (1966).

[13] Zolotarev V.M., One-dimensional Stable Distributions, MM, Vol. 65, AMS (1986).

[14] Miller, K.S., An Introduction to the Fractional Calculus and Fractional Differential Equations, J. Wiley & sons, New York (1993).

[15] Paulauskas V.J. (1976). Some remarks on Multivariate Stable Distributions. *J. Multivariate Anal.*,**6** 356-368

[16] Nikias C.L. and M. Shao (1995). *Signal Processing with alpha-stable distributions and applications*, John Wiley and Sons, New-York.

[17] Schertzer, D., M. Larchevêque, J. Duan, S. Lovejoy, Submitted to J. Multivariate Anal., July 27 1999.

[18] Oksendal, B. Stochastic Differential Equations, Springer-Verlag, 5th ed., 324 pp. (1998).