Statistical stability for robust classes of maps with non-uniform expansion

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Abstract

We consider open sets of maps in a manifold M exhibiting non-uniform expanding behaviour in some domain $S \subset M$. Assuming that there is a forward invariant region containing S where each map has a unique SRB measure, we prove that under general uniformity conditions, the SRB measure varies continuously in the L^1-norm with the map.

As a main application we show that the open class of maps introduced in [V] fits to this situation, thus proving that the SRB measures constructed in [A] vary continuously with the map.

1 Introduction

In general terms, Dynamics has a twofold aim: to describe, for the majority of dynamical systems, the typical behaviour of trajectories, specially as time goes to infinity; to understand how this behaviour changes when the system is modified, and to what extent it is *stable* under small modifications. In this work we are primarily concerned with the latter problem.

A first fundamental concept of stability, *structural stability*, was formulated by Andronov and Pontryagin [AP]. It requires that the whole orbit structure remain unchanged under any small perturbation of the dynamical system: there exists a homeomorphism of the ambient manifold mapping trajectories of the initial system onto

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trajectories of the perturbed one, preserving the direction of time. In the early sixties, Smale introduced the notion of *uniformly hyperbolic* (or *Axiom A*) system, having as one of his main goals to obtain a characterization of structural stability. Such a characterization was conjectured by Palis and Smale in [PS]: a diffeomorphism (or a flow) is structurally stable if and only if it is uniformly hyperbolic and satisfies the so-called strong transversality condition. Before that, structural stability had been proved for certain classes of systems, including Anosov and Morse-Smale systems. The "if" part of the conjecture was proved by Robbin, de Melo, Robinson in the mid-seventies. The converse remained a major open problem for yet another decade, until it was settled by Mañé for C^1 diffeomorphisms (perturbations are small with respect to the C^1 norm). The flow case was recently solved by Hayashi, also in the C^1 category (the C^k case, $k > 1$, is still open both for diffeomorphisms and for flows). See e.g. the book of Palis and Takens [PT] for precise definitions, references and a detailed historical account.

Despite these remarkable successes, structural stability proved to be too strong a requirement for many applications. Several important models, including e.g. Lorenz flows and Hénon maps, are not stable in the structural sense, yet key aspects of their dynamical behaviour clearly persist after small modifications of the system. Weaker notions of stability, with a similar topological flavour, were proposed throughout the sixties and the seventies, but they all turned out to be too restrictive.

More recently, increasing emphasis has been put on expressing stability in terms of persistence of *statistical properties* of the system. A natural formulation, the one that concerns us most in this work, corresponds to continuous variation of physical measures as a function of the dynamical system. Let us explain this in precise terms. We consider discrete-time systems, namely, smooth transformations $\varphi : M \to M$ on a manifold M. A Borel probability measure μ on M is a *Sinai-Ruelle-Bowen (SRB) measure* (or a *physical measure*), if there exists a positive Lebesgue measure set of points $z \in M$ for which

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(\varphi^j(z)) = \int f \, d\mu \tag{1}$$

for any continuous function $f : M \to \mathbb{R}$. In other words, time averages of all continuous functions are given by the corresponding spatial averages computed with respect to μ, at least for a large set of initial states $z \in M$.

Let us suppose that φ admits a forward invariant region $U \subset M$, meaning that $\varphi(U) \subset U$, and there exists a (unique) SRB measure $\mu = \mu_\varphi$ supported in U such that (1) holds for Lebesgue almost every point $z \in U$. We say that φ is *statistically stable* (restricted to U) if similar facts are true for any C^k nearby map ψ, for some $k \geq 1$, and the map $\psi \mapsto \mu_\psi$, associating to each ψ its SRB measure μ_ψ, is continuous at $\psi = \varphi$. For this definition, we consider in the space of Borel measures the usual weak* topology: two measures are close to each other if they assign close-by integrals to each continuous function. Thus, this notion of stability really means that time averages of continuous functions are only slightly affected when the system is perturbed.

Uniformly expanding smooth maps are well known to be statistically stable, and so are Axiom A diffeomorphisms, restricted to the basin of each attractor. On the other hand, not much is known in this regard outside the uniformly hyperbolic context. In the present work we propose an approach to proving statistical stability for certain robust (open) classes of non-uniformly expanding maps. Precise conditions will be given in the next subsection. For the time being, we just mention that our maps φ exhibit asymptotic expansion,

$$\lim_{n\to+\infty} \frac{1}{n} \log \|D\varphi^n(z)v\| > 0 \quad \text{for every } v \in T_zM,$$

at Lebesgue almost every point z in some forward invariant region U (but they are not uniformly expanding). Moreover, they admit a unique SRB measure which is an ergodic invariant measure absolutely continuous with respect to Lebesgue measure in U. These properties remain valid in a neighbourhood of the initial map, and we prove that the SRB measure varies continuously with the mapping in this neighbourhood. In fact, our approach proves statistical stability in a strong sense: the density (Radon-Nikodym derivative with respect Lebesgue measure m) of the SRB measure, $d\mu_\varphi/dm$, varies continuously with φ as an L^1-function.

To the best of our knowledge this is the first result of statistical stability for maps with non-uniform expansion. An application, and the example we had in mind when we started this work, are the maps with multidimensional non-uniform expansion introduced in [V1], and whose SRB measures were constructed in [A]. In this context we mention the important work of Dolgopyat [D], where statistical stability (and other ergodic properties) were proved for some open classes of diffeomorphisms having partially hyperbolic attractors whose central direction is *mostly contracting* (negative Lyapunov exponents). In that situation, cf. also Bonatti-Viana [BV], SRB measures are absolutely continuous with respect to Lebesgue measure along the strong-unstable (uniformly expanding) foliation of the attractor. Our systems in the present work are closer in spirit to partially hyperbolic attractors with *mostly expanding* central direction, in the sense of Alves-Bonatti-Viana [ABV]. Statistical stability for the latter systems has not yet been proved.

1.1 Statement of results

Let $\varphi\colon M \to M$ be a map from some $d-$dimensional manifold into itself, S be some region in M, and $\phi : S \to S$ be a return map for φ in S. That is, there exists a countable partition $\mathcal{R} = \{R_i\}_i$ into subsets of S, and there exists a function $h\colon \mathcal{R} \to \mathbb{Z}^+$ such that

$$\phi \mid R = \varphi^{h(R)} \mid R \quad \text{for each} \quad R \in \mathcal{R}.$$

For simplicity, we will assume that S is diffeomorphic to some bounded region $\tilde{S}$ of $\mathbb{R}^n$ (but similar arguments hold in general, using local charts). Then we can pretend that $S \subset \mathbb{R}^n$, through identifying it with $\tilde{S}$, and we do so.

We say that ϕ is a C^2 *piecewise expanding map* if the following conditions hold:

(1) The boundary of each R_i is piecewise C^2 (a countable union of C^2 hyper-surfaces) and has finite $(d-1)$-dimensional volume.

(2) Each $\phi_i \equiv \phi \mid R_i$ is a C^2 bijection from the interior of R_i onto its image, admitting a C^2 extension to the closure of R_i.

(3) There is $0 < \sigma < 1$ such that $\|D\phi_i^{-1}\| < \sigma$ for every $i \geq 1$.

We say that ϕ satisfies a *bounded distortion* property if:

(4) There is some $K > 0$ such that for every $i \geq 1$

$$\frac{\left\|D\left(J \circ \phi_i^{-1}\right)\right\|}{\left|J \circ \phi_i^{-1}\right|} < K,$$

where J is the Jacobian of ϕ.

Moreover, we assume that the images under ϕ of all the elements of the partition $\mathcal{R}$ satisfy the following *bounded geometry* condition:

(5) There are constants $1 \geq \beta > \sigma/(1-\sigma)$ and $\rho > 0$ such that the boundary of each R_i has a tubular neighborhood of size ρ inside R_i, and the C^2 components of the boundary of each R_i meet at angles greater than $\arcsin(\beta) > 0$.

It was shown in [A, Section 5] that conditions (1)–(5) imply that the map ϕ has some invariant probability measure μ absolutely continuous with respect to Lebesgue measure on S (henceforth denoted m and assumed to be normalized). Then

$$\mu^* = \sum_{j=0}^{\infty} \varphi_*^j \left(\mu \mid \{h > j\}\right)$$

is an absolutely continuous invariant measure for φ. Moreover, the density $d\mu/dm$ of μ is in $L^p(S)$ for $p = d/d-1$. As a consequence, the measure μ^* is finite, as long as we have:

(6) The function h is in $L^q(S)$ for $q = d$ (this is taken so that $1/p + 1/q = 1$).

It was also observed in [A, Sections 5 and 6] that the absolutely continuous invariant measure μ^* may be taken ergodic (which implies that it is an SRB measure for φ) and, moreover, φ has finitely many such ergodic measures.

Now we state our first main result. Let $k \geq 1$ be fixed, and $\mathcal{U}$ be an open set of C^k transformations on M admitting a forward invariant compact region U. We endow $\mathcal{U}$ with the C^k topology. Assume that we may associate to each $\varphi \in \mathcal{U}$ a map $\phi_\varphi \colon S \to S$, a partition $\mathcal{R}_\varphi$ of $S \subset U$, and a function $h_\varphi \colon \mathcal{R}_\varphi \to \mathbb{Z}^+$, satisfying properties (1) to (6) above. We consider elements φ_0 of $\mathcal{U}$ satisfying the following uniformity conditions:

(U1) Given any integer $N \geq 1$ and any $\epsilon > 0$, there is $\delta = \delta(\epsilon, N) > 0$ such that for $j = 1, \ldots, N$

$$\|\varphi - \varphi_0\|_{C^k} < \delta \Rightarrow m\big(\{h_\varphi = j\}\Delta\{h_{\varphi_0} = j\}\big) < \epsilon,$$

where Δ represents symmetric difference of two sets.

(U2) Given $\epsilon > 0$, there are $N \geq 1$ and $\delta = \delta(\epsilon, N) > 0$ for which

$$\|\varphi - \varphi_0\|_{C^k} < \delta \Rightarrow \Big\| \sum_{j=N}^{\infty} \chi_{\{h_\varphi > j\}} \Big\|_q < \epsilon.$$

(U3) Constants σ, K, β, ρ as above may be chosen uniformly in a C^k neighborhood of φ_0.

Theorem A. *Let $\mathcal{U}$ be as above, and suppose that every $\varphi \in \mathcal{U}$ admits a unique SRB measure μ_φ in S. Then*

1. *μ_φ is absolutely continuous with respect to the Lebesgue measure m;*
2. *if $\varphi_0 \in \mathcal{U}$ satisfies (U1), (U2), (U3) then φ_0 is statistically stable in a strong sense: the map*
$$\mathcal{U} \ni \varphi \mapsto \frac{d\mu_\varphi}{dm}$$
is continuous, with respect to the L^1-norm, at $\varphi = \varphi_0$.

We observe that under assumption (U1), condition (U2) can be reformulated in equivalent terms as:

(U2') Given $\epsilon > 0$, there is $\delta > 0$ for which

$$\|\varphi - \varphi_0\|_{C^k} < \delta \Rightarrow \|h_\varphi - h_{\varphi_0}\|_q < \epsilon.$$

A simple proof of this equivalence will be given in Section 3 (just before Proposition 3.3).

Our next results state that the assumptions of Theorem A do correspond to robust classes of smooth maps in some manifolds.

Theorem B. *There exists a non-empty open set $\mathcal{N}$ in the space of C^3 transformations from $S^1 \times I$ into itself such that conditions (1)–(6) and (U1)–(U3) are satisfied by every element of $\mathcal{N}$.*

The open set $\mathcal{N}$ we exhibit for the proof of this result is the one constructed in [V1]. As pointed out in that paper, the choice of the cylinder $S^1 \times I$, $I = [0, 1]$, as ambient space is rather arbitrary, the construction extends easily to more general manifolds. In what follows we briefly describe the set $\mathcal{N}$, referring the reader to [V1] and Section 4 for more details.

Let d be some large integer: $d \geq 16$ suffices, but this is far from being optimal. Let $a_0 \in (1, 2)$ be such that the critical point $x = 0$ is pre-periodic under iteration by the quadratic map $q(x) = a_0 - x^2$ (again, this is far too strong a requirement on the parameter a_0). Let $b : S^1 \to \mathbb{R}$ be a Morse function, for instance, $b(t) = \sin(2\pi t)$. Note that $S^1 = \mathbb{R}/\mathbb{Z}$. For each $\alpha > 0$, consider the map $\varphi_\alpha : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ given by $\varphi_\alpha(\theta, x) = (\hat{g}(\theta), \hat{f}(\theta, x))$, where $\hat{g}$ is the uniformly expanding map of the circle defined by $\hat{g}(\theta) = d\theta \pmod{\mathbb{Z}}$, and $\hat{f}(\theta, x) = a(\theta) - x^2$ with $a(\theta) = a_0 + \alpha b(\theta)$. We shall take $\mathcal{N}$ to be a small C^3 neighborhood of φ_α, for some (fixed) sufficiently small α.

It is easy to check that for α small enough there is an interval $I \subset (-2, 2)$ for which $\varphi_\alpha(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any map φ close to φ_α in the C^0 topology has $U = S^1 \times I$ as a forward invariant region, and so φ has an attractor inside this invariant region, which is precisely the set

$$\Lambda = \bigcap_{n \geq 0} \varphi^n(U).$$

As we mentioned before, properties (1)–(6) imply that the maps $\varphi \in \mathcal{N}$ admit (finitely many) SRB measures, which are ergodic absolutely continuous invariant measures. In order to be able to apply Theorem A to this open set $\mathcal{N}$, we also have to show that the SRB measure is unique for each $\varphi \in \mathcal{N}$. This will follow from a stronger fact, stated in Theorem C below.

Let us say that φ is *topologically mixing* if for every open set $A \subset S^1 \times I$ there is some $n = n(A) \in \mathbb{Z}^+$ for which $\varphi^n(A) = \Lambda$, and say that φ is *ergodic with respect to Lebesgue measure* if for every Borel subset $B \subset S^1 \times I$ such that $\varphi^{-1}(B) = B$, either B or $(S^1 \times I) \setminus B$ have Lebesgue measure equal to zero.

Theorem C. *Let $\mathcal{N}$ be as described above. Then the transformations $\varphi \in \mathcal{N}$ are topologically mixing and ergodic with respect to Lebesgue measure.*

2 Absolute continuity

A main ingredient in our arguments, as well as in [A], is the notion of variation for functions in higher dimensions. For $f \in L^1(\mathbb{R}^d)$ with compact support we define the *variation* of f as

$$\text{var}\,(f) = \sup\left\{ \int_{\mathbb{R}^d} f \,\text{div}(g)\, dm \colon g \in C_0^1(\mathbb{R}^d, \mathbb{R}^d),\ \|g\|_0 \le 1 \right\},$$

where $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ is the set of C^1 maps from $\mathbb{R}^d$ to $\mathbb{R}^d$ with compact support and $\|\ \|_0$ is the supremum norm in $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$. We observe that in the case of f be a C^1 map, then var (f) coincides with $\int \|Df\| dm$ (see e.g. [G, Example 1.2]). We consider the space of *bounded variation* functions

$$BV(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d) \colon \text{var}\,(f) < +\infty \right\}.$$

We will make use of the following results concerning bounded variation functions:

Proposition 2.1. *Given $f \in BV(\mathbb{R}^d)$, there is a sequence $(f_n)_n$ of C^∞ maps such that*

$$\lim_{n\to\infty} \int |f - f_n| dm = 0 \quad \textit{and} \quad \lim_{n\to\infty} \int \|Df_n\| dm = var(f).$$

Proof. See [G, Theorem 1.17]. □

Proposition 2.2. *If $(f_k)_k$ is a sequence of functions in $BV(\mathbb{R}^d)$ such that there is a constant $K_0 > 0$ for which*

$$var\,(f_k) \le K_0 \quad \textit{and} \quad \int |f_k| dm \le K_0 \quad \textit{for every } k,$$

then $(f_k)_k$ has a subsequence converging in the L^1-norm to an f_0 with $var\,(f_0) \le K_0$.

Proof. See [G, Theorem 1.19]. ☐

Proposition 2.3. *Let $f \in BV(\mathbb{R}^d)$ and take $p = d/(d-1)$. Then*

$$\|f\|_p \leq K_1 var\,(f),$$

where $K_1 > 0$ is a constant depending only on d.

Proof. See [G, Theorem 1.28]. ☐

Now we introduce the linear transfer operator associated to ϕ,

$$\mathcal{L}_\phi : L^1(S) \longrightarrow L^1(S)$$

defined as

$$\mathcal{L}_\phi f = \sum_{i=1}^{\infty} \frac{f \circ \phi_i^{-1}}{|J \circ \phi_i^{-1}|} \chi_{\phi(R_i)}.$$

It is well-known that each fixed point of $\mathcal{L}_\phi$ is the density of an absolutely continuous ϕ-invariant finite measure. The next lemma gives a Lasota-Yorke type inequality for maps in $BV(\mathbb{R}^d)$, which plays a crucial role in the proof of the existence of fixed points for $\mathcal{L}_\phi$.

Lemma 2.4. *There are constants $0 < \lambda < 1$ and $K_2 > 0$ such that for every $f \in BV(\mathbb{R}^d)$,*

$$var\,(\mathcal{L}_\phi f) \leq \lambda var\,(f) + K_2 \int |f| dm.$$

Proof. See [A, Lemma 5.4]. ☐

Remark 2.5. *It follows from the proof of [A, Lemma 5.4] that the constant λ is equal to $\sigma(1+1/\beta)$. Hence, by assumption (U3), λ may be taken uniformly smaller than one in a whole neighborhood of φ. It also follows from the proof of [A, Lemma 5.4] that the constant K_2 coincides with $K + 1/(\beta\rho) + K\beta$, which may also be taken uniform in a neighborhood of the map φ. Hence, the constants $0 < \lambda < 1$ and $K_2 > 0$ may be taken in such a way that the Lasota-Yorke type inequality in the previous lemma holds for every map in a neighborhood of $\varphi \in \mathcal{U}$.*

Consider for each $k \geq 1$ the function

$$f_k = \frac{1}{k} \sum_{j=0}^{k-1} \mathcal{L}_\phi^j 1.$$

We have

$$\int |f_k| dm = 1 \quad \text{for } k \geq 1,$$

and it follows from Lemma 2.4 that

$$\text{var}\,(f_k) \leq K_3 \quad \text{for } k \geq 1,$$

where $K_3 = \text{var}\,(\chi_S) + K_2 \sum_{k=0}^{\infty} \lambda^k + 1$. It follows from Proposition 2.2 that $(f_k)_k$ has a subsequence converging in the L^1-norm to some ρ with $\text{var}\,(\rho) \leq K_3$. Hence, $\mu_\phi = \rho m$ is an absolutely continuous ϕ-invariant probability measure. From this it is deduced in [A, Section 6] that

$$\mu_\varphi^* = \sum_{j=0}^{\infty} \varphi_*^j \left(\mu_\phi \mid \{h_\varphi > j\}\right)$$

is an absolutely continuous φ-invariant finite measure.

Lemma 2.6. *Given any ϕ-invariant set $A \subset S$ with positive Lebesgue measure, there is an absolutely continuous ϕ-invariant probability measure $\mu_A = f_A m$ for which $\mu_A(A) = 1$. Moreover, f_A may be taken with var $(f_A) \leq 4K_2$.*

Proof. We start by proving that given any $f \in L^1(\mathbb{R}^d)$, the sequence $1/n \sum_{j=0}^{n-1} \mathcal{L}_\phi^j f$ has accumulation points (in the L^1-norm) in $BV(\mathbb{R}^d)$. Let $f \in L^1(\mathbb{R}^d)$ and take a sequence $(f_n)_n$ in $BV(\mathbb{R}^d)$ converging to f in the L^1-norm. It is no restriction to assume that $\|f_n\|_1 \leq 2\|f\|_1$ for every $n \geq 1$ and we do it. For each $n \geq 1$ we have

$$\text{var}\,(\mathcal{L}_\phi^j f_n) \leq \lambda^j \text{var}\,(f_n) + K_2 \|f_n\|_1 \leq 3K_2 \|f\|_1$$

for large j. So, taking k large enough we have

$$\text{var}\left(\frac{1}{k}\sum_{j=0}^{k-1} \mathcal{L}_\phi^j f_n\right) \leq 4K_1 \|f\|_1.$$

Using the well known fact that the transfer operator does not expand L^1-norms, we also have

$$\Big\|\frac{1}{k}\sum_{j=0}^{k-1} \mathcal{L}_\phi^j f_n\Big\|_1 \leq \frac{1}{k}\sum_{j=0}^{k-1} \|\mathcal{L}_\phi^j f_n\|_1 \leq 2\|f\|_1$$

for every $j \geq 1$. It follows from Proposition 2.2 that there exists some $\hat{f}_n \in BV(\mathbb{R}^d)$ and a sequence $(k_i)_i$ for which

$$\lim_{i\to\infty} \Big\| \frac{1}{k_i} \sum_{j=0}^{k_i-1} \mathcal{L}_\phi^j f_n - \hat{f}_n \Big\|_1 = 0$$

and, moreover, $\operatorname{var}(\hat{f}_n) \leq 4K_2\|f\|_1$. Now we apply the same argument to the sequence $(\hat{f}_n)_n$ in order to obtain a subsequence $(n_l)_l$ such that $(\hat{f}_{n_l})_l$ converges in the L^1-norm to some $\hat{f}$ with $\operatorname{var}(\hat{f}) \leq 4K_2\|f\|_1$. Since

$$\Big\| \frac{1}{k} \sum_{j=0}^{k-1} \mathcal{L}_\phi^j f_{n_l} - \hat{f} \Big\|_1 \leq \Big\| \frac{1}{k} \sum_{j=0}^{k-1} \mathcal{L}_\phi^j f_{n_l} - \hat{f}_{n_l} \Big\|_1 + \|\hat{f}_{n_l} - \hat{f}\|_1,$$

there is some sequence $(k_l)_l$ for which

$$\lim_{l\to\infty} \Big\| \frac{1}{k_l} \sum_{j=0}^{k_l-1} \mathcal{L}_\phi^j f_{n_l} - \hat{f} \Big\|_1 = 0.$$

On the other hand,

$$\Big\| \frac{1}{k_l} \sum_{j=0}^{k_l-1} \big(\mathcal{L}_\phi^j f_{n_l} - \mathcal{L}_\phi^j f \big) \Big\|_1 \leq \frac{1}{k_l} \sum_{j=0}^{k_l-1} \|f_{n_l} - f\|_1 = \|f_{n_l} - f\|_1$$

and this last term goes to 0 as $l \to \infty$. This finally implies that

$$\lim_{l\to\infty} \Big\| \frac{1}{k_l} \sum_{j=0}^{k_l-1} \mathcal{L}_\phi^j f - \hat{f} \Big\|_1 = 0,$$

thus proving that the sequence $1/n \sum_{j=0}^{n-1} \mathcal{L}_\phi^j f$ has accumulation points in $BV(\mathbb{R}^d)$.

Now let $A \subset S$ be a ϕ-invariant set with positive Lebesgue measure. Considering $f = \chi_A \in L^1(\mathbb{R}^d)$ in the previous argument, we obtain some sequence $(k_l)_l$ and $f_A \in BV(\mathbb{R}^d)$ for which

$$\lim_{l\to\infty} \Big\| \frac{1}{k_l} \sum_{j=0}^{k_l-1} \mathcal{L}_\phi^j \chi_A - f_A \Big\|_1 = 0$$

Moreover,

$$\operatorname{var}(f_A) \leq 4K_2\|\chi_A\|_1 \leq 4K_2 \quad \text{and} \quad \|f_A\|_1 > 0$$

(here we use that $m \mid S$ is normalized). Considering f_A already multiplied by a constant factor in order to have L^1-norm equal to 1, we take $\mu_A = f_A m$. Since

$$\mu_A(S \setminus A) = \lim_{l\to\infty} \frac{1}{k_l} \sum_{j=0}^{k_l-1} \int_{S\setminus A} \mathcal{L}_\phi^j \chi_A \, dm = \lim_{l\to\infty} \frac{1}{k_l} \sum_{j=0}^{k_l-1} \int_S \chi_{(S\setminus A)} \circ \phi^j \cdot \chi_A \, dm = 0$$

we have that μ_A gives full weight to A, thus concluding the proof of the result. □

Corollary 2.7. *There is a constant $\widehat{K}(d) > 0$ such that if $A \subset S$ is a ϕ-invariant set with positive Lebesgue measure, then $m(A) \geq \widehat{K}(d)$.*

Proof. Let $A \subset S$ be a ϕ-invariant set with positive Lebesgue measure and $\mu_A = f_A m$ a measure as in Lemma 2.6. Since $f_A \in BV(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ (recall Proposition 2.3) and μ_A gives full weight to A, it follows from Minkowski's inequality that

$$1 = \int f_A dm \leq \|f_A\|_p \cdot \|\chi_A\|_q \leq K_1 4 K_2 \, m(A)^{1/d}.$$

We take $\widehat{K}(d) = (K_1 4 K_2)^{-d}$. □

The proposition below gives the first item of Theorem A.

Proposition 2.8. *If $\varphi \in \mathcal{U}$ has a unique SRB measure μ_φ in U, then $\mu_\varphi^* = \mu_\varphi$.*

Proof. We will prove that μ_φ^* is ergodic, and so an SRB measure for φ in U. Let $A \subset U$ be any φ-invariant set with $m(A \cap S) > 0$. We have

$$\phi^{-1}(A \cap S) = \big\{x \in S \colon \phi(x) \in A\big\} = \bigcup_{j\geq 1} \big(\varphi^{-j}(A) \cap \{h_\varphi = j\}\big) = A \cap S,$$

and so the set $A \cap S$ is also ϕ-invariant. Since we are taking A with $m(A \cap S) > 0$, it follows from Corollary 2.7 that $m(A \cap S) \geq \widehat{K}(d)$. As a consequence, S can be covered by a finite number of φ-invariant sets $A_1, \ldots, A_r$ intersecting S in a positive Lebesgue measure set, and which are minimal: for $1 \leq i \leq r$ there is no φ-invariant set $B_i \subset A_i$ with $m(B_i \cap S) > 0$.

It follows from Lemma 2.6 that for each $i = 1, \ldots, r$ there is an absolutely continuous ϕ-invariant measure μ_i giving full weight to $A_i \cap S$. Take

$$\mu_i^* = \sum_{j=0}^{\infty} \varphi_*^j \left(\mu_i \mid \{h_\varphi > j\}\right)$$

and denote $A_i^c = U \setminus A_i$. Since A_i is φ-invariant, we have that A_i^c is also φ-invariant, and so

$$\mu_i^*(A_i^c) = \sum_{j=0}^{\infty} \mu_i \left(\varphi^{-j}\left(A_i^c\right) \cap \{h_\varphi > j\} \right) = \sum_{j=0}^{\infty} \mu_i \left(A_i^c \cap \{h_\varphi > j\} \right) = 0$$

Hence, assuming μ_i^* normalized we have that each μ_i^* is an absolutely continuous φ-invariant probability measure giving full weight to A_i. The minimality of each A_i implies that μ_i^* is ergodic for $1 \le i \le r$, and so it coincides with the SRB measure μ_φ. This in particular implies that $r = 1$. The fact that A_1 is a minimal φ-invariant set that contains S implies that μ_φ^* is an ergodic absolutely continuous φ-invariant probability measure, thus coinciding with μ_φ. □

Remark 2.9. *The proof of Proposition 2.8 gives also that under the hypothesis of φ having a unique SRB measure in U, the region S intersects a unique φ-invariant minimal set and, consequently, is contained in it. However, if we do not assume uniqueness of the SRB measure in U, we may write $\mu_\varphi^* = \sum_i \mu_\varphi^*(A_i)\mu_i^*$, where the sum is over the values of i for which $\mu_\varphi^*(A_i) > 0$ ($A_1, \cdots A_r$ are the minimal sets given by the proof of Proposition 2.8) and each μ_i^* is the normalized restriction of μ_φ^* to A_i, thus an SRB measure. More generally, if μ is an absolutely continuous φ-invariant probability measure giving full weight to $A_1 \cup \cdots \cup A_r$, then $\mu = \sum_i \mu(A_i)\mu_i$ where each μ_i is an SRB measure defined in the same way as μ_i^* above.*

3 Statistical stability

Now we prove that under the assumptions of Theorem A the density of the measure μ_φ^* varies continuously in the L^1-norm with the map φ. Let φ_0 be some map in $\mathcal{U}$ and $(\varphi_n)_n$ a sequence of maps in $\mathcal{U}$ converging to φ_0 in the C^k topology. As described above, we make for each on the maps φ_n and φ_0 the following choices:

$$\begin{tikzcd}
\varphi_n \arrow[r, mapsto] \arrow[d] & h_n \arrow[r, mapsto] & \phi_n \arrow[r, mapsto] & \rho_n \arrow[r, mapsto] & \mu_n \arrow[r, mapsto] & \mu_n^* \arrow[d, "?"] \\
\varphi_0 \arrow[r, mapsto] & h_0 \arrow[r, mapsto] & \phi_0 \arrow[r, mapsto] & \rho_0 \arrow[r, mapsto] & \mu_0 \arrow[r, mapsto] & \mu_0^*
\end{tikzcd}$$

It follows from the way we obtain each ρ_n that

$$\operatorname{var}(\rho_n) \le K_3 \quad \text{and} \quad \int \rho_n dm \le 1$$

for every $n \geq 1$ (recall also Remark 2.5). Thus, we may apply Proposition 2.2 to the sequence of densities $(\rho_n)_n$ and deduce that it has some subsequence $(\rho_{n_i})_{n_i}$ converging in the L^1-norm to some ρ_∞ with var $(\rho_\infty) \leq K_3$. We consider $\mu_\infty = \rho_\infty m$ and define

$$\mu_\infty^* = \sum_{j=0}^{\infty} \varphi_*^j \left(\mu_\infty \mid \{h_0 > j\}\right).$$

The goal of the results below is to show that the densities of μ_n^* with respect to the Lebesgue measure converge in the L^1-norm to the density of μ_∞^* and, moreover, the measure μ_∞^* coincides with μ_0^*. We start with some auxiliary lemmas.

Lemma 3.1. *There is $K_4 > 0$ (depending only on the dimension d) such that for every $f \in BV(\mathbb{R}^d)$ and $\psi \colon D \to \mathbb{R}^d$ a C^1 diffeomorphism from a compact $D \subset \mathbb{R}^d$ onto its image*

$$\int_D |f \circ \psi - f| dm \leq K_4 \|\psi - id\|_0^d var\,(f).$$

Proof. We start by proving the result in the case of f being a continuous piecewise affine map, i.e. letting Δ be the support of f, there is a finite number of domains $\Delta_1, \ldots, \Delta_N$ for which $\Delta = \cup_{i=1}^N \Delta_i$ and ∇f (the gradient of f) is constant on each Δ_i. We define $D_1 = \{(x,z) \in \mathbb{R}^{d+1} \colon x \in D \quad \text{and} \quad z \in [f(x), f(\psi(x))]\}$ and D_2 the horizontal $\|\psi - id\|_0$-neighborhood of the graph of f. That is, D_2 is equal to the set of points $(x,z) \in \mathbb{R}^{d+1}$ for which there is $t \in \mathbb{R}^d$ with $\|t\| \leq \|\psi - id\|_0$ and $y \in \mathbb{R}^d$ such that $x = y + t$ and $z = f(y)$. We observe that $D_1 \subset D_2$. Indeed, given $(x,z) \in D_1$, and since $z \in [f(x), f(\psi(x))]$, by the continuity of f there is $y \in [x, \psi(x)]$ such that $z = f(y)$. Taking $t = y - x$ we have $\|t\| \leq \|\psi(x) - x\|$. Hence

$$\int_D |f \circ \psi - f| dm = \int_D \int_{[f(x), f(\psi(x))]} 1\, dz dm(x) = \text{vol}(D_1) \leq \text{vol}(D_2).$$

For each $i = 1, \cdots, N$ we define $G_i = \text{graph}(f \mid \Delta_i)$ and H_i the horizontal $\|\psi - id\|_0$-neighborhood of G_i. We have

$$\text{vol}(D_2) \leq \sum_{i=1}^{N} \text{vol}(H_i).$$

Letting $\nabla_i f$ denote the gradient (constant) vector of $f \mid \Delta_i$, we have that $(-\nabla_i f, 1)$ is orthogonal to G_i. Taking $\partial_z = (0, \ldots, 0, 1) \in \mathbb{R}^{d+1}$ we have

$$\text{vol}(H_i) \leq K_4 \|\psi - id\|_0^d \sin \angle\big((-\nabla_i f, 1), \partial_z\big)\, \text{vol}(G_i),$$

where $K_4 > 0$ is a constant depending only on the volume of the unit ball in $\mathbb{R}^d$. We have

$$\sin\angle\big((-\nabla_i f, 1), \partial_z\big) = \sqrt{1-\cos^2\angle\big((-\nabla_i f, 1), \partial_z\big)} = \frac{\|\nabla_i f\|}{\sqrt{1+\|\nabla_i f\|^2}}$$

and

$$\operatorname{vol}(G_j) = \sqrt{1+\|\nabla_i f\|^2}\operatorname{vol}(\Delta_i).$$

Altogether this yields

$$\int_D |f\circ\psi - f|dm \leq K_4\|\psi - id\|_0^d \sum_{i=1}^N \|\nabla_i f\| \operatorname{vol}(\Delta_i).$$

Taking into account that in this case

$$\sum_{i=1}^N \|\nabla_i f\| \operatorname{vol}(\Delta_i) = \int \|\nabla f\| dm = \operatorname{var}(f),$$

we obtain the result for any continuous piecewise affine map.

The next step is to deduce the result for any C^1 map f. In this case, we may take a sequence $(f_n)_n$ of continuous piecewise affine maps such that

$$\|f - f_n\|_0 \to 0 \quad \text{and} \quad \|Df - Df_n\|_0 \to 0 \quad \text{as} \quad n\to\infty$$

(here we are take derivatives only in the interior points of the elements of the partition associated to each piecewise affine map). This implies that

$$\int_D |f\circ\psi - f|dm = \lim_{n\to\infty}\int_D |f_n\circ\psi - f_n|dm$$

and

$$\operatorname{var}(f) = \int \|Df\|dm = \lim_{n\to\infty}\int \|Df_n\|dm = \lim_{n\to\infty}\operatorname{var}(f_n).$$

Taking into account the case we have seen before, this implies the result also for the case of f being a C^1 map.

For the general case, we know by Proposition 2.1 that given $f \in BV(\mathbb{R}^d)$ there is a sequence $(f_n)_n$ of C^1 maps for which

$$\lim_{n\to\infty}\int |f - f_n|dm = 0 \quad \text{and} \quad \lim_{n\to\infty}\operatorname{var}(f_n) = \operatorname{var}(f). \tag{2}$$

We have

$$\int_D |f\circ\psi - f|dm \le \int_D |f\circ\psi - f_n\circ\psi|dm + \int_D |f_n\circ\psi - f_n|dm + \int_D |f_n - f|dm.$$

Since

$$\int_D |f_n\circ\psi - f\circ\psi|dm = \int_{\psi(D)} |f_n - f|\cdot\Psi dm \le \|\Psi\|_0 \int |f_n - f|dm$$

where $\Psi = 1/|\det D\psi|\circ\psi^{-1}$, the result for general $f\in BV(\mathbb{R}^d)$ follows from (2) and the previous case. $\square$

At this point we also introduce the transfer operator $\mathcal{L}_\varphi$ associated to $\varphi\in\mathcal{U}$, defined for each $f\in L^1(\mathbb{R}^d)$ as

$$\mathcal{L}_\varphi f(y) = \sum_{x\in\varphi^{-1}(y)} \frac{f(x)}{|\det D\varphi(x)|}. \tag{3}$$

For our purposes the value of $\mathcal{L}_\varphi f(y)$ for y a critical value of φ is rather unimportant. $\mathcal{L}_\varphi$ is defined in such a way that

$$\int (\mathcal{L}_\varphi f) g\, dm = \int f(g\circ\varphi)\, dm$$

for every $f, g\in L^1(\mathbb{R}^d)$ wherever these integrals make sense. Let us say that $\mathcal{L}_\varphi$ is being introduced just for the sake of notational simplicity, and so we stay away from rigorous formalities.

Lemma 3.2. *Given $\epsilon > 0$ there is $\delta > 0$ such that if $\|\varphi - \varphi_0\|_{C^1} < \delta$, then for every $f\in BV(\mathbb{R}^d)$ with support contained in S*

$$\int |\mathcal{L}_\varphi f - \mathcal{L}_{\varphi_0} f|dm < \epsilon var(f).$$

Proof. Take some small $\epsilon_1 > 0$ and define $\mathcal{C}(\epsilon_1)$ the ϵ_1-neighborhood of the critical set of φ_0 in S. We divide $S\setminus\mathcal{C}(\epsilon_1)$ into a finite number of domains of injectivity of φ_0 whose collection we call $\mathcal{D}(\varphi_0)$. We observe that if φ is close enough to φ_0, then $\mathcal{C}(\epsilon_1)$ also contains the critical set of φ, and so we may define an analogous $\mathcal{D}(\varphi)$ for φ in such a way that for each $D_0\in\mathcal{D}(\varphi_0)$ there is one (and only one) $D\in\mathcal{D}(\varphi)$ for which the Lebesgue measure of $D\Delta D_0$ is small. For each $D_0 \in \mathcal{D}(\varphi_0)$ let D be the element in $\mathcal{D}(\varphi_0)$ naturally associated to D_0, and define

$$\widehat{D}_0 = \varphi_0^{-1}\big(\varphi_0(D_0) \cap \varphi(D)\big) = D_0 \cap \varphi_0^{-1} \circ \varphi(D).$$

We have

$$\begin{aligned}
\int |\mathcal{L}_\varphi f - \mathcal{L}_{\varphi_0} f| dm \;\; \leq \;\; & \int_{\varphi_0(\mathcal{C}(\epsilon_1))} |\mathcal{L}_\varphi f - \mathcal{L}_{\varphi_0} f| dm & (4)\\
& + \sum_{D_0 \in \mathcal{D}(\varphi_0)} \int_{\varphi_0(D_0)\cap\varphi(D)} |\mathcal{L}_\varphi f - \mathcal{L}_{\varphi_0} f| dm & (5)\\
& + \sum_{D_0 \in \mathcal{D}(\varphi_0)} \int_{\varphi_0(D_0)\setminus\varphi(D)} |\mathcal{L}_{\varphi_0} f| dm & (6)\\
& + \sum_{D_0 \in \mathcal{D}(\varphi_0)} \int_{\varphi(D)\setminus\varphi_0(D_0)} |\mathcal{L}_{\varphi} f| dm & (7)
\end{aligned}$$

Let us estimate the expressions on the right hand side of the inequality above. For the first one we have

$$\begin{aligned}
\int_{\varphi_0(\mathcal{C}(\epsilon_1))} |\mathcal{L}_\varphi f - \mathcal{L}_{\varphi_0} f| dm \;\; & \leq \;\; \int_{\varphi_0(\mathcal{C}(\epsilon_1))} |\mathcal{L}_\varphi f| dm + \int_{\varphi_0(\mathcal{C}(\epsilon_1))} |\mathcal{L}_{\varphi_0} f| dm \\
& \leq \;\; \int \chi_{\varphi_0(\mathcal{C}(\epsilon_1))} \mathcal{L}_\varphi |f| dm + \int \chi_{\varphi_0(\mathcal{C}(\epsilon_1))} \mathcal{L}_{\varphi_0} |f| dm \\
& \leq \;\; \int \chi_{\varphi_0(\mathcal{C}(\epsilon_1))} \circ \varphi |f| dm + \int \chi_{\varphi_0(\mathcal{C}(\epsilon_1))} \circ \varphi_0 |f| dm
\end{aligned}$$

Since $f \in L^p(\mathbb{R}^d)$ (recall Proposition 2.3), it follows from Minkowski's inequality that

$$\int \chi_{\varphi_0(\mathcal{C}(\epsilon_1))} \circ \varphi |f| dm \leq m\big(\varphi^{-1}\varphi_0(\mathcal{C}(\epsilon_1))\big)^{1/q} \|f\|_p$$

and

$$\int \chi_{\varphi_0(\mathcal{C}(\epsilon_1))} \circ \varphi_0 |f| dm \leq m\big(\varphi_0^{-1}\varphi_0(\mathcal{C}(\epsilon_1))\big)^{1/q} \|f\|_p.$$

Let $\epsilon_2 > 0$ be some small constant (to be determined later in terms of ϵ). Using Proposition 2.3 and taking ϵ_1 and δ sufficiently small we can make

$$\int_{\varphi_0(\mathcal{C}(\epsilon_1))} |\mathcal{L}_\varphi f - \mathcal{L}_{\varphi_0} f| dm \leq \epsilon_2 \mathrm{var}\,(f). \tag{8}$$

By a change of variables induced by φ_0 we deduce for each $D_0 \in \mathcal{D}(\varphi_0)$ and $D \in \mathcal{D}(\varphi)$ associated to D_0

$$\int_{\varphi_0(D_0)\cap\varphi(D)} |\mathcal{L}_\varphi f - \mathcal{L}_{\varphi_0} f| dm =$$
$$= \int_{\widehat{D}_0} \left| \frac{f}{|\det D\varphi|} \circ \varphi^{-1} \circ \varphi_0 - \frac{f}{|\det D\varphi_0|} \right| \cdot |\det D\varphi_0| dm,$$

and this last expression is bounded from above by

$$\int_{\widehat{D}_0} \left(|f \circ \varphi^{-1} \circ \varphi_0 - f| \cdot \frac{|\det D\varphi_0|}{|\det D\varphi| \circ \varphi^{-1} \circ \varphi_0} + |f| \cdot \left| \frac{|\det D\varphi_0|}{|\det D\varphi| \circ \varphi^{-1} \circ \varphi_0} - 1 \right| \right) dm.$$

Choosing $\delta > 0$ sufficiently small, then $\|\varphi - \varphi_0\|_{C^1} < \delta$ implies that

$$\frac{|\det D\varphi_0|}{|\det D\varphi| \circ (\varphi^{-1} \circ \varphi_0)} \leq 2 \quad \text{and} \quad \left| \frac{|\det D\varphi_0|}{|\det D\varphi| \circ (\varphi^{-1} \circ \varphi_0)} - 1 \right| \leq \epsilon_2$$

on $S \setminus \mathcal{C}(\epsilon_1)$ (which contains $\widehat{D}_0$). Hence

$$\int_{\varphi_0(D_0)\cap\varphi(D)} |\mathcal{L}_\varphi f - \mathcal{L}_{\varphi_0} f| dm \leq 2 \int_{\widehat{D}_0} |f \circ \varphi^{-1} \circ \varphi_0 - f| dm + \epsilon_2 \int |f| dm.$$

Thus, applying Lemma 3.1 and choosing $\delta > 0$ sufficiently small we obtain

$$\int_{\varphi_0(D_0)\cap\varphi(D)} |\mathcal{L}_\varphi f - \mathcal{L}_{\varphi_0} f| dm \leq \epsilon_2 \mathrm{var}\,(f). \tag{9}$$

Let us finally estimate the terms involved in (6) (the same method can be applied to obtain a similar estimate for (7)).

$$\begin{aligned} \int_{\varphi_0(D_0)\setminus\varphi(D)} |\mathcal{L}_{\varphi_0} f| dm &\leq \int_{\varphi_0(D_0)\setminus\varphi(D)} \frac{|f|}{|\det D\varphi_0|} \circ \varphi_0^{-1} dm \\ &= \int_{D_0 \setminus \varphi_0^{-1}(\varphi(D))} |f| dm \\ &\leq m(D \setminus \widehat{D}_0)^{1/q} \|f\|_p \end{aligned}$$

Using Proposition 2.3 and taking δ is sufficiently small, then

$$\int_{\varphi_0(D_0)\setminus\varphi(D)} |\mathcal{L}_{\varphi_0} f| dm + \int_{\varphi(D)\setminus\varphi_0(D_0)} |\mathcal{L}_{\varphi_0} f| dm \leq \epsilon_2 \mathrm{var}\,(f). \tag{10}$$

Putting estimates (8), (9), (10) above together we obtain

$$\int |\mathcal{L}_{\varphi} f - \mathcal{L}_{\varphi_0} f| dm \le \big(\epsilon_2 + 3\#\mathcal{D}(\varphi_0)\epsilon_2\big) \mathrm{var}\,(f).$$

So we only have to take ϵ_2 in such a way that $\epsilon_2 + 3\#\mathcal{D}(\varphi_0)\epsilon_2 < \epsilon$. $\square$

At this point we prove that conditions (U2') and (U2) are equivalent if we assume (U1). First we prove that (U2') implies (U2). Let $\epsilon > 0$ be some small number and take $N \ge 1$ in such a way that $\| \sum_{j=N}^{\infty} \chi_{\{h_{\varphi_0} > j\}} \|_q < \epsilon/3$. We have

$$\begin{aligned} \Big\| \sum_{j=N}^{\infty} \chi_{\{h_{\varphi} > j\}} \Big\|_q &= \Big\| h_{\varphi} - h_{\varphi_0} + h_{\varphi_0} - \sum_{j=0}^{N-1} \chi_{\{h_{\varphi_0} > j\}} + \sum_{j=0}^{N-1} \chi_{\{h_{\varphi_0} > j\}} - \sum_{j=0}^{N-1} \chi_{\{h_{\varphi} > j\}} \Big\|_q \\ &\le \|h_{\varphi} - h_{\varphi_0}\|_q + \Big\| \sum_{j=N}^{\infty} \chi_{\{h_{\varphi_0} > j\}} \Big\|_q + \sum_{j=0}^{N-1} \big\| \chi_{\{h_{\varphi_0} > j\}} - \chi_{\{h_{\varphi} > j\}} \big\|_q, \end{aligned}$$

and so, if we take $\delta = \delta(N, \epsilon) > 0$ sufficiently small then, under assumptions (U1) and (U2'), the first and third terms in the sum above can be made smaller than $\epsilon/3$. This gives the conclusion of condition (U2).

For the other implication, let $\epsilon > 0$ be some small number, and $N \ge 1$ and $\delta = \delta(N, \epsilon) > 0$ be taken in such a way that the conclusion of (U2) holds for $\epsilon/3$. We have

$$\begin{aligned} \|h_{\varphi} - h_{\varphi_0}\|_q &= \Big\| h_{\varphi} - \sum_{j=0}^{N-1} \chi_{\{h_{\varphi} > j\}} + \sum_{j=0}^{N-1} \big(\chi_{\{h_{\varphi} > j\}} - \chi_{\{h_{\varphi_0} > j\}} \big) + \sum_{j=0}^{N-1} \chi_{\{h_{\varphi_0} > j\}} - h_{\varphi_0} \Big\|_q \\ &\le \Big\| \sum_{j=N}^{\infty} \chi_{\{h_{\varphi} > j\}} \Big\|_q + \sum_{j=0}^{N-1} \| \chi_{\{h_{\varphi} > j\}} - \chi_{\{h_{\varphi_0} > j\}} \|_q + \Big\| \sum_{j=0}^{N-1} \chi_{\{h_{\varphi_0} > j\}} \Big\|_q. \end{aligned}$$

By the choices of N and δ, the first and third terms in the last sum above are smaller than $\epsilon/3$. Since condition (U1) is verified, the second term can also be made smaller than $\epsilon/3$ for $\delta = \delta(\epsilon, N) > 0$ small enough.

Proposition 3.3. $\dfrac{d\mu_{n_i}^*}{dm}$ *converges to* $\dfrac{d\mu_{\infty}^*}{dm}$ *in the* L^1*-norm.*

Proof. Fixing some small $\epsilon > 0$, we are going to prove that there is some $\delta > 0$ for which

$$\left\| \frac{d\mu_{n_i}^*}{dm} - \frac{d\mu_{\infty}^*}{dm} \right\|_1 < \epsilon \quad \text{whenever} \quad \|\varphi_{n_i} - \varphi_0\|_{C^1} < \delta.$$

We have

$$\mu_\infty^* = \sum_{j=0}^{\infty} (\varphi_0^j)_* \left(\mu_\infty \mid \{h_0 > j\}\right) \quad \text{and} \quad \mu_{n_i}^* = \sum_{j=0}^{\infty} (\varphi_{n_i}^j)_* \left(\mu_{n_i} \mid \{h_{n_i} > j\}\right). \tag{11}$$

By (U2) we know that there is an integer $N \geq 1$ and $\delta = \delta(\epsilon, N) > 0$ for which

$$\|\varphi - \varphi_0\|_{C^1} < \delta \Rightarrow \Big\| \sum_{j=N}^{\infty} \chi_{\{h_\varphi > j\}} \Big\|_q < \frac{\epsilon}{4K_1K_3}. \tag{12}$$

Now we take $i \geq 1$ sufficiently large in such a way that $\|\varphi_{n_i} - \varphi_0\| < \delta$. We split each one of the sums in (11) into two sums and write

$$\mu_\infty^* = \sum_{j=0}^{N} \nu_{\infty,j} + \eta_{\infty,N} \quad \text{and} \quad \mu_{n_i}^* = \sum_{j=0}^{N} \nu_{n_i,j} + \eta_{n_i,N}, \tag{13}$$

where

$$\nu_{\infty,j} = \varphi_*^j \left(\mu_0 \mid \{h_0 > j\}\right), \quad \eta_{\infty,N} = \sum_{j=N+1}^{\infty} \varphi_*^j \left(\mu_\infty \mid \{h_0 > j\}\right)$$

and

$$\nu_{n_i,j} = (\varphi_{n_i}^j)_* \left(\mu_{n_i} \mid \{h_{n_i} > j\}\right), \quad \eta_{n_i,N} = \sum_{j=N+1}^{\infty} (\varphi_{n_i}^j)_* \left(\mu_{n_i} \mid \{h_{n_i} > j\}\right).$$

We have

$$\eta_{\infty,N}(M) = \sum_{j=N}^{\infty} \mu_\infty(\{h_0 > j\}) = \sum_{j=N}^{\infty} \int \rho_\infty \chi_{\{h_0 > j\}} dm \leq \|\rho_\infty\|_p \cdot \Big\| \sum_{j=N}^{\infty} \chi_{\{h_0 > j\}} \Big\|_q,$$

and

$$\eta_{n_i,N}(M) = \sum_{j=N}^{\infty} \mu_{n_i}(\{h_{n_i} > j\}) = \sum_{j=N}^{\infty} \int \rho_{n_i} \chi_{\{h_{n_i} > j\}} dm \leq \|\rho_{n_i}\|_p \cdot \Big\| \sum_{j=N}^{\infty} \chi_{\{h_{n_i} > j\}} \Big\|_q$$

which together with Proposition 2.3 and (12) yield

$$\left\| \frac{d\eta_{n_i,N}}{dm} - \frac{d\eta_{\infty,N}}{dm} \right\|_1 \leq \eta_{n_i,N}(M) + \eta_{\infty,N}(M) < \epsilon/2. \tag{14}$$

On the other hand, we have for $j = 1, \dots, N$

$$\left\| \frac{d\nu_{n_i,j}}{dm} - \frac{d\nu_{\infty,j}}{dm} \right\|_1 = \left\| \mathcal{L}_{\varphi^j_{n_i}}(\rho_{n_i}\chi_{\{h_{n_i}>j\}}) - \mathcal{L}_{\varphi^j_0}(\rho_\infty \chi_{\{h_0>j\}}) \right\|_1$$

which is bounded from above by

$$\underbrace{\|\mathcal{L}_{\varphi^j_{n_i}}(\rho_{n_i}\chi_{\{h_{n_i}>j\}}) - \mathcal{L}_{\varphi^j_{n_i}}(\rho_\infty\chi_{\{h_0>j\}})\|_1}_{A} + \underbrace{\|\mathcal{L}_{\varphi^j_{n_i}}(\rho_\infty\chi_{\{h_0>j\}}) - \mathcal{L}_{\varphi^j_0}(\rho_\infty\chi_{\{h_0>j\}})\|_1}_{B}.$$

Here we also consider the transfer operator for the iterated maps $\varphi^j_{n_i}$ and φ^j_0 defined analogously as for φ in (3). We have

$$\begin{aligned} A &\le \|\rho_{n_i}\chi_{\{h_{n_i}>j\}} - \rho_\infty\chi_{\{h_0>j\}}\|_1 \\ &\le \|\rho_{n_i}\chi_{\{h_{n_i}>j\}} - \rho_\infty\chi_{\{h_{n_i}>j\}}\|_1 + \|\rho_\infty\chi_{\{h_{n_i}>j\}} - \rho_\infty\chi_{\{h_0>j\}}\|_1 \\ &\le \|\rho_{n_i} - \rho_\infty\|_1 + \|\rho_\infty(\chi_{\{h_{n_i}>j\}} - \chi_{\{h_0>j\}})\|_1 \end{aligned}$$

and

$$\|\rho_\infty(\chi_{\{h_{n_i}>j\}} - \chi_{\{h_0>j\}})\|_1 \le \|\rho_\infty\|_p \|\chi_{\{h_{n_i}>j\}} - \chi_{\{h_0>j\}}\|_q.$$

Taking into account (U1) we can make $A \le \epsilon/(4N)$ if i is sufficiently large. Using Proposition 3.2 we can also make $B \le \epsilon/(4N)$, as long as we take i large enough. This completes the proof of the proposition. □

Proposition 3.4. *μ^*_∞ is a φ_0-invariant measure.*

Proof. It follows from Proposition 3.3 that $(\mu^*_{n_i})_i$ converges to μ^*_∞ in the weak* topology. Hence, given any $f\colon M \to \mathbb{R}$ continuous we have

$$\int f d\mu^*_{n_i} \to \int f d\mu^*_\infty \quad \text{when} \quad i \to \infty.$$

On the other hand, since $\mu^*_{n_i}$ is φ_{n_i}-invariant we have

$$\int f d\mu^*_{n_i} = \int f \circ \varphi_{n_i} d\mu^*_{n_i} \quad \text{for every } i.$$

It suffices to prove that

$$\int f \circ \varphi_{n_i} d\mu^*_{n_i} \to \int f \circ \varphi_0 d\mu^*_\infty \quad \text{when} \quad i \to \infty. \tag{15}$$

We have

$$\Big|\int f\circ\varphi_{n_i}d\mu^*_{n_i}-\int f\circ\varphi_0 d\mu^*_\infty\Big|\le \\ \Big|\int f\circ\varphi_{n_i}d\mu^*_{n_i}-\int f\circ\varphi_0 d\mu^*_{n_i}\Big|+\Big|\int f\circ\varphi_0 d\mu^*_{n_i}-\int f\circ\varphi_0 d\mu^*_\infty\Big|.$$

Since $f\circ\varphi_{n_i}-f\circ\varphi_0$ is uniformly close to zero when i is large (at least in the compact set U that contains the supports of the measures), we have that the first term in the sum above is close to zero for i sufficiently large. On the other hand, since $(\mu^*_{n_i})_i$ converges to μ^*_∞ in the weak* topology we also have that the second term in the sum above is close to zero if i is taken large enough. ☐

It follows from this last result and the uniqueness of the absolutely continuous φ-invariant measure that necessarily $\mu^*_\infty=\mu^*_0$, thus proving that the measures $\mu^*_{n_i}$ have densities converging in the L^1-norm to the density of μ^*_0. Moreover, the argument shows that this happens with the densities of any convergent subsequence of $(\mu^*_n)_n$. This completes the proof of Theorem A.

4 Hyperbolic returns

The main goal of this section is to introduce a notion of hyperbolic returns, which allows us to improve some of the estimates in [A] and [V1] that are useful in the proofs of Theorem B and Theorem C. For the sake of clearness, we start by assuming that the map φ has the special form

$$\varphi(\theta,x)=(g(\theta),f(\theta,x)),\quad\text{with}\quad \partial_x f(\theta,x)=0\quad\text{if and only if}\quad x=0,\tag{16}$$

and prove the conclusions of Theorems A and B for every C^2 map φ satisfying

$$\|\varphi-\varphi_\alpha\|_{C^2}\le\alpha\quad\text{on}\quad S^1\times I.\tag{17}$$

Later we explain how the conclusions extend to general case, using the existence of a central invariant foliation in the same way as in [V1] and [A].

Our estimates on the derivative depend in an important way on the returns of orbits to the neighborhood $S^1\times(-\sqrt{\alpha},\sqrt{\alpha})$ of the critical set $\{x=0\}$. For this, we introduce a partition $\mathcal{Q}$ of I (modulo a zero Lebesgue measure set) into the following intervals:

$$I_r=(\sqrt{\alpha}e^{-r},\sqrt{\alpha}e^{-(r-1)})\quad\text{for}\quad r\ge 1,$$

$$I_r = -I_{-r} \quad \text{for} \quad r \leq -1,$$
$$I_{0^+} = \big(I \setminus [-\sqrt{\alpha}, \sqrt{\alpha}]\big) \cap \mathbb{R}^+ \qquad \text{and} \qquad I_{0^-} = \big(I \setminus [-\sqrt{\alpha}, \sqrt{\alpha}]\big) \cap \mathbb{R}^- .$$

This partition induces in a natural way analogous ones at each fiber of the type $\{\theta\} \times I$. For the sake simplicity in the notation no specification will be made in which fiber they are on, since this will be always clear in all our settings.

In what follows we assume that $\alpha > 0$ is a sufficiently small number independent of any other constant involved in the arguments. Furthermore, for each new constant appearing we will always specify whether it depends on α or not. Given $(\theta, x) \in S^1 \times I$ and $j \geq 0$ we define $(\theta_j, x_j) = \varphi^j(\theta, x)$. Following [V1], for the next lemma we take η a positive constant smaller than $1/4$ depending only on the quadratic map q.

Lemma 4.1. *There are constants $C_1 > 1$ such that for every small α we have an integer $N(\alpha)$ satisfying:*

1. *If $|x| < 2\sqrt{\alpha}$, then $\prod_{j=0}^{N(\alpha)-1} |\partial_x f(\theta_j, x_j)| \geq |x|\alpha^{-1+\eta}$.*
2. *If $|x| < 2\sqrt{\alpha}$, then $|x_j| > \sqrt{\alpha}$ for every $j = 1, \ldots, N(\alpha)$.*
3. *$C_1^{-1} \log(1/\alpha) \leq N(\alpha) \leq C_1 \log(1/\alpha)$.*

Proof. See [V1, Lemma 2.4] and [A, Lemma 2.1]. □

Lemma 4.2. *There are $\tau > 1$, $C_2 > 0$ and $\delta > 0$ such that for $(\theta, x) \in S^1 \times I$ and $k \geq 1$ the following holds:*

1. *If $|x_0|, \ldots, |x_{k-1}| \geq \sqrt{\alpha}$, then $\prod_{j=0}^{k-1} |\partial_x f(\theta_j, x_j)| \geq C_2\sqrt{\alpha}\tau^k$.*
2. *If $|x_0|, \ldots, |x_{k-1}| \geq \sqrt{\alpha}$ and $|x_k| < \delta$, then $\prod_{j=0}^{k-1} |\partial_x f(\theta_j, x_j)| \geq C_2\tau^k$.*

Proof. See [V1, Lemma 2.5]. □

For each integer $j \geq 0$ we define

$$r_j(\theta, x) = \begin{cases} |r| & \text{if } \varphi^j(\theta, x) \in I_r \text{ with } |r| \geq 1; \\ 0 & \text{if } \varphi^j(\theta, x) \in I \setminus [-\sqrt{\alpha}, \sqrt{\alpha}]. \end{cases} \tag{18}$$

We say that $\nu \geq 0$ is a *return* for (θ, x) if $r_\nu(\theta, x) \geq 1$. Let n be some positive integer and $0 \leq \nu_1 \leq \cdots \leq \nu_s \leq n$ the returns of (θ, x) from 0 to n. It follows from Lemma 4.1 that for each $1 \leq i \leq s$

$$\prod_{j=\nu_i}^{\nu_i+N-1} |\partial_x f(\theta_j, x_j)| \geq e^{-r_{\nu_i}(\theta,x)}\alpha^{-1/2+\eta},$$

and from Lemma 4.2

$$\prod_{j=0}^{\nu_1-1} |\partial_x f(\theta_j, x_j)| \geq C_2 \tau^{\nu_1} \quad \text{and} \quad \prod_{j=\nu_i+N}^{\nu_{i+1}-1} |\partial_x f(\theta_j, x_j)| \geq C_2 \tau^{\nu_{i+1}-\nu_i-N}.$$

For the last piece of orbit (if it exists) we use again Lemma 4.2 and obtain

$$\prod_{j=\nu_s}^{n-1} |\partial_x f(\theta_j, x_j)| \geq |\partial_x f(\theta_{\nu_s}, x_{\nu_s})| C_2 \sqrt{\alpha} \tau^{n-\nu_s}. \tag{19}$$

Considering

$$G_n(\theta, x) = \left\{ 1 \leq \nu_i \leq n-1 : r_{\nu_i}(\theta, x) \geq \left(\frac{1}{2} - 2\eta \right) \log \frac{1}{\alpha} \right\}, \tag{20}$$

the estimates above yield (see [A, Section 2])

$$\prod_{j=0}^{n-1} |\partial_x f(\theta_j, x_j)| \geq \exp \left(4cn - \sum_{j \in G_n(\theta,x)} r_j(\theta, x) - \log C_2 - \frac{3}{2} \log \frac{1}{\alpha} \right) \tag{21}$$

for every $n \geq 1$ and α sufficiently small, where $c > 0$ is some constant depending only on the map q. The term $(3/2)\log(1/\alpha)$ appears if a last piece of orbit has to be considered whenever n is not a return for (θ, x) (estimate (19) above). Hence, if n is a return for (θ, x) we can improve estimate (21) and deduce that

$$\prod_{j=0}^{n-1} |\partial_x f(\theta_j, x_j)| \geq C_2^{-1} \exp \left(4cn - \sum_{j \in G_n(\theta,x)} r_j(\theta, x) \right) \tag{22}$$

A key fact underlying our construction is that the exponent on the right hand side is positive, except for a set of initial points (θ, x) whose measure decreases very rapidly with n. More precisely, let us define

$$E_n = \big\{ (\theta, x) \in S^1 \times I : \sum_{j \in G_n(\theta,x)} r_j(\theta, x) > 2n \big\}. \tag{23}$$

Then, cf. (16) and (17) in [V1, Section 2.4], there are constants $C, \gamma > 0$ such that

$$m\big(E_n\big) \leq C e^{-\gamma\sqrt{n}} \tag{24}$$

for every sufficiently large n, only depending on α. Note that (22) gives

$$\|D\varphi^n(\theta,x)(0,1)\| \geq e^{cn} \quad \text{for } (\theta,x) \in (S^1\times I)\setminus E_n$$

and n sufficiently large.

One of the basic ingredients in the proof of the existence of the SRB measures is the notion of hyperbolic times. Following [A] we fix $0<\epsilon<c/2$ and say that $n\geq 1$ is a *hyperbolic time* for $(\theta,x)\in S^1\times I$ if

$$\sum_{\substack{i\,\in\, G_n(\theta,x)\\ k\,\leq\, i\,<\,n}} r_i(\theta,x) < (c+\epsilon)(n-k) \quad \text{for every} \quad 0\leq k<n.$$

We say that $n\geq 1$ is a *hyperbolic return* for $(\theta,x)\in S^1\times I$ if n is both a hyperbolic time and a return for (θ,x). It follows from [A, Proposition 2.5] (see also [A, Remark 2.6]) that Lebesgue almost every point in $S^1\times I$ has infinitely many hyperbolic times. This in particular implies that Lebesgue almost every point in $S^1\times I$ also has infinitely many hyperbolic returns. Indeed, if n is a hyperbolic time for (θ,x) and $l>n$ is the next return for (θ,x) after n, then since $r_j(\theta,x)=0$ for $j=n+1,\cdots,l-1$, it easily follows that l is a hyperbolic return for (θ,x).

Fixing an integer $p\geq 1$ (whose value will be made precise below in terms of the expansion rates of the maps $\hat{g}$ and $\hat{f}$), let H be the set of points that has at least one hyperbolic time greater or equal to p. Decompose $H=\cup_{n\geq p}H_n$, where each H_n is the set of points whose first hyperbolic time greater or equal to p is n. It follows from [A, Proposition 2.5] that there is a positive integer $n_0=n_0(p,\epsilon)\geq p$ such that

$$(S^1\times I)\setminus(H_p\cup\cdots\cup H_n)\subset E_n \quad \text{for every} \quad n\geq n_0\,. \tag{25}$$

Now we briefly describe how in [A, Section 3] is defined a partition $\mathcal{R}$ into rectangles of $S^1\times I$ (modulo a zero Lebesgue measure set). For this, we consider the partition $\mathcal{Q}$ of I described above, and introduce a sequence of Markov partitions of S^1: assume that $S^1=\mathbb{R}/\mathbb{Z}$ has the orientation induced by the usual order in $\mathbb{R}$ and let θ_0 be the fixed point of g close to $\theta=0$. We define Markov partitions $\mathcal{P}_n$, $n\geq 1$, of S^1 in the following way:

- $\mathcal{P}_1=\{[\theta_{j-1},\theta_j):1\leq j\leq d\}$, where $\theta_0,\theta_1,\ldots,\theta_d=\theta_0$ are the pre-images of θ_0 under g (ordered according to the orientation of S^1).
- $\mathcal{P}_{n+1}=\{$ connected components of $g^{-1}(\omega):\omega\in\mathcal{P}_n\}$ for each $n\geq 1$.

The partition $\mathcal{R}$ is obtained by successive divisions of the rectangles in an initial partition $\mathcal{P}_p \times \mathcal{Q}$ of $S^1 \times I$, for some fixed large integer p, according to the itineraries of points through the horizontal strips $S^1 \times I_*$ with $I_* \in \mathcal{Q}$ and their hyperbolic times. This partition may be written as a union $\mathcal{R} = \cup_{n \geq p} \mathcal{R}_n$ with the sets $\mathcal{R}_n$ defined inductively and satisfying

$$H_n \subset \bigcup_{R \in \mathcal{R}_n} R \quad \text{and} \quad R \cap H_n \neq \emptyset \quad \text{for every} \quad R \in \mathcal{R}_n. \tag{26}$$

For $n \geq p$ rectangles in $\mathcal{R}_n$ always have the form $\omega \times J$, with ω belonging to $\mathcal{P}_n$ and J a subinterval of I_* for some $I_* \in \mathcal{Q}$.

Proposition 4.3. *There is some constant $\Delta > 1$ such that for every $n \geq p$, $R \in \mathcal{R}_n$ and $(\theta, x), (\sigma, y) \in R$ we have*

$$\frac{1}{\Delta} \leq \left| \frac{J(\theta, x)}{J(\sigma, y)} \right| \leq \Delta,$$

where J is the Jacobian of $\varphi^n \mid R$.

Proof. Fix some $R \in \mathcal{R}_n$ with $n \geq p$ and let $\phi = \varphi^n \mid R$. We have

$$\left| \frac{J(\theta, x)}{J(\sigma, y)} \right| = \exp\left(\log \left| \left(J \circ \phi^{-1} \right) \left(\phi(\theta, x) \right) \right| - \log \left| \left(J \circ \phi^{-1} \right) \left(\phi(\sigma, y) \right) \right| \right)$$

and

$$\left| \log \left| \left(J \circ \phi^{-1} \right) \left(\phi(\theta, x) \right) \right| - \log \left| \left(J \circ \phi^{-1} \right) \left(\phi(\sigma, y) \right) \right| \right| \leq \left\| D \left(\log \left| J \circ \phi^{-1} \right| \right) (\tau, z) \right\| \cdot C,$$

for some $(\tau, z) \in \phi(S)$ and $C > 0$ depending only on the diameter of $S^1 \times I$. Now, since

$$\left\| D \left(\log \left| J \circ \phi^{-1} \right| \right) (\tau, z) \right\| = \frac{\left\| D \left(J \circ \phi^{-1} \right) (\tau, z) \right\|}{\left| (J \circ \phi^{-1})(\tau, z) \right|},$$

the result follows from [A, Proposition 4.2]. □

Now we are going to prove that the Lebesgue measure of the set of points that have no hyperbolic returns smaller than some large integer n decays at least sub-exponentially fast with n. Similarly to what we have done for hyperbolic times, let $p \geq 1$ be some fixed large integer, and define H^* the set of points that has at least one hyperbolic return greater or equal to p. We decompose $H^* = \cup_{n \geq p} H_n^*$, where each H_n^* is the set of points whose first hyperbolic return greater or equal to p is n.

Proposition 4.4. *There is a positive integer* $n_1 = n_1(p,\epsilon) \geq p$ *and constants* $C_0, \gamma_0 > 0$ *such that for each* $n \geq n_1$

$$m\big((S^1 \times I) \setminus (H_p^* \cup \cdots \cup H_n^*)\big) \leq C_0 e^{-\gamma_0 \sqrt{n}}.$$

Proof. Take $n \geq \max\{2p, n_0\}$ and let $l = [n/2]$. The set of points $(\theta, x) \in H_l$ for which there is some $1 \leq k \leq l$ that is a return for $\varphi^l(\theta, x)$ is contained in $H_p^* \cup \cdots \cup H_n^*$. Hence, defining

$$B_l = \bigcup_{k=p}^{l} \{(\theta, x) \in H_k \colon \varphi^k(\theta, x) \text{ has no returns from time 1 to } l\}$$

we have

$$(H_p \cup \cdots \cup H_l) \cap \big((S^1 \times I) \setminus B_l\big) \subset (H_p^* \cup \cdots \cup H_n^*)$$

and so

$$m\big((S^1 \times I) \setminus (H_p^* \cup \cdots \cup H_n^*)\big) \leq m\big((S^1 \times I) \setminus (H_p \cup \cdots \cup H_l)\big) + m(B_l),$$

Taking into account estimates (25) and (24) above, it suffices to study the decay of $m(B_l)$ with n. We define for each $k \geq p$ and $R_k \in \mathcal{R}_k$

$$R_k(l) = \{(\theta, x) \in R_k \colon \varphi^k(\theta, x) \text{ has no returns from time 1 to } l\}.$$

Using (26) we obtain

$$m(B_l) \leq \sum_{k=p}^{l} \sum_{R_k \in \mathcal{R}_k} m\big(R_k(l)\big). \tag{27}$$

Fixing some $R_k \in \mathcal{R}_k$ and $(\theta_0, x_0) \in R_k$ we deduce from Proposition 4.3

$$\begin{aligned}
m\left(\varphi^k\left(R_k(l)\right)\right) &= \int_{R_k(l)} |J(\theta, x)| \, dm(\theta, x) \\
&\geq \frac{1}{\Delta} \int_{R_k(l)} |J(\theta_0, x_0)| \, dm(\theta, x) \\
&\geq \frac{1}{\Delta} |J(\theta_0, x_0)| \, m\big(R_k(l)\big).
\end{aligned}$$

Similarly we prove that

$$m\left(\varphi^k(R_k)\right) \leq \Delta \, |J(\theta_0, x_0)| \, m(R_k).$$

Hence

$$\frac{m\big(R_k(l)\big)}{m(R_k)} \le \Delta^2 \frac{m\left(\varphi^k\big(R_k(l)\big)\right)}{m\left(\varphi^k(R_k)\right)}. \tag{28}$$

It follows from the definition of $R_k(l)$ that the iterates of points in $\varphi^k\big(R_k(l)\big)$ do not hit the critical region $S^1 \times [-\sqrt{\alpha}, \sqrt{\alpha}]$ from time 1 to l. From Lemma 4.2 we deduce that there is some constant $C > 0$ for which

$$m\left(\varphi^k\big(R_k(l)\big)\right) \le C\tau^{-l}. \tag{29}$$

On the other hand, it follows from [A, Proposition 3.8] that there is some absolute constant $\delta > 0$ for which

$$m\left(\varphi^k\big(R_k\big)\right) \ge \delta. \tag{30}$$

From (28), (29) and (30) we obtain

$$m\big(R_k(l)\big) \le \frac{\Delta^2 C}{\delta}\tau^{-l} m(R_k)$$

which together with (27) gives

$$m(B_l) \le \sum_{k=p}^{l} \sum_{R_k \in \mathcal{R}_k} \frac{\Delta^2 C}{\delta}\tau^{-l} m(R_k) \le (l-p)\frac{\Delta^2 C}{\delta}\tau^{-l} \le n\frac{\Delta^2 C}{\delta}\tau^{-n/2}.$$

□

Remark 4.5. *It follows from the proof of Proposition 4.4 that the constants C_0 and γ_0 only depend on the constants C, γ and absolute constants associated to the quadratic map q. Moreover, the integer n_1 only depends on the previous constants and the integer $p \ge 1$. At the end of this section we will see that p may be chosen independent of the map $\varphi \in \mathcal{N}$.*

Hyperbolic times play a crucial role in [A, Proposition 3.8] in order to obtain that the images $\varphi^n(R)$ of rectangles $R \in \mathcal{R}_n$ have sizes uniformly bounded away from zero. However, the uniform constant that bounds such sizes from below depends on α, and that is still inconvenient for proving the mixing and ergodic properties. To bypass this difficulty we are going to consider hyperbolic returns in the place of hyperbolic times. We proceed as in [A, Section 3] and define a partition $\mathcal{R}$ into rectangles of $S^1 \times I$ (modulo a zero Lebesgue measure set) exactly in the same way with the sets

H_n^* playing the role of the sets H_n. In particular, this partition may also be written as a union $\mathcal{R} = \cup_{n\ge p}\mathcal{R}_n$ with the sets $\mathcal{R}_n$ defined inductively and satisfying

$$H_n^* \subset \bigcup_{R\in\mathcal{R}_n} R \quad \text{and} \quad R\cap H_n^* \neq \emptyset \quad \text{for every} \quad R\in\mathcal{R}_n.$$

Furthermore, for each $n\ge p$, rectangles in $\mathcal{R}_n$ also have the form $\omega\times J$, with ω belonging to $\mathcal{P}_n$ and J a subinterval of I_* for some $I_*\in\mathcal{Q}$. We define a map $h:\mathcal{R}\to\mathbb{Z}^+$, by putting $h(R)=n\ge p$ for each $R\in\mathcal{R}_n$.

Lemma 4.6. *Let $(\theta,x)\in R$ for some $R\in\mathcal{R}$. Then for every $j=0,\cdots,h(R)-1$ we have*

$$\prod_{i=j}^{h(R)-1} |\partial_x f(\theta_i,x_i)| \ge C_2^{-1}\exp\big((2c-\epsilon)(h(R)-j)\big).$$

Proof. The same proof of [A, Lemma 3.7] with the improved estimate (22) in the place of (21). ☐

It follows from assumption (16) that for each $n\ge 1$ there is a map $F_n\colon S^1\times I\to I$ such that $\varphi^n(\theta,x)=(g^n(\theta),F_n(\theta,x))$ for every $(\theta,x)\in S^1\times I$. Let (θ,x) belong to $R\in\mathcal{R}$ and $h=h(R)$. Then

$$D\varphi^h(\theta,x)=\begin{pmatrix}\partial_\theta g^h(\theta) & 0\\ \partial_\theta F_h(\theta,x) & \partial_x F_h(\theta,x)\end{pmatrix},$$

and so

$$\begin{aligned}(D\varphi^h(\theta,x))^{-1} &= \frac{1}{\partial_\theta g^h(\theta)\partial_x F_h(\theta,x)}\begin{pmatrix}\partial_x F_h(\theta,x) & 0\\ -\partial_\theta F_h(\theta,x) & \partial_\theta g^h(\theta)\end{pmatrix}\\ &= \begin{pmatrix}(\partial_\theta g^h(\theta))^{-1} & 0\\ -\partial_\theta F_h(\theta,x)\big(\partial_\theta g^h(\theta)\partial_x F_h(\theta,x)\big)^{-1} & (\partial_x F_h(\theta,x))^{-1}\end{pmatrix}.\end{aligned}$$

It follows from [A, Lemma 4.1] that there is some constant $C_3>0$ such that for every $(\theta,x)\in S^1\times I$ we have $|\partial_\theta F_h(\theta,x)|\le C_3|\partial_\theta g^h(\theta)|$. Then

$$\|D\varphi^{-h}(\varphi^h(\theta,x))\| \le \max\left\{|\partial_\theta g^h(\theta)|^{-1}+C_3|\partial_x F_h(\theta,x)|^{-1}, |\partial_x F_h(\theta,x)|^{-1}\right\}.$$

We have

$$|\partial_\theta g^h(\theta)|^{-1}\le (d-\alpha)^{-h}$$

and from Lemma 4.6

$$|\partial_x F_h(\theta, x)|^{-1} \leq C_2 \exp(-(2c - \epsilon)h).$$

Hence,

$$\|D\varphi^{-h}(\varphi^h(\theta, x))\| \leq (d - \alpha)^{-h} + (1 + C_3)C_2 \exp(-(2c - \epsilon)h), \tag{31}$$

At this point we can specify the choice of the integer p: we take $p \geq 1$ large enough in such that the induced map ϕ associated to φ is an expanding map in the sense of the definition given in Subsection 1.1 (recall that $h \geq p$). Note that this choice of p only depends on the expansion rates of the maps $\hat{g}$ and $\hat{f}$, thus p may be taken independent of the map $\varphi \in \mathcal{N}$.

5 Uniformity conditions

An important feature of this construction, cf. [V1, Section 2.5], is that it remains valid for any map ψ close enough to φ, with uniform bounds on the measure of the exceptional sets $E_n(\psi)$:

$$m(E_n(\psi)) \leq Ce^{-\gamma\sqrt{n}} \quad \text{for every } n \geq 1$$

where C and γ may be taken uniform (that is, constant) in a whole C^3 neighbourhood of φ. Let us explain this last point, since it is not explicitly addressed in the previous papers. One consequence is that Proposition 4.4 holds in the whole open set $\mathcal{N}$, with uniform constants C_0 and γ_0 (recall Remark 4.5).

As explained in [V1, Section 2.5], it follows from the methods of [HPS] that any map ψ sufficiently close to φ admits a unique invariant central foliation $\mathcal{F}^c$ of $S^1 \times I$ by smooth curves uniformly close to vertical segments. This is because the vertical foliation is invariant and normally expanding for the map φ. In addition, the space of leaves of $\mathcal{F}^c$ is homeomorphic to a circle, and the map induced by ψ in it topologically conjugate to $\hat{g}$. The previous analysis can then be carried out in terms of the expansion of ψ along this central foliation $\mathcal{F}^c$. More precisely, $|\partial_x f(\theta, x)|$ is replaced by

$$|\partial_c f(\theta, x)| \equiv |D\psi(\theta, x)v_c(\theta, x)|,$$

where $v_c(\theta, x)$ represents a norm 1 vector tangent to the foliation at (θ, x). The previous observations imply that v_c is uniformly close to $(0, 1)$ if ψ is close to φ. Moreover, cf. [V1, Section 2.5], it is no restriction to suppose $|\partial_c f(\theta, 0)| \equiv 0$ (incidentally, this is the only place where we need our maps to be C^3), so that $\partial_c f(\theta, x) \approx |x|$, as in the unperturbed case; recall (16). Defining $r_j(\theta, x)$ and $E_n = E_n(\psi)$ in the same way as before, cf. (18), we obtain an analog of (22).

$$\|D\psi^n(\theta, x)v_c(\theta, x)\| = \prod_{j=0}^{n-1} |\partial_c f(\theta_j, x_j)| \geq C_2^{-1} \exp\left(4cn - \sum_{j \in G_n(\theta,x)} r_j(\theta, x)\right),$$

for every (θ, x). We define $E_n(\psi)$ in the same way as $E_n = E_n(\varphi)$, recall (23), and then

$$\|D\psi^n(\theta, x)v_c(\theta, x)\| \geq e^{cn} \quad \text{for all } (\theta, x) \in (S^1 \times I) \setminus E_n.$$

The arguments in [V1, Section 2.4] apply with $|\partial_c f|$ in the place of $|\partial_x f|$, proving that the Lebesgue measure of $E_n(\psi)$ satisfies the bound in (24). The constants C and γ produced by these arguments depend only on α, which is fixed, and on estimates obtained in the previous sections of that paper. So, to see that these constants are indeed uniform in a neighbourhood of φ, it suffices to check that the same is true for those preparatory estimates. This is clear in the case of the results of Section 2.1 (Lemmas 2.1 and 2.2, and Corollary 2.3), because they only involve one iterate of the map. Let us point out that the definition of admissible curve for ψ is just the same as for the unperturbed map φ. A continuity argument can be applied also to Section 2.2, but it is more subtle. The key observation is that, although the statements of Lemmas 2.4 and 2.5 involve an unbounded number of iterates, their proofs are based on analyzing bounded stretches of orbits. Finally, the results in Section 2.4 (Lemmas 2.6 and 2.7), involve not more than $M \approx \log(1/\alpha)$ iterates. So, once more by continuity, their estimates remain valid in a neighbourhood of φ. We have concluded the observation that the bound (24) on the Lebesgue measure of the exceptional set E_n holds uniformly in a neighbourhood of the map.

Now we are able to show that conditions (U1)-(U3) are satisfied by every element of $\mathcal{N}$, as long as we take the open set $\mathcal{N}$ sufficiently small.

(U1) The construction of the partition that leads to the map h_φ is based on the itineraries of points through the horizontal strips $S^1 \times I_*$ with $I_* \in \mathcal{Q}$, according to the expanding behaviour of the iterates of φ at hyperbolic returns. Since these hyperbolic returns depend only on a finite number of iterates of the map φ, by continuity, we can perform the construction of the partition in such a way that for some fixed integer N the Lebesgue measure of $\{h_\varphi = j\}$ varies continuously with the map φ for $j \leq N$.

(U2) We have for every $\varphi \in \mathcal{N}$ and any fixed large integer $N \geq 1$

$$\| \sum_{j \geq N} \chi_{\{h_\varphi > j\}} \|_q^q = \sum_{j \geq N} j^q m\big(\{h_\varphi = N + j\}\big) \leq \sum_{j \geq N} j^q m\big(\{h_\varphi \geq N + j\}\big)$$

Taking into account Proposition 4.4 we deduce that

$$\| \sum_{j \geq N} \chi_{\{h_\varphi > j\}} \|_q^q \leq \sum_{j \geq N} j^q C_0 e^{-\gamma_0 \sqrt{N+j}}$$

which can be made uniformly small if N is taken sufficiently large.

(U3) The constant K that bounds the distortion is given by [A, Proposition 4.2], which may be taken uniform in the whole $\mathcal{N}$.

The constant σ is given by (31), which may be taken uniformly smaller than one for every $\varphi \in \mathcal{N}$.

It follows from [A, Corollary 3.3] that β is uniformly bounded away from zero, as long as α and the open set $\mathcal{N}$ are taken small enough.

Finally, the proof [A, Proposition 3.8] shows that ρ may be taken bounded from below by a constant only depending on α (see also Step 1 in the proof of Proposition 6.1 below).

Remark 5.1. *The following comments are meant to help clarify the presentation of [V1, Lemma 2.6], they are not used elsewhere in the present work. We refer the reader to [V1] for the setting and notations. The conclusion of the lemma is contained in Corollary 2.3 of [V1], when r is large enough so that $|J(r-2)| \ll \sqrt{\alpha}$. In particular, it is enough to prove the lemma for values of r smaller than $(1/2 + 2\eta)\log(1/\alpha)$ (and larger than $(1/2 - 2\eta)\log(1/\alpha)$, cf. statement of the lemma). By definition, the function $k(r)$ defined in page 73 of [V1] can not exceed $M \approx \log(1/\alpha)$. So, the arguments at the end of page 73 actually prove that either $k(r) \geq const\, r$, or $k(r) \approx M$. However, under the above restriction on r, the latter possibility also implies $k(r) \geq const\, r$. In this way, the conclusion of the lemma follows in all the cases.*

6 Topological mixing

In this section we prove that the maps in $\mathcal{N}$ are topologically mixing. For that, let us start by giving a good description of the attractor of a map $\varphi \in \mathcal{N}$ inside the forward invariant region $S^1 \times I$. We claim that the attractor of φ inside this invariant region, defined as the intersection

$$\Lambda = \bigcap_{n \geq 0} \varphi^n(S^1 \times I)$$

of all forward images of $S^1 \times I$, is just $\Lambda = \varphi^2(S^1 \times I)$, as long as the interval I is properly chosen. Indeed, for the one-dimensional map q we may take $I \subset (-2, 2)$ in such a way that $q(I)$ is contained in the interior of I and

$$\bigcap_{n \geq 0} q^n(I) = q^2(I).$$

For each $\theta \in S^1$ the map $\varphi \mid (\{\theta\} \times I)$ may be thought of as a one-dimensional map from I into itself, close to q. Then, our claim follows by continuity.

Before we go into the main proposition of this section, let us remark that inequality (31) shows in particular that the diameter of the partition $\mathcal{R}$ of $S^1 \times I$, defined as

$$\operatorname{diam}(\mathcal{R}) = \sup \{\operatorname{diam}(R) : R \in \mathcal{R}\},$$

is small when p is large. Thus, taking arbitrarily large integers p we may define a sequence of partitions $(\mathcal{S}_n)_n$ in $S^1 \times I$ in such a way that

$$\lim_{n \to +\infty} \operatorname{diam}(\mathcal{S}_n) = 0.$$

Moreover, for each $n \geq 1$ we may define a map $h_n : \mathcal{S}_n \to \mathbb{Z}^+$ in the same way as we did for $h : \mathcal{R} \to \mathbb{Z}^+$.

Proposition 6.1. *There is integer $M = M(\alpha)$ such that for every $n \geq 1$ and $\omega \times J \in \mathcal{S}_n$,*

$$\left| \varphi^{h_n(\omega \times J)+M}(\omega \times J) \right| = \Lambda.$$

Proof. The proof of this proposition will be made in four steps. In the first one we prove that the image of $\omega \times J \in \mathcal{S}_n$ by $\varphi^{h_n(\omega \times J)}$ has height bounded from below by a constant of order $\alpha^{1-2\eta}$. In the second step we prove that a vertical segment of order $\alpha^{1-2\eta}$ becomes, after a finite number of iterates, an interval with height bounded from below by a constant of order $\sqrt{\alpha}$. In the third step we show that iterating vertical segments of order $\sqrt{\alpha}$ they become segments with length bounded from below by a constant not depending on α. In the final step we make use of the properties of the quadratic map q to obtain the result.

Step 1. *There is a constant* $\Delta_1 > 0$ *such that for every* $n \geq 1$, $\omega \times J \in \mathcal{S}_n$ *and* $\theta \in \omega$,

$$\left|\varphi^{h_n(\omega\times J)}(\{\theta\} \times J)\right| \geq \Delta_1 \alpha^{1-2\eta}.$$

This follows from [A, Proposition 3.8]. It is easy to check that the estimate in Lemma 4.6 above in the place of the estimate of [A, Lemma 3.7] is enough to yield this dependence on α.

Step 2. *There is a constant* $\Delta_2 > 0$ *and an integer* $M_1 = M_1(\alpha)$ *such that if* $J \subset I$ *is an interval with* $|J| \geq \Delta_1\alpha^{1-2\eta}$*, then for every* $\theta \in S^1$

$$\left|\varphi^{M_1}(\{\theta\} \times J)\right| \geq \Delta_2\sqrt{\alpha}.$$

We start by remarking that we may assume that J intersects $(-\sqrt{\alpha}, \sqrt{\alpha})$. (If this is not the case, we take $R \geq 1$ the first integer for which $\varphi^R(\{\theta\} \times J)$ intersects $(-\sqrt{\alpha}, \sqrt{\alpha})$. It follows from Lemma 4.2 that $R \leq C\log(1/\alpha)$ for some $C > 0$, and so we may start with $\varphi^R(\{\theta\} \times J)$). Take J_1 a subinterval of J such that

$$J_1 \subset (-2\sqrt{\alpha}, 2\sqrt{\alpha}), \quad J_1 \cap \left(-\frac{\Delta_1}{4}\alpha^{1-2\eta}, \frac{\Delta_1}{4}\alpha^{1-2\eta}\right) = \emptyset \quad \text{and} \quad |J_1| \geq \frac{\Delta_1}{4}\alpha^{1-2\eta}.$$

It follows from Lemma 4.1 that for $N = N(\alpha)$

$$\left|\varphi^N(\{\theta\} \times J_1)\right| \geq \frac{\Delta_1}{4}\alpha^{1-2\eta}\ \alpha^{-1+\eta}|J_1| \geq \frac{\Delta_1}{4}\alpha^{1-3\eta}. \tag{32}$$

Let $R \geq 1$ be the first integer for which $\varphi^{N+R}(\{\theta\} \times J_1)$ intersects $(-\sqrt{\alpha}, \sqrt{\alpha})$. It follows from Lemma 4.2 that

$$\left|\varphi^{N+R}(\{\theta\} \times J_1)\right| \geq \frac{C_2\Delta_1}{4}\alpha^{1-3\eta}.$$

Now we proceed inductively and prove that for each $l \geq 1$ there is an interval $J_l \subset J$ and a sequence of integers $3 = k_1 < k_2 < \cdots < k_l$ for which

$$\left|\varphi^{lN+(l-1)R}(\{\theta\} \times J_l)\right| \geq \frac{C_2^{l-1}\Delta_1}{4^l}\alpha^{1-k_l\eta}.$$

We stop when $1 - k_l\eta \leq 1/2$, and take $\Delta_2 = C_2^{l-1}\Delta_1/4^l$ and $M_1 = lN + (l-1)R$ (note that l only depends on η which does not depend on α).

Step 3. *There is a constant $\Delta_3 > 0$ and an integer $M_2 = M_2(\alpha)$ such that if $J \subset I$ is an interval with $|J| \geq \Delta_2\sqrt{\alpha}$, then for every $\theta \in S^1$*

$$\left|\varphi^{M_2}(\{\theta\} \times J)\right| \geq \Delta_3.$$

Arguing as in Step 2 we can prove an analog to (32)

$$\left|\varphi^N(\{\theta\} \times J)\right| \geq \frac{\Delta_2}{4}\sqrt{\alpha}\, \alpha^{-1+\eta}|J| \geq \frac{\Delta_2}{4}\alpha^\eta.$$

Letting $R \geq 1$ be the first integer for which $\varphi^{N+R}(\{\theta\} \times J)$ intersects $[-\sqrt{\alpha}, \sqrt{\alpha}]$ we have

$$\left|\varphi^{N+R}(\{\theta\} \times J)\right| \geq \begin{cases} C_2\tau^R\alpha^\eta & \text{if } \varphi^{N+R}(\{\theta\} \times J) \subset (S^1 \times [-\delta, \delta]), \\ \delta - \alpha^\eta & \text{otherwise.} \end{cases}$$

In both cases we have that the x-component of $\varphi^{N+R}(\{\theta\} \times J)$ contains some interval L not intersecting $(-\sqrt{\alpha}, \sqrt{\alpha})$ with an end point at $-\sqrt{\alpha}$ or $\sqrt{\alpha}$ and whose length is at least $C_2\alpha^\eta$.

From now on we use C to denote any large constant depending only on the map q. Take $l \geq 1$ the smallest integer for which $z = q^l(0)$ is a periodic point for q and let $k \geq 1$ be its period. Denote $\rho^k = |(q^k)'(z)|$ and note that by [S] we must have $\rho > 1$. Fix $\rho_1, \rho_2 > 0$ with $\rho_1 < \rho < \rho_2$ and $\rho_1 > \rho_2^{1-\eta/2}$, and take $\delta_0 > 0$ small enough in order to obtain

$$\rho_1^k < \prod_{j=0}^{k-1} \left|\partial_x f(\varphi^i(\sigma, y))\right| < \rho_2^k, \quad \text{whenever} \quad |y - z| < \delta_0$$

(and α sufficiently small). Since 0 is pre-periodic for q, there exists some constant $\epsilon > 0$ such that $|q^j(0)| > \epsilon$ for every $j > 0$. From this we deduce

$$|x_1|, \ldots, |x_{l-1}| > \frac{\epsilon}{2}, \quad \text{whenever} \quad x \in L, \tag{33}$$

as long as α is sufficiently small. By (16) and (17) we may write $\partial_x f(\theta, x) = x\psi(\theta, x)$ with $|\psi + 2| < \alpha$ at every point $(\theta, x) \in S^1 \times I$. This, together with (33), gives for every $x \in L$

$$\prod_{j=0}^{l-1} |\partial_x f(\theta_j, x_j)| \geq \frac{1}{C}|x|, \tag{34}$$

and so we have for some $x \in L$

$$\left|\varphi^{N+R+l}(\{\theta\} \times J)\right| \geq \prod_{j=0}^{l-1} |\partial_x f(\theta_{N+R+j}, x_j)| \cdot |L| \geq \frac{1}{C}|x|\alpha^{\eta} \geq \frac{1}{C}\alpha^{1/2+\eta}.$$

For $(\theta, x) \in S^1 \times I$ and $i \geq 0$ we denote $d_i = |x_{l+ki} - z|$. Take $\delta_1 > 0$ and α sufficiently small in such a way that

$$|x| < \delta_1 \ \Rightarrow \ d_0 \leq Cx^2 + C\alpha < \delta_0.$$

If (θ, x) and $i \geq 1$ are such that $|x| < \delta_1$ and $d_0, \ldots, d_{i-1} < \delta_0$, then $d_i \leq \rho_2^k d_{i-1} + C\alpha$ and so, inductively,

$$d_i \leq (1 + \rho_2^k + \cdots + \rho_2^{k(i-1)})C\alpha + \rho_2^{ki} d_0 \leq \rho_2^{ki}(C\alpha + Cx^2).$$

In particular for the points $x = \pm\sqrt{\alpha}$ we have $d_i \leq \rho_2^{ki} C\alpha$. Now we take $N_0 = N_0(\alpha) \geq 1$ the smallest integer for which $\rho_2^{kN_0(\alpha)} C\alpha \geq \delta_0/2$. This choice of N_0 implies

$$d_i < \delta_0/2 \quad \text{for} \quad i = 0, \ldots, N_0 - 1. \tag{35}$$

Now we consider the following two possible cases:

1. *$\varphi^{l+ki}(\{\theta_{N+R}\} \times L) \subset (z - \delta_0, z + \delta_0)$ for every $i \in \{0, \cdots, N_0 - 1\}$.*

   This implies that

$$\begin{aligned} \left|\varphi^{l+kN_0}(\{\theta_{N+R}\} \times L)\right| &\geq \rho_1^{kN_0} \left|\varphi^{l}(\{\theta_{N+R+kN_0}\} \times L)\right| \\ &\geq \rho_2^{(1-\eta/2)kN_0} \frac{1}{C}\alpha^{1/2+\eta} \\ &\geq \frac{1}{C}\alpha^{-1+\eta/2}\frac{1}{C}\alpha^{1/2+\eta} \\ &\geq \frac{1}{C}\alpha^{-1/2+3\eta/2} \\ &\gg \frac{1}{C} \end{aligned}$$

   (recall that $\eta < 1/3$).

2. *$\varphi^{l+ki}(\{\theta_{N+R}\} \times L) \not\subset (z - \delta_0, z + \delta_0)$ for some $i \in \{0, \cdots, N_0 - 1\}$.*

   Since $d_i \leq \delta_0/2$, it follows that

$$\left|\varphi^{l+ki}(\{\theta_{N+R}\} \times L)\right| \geq \delta_0 - \delta_0/2 = \delta_0/2.$$

In both cases we have some integer $N_1 \leq N_0$ for which

$$\left|\varphi^{l+kN_1}(\{\theta_{N+R}\} \times L)\right| \geq \frac{1}{C}.$$

Thus, taking $M_2 = N + R + l + kN_1$ we have

$$\left|\varphi^{M_2}(\{\theta\} \times J)\right| \geq \frac{1}{C}.$$

Step 4. *There is an integer* $M_3 = M_3(\alpha)$ *such that if* $J \subset I$ *is an interval with* $|J| \geq \Delta_3$*, then for every* $\theta \in S^1$

$$\left|\varphi^{M_3}(\{\theta\} \times J)\right| = (\{\theta_{M_3}\} \times I) \cap \Lambda.$$

Since we are taking a_0 a Misiurewicz parameter, it follows that the pre-orbit of the repelling fixed point P of q is dense. So, there is some integer $n_1(\Delta_3) \geq 1$ such that for every interval $J \subset I$ with $|J| \geq \Delta_3$ we have that $q^{n_1(\Delta_3)}(J)$ covers a neighbourhood of P with a definite size (depending only on Δ_3). By a finite number of iterates $n_2(\Delta_3)$ we transform this neighbourhood in the whole interval $q^2(I)$. Hence, taking $M_3 = n_1(\Delta_3) + n_2(\Delta_3) + 1$ we have by continuity $\varphi^{M_3}(\{\theta\} \times J) = (\{\theta_{M_3}\} \times I) \cap \Lambda$ for sufficiently small α.

Now it suffices to take $M(\alpha) = M_1(\alpha) + M_2(\alpha) + M_3(\alpha)$ and we complete the proof of Proposition 6.1. ☐

Now we are in conditions to prove that the maps $\varphi \in \mathcal{N}$ are topologically mixing. Let A be an open set in $S^1 \times I$. Since the partitions $\mathcal{S}_n$ have diameters converging to zero when n goes to infinity, there must be some $n \geq 1$ and $S \in \mathcal{S}_n$ for which $S \subset A$. Hence, taking $n(A) = h_n(S) + M$ (M given by Proposition 6.1) it follows from Proposition 6.1 that $\varphi^{n(A)}(A) = \Lambda$.

7 Ergodicity

In this section we prove the ergodicity of the maps $\varphi \in \mathcal{N}$ with respect to Lebesgue measure. We start by proving some auxiliary results.

Lemma 7.1. *Let* B *be a Borel subset of* $S^1 \times I$ *such that* $\varphi^{-1}(B) = B$.

1. *If* $m(B \cap \Lambda) = 0$ *then* $m(B) = 0$.

2. If $\varphi^n(R) = \Lambda$ for some $n \geq 1$ and $R \subset S^1 \times I$, then $B \cap \Lambda \subset \varphi^n(B \cap R)$.

Proof. Since we have $\Lambda = \varphi^2(S^1 \times I)$, it follows that

$$A = \varphi^{-2}(A) = \varphi^{-2}\left(A \cap \varphi^2(S^1 \times I)\right) = \varphi^{-2}\left(A \cap \Lambda\right).$$

Thus, if $m(A \cap \Lambda) = 0$, then $m(A) = 0$, and so we have proved the first item.

Now let $x \in B \cap \Lambda$. Since $\varphi^n(R) = \Lambda$, there must be some $z \in R$ for which $\varphi^n(z) = x$. On the other hand, since $\varphi^{-1}(B) = B$ we have $\varphi^{-n}(B) = B$, and so z belongs to B. Hence $x \in \varphi^n(B \cap B)$. □

Now we prove a general result that will play an important role in the proof of the ergodicity with respect to Lebesgue measure.

Proposition 7.2. *Let X be a metric space, μ be a Borel measure on X, and $\mathcal{P} = \{P_1, \ldots, P_r\}$ be a partition of X into Borel subsets. Assume that $(\mathcal{S}_n)_{n\geq 1}$ are partitions of X such that diam$\,(\mathcal{S}_n) \to 0$ when $n \to \infty$. Then, for each $n \geq 1$ there is a partition $\{Q_1^n, \ldots, Q_r^n\}$ of X such that for $i = 1, \ldots, r$*

1. *Q_i^n is a union of atoms of $\mathcal{S}_n$.*
2. *$\lim_{n\to\infty} \mu(Q_i^n \triangle P_i) = 0$.*

Proof. Take an arbitrary $\epsilon > 0$. Since μ is a regular measure, there are compact sets $K_1, \ldots, K_r \subset X$ with

$$K_i \subset P_i \quad \text{and} \quad \mu(P_i \setminus K_i) < \epsilon$$

for $i = 1, \ldots, m$. Let

$$\delta = \inf_{i \neq j} d(K_i, K_j) > 0$$

and take $n_0 \geq 1$ such that

$$\operatorname{diam}(\mathcal{S}_n) < \delta/2 \quad \text{for} \quad n \geq n_0.$$

For $n \geq n_0$ we divide $\mathcal{S}_n$ into r groups, whose unions we call $Q_1^n, \ldots, Q_r^n$, by putting $S \subset Q_i^n$ if $S \in \mathcal{S}_n$ intersects K_i. Note that each $S \in \mathcal{S}_n$ intersects at most one K_i. If it does not intersect any K_i, then we include it arbitrarily in some Q_i^n. We have

$$\begin{aligned}
\mu(Q_i^n \triangle P_i) &= \mu(Q_i^n \setminus P_i) + \mu(P_i \setminus Q_i^n) \\
&\leq \mu(X \setminus \cup_{i=1}^r K_i) + \mu(P_i \setminus K_i) \\
&\leq (r+1)\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary and r is fixed, we have proved the result. □

The following corollary may be thought as a similar result to the Lebesgue density theorem for balls in Euclidean spaces.

Corollary 7.3. *Let B be a Borel subset of X with $\mu(B) > 0$. Then, for every $\epsilon > 0$ there is an integer $n_\epsilon \geq 1$ such that for each $n \geq n_\epsilon$ there is $S \in \mathcal{S}_n$ with*

$$\mu(B^c \cap S) < \epsilon \mu(S).$$

Proof. Assume by contradiction that there are $\epsilon_0 > 0$ and a sequence of integers $(n_k)_{k\geq 1}$ going to $+\infty$ such that

$$\forall\, k \geq 1\; \forall\, S_{n_k} \in \mathcal{S}_{n_k} \;:\; \mu(B^c \cap S_{n_k}) \geq \epsilon_0 \mu(S_{n_k}). \tag{36}$$

We know from Lemma 7.2 that for every $k \geq 1$ there is a partition $\{Q_1^{n_k}, Q_2^{n_k}\}$ of X such that $Q_1^{n_k}$, $Q_2^{n_k}$ are unions of atoms of $\mathcal{S}_{n_k}$, and

$$\lim_{k\to+\infty} \mu(Q_1^{n_k} \triangle B) = 0, \qquad \lim_{k\to+\infty} \mu(Q_2^{n_k} \triangle B^c) = 0.$$

Take $\epsilon = \epsilon_0 \mu(B)/(1+\epsilon_0)$ and $k_0 \geq 1$ sufficiently large in order to

$$\mu(Q_1^{n_{k_0}} \triangle B) < \epsilon \quad \text{and} \quad \mu(Q_2^{n_{k_0}} \triangle B^c) = \epsilon. \tag{37}$$

Letting $\mathcal{S}_{n_{k_0}} = \{S_i\}_{i\in\mathbb{N}}$, we know that there is some $\mathbb{I} \subset \mathbb{N}$ for which

$$Q_1^{n_{k_0}} = \bigcup_{i\in\mathbb{I}} S_i \quad \text{and} \quad Q_2^{n_{k_0}} = \bigcup_{i\in\mathbb{N}\setminus\mathbb{I}} S_i.$$

From (36) we have in particular

$$\mu(B^c \cap S_i) \geq \epsilon_0 \mu(S_i)$$

for every $i \in \mathbb{I}$, and so summing over all $i \in \mathbb{I}$ we have

$$\mu(B^c \cap Q_1^{n_{k_0}}) \geq \epsilon_0 \mu(Q_1^{n_{k_0}}). \tag{38}$$

Finally, from (37) and (38) we get

$$\epsilon > \mu(B^c \cap Q_1^{n_{k_0}}) \geq \epsilon_0 \mu(Q_1^{n_{k_0}}) \geq \epsilon_0 \left(\mu(B) - \mu(B \setminus Q_1^{n_{k_0}})\right) > \epsilon_0(\mu(B) - \epsilon),$$

which gives a contradiction (recall our choice of ϵ). ☐

It will be useful to have the following distortion result, whose role is essentialy to state the non dependence on the partition $\mathcal{S}_n$ of the constant in Proposition 4.3 .

Proposition 7.4. *There is some constant $\Delta > 1$ such that for every $n \geq 1$, $S \in \mathcal{S}_n$ and $(\theta, x), (\sigma, y) \in S$ we have*

$$\frac{1}{\Delta} \leq \left| \frac{J_n(\theta, x)}{J_n(\sigma, y)} \right| \leq \Delta,$$

where J_n is the Jacobian of $\varphi^{h_n(S)}|S$.

Proof. We observe that the constant in [A, Proposition 4.2] that bounds the distortion does not depend on the integer p that we have used for starting the construction of the partition (it essentially depends on the expansion rates of the maps g and f). This means that the same proof of Proposition 4.3 applies to this situation with the constant $\Delta > 0$ not depending on n. $\square$

Now we are in conditions to prove the ergodicity of the maps $\varphi \in \mathcal{N}$ with respect to the Lebesgue measure. Let B be a Borel subset of $S^1 \times I$ with $\varphi^{-1}(B) = B$ and having positive Lebesgue measure. We need to prove that the Lebesgue measure of $B^c = (S^1 \times I) \setminus B$ is equal to zero. From the first item of Lemma 7.1 it suffices to prove that $m(B^c \cap \Lambda) = 0$. Take any $\epsilon > 0$ small. It follows from Corollary 7.3 that there are $n_\epsilon \geq 1$ and $S \in \mathcal{S}_{n_\epsilon}$ for which

$$m(B^c \cap S) < \epsilon m(S).$$

Letting $h = h_{n_\epsilon}(S)$ we have from Proposition 6.1 that $\varphi^{h+M}(S) = \Lambda$. Thus, applying the second item of Lemma 7.1 we obtain

$$m(B^c \cap \Lambda) \leq m\left(\varphi^{h+M}(B^c \cap S)\right).$$

Fixing some $(\theta_0, x_0) \in S$, we deduce from Proposition 7.4

$$\begin{aligned} m\left(\varphi^{h}(B^c \cap S)\right) &= \int_{B^c \cap S} |J_{n_\epsilon}(\theta, x)|\, dm(\theta, x) \\ &\leq \Delta \int_{B^c \cap S} |J_{n_\epsilon}(\theta_0, x_0)|\, dm(\theta, x) \\ &\leq \Delta\, |J_{n_\epsilon}(\theta_0, x_0)|\, m\left(B^c \cap S\right). \end{aligned}$$

Similarly we prove that

$$m\left(\varphi^{h}(S)\right) \geq \frac{1}{\Delta} |J_{n_\epsilon}(\theta_0, x_0)|\, m\left(S\right).$$

Hence

$$m\left(\varphi^h(B^c \cap S)\right) \leq \frac{m\left(\varphi^h(B^c \cap S)\right)}{m\left(\varphi^h(S)\right)} \leq \Delta^2 \frac{m\left(B^c \cap S\right)}{m\left(S\right)} \leq \Delta^2 \epsilon,$$

which finally gives

$$m\left(\varphi^{h+M}(B^c \cap S)\right) \leq (d+\alpha)^M 4^M m\left(\varphi^h(B^c \cap S)\right) \leq (d+\alpha)^M 4^M \Delta^2 \epsilon.$$

Since this holds for ϵ arbitrarily small and M is fixed, the proof is complete.

References

[A] J. F. Alves, *SRB measures for non-hyperbolic systems with multidimensional expansion*, to appear in Ann. Sci. de l'ENS.

[ABV] J. F. Alves, C. Bonatti, M. Viana, *SRB measures for partially hyperbolic systems with mostly expanding central direction*, to appear in Invent. Math.

[AP] A. Andronov and L. Pontryagin, *Systèmes grossiers*, Dokl. Akad. Nauk. USSR, 14 (1937), 247–251.

[BV] C. Bonatti, M. Viana, *SRB measures for partially hyperbolic systems with mostly contracting central direction*, to appear Israel J. Math.

[D] D. Dolgopyat, *On dynamics of mostly contracting diffeomorphisms*, preprint 1998.

[G] E. Giusti, *Minimal surfaces and functions of bounded variation*, Birkäuser Verlag, Basel-Boston, Mass., 1984.

[HPS] M. Hirsch, C. Pugh, M. Shub, *Invariant manifolds*, Lect. Notes in Math. 583, Springer Verlag, 1977.

[LY] A. Lasota and J.A. Yorke, *On the existence of invariant measures for piecewise monotonic maps*, Trans. Amer. Math. Soc. **186** (1973), 481-488.

[PS] J. Palis and S. Smale, *Structural stability theorems*, in *Global Analysis*, Proc. Sympos. Pure Math. XIV (Berkeley 1968), Amer. Math. Soc., 223–232, 1970.

[PT] J. Palis and F. Takens, *Hyperbolicity and sensitive-chaotic dynamics at homoclinic bifurcations*, Cambridge University Press, 1993.

[S] D. Singer, *Stable orbits and bifurcations of maps of the interval*, SIAM J. Appl. Math. **35** (1978), 260-267.

[V1] M. Viana, *Multidimensional nonhyperbolic attractors*, Publ. Math. IHES **85** (1997), 63-96.

[V2] M. Viana, *Stochastic dynamics of deterministic systems*, Lect. Notes XXI Braz. Math Colloq., IMPA, 1997.

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