ORNSTEIN-ZERNIKE THEORY FOR THE BERNOULLI BOND PERCOLATION ON $\mathbb{Z}^d$

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Abstract. We derive precise Ornstein-Zernike asymptotic formula for the decay of the two-point function $\mathbb{P}_p(0 \longleftrightarrow x)$ of the Bernoulli bond percolation on the integer lattice $\mathbb{Z}^d$ in any dimension $d \geqslant 2$, in any direction x and for any sub-critical value of $p < p_c(d)$.

Contents

1. Introduction and Results

1.1. Ornstein-Zernike theory.

The Ornstein-Zernike theory [15] gives a sharp asymptotic description of density correlation functions in classical fluids away from the critical point. In their original work Ornstein and Zernike clearly perceived the mathematical structure of the model and, as it often happens in the physical literature, gave a convincing derivation of their asymptotic

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result, assuming, though, the most formidable issue to be proven. We refer to Chapter 5 of [17] for a very clean and stimulating discussion of the physical background.
The abovementioned crucial property is a certain mass-gap condition (see Subsection 1.3), and the focal point of our work here is to establish it in the context of the sub-critical Bernoulli bond percolation on $\mathbb{Z}^d$, which, for the matter, could be thought as a spatially discretized model of fluids. An excellent reference for the percolation models is [10]. To set up notation, let $\{\eta(b)\}$ be a family of Bernoulli i.i.d random variables, indexed by the nearest neighbour bounds of the integer lattice $\mathbb{Z}^d$. We use $\mathbb{P}_p$ to denote the corresponding joint probability distribution

$$\mathbb{P}_p\left(\eta(b)=1\right) \;=\; p.$$

Given a realization η, we say that a bond b is open if $\eta(b)=1$, otherwise we shall call it closed. Two points $x,y\in\mathbb{Z}^d$ are said to be connected, $\{x\longleftrightarrow y\}$, if there exists a chain of open bonds leading from x to y. The event $\{x\longleftrightarrow y\}$ is, obviously, measurable and, due to the symmetries of the lattice;

$$\mathbb{P}_p\left(x\longleftrightarrow y\right) \;=\; \mathbb{P}_p\left(0\longleftrightarrow y-x\right).$$

In the fluid interpretation, the connectivity function $\mathbb{P}_p\left(x\longleftrightarrow y\right)$ is supposed to describe the truncated correlation function between the particle densities at x and y. Short range order of fluctuations corresponds, then, to the requirement that the influence from the origin does not propagate along the lattice:

$$\mathbb{P}_p\left(0\longleftrightarrow\infty\right) \;=\; 0, \tag{1.1.1}$$

where the event $\{0\longleftrightarrow\infty\}$ naturally means that the open cluster of the origin is infinite. The percolation threshold $p_c=p_c(d)$ is defined as

$$p_c \;=\; \sup\left\{p:\ (1.1.1)\text{ holds}\right\}.$$

We say that the Bernoulli bond percolation model on $\mathbb{Z}^d$ is sub-critical if $p<p_c(d)$.
A fundamental result by Menshikov [14] and Aizenman-Barsky [1] states that sub-criticality is always reinforced with strong decay properties of connectivities. Namely, in any dimension d;

$$\chi_d(p) \;\overset{\Delta}{=}\; \sum_{x\in\mathbb{Z}^d}\mathbb{P}_p\left(0\longleftrightarrow x\right) \;<\; \infty, \tag{1.1.2}$$

whenever $p<p_c(d)$. An application of the BK inequality (c.f. the proof of Hammersley's Theorem 5.1 in [10]) shows that (1.1.2) actually implies exponential decay of connectivities: for every $p<p_c$ there exists $c_1=c_1(p)>0$, such that

$$\mathbb{P}_p\left(0\longleftrightarrow x\right) \;\leqslant\; \mathrm{e}^{-c_1\|x\|}. \tag{1.1.3}$$

On the other hand, the FKG property of the Bernoulli bond percolation implies that the inverse correlation length ξ_{p};

$$\xi_{\mathrm{p}}(x) \;\overset{\Delta}{=}\; -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\,0\longleftrightarrow[nx]\right) \qquad\text{or}\qquad \mathbb{P}_p\left(\,0\longleftrightarrow[x]\right)\asymp\mathrm{e}^{-\xi_{\mathrm{p}}(x)}, \tag{1.1.4}$$

is always defined and, moreover, is a finite, convex and homogeneous of order one function on $\mathbb{R}^d$. By the sub-additivity argument, which again follows from the FKG property of $\mathbb{P}_p$;

$$\mathbb{P}_p\left(\,0\longleftrightarrow x\right) \;\leqslant\; \mathrm{e}^{-\xi_{\mathrm{p}}(x)}, \tag{1.1.5}$$

for every $x\in\mathbb{Z}^d$. Thus, the Hammersley's estimate (1.1.3) asserts that for every $p<p_c$, the inverse correlation length ξ_{p} is a strictly positive function on $\mathbb{R}^d\setminus\{0\}$. In other words, for sub-critical percolation models the inverse correlation length ξ_{p} captures the non-trivial leading asymptotics of decay of point-to-point connectivities $\mathbb{P}_p\left(\,0\longleftrightarrow x\right)$ on the logarithmic scale.

In this paper we derive a rigorous version of the Ornstein-Zernike theory which gives precise asymptotic description of connectivities up to the zero-order terms.

Theorem A. *Let $d \geqslant 2$ and $p < p_c(d)$. Then, uniformly in $x \in \mathbb{Z}^d$, $\|x\| \to \infty$;*

$$\mathbb{P}_p\left(0 \longleftrightarrow x\right) \;=\; \frac{\Psi_p(\vec{\mathfrak{n}}(x))}{\sqrt{(2\pi\|x\|)^{d-1}}} e^{-\xi_p(x)}\left(1+o(1)\right), \tag{1.1.6}$$

where $\vec{\mathfrak{n}}(x)$ is the unit vector in the direction of x and Ψ_p is a positive real analytic function on $\mathbb{S}^{d-1}$.

Remark. The explicit expression for Ψ_p will be given in Subsection 3.5. The coefficient $\sqrt{(2\pi)^{d-1}}$ in the denominator of the pre-factor is, of course, superfluous - we put it there only in order to stress that the result is a local limit-type theorem for connectivities. □

The relevant local limit behaviour will be read from peculiar renewal structures of the probabilities $\mathbb{P}_p\left(0 \longleftrightarrow x\right)$. The corresponding analytic properties of multi-dimensional moment generating functions already lie in the heart of the original paper [15]. In our setup we follow [7], where an analog of Theorem A has been established for on-axis directions of x. The lattice symmetries, therefore, played an essential role in the latter work. In order to treat off-axis directions we rely on general results on the local limit structure of multi-dimensional renewal arrays, as developed in [11].

As we have already mentioned, the whole Ornstein-Zernike theory hinges on a validity of a certain mass-gap condition. In the case of self-avoiding walks [11] such condition happens to be technically insensitive to various tilting of connectivities needed to explore decays in off-axis directions. . Thus it could be verified by an almost literal application of "on-axis" methods, which have been earlier developed in [8] and [13].

Contrary to this, in the case of percolation the verification of the on-axis mass-gap condition already required tedious computation (see Section 5 of [7]); an introduction of an additional tilt would have complicated the method employed there beyond reason.

A very different approach to the problem of refining sub-additive bounds on the connectivity-type functions has been developed in a series of papers by Alexander [2],[3],[4]. The renormalization ideas he has introduced in these works were not designed to furnish exact local limit descriptions as in Theorem A, but helped to illuminate the coarse-grained structure of the model to an extent of providing complementary lower bounds with correct order of pre-factors near the decay exponents. It should be mentioned that, unlike the refined renewal methods, Alexander's techniques require much less structure and apply for a large variety of other models.

Our main observation in this work could be now formulated as follows: the mass- gap condition in question is a coarse property in the realm of renormalization estimates. Furthermore, an appropriate modification of the renormalization ideas of Alexander leads to a relatively short proof. In other words, our version of the Ornstein-Zernike theory comprises two steps - on the first stage "heavy-duty" renormalization techniques are used to clean up the model from exponentially improbable events. Then the restructured model is tuned up with the help of more delicate local limit-type methods, based on the specific renewal properties of the Bernoulli bond percolation.

Notation. The constants $c_1, c_2, ... > 0$ are updated with each section. We use $\|\cdot\|_d$ and $\|\cdot\|_{d-1}$ to denote the Euclidean norm on $\mathbb{R}^d$ and, respectively, $\mathbb{R}^{d-1}$. Similarly $(\cdot,\cdot)_d$ and $(\cdot,\cdot)$ are used to denote the corresponding scalar products.

1.2. **Renewal structure of connectivities.** The aforementioned renewal properties could be recorded in several different ways. In fact, we give here two alternative proofs of Theorem A, which correspond two two different renewal setups. The first approach is a "parameterized" one, and it has been previously introduced in [7], and developed, to the state we are using it this work, in [11]. An advantage of the parameterized approach is that it contains an explicit treatment of the relevant $(d-1)$-dimensional local limit result. Also it illuminates several related geometric issues (see Subsection 1.4) in a natural way. Most importantly, the parameterized approach is well suited for studying various related problems, such as scaling analysis of percolation paths, or, in the case of two dimensions, the refined fluctuation analysis of phase boundaries. The corresponding results will appear elsewhere.

In the concluding Section 4 we work out an alternative "direct" proof. In this direct approach both the underlying $(d-1)$-dimensional local limit structure and the intrinsic geometry of shapes are implicit. However, the proof itself is technically more straightforward, and, moreover, it clearly illustrates the generality and the limitations of the method. In particular, whereas the independence of bond variables and, to a lesser extent, shift invariance are crucial for the whole approach, the nearest neighbour structure of the bonds and the lattice symmetries of bond variables play no role. We could have generalized our techniques to the case of shift invariant Bernoulli percolation on $\mathbb{Z}^d$ with, for example, finite range of connecting bonds. For the sake of the clarity of the exposition, however, we refrain from such an exercise.

The renormalization estimates, which are crucial for both approaches are developed in Section 2.

Let us proceed and set up the notation for the parameterized approach. It would be enough to prove the result for the lattice cone of $x \in \mathbb{Z}^d$ satisfying $x_1 \geqslant \|x\|_d/\sqrt{d}$. Hence the motivation for a parameterization: fix a unit axis direction $\vec{e}_1$ and write $\mathbb{Z}^d = \mathbb{Z} \times \mathbb{Z}^{d-1}$. Accordingly, we write $x = (n,k)$ for a point $x \in \mathbb{Z}^d$. We shall prove that for every $\alpha > 0$, the claim of Theorem A holds uniformly over x belonging to the cone $\mathcal{C}_\alpha$;

$$\mathcal{C}_\alpha \ \stackrel{\Delta}{=}\ \{x = (n,k) :\ \|k\|_{d-1} \ \leqslant\ \alpha n\}\,. \tag{1.2.1}$$

In view of the above parameterization, it happens to be convenient to establish an analog of Theorem A first for modified connectivities, the so called cylindrical ones. We follow [7] and [11] for the notation and general setup.
Define lattice $(d-1)$-dimensional hyperplanes $\mathcal{H}_n$; $n = 0, 1, \dots$, as

$$\mathcal{H}_n \ \stackrel{\Delta}{=}\ \left\{x \in \mathbb{Z}^d :\ x = (n,k)\right\}. \tag{1.2.2}$$

Similarly, given $m, n \in \mathbb{N}$, define lattice strip $\mathcal{S}_{m,n}$ as

$$\mathcal{S}_{m,n} \ \stackrel{\Delta}{=}\ \bigcup_{r=m}^{n} \mathcal{H}_r \ =\ \left\{x \in \mathbb{Z}^d :\ x = (r,k) \text{ with } m \ \leqslant\ r\ \leqslant\ n\right\}. \tag{1.2.3}$$

Definition. We say that a point $x \in \mathcal{H}_n$ is h-connected to the origin; $\left\{0 \stackrel{h}{\longleftrightarrow} x\right\}$, if

1. The point x is connected with 0 in the restriction of the percolation configuration to the strip $\mathcal{S}_{0,n}$.
2. Let $\mathbf{C}^n_{\{0,x\}}$ be the corresponding common open cluster of x and 0 in $\mathcal{S}_{0,n}$. Then

$$\mathbf{C}^n_{\{0,x\}} \cap \mathcal{H}_0 \ =\ \{0\} \qquad \text{and} \qquad \mathbf{C}^n_{\{0,x\}} \cap \mathcal{H}_n \ =\ \{x\}\,.$$

□

For every $n \in \mathbb{N}$ and every $x = (n,k) \in \mathcal{H}_n$, define;

$$h(n,k) \ \stackrel{\Delta}{=}\ \mathbb{P}_p\left(0 \stackrel{h}{\longleftrightarrow} (n,k)\right).$$

Of course, $h(n,k) \ <\ \mathbb{P}_p\left(0 \longleftrightarrow x\right)$, for every $n \in \mathbb{N}$ and $x = (n,k) \in \mathcal{H}_n$. On the other hand, it takes a soft argument (see Proposition 3.2 in [11]) to show that both $h(n,k)$ and $\mathbb{P}_p\left(0 \longleftrightarrow x\right)$ have the same leading asymptotics on the logarithmic scale: Namely,

$$\xi_p(1, k/n) \ +\ \frac{1}{n}\log h(n,k) \ =\ o(1), \tag{1.2.4}$$

uniformly (as $n \to \infty$) in $x = (n,k) \in \mathcal{C}_\alpha$, where $\mathcal{C}_\alpha$ is the cone defined in (1.2.1).
In fact, as we shall prove in Subsection 3.5, h-connectivities approximate the full ones in a much more stringent way:

Lemma 1.1. *Let d, $p < p_c(d)$ and $\alpha \in \mathbb{R}_+$ be fixed. Then, uniformly in $x = (n,k) \in \mathcal{C}_\alpha$;*

$$\mathbb{P}_p\left(0 \longleftrightarrow x\right) \;=\; \Phi_p\left(\vec{\mathfrak{n}}(x)\right) h(n,k)\left(1 + o(1)\right), \tag{1.2.5}$$

where Φ_p is a positive real analytic function on $\mathbb{S}_+^{d-1} \triangleq \left\{\vec{\mathfrak{n}} = (\mathfrak{n}_1, ..., \mathfrak{n}_d) \in \mathbb{S}^{d-1} :\ \mathfrak{n}_1 > 0\right\}$.

Thus, by the above lemma, and in view of the $\mathbb{Z}^d$-lattice symmetries of $\mathbb{P}_p$, it would be enough to restrict attention to the asymptotic behaviour of h-connectivities. For every $n \in \mathbb{N}$ these induce a probability distribution $\mathbb{P}_n$ on $\mathbb{Z}^d$ specified by the weights $h(n,k)$:

$$\mathbb{P}_n(k) \;\triangleq\; \frac{h(n,k)}{\sum_{j \in \mathbb{Z}^{d-1}} h(n,j)}.$$

Our study of the local limit properties of $\mathbb{P}_n$ is based on the specific renewal structure of h-connectivities, which, following [7], we proceed to describe:

Definition. Given a percolation configuration η on $\mathcal{S}_{m,n}$ and a point $x = (n,k) \in \mathcal{H}_n$ let us say that x is f-connected to the origin; $\left\{0 \overset{f}{\longleftrightarrow} x\right\}$, if:

1. x is h connected to the origin.
2. For every $m = 1, ..., n-1$;

$$\#\left(\mathbf{C}^n_{\{0,x\}} \cup \mathcal{H}_m\right) \;>\; 1.$$

$\square$

Define;

$$f(n,k) \;\triangleq\; \mathbb{P}_p\left(0 \overset{f}{\longleftrightarrow} x\right).$$

The event $\left\{0 \overset{h}{\longleftrightarrow} x\right\}$ depends only on the percolation configuration inside the strip $\mathcal{S}_{0,n}$. Using the disjoint decomposition of $\left\{0 \overset{h}{\longleftrightarrow} x\right\}$ with respect to the smallest index m satisfying $\#\left(\mathbf{C}^n_{\{0,x\}} \cup \mathcal{H}_m\right) = 1$, we, in view of the shift invariance of $\mathbb{P}_p$, obtain [7]:

$$\begin{cases} h(n,k) \;=\; \dfrac{1}{(1-p)^{2(d-1)}} \displaystyle\sum_{m=1}^{n} \sum_{l \in \mathbb{Z}^d} f(m,l) h(n-m, k-l) \\ h(0,k) \;=\; (1-p)^{2(d-1)} \delta_0(k) \end{cases} \tag{1.2.6}$$

Normalizing $\tilde{h} = h/(1-p)^{2(d-1)}$ and $\tilde{f} = f/(1-p)^{2(d-1)}$, we arrive to the usual $(d-1)$-dimensional renewal relation

$$\tilde{h}(n,k) \;=\; \sum_{m=1}^{n} \sum_{l \in \mathbb{Z}^d} \tilde{f}(m,l)\tilde{h}(n-m, k-l) \qquad \text{and} \qquad \tilde{h}(0,k) \;=\; \delta_0(k). \tag{1.2.7}$$

1.3. **Local limit results and separation of masses.** An appropriate general local limit theorem for d-dimensional renewal arrays (1.2.7) has been established in [11]. Given $\hat{t} \in \mathbb{R}^{d-1}$, define moment generating functions

$$\mathbb{H}_n(\hat{t}) \;\triangleq\; \sum_{k \in \mathbb{Z}^{d-1}} \tilde{h}(n,k) \mathrm{e}^{(\hat{t},k)_{d-1}} \;\text{ and }\; \mathbb{F}_n(\hat{t}) \;\triangleq\; \sum_{k \in \mathbb{Z}^{d-1}} \tilde{f}(n,k) \mathrm{e}^{(\hat{t},k)_{d-1}}.$$

Of course, $\mathbb{H}_n(\hat{t}) > \mathbb{F}_n(\hat{t})$, and both sums above diverge for sufficiently large values of $\|\hat{t}\|_{d-1}$. In any case, however;

$$\mathbb{H}_n(\hat{t}) \;=\; \sum_{m=1}^{n} \mathbb{F}_m(\hat{t}) \mathbb{H}_{n-m}(\hat{t}). \tag{1.3.1}$$

Definition. For every $\hat{t} \in \mathbb{R}^{d-1}$ the masses $m_{\mathbb{H}}(\hat{t})$ and $m_{\mathbb{F}}(\hat{t})$ are defined as

$$m_{\mathbb{H}}(\hat{t}) \;\stackrel{\Delta}{=}\; \lim_{n\to\infty} \frac{1}{n} \log \mathbb{H}_n(\hat{t}) \quad \text{and} \quad m_{\mathbb{F}}(\hat{t}) \;\stackrel{\Delta}{=}\; \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{F}_n(\hat{t}).$$

Notice that by the renewal property (1.3.1), the limit in the above definition of $m_{\mathbb{H}}$ always exists (though, could be infinite). Furthermore, both $m_{\mathbb{H}}$ and $m_{\mathbb{F}}$ are convex functions on $\mathbb{R}^{d-1}$, and, of course, $m_{\mathbb{F}} \leqslant m_{\mathbb{H}}$. Let us use $\mathcal{D}_{\mathbb{H}}$ to denote the effective domain of $m_{\mathbb{H}}$:

$$\mathcal{D}_{\mathbb{H}} \;\stackrel{\Delta}{=}\; \left\{\hat{t} \in \mathbb{R}^{d-1} \,:\, m_{\mathbb{H}}(\hat{t}) < \infty\right\}.$$

Because of (1.1.3), (1.1.4) and an obvious bound ($x = (n,k)$)

$$\mathrm{e}^{-c_2\|x\|_d} \;\leqslant\; h(n,k) \;\leqslant\; \mathbb{P}_p\left(0 \longleftrightarrow x\right),$$

the convex set $\mathcal{D}_{\mathbb{H}}$ is bounded and has a non-empty interiour $\mathrm{int}\,(\mathcal{D}_{\mathbb{H}})$; $0 \in \mathrm{int}\,(\mathcal{D}_{\mathbb{H}})$.
We finally formulate the separation of masses type condition to which we have referred on several occasion in the first subsection.
Definition. Let us say the the mass-gap condition is satisfied at a point $\hat{t} \in \mathrm{int}\,(\mathcal{D}_{\mathbb{H}})$, if

$$m_{\mathbb{H}}(\hat{t}) \;>\; m_{\mathbb{F}}(\hat{t}). \tag{1.3.2}$$

□

We rely on the following local limit theorem [11]:

Theorem B. *Assume that for every point* $\hat{t} \in$ *int* $(\mathcal{D}_{\mathbb{H}})$

1. *The mass-gap condition*(1.3.2) *is satisfied.*
2. *The Hessian* $\mathrm{D}^2 m_{\mathbb{H}}(\hat{t})$ *is non-degenerate:*

$$\det\left(\mathrm{D}^2 m_{\mathbb{H}}(\hat{t})\right) \;\neq\; 0. \tag{1.3.3}$$

Then, for every $\alpha \in \mathbb{R}_+$*;*

$$h(n,k) \;=\; \frac{\Lambda_p(\vec{\mathfrak{n}}(x))}{\sqrt{(2\pi\|x\|_d)^{d-1}}} e^{-\xi_p(x)}\left(1 + o(1)\right). \tag{1.3.4}$$

uniformly in $x = (n,k) \in \mathcal{C}_\alpha$*. As before* $\vec{\mathfrak{n}}(x)$ *is the unit vector in the direction of* x*, and* Λ_p *is a positive real analytic function on* $\mathbb{S}^{d-1}_{+}$*.*

lattice symmetries there). □

The main impact of the mass-gap condition (1.3.2) at $\hat{t} \in \mathrm{int}\,(\mathcal{D}_{\mathbb{H}})$ is the validity of the Lee-Yang type analyticity structure of moment generating functions $\mathbb{H}_n$ in a $(d-1)$-dimensional complex neighbourhood of $\hat{t}$. In particular [11], for every $\hat{t} \in \mathrm{int}\,(\mathcal{D}_{\mathbb{H}})$;

$$m_{\mathbb{H}}(\hat{t}) \;>\; m_{\mathbb{F}}(\hat{t}) \;\Longrightarrow\; m_{\mathbb{H}}(\hat{z}) = \lim_{n\to\infty} \frac{1}{n} \mathbb{H}_n(\hat{z}), \tag{1.3.5}$$

in the sense of analytic functions on a $\mathbb{C}^{d-1}$-neighbourhood of $\hat{t}$. Thus, $m_{\mathbb{H}}$ is real analytic in a $\mathbb{R}^{d-1}$-neighbourhood of $\hat{t}$, as soon as the mass- gap condition (1.3.2) is satisfied. Together with the non-degeneracy condition (1.3.3) such results enable the classical local limit analysis of the $\hat{t}$-tilted measure (c.f. Section 2 of [11]).
The main technical result of this paper states:

Theorem C. *Let* $d \;\geqslant\; 2$ *and* $p \;<\; p_c(d)$*. Then both the mass gap condition* (1.3.2) *and the non-degeneracy condition* (1.3.3) *hold for every* $\hat{t} \in$ *int* $(\mathcal{D}_{\mathbb{H}})$*.*

The crux of the matter is to prove the mass-gap. Once this is accomplished, the non-degeneracy follows by a simple conditional variance bound.

1.4. **Geometry of Wulff shapes and equi-decay profiles.** Analytic properties of connectivities have useful geometric counterparts: By (1.1.3), ξ_{p} is an equivalent norm on $\mathbb{R}^d$;

$$0 \;<\; \min_{x\in\mathbb{S}^{d-1}} \xi_{\mathrm{p}}(x) \;\leqslant\; \max_{x\in\mathbb{S}^{d-1}} \xi_{\mathrm{p}}(x) \;<\; \infty. \tag{1.4.1}$$

Let us denote the corresponding ξ_{p}-unit ball as $\mathbf{U}^{\mathrm{p}}$:

$$\mathbf{U}^{\mathrm{p}} \;=\; \left\{x\in\mathbb{R}^d \,:\, \xi_{\mathrm{p}}(x) \;\leqslant\; 1\right\}.$$

We use the term equi-decay profiles $\partial(a\mathbf{U}^{\mathrm{p}})$ for the boundaries of the ξ_{p}-balls

$$a\mathbf{U}^{\mathrm{p}} \;=\; \left\{x\in\mathbb{R}^d \,:\, \xi_{\mathrm{p}}(x) \;\leqslant\; a\right\}.$$

Similarly, because of (1.4.1) ξ_{p} is the support function of the compact convex set $\mathbf{K}^{\mathrm{p}}$:

$$\mathbf{K}^{\mathrm{p}} \;\stackrel{\Delta}{=}\; \bigcap_{n\in\mathbb{S}^{d-1}} \left\{t\in\mathbb{R}^d \,:\, (t,n)_d \;\leqslant\; \xi_{\mathrm{p}}(n)\right\},$$

with non-empty interiour $\mathrm{int}\,\{\mathbf{K}^{\mathrm{p}}\}$; $0\in\mathrm{int}\,\{\mathbf{K}^{\mathrm{p}}\}$.
Furthermore, the function ξ_{p} is symmetric with respect to permutations and reflections across coordinate hyperplanes:

$$\xi_{\mathrm{p}}(\epsilon_1 x_{\pi(1)},\dots,\epsilon_d x_{\pi(d)}) \;=\; \xi_{\mathrm{p}}(x_1,\dots,x_d), \tag{1.4.2}$$

for every $x=(x_1,\dots,x_d)\in\mathbb{R}^d$, every permutation $\{\pi(1),\dots,\pi(d)\}$ of $\{1,\dots,d\}$ and every collection of numbers $\epsilon_i=\pm$; $i=1,..,d$. Consequently, both the polar shape $\mathbf{K}^{\mathrm{p}}$ and the equi-decay profile $\mathbf{U}^{\mathrm{p}}$ enjoy the same symmetries as ξ_{p}. These symmetries will substantially facilitate some of our arguments below.

We shall refer to $\mathbf{K}^{\mathrm{p}}$ as to the polar body or as to to the Wulff shape (which it really is in the case of two dimensions). The first appellation is justified by the fact that the convex bodies $\mathbf{U}^{\mathrm{p}}$ and $\mathbf{K}^{\mathrm{p}}$ are in the polar relation: for every $t\in\partial\mathbf{K}^{\mathrm{p}}$ and $x\in\partial\mathbf{U}^{\mathrm{p}}$,

$$1 \;=\; \max_{y\in\mathbf{U}^{\mathrm{p}}}(t,y)_d \;=\; \max_{s\in\mathbf{K}^{\mathrm{p}}}(s,x)_d. \tag{1.4.3}$$

Given $x\in\mathbb{R}^d\setminus\{0\}$, let us say that a point $t\in\partial\mathbf{K}^{\mathrm{p}}$ is polar to x if

$$(t,x)_d \;=\; \xi_{\mathrm{p}}(x) \;=\; \max_{s\in\partial\mathbf{K}^{\mathrm{p}}}(s,x)_d. \tag{1.4.4}$$

Geometrically, t is orthogonal to a tangent hyperplane to the equi-decay profile $\xi_{\mathrm{p}}(x)\in\partial\mathbf{U}^{\mathrm{p}}$ passing through x. A priori x might have many different polar points. The collection of these points, however, always forms a convex set. Therefore non-unicity of polar points at x is tantamount to an existence of a flat facet on $\partial\mathbf{K}^{\mathrm{p}}$.
There is, of course, an intimate relation between the geometry of polar sets and the mass $m_{\mathbb{H}}$.

Proposition 1.2. *If* $\hat t\in \mathit{int}(\mathcal{D}_{\mathbb{H}})\subset\mathbb{R}^{d-1}$, *then* $t=\left(-m_{\mathbb{H}}(\hat t),\hat t\right)\in\partial\mathbf{K}^p\subset\mathbb{R}^d$.

Proof. First of all, by the usual LD-style application of the Hölder inequality, there exists $\delta>0$ and a constant $A_\delta<\infty$ such that

$$\sum_{\|k\|\;\geqslant\; A_\delta n} \mathrm{e}^{(k,\hat t)_{d-1}} h(n,k) \;\leqslant\; \mathrm{e}^{n(m_{\mathbb{H}}(\hat t)-\delta)}. \tag{1.4.5}$$

Thus, using (1.2.4) with $\alpha=A_\delta$, we obtain:

$$\lim_{n\to\infty}\frac{1}{n}\log\left\{\sum_{\|k\|\;\leqslant\; A_\delta n} \mathrm{e}^{-\xi_{\mathrm{p}}(n,k)+(k,\hat t)_{d-1}-nm_{\mathbb{H}}(\hat t)}\right\} \;=\; 0.$$

It follows that there exists $x \in \mathbb{R}^d$ with

$$\sqrt{\sum_{i=2}^{d} x_i^2} \ \leqslant \ A_\delta x_1,$$

satisfying

$$\xi_{\mathrm{p}}(x) = (t, x)_d, \qquad \text{where } t = (-m_{\mathbb{H}}(\hat{t}), \hat{t}).$$

Hence, $t \in \partial \mathbf{K}^{\mathrm{p}}$. □

Remark. Notice that because of the lattice symmetries (1.4.2) the above proposition implies that the point $\tilde{t} \ \overset{\Delta}{=} \ (0, \hat{t})$ belongs to the convex set $\mathbf{K}^{\mathrm{p}}$ whenever $\hat{t} \in \operatorname{int}(\mathcal{D}_{\mathbb{H}})$. Since, by (1.1.3), $\mathbf{K}^{\mathrm{p}}$ has a non-empty interiour, we conclude;

$$\hat{t} \in \operatorname{int}(\mathcal{D}_{\mathbb{H}}) \iff \tilde{t} \ \overset{\Delta}{=} \ (0, \hat{t}) \in \operatorname{int}(\mathbf{K}^{\mathrm{p}}) \tag{1.4.6}$$

In particular, $-m_{\mathbb{H}}(\hat{t}) > 0$ for every interiour point $\hat{t} \in \operatorname{int}(\mathcal{D}_{\mathbb{H}})$. □

Local validity of the assumptions of Theorem A leads to nice analytic properties of the boundary $\partial \mathbf{K}^{\mathrm{p}}$:

Proposition 1.3. *Let $\hat{t} \in int(\mathcal{D}_{\mathbb{H}})$. Assume that $m_{\mathbb{H}}(\hat{t}) > m_{\mathbb{F}}(\hat{t})$ and that* $\det\left(\mathrm{D}^2 m_{\mathbb{H}}(\hat{t})\right) \neq 0$.. *Then $\partial \mathbf{K}^p$ is analytic and strictly convex in a neighbourhood of $t = (-m_{\mathbb{H}}(\hat{t}), \hat{t})$.*

As in [11], Theorem C implies the following result on the geometry of sets $\mathbf{K}^{\mathrm{p}}$ and $\mathbf{U}^{\mathrm{p}}$:

Theorem D. *Assume that the assumptions of Theorem B are satisfied. Then $m_{\mathbb{H}}$ is a real analytic function on $int(\mathcal{D}_{\mathbb{H}})$. In addition, $m_{\mathbb{H}}$ is strictly convex and steep;*

$$\bigcup_{\hat{t} \in int(\mathcal{D}_{\mathbb{H}})} \nabla m_{\mathbb{H}}(\hat{t}) \ = \ \mathbb{R}^{d-1}. \tag{1.4.7}$$

Furthermore, both $\mathbf{K}^p$ and $\mathbf{U}^p$ are strictly convex bodies with analytic boundaries $\partial \mathbf{K}^p$ and $\partial \mathbf{U}^p$. Furthermore, the Gaussian curvatures of both $\partial \mathbf{U}^p$ and $\partial \mathbf{K}^p$ are everywhere strictly positive.

Remark. In particular, Theorem D implies that the two dimensional Wulff shape is strictly convex and analytic for every super-critical (dual) value $p^* > 1/2$. This enables to refine results of [6] and [5] along the lines of [12].

2. COARSE GRAINING

Assume that 0 and x are connected. We use $\mathbf{C}_{\{0,x\}}$ to denote the corresponding common cluster. In this section we study the coarse-grained structure of $\mathbf{C}_{\{0,x\}}$. The coarse graining is constructed around a self-avoiding path $\gamma : 0 \to x$ (trunk) lying inside $\mathbf{C}_{\{0,x\}}$ and disjoint leaves growing from this trunk, also inside $\mathbf{C}_{\{0,x\}}$. What we roughly show is that on appropriate renormalization scales the corresponding tree-skeleton has, with an overwhelming probability, a nice localized structure, in the sense that the trunk skeleton of γ goes in a more or less straight way, with only negligible number of backtracks, from 0 to x, whereas the renormalized leaves become very sparse.

2.1. Alexander's surcharge function.

Given $t \in \partial \mathbf{K}^{\mathrm{p}}$ define the surcharge function $\mathfrak{s}_t$ in the direction t as

$$\mathfrak{s}_t(x) \ \overset{\Delta}{=} \ \xi_{\mathrm{p}}(x) - (t, x)_d.$$

By (1.4.3) $\mathfrak{s}_t$ is always non-negative and $\mathfrak{s}_t(x) = 0$ only if t is polar to x. Notice that our definition here differs from the original one given in [2], but our approach is definitely inspired by Alexander's point of view on the relevant renormalization procedures.

Proposition 2.1. *Let $x \in \mathbb{Z}^d$. Set $y_0 = 0$ and $y_n = x$. Then for every $t \in \partial\mathbf{K}^p$ and for any collection $\{y_1, ..., y_{n-1}\}$ of points from $\mathbb{Z}^d$;*

$$\begin{aligned}&\mathbb{P}_p\left(0 \longleftrightarrow y_1 \circ y_1 \longleftrightarrow y_2 \circ ... \circ y_{n-1} \longleftrightarrow x\right)\\ &\qquad \leqslant \ \exp\left\{-\sum_{k=0}^{n-1}\mathfrak{s}_t(y_{k+1}-y_k) \ -(t,x)_d\right\}.\end{aligned} \tag{2.1.1}$$

Proof. By the BK-inequality

$$\begin{aligned}&\mathbb{P}_p\left(0 \longleftrightarrow y_1 \circ y_1 \longleftrightarrow y_2 \circ ... \circ y_{n-1} \longleftrightarrow x\right)\\ &\qquad \leqslant \ \exp\left\{-\sum_{k+0}^{n}\xi_{\mathrm{p}}(y_{k+1}-y_k)\right\}\\ &\qquad = \exp\left\{-\sum_{k+0}^{n-1}\mathfrak{s}_t(y_{k+1}-y_k)-(t,x)_d\right\}.\end{aligned}$$

□

2.2. **Coarse graining of a self-avoiding path** γ**.** Fix a number $k \in \mathbb{R}_+$. Given $x \in \mathbb{Z}^d$ and a self-avoiding lattice path $\gamma : 0 \longleftrightarrow x$;

$$\gamma \ = \ \{\gamma(1), ...\gamma(n)\}; \qquad \gamma(0) = 0 \text{ and } \gamma(n) = x,$$

we construct the k-skeleton $\gamma^{(k)} = \{y_0, ..., y_m\}$ of γ as follows:
Step 0 $y_0 = 0$.
Step 1 $n_1 = \min\{l : \gamma(l) \not\in k\mathbf{U}^{\mathrm{p}}(y_0)\} \wedge n$. If $n_1 = n$, then $m = 1$, $y_m = x$ and the process terminates. Otherwise set $y_1 = \gamma(n_1)$ and proceed to the next step.
Step i $n_i = \min\{l > n_{i-1} : \gamma(l) \not\in k\mathbf{U}^{\mathrm{p}}(y_{k-1})\} \wedge n$. If $n_i = n$, then $m = i$, $y_m = x$ and the process terminates. Otherwise set $y_i = \gamma(n_i)$, and proceed to **Step (i+1)** .

Above we have used the following notation: for a set $B \subset \mathbb{R}^d$ and a point $z \in \mathbb{R}^d$

$$B(z) \ \stackrel{\Delta}{=} \ z + B.$$

Let us use the symbol $\gamma \sim \gamma^{(k)}$ to denote the fact that $\gamma^{(k)}$ is the k-skeleton of γ. Given a k-skeleton $\gamma^{(k)}$ define the event

$$\left\{0 \stackrel{\gamma^{(k)}}{\longleftrightarrow} x\right\} \ \stackrel{\Delta}{=} \ \left\{0 \text{ is connected to } x \text{ by a self-avoiding path } \gamma;\ \gamma \sim \gamma^{(k)}\right\}.$$

As it immediately follows from the Proposition 2.1, for every $x \in \mathbb{Z}^d$, every $t \in \partial\mathbf{K}^{\mathrm{p}}$, each renormalization scale $k \in \mathbb{R}_+$ and for every k-skeleton $\gamma^{(k)}$

$$\mathbb{P}_p\left(0 \stackrel{\gamma^{(k)}}{\longleftrightarrow} x\right) \ \leqslant \ \exp\left\{-\sum_{l=1}^{m}\mathfrak{s}_t(y_{l+1}-y_l)-(t,x)_d\right\}. \tag{2.2.2}$$

2.3. **Surcharge cones and typical** k**-trunks.** Given $t \in \partial\mathbf{K}^{\mathrm{p}}$ and $\epsilon > 0$ let us define the surcharge cone $\mathcal{C}_\epsilon(t)$ as

$$\mathcal{C}_\epsilon(t) \ \stackrel{\Delta}{=} \ \left\{x \in \mathbb{R}^d : \ \mathfrak{s}_t(x) \ \leqslant \ \epsilon\xi_{\mathrm{p}}(x)\right\} \ = \ \left\{x \in \mathbb{R}^d : \ (t,x) \ \geqslant \ (1-\epsilon)\xi_{\mathrm{p}}(x)\right\}. \tag{2.3.1}$$

We quantify k-skeletons $\gamma^{(k)} : 0 \stackrel{\gamma^{(k)}}{\longleftrightarrow} x$; $\gamma^{(k)} = \{y_0, ..., y_m\}$ by the number of k-increments of $\gamma^{(k)}$

$$\mathfrak{g}^{(k)} \ \stackrel{\Delta}{=} \ m \ = \ \#\left(\gamma^{(k)}\right),$$

and by the number of costly ϵ-backtracking full increments with respect to the surcharge cone $\mathcal{C}_\epsilon(t)$

$$\#_{t,\epsilon}\left(\gamma^{(k)}\right) \;=\; \#\left\{1 \;\leqslant\; l \;\leqslant\; \mathfrak{g}^{(k)} - 1 :\; y_l - y_{l-1} \not\in \mathcal{C}_\epsilon(t)\right\}.$$

Notice that on k-th renormalization scale "bad" increment $y_l - y_{l-1} \not\in \mathcal{C}_\epsilon(t)$ automatically satisfies

$$\mathfrak{s}_t(y_l - y_{l-1}) \;\geqslant\; \epsilon k. \tag{2.3.2}$$

Lemma 2.2. *For every $x \in \mathbb{Z}^d$, $\epsilon > 0$, $t \in \partial \mathbf{K}^p$, $\delta \in \mathbb{R}_+$ and every coarse-graining scale $k \in \mathbb{R}_+$;*

$$\begin{aligned}\mathbb{P}_p\left(\gamma^{(k)} :\; \#_{t,\epsilon}\left(\gamma^{(k)}\right) \;\geqslant\; \frac{\delta\|x\|_d}{k} \;:\; 0 \overset{\gamma^{(k)}}{\longleftrightarrow} x\right)&\\ \leqslant\; \exp\left\{c_1 \frac{\log k}{k}\|x\|_d - \delta\epsilon\|x\|_d - (x,t)_d\right\}&.\end{aligned} \tag{2.3.3}$$

Proof. By the very construction of k-skeletons and by the BK inequality (2.2.2)

$$\mathbb{P}_p\left(0 \overset{\gamma^{(k)}}{\longleftrightarrow} x\right) \;\leqslant\; \exp\left\{-\mathfrak{g}^{(k)}k\right\}$$

uniformly in k-skeletons $\gamma^{(k)}$.
On the other hand, the number of skeletons

$$\#\left\{\gamma^{(k)} :\; 0 \overset{\gamma^{(k)}}{\longleftrightarrow} x \text{ and } \#\left(\gamma^{(k)}\right) = m\right\}$$

is bounded above by

$$\left(c_1 k^{d-1}\right)^m \;=\; \exp\left\{c_2 m \log k\right\}. \tag{2.3.4}$$

Consequently,

$$\mathbb{P}_p\left(\gamma^{(k)} :\; 0 \overset{\gamma^{(k)}}{\longleftrightarrow} x;\; \mathfrak{g}^{(k)} \;\geqslant\; m\right) \;\leqslant\; \exp\left\{-\frac{m}{2}k\right\},$$

as soon as k is sufficiently large. Since, by (1.4.1), $\xi_{\mathrm{p}}(x) \;\leqslant\; c_3\|x\|_d$, we infer that there exist two positive constants $c_4 = c_4(d,p)$ and $c_5 = c_5(d,p)$, such that

$$\mathbb{P}_p\left(\gamma^{(k)} :\; 0 \overset{\gamma^{(k)}}{\longleftrightarrow} x;\; \mathfrak{g}^{(k)} \;\geqslant\; c_4\frac{\|x\|_d}{k}\right) \;\leqslant\; \exp\left\{-c_5\|x\|_d - \xi_{\mathrm{p}}(x)\right\} \;\leqslant\; \exp\left\{-c_5\|x\|_d - (x,t)_d\right\} \tag{2.3.5}$$

uniformly in $x \in \mathbb{Z}^d$ and $t \in \partial\mathbf{K}^{\mathrm{p}}$. As a result we can restrict attention only to the k-skeletons $\gamma^{(k)}$ satisfying

$$\mathfrak{g}^{(k)} = \#\left(\gamma^{(k)}\right) \;\leqslant\; c_4\|x\|_d/k. \tag{2.3.6}$$

By (2.3.4) the number of such skeletons is bounded above by

$$\exp\left\{c_6\frac{\log k}{k}\|x\|_d\right\}.$$

Finally, (2.2.2) and the lower bound (2.3.2) on the surcharge value of bad increments imply that for any $t \in \partial\mathbf{K}^{\mathrm{p}}$, $\epsilon > 0$ and any skeleton $\gamma^{(k)}$;

$$\mathbb{P}_p\left(0 \overset{\gamma^{(k)}}{\longleftrightarrow} x\right) \;\leqslant\; \exp\left\{-c_7\epsilon k\#_{t,\epsilon}\left(\gamma^{(k)}\right) - (t,x)_d\right\}.$$

Patching the latter two estimates together we arrive to the conclusion (2.3.3) of Lemma 2.2 . ☐

2.4. **Coarse-graining of clusters $\mathbf{C}_{\{0,x\}}$.** Recall that if 0 is connected to x, then we denote the corresponding common connected cluster as $\mathbf{C}_{\{0,x\}}$. With each realization of $\mathbf{C}_{\{0,x\}}$ we associate, on every coarse graining scale k, a tree like subset $\mathbf{C}^{(k)}_{\{0,x\}} \subset \mathbb{Z}^d$, such that $\mathbf{C}_{\{0,x\}}$ lies inside the k-neighbourhood of $\mathbf{C}^{(k)}_{\{0,x\}}$ in the sense of the $\xi_{\rm p}$-distance, that is

$$\forall\, y \in \mathbf{C}_{\{0,x\}} \qquad \min_{z\in \mathbf{C}^{(k)}_{\{0,x\}}} \xi_{\rm p}(y-z) \ \leqslant\ k. \tag{2.4.1}$$

The sets $\mathbf{C}^{(k)}_{\{0,x\}}$ will be always composed of shifts of the $\xi_{\rm p}$-ball $k\mathbf{U}^{\rm p}$:

$$\mathbf{C}^{(k)}_{\{0,x\}} \ =\ \bigcup_{y\in\mathfrak{T}^{(k)}} k\mathbf{U}^{\rm p}(y). \tag{2.4.2}$$

I would be convenient to describe the construction of $\mathbf{C}^{(k)}_{\{0,x\}}$ or, equivalently, of its tree-skeleton $\mathfrak{T}^{(k)}$ in an algorithmic way. For technical reasons we prefer to construct $\mathfrak{T}^{(k)}$ in an unambiguous way, later on this will enable a disjoint splitting of the relevant percolation events with respect to different possible tree-skeletons.

Choosing the self-avoiding trunk $\gamma^{(k)}$ Consider all possible self-avoiding paths γ leading from 0 to x within $\mathbf{C}_{\{0,x\}}$, and let $\gamma^{(k)} = \{y_0, ..., y_m\}$ be the corresponding k-skeletons of γ. Of all these $\gamma^{(k)}$ we first choose skeletons of minimal length $\mathfrak{g}^{(k)}$, and, provided that there are several such minimal length skeletons, we further choose the minimal one of them, say in the sense of the lexicographical order.

We shall refer to the resulting $\gamma^{(k)} = (y_0, ..., y_{\mathfrak{g}^{(k)}})$ as to the self- avoiding trunk of $\mathbf{C}_{\{0,x\}}$ on the k-th renormalization scale.

Define $\mathfrak{T}^{(k)} = \{y_0, ..., y_m\}$ and, accordingly, define $\mathbf{C}^{(k)}_{\{0,x\}}$ by (2.4.2). If (2.4.1) is already satisfied, then stop.

Otherwise proceed to the following update step:

Update step Reorder all the sites of $\mathfrak{T}^{(k)}$, for instance again according to the lexicographical order; $\mathfrak{T}^{(k)} = \{z_1, ..., z_{\mathsf{t}^{(k)}}\}$, where $\mathsf{t}^{(k)}$ is used to denote the cardinality of $\mathfrak{T}^{(k)}$:

$$\mathsf{t}^{(k)} \ \triangleq\ \#\left\{\mathfrak{T}^{(k)}\right\}.$$

Set $l := 1$.

Step l ($l \ \leqslant\ \mathsf{t}^{(k)}$) Screen the points of $k\partial\mathbf{U}^{\rm p}(z_l)$ in the lexicographical order. If there exists $z \in k\partial\mathbf{U}^{\rm p}(z_l)$ such that one can find a self-avoiding open path γ_z leading from z to $k\partial\mathbf{U}^{\rm p}(z)$ inside $\mathbb{Z}^d \setminus \mathbf{C}^{(k)}_{\{0,x\}}$, then add z to $\mathfrak{T}^{(k)}$, i.e. set

$$\mathfrak{T}^{(k)} \ :=\ \mathfrak{T}^{(k)} \cup \{z\} \qquad \text{and} \qquad \mathbf{C}^{(k)}_{\{0,x\}} \ :=\ \mathbf{C}^{(k)}_{\{0,x\}} \cup k\mathbf{U}^{\rm p}(z),$$

and return to the Update step.

Otherwise set $l := l + 1$ and proceed to the Step l.

Step $(\mathsf{t}^{(k)} + 1)$ Stop. We shall say that the resulting $\mathfrak{T}^{(k)}$ is the tree-skeleton of $\mathbf{C}_{\{0,x\}}$ on the k-th renormalization scale, and denote the corresponding percolation event either as $\mathfrak{T}^{(k)} \sim \mathbf{C}_{\{0,x\}}$ or as $\left\{0 \overset{\mathfrak{T}^{(k)}}{\longleftrightarrow} x\right\}$ □

Clearly, once Step $(\mathsf{t}^{(k)} + 1)$ is reached, condition (2.4.1) is satisfied.

We should still provide an argument that with the probability 1 the process terminates in a finite number of steps. This will follow from a much more precise estimate on the number $\mathfrak{t}^{(k)}$, which we proceed to derive:

2.5. **Renormalization: typical tree-skeletons.** By the construction, the tree-skeleton $\mathfrak{T}^{(k)}$ is composed of the original self-avoiding trunk $\gamma^{(k)}$ and of the set of additional leaves $\mathfrak{L}^{(k)}$:

$$\mathfrak{T}^{(k)} \ = \ \gamma^{(k)} \bigcup \mathfrak{L}^{(k)}.$$

Thus, the corresponding set $\mathbf{C}^{(k)}_{\{0,x\}}$ (defined in (2.4.2)) contains:

1. A self-avoiding path $\gamma : 0 \longleftrightarrow x$ with a k-trunk $\gamma^{(k)}$.
2. For each leave $z \in \mathfrak{L}^{(k)}$, a self-avoiding path γ_z leading from z to $k\partial \mathbf{U}^{\rm p}(z)$.
3. By the construction all these γ_z are disjoint and, moreover, each such γ_z is disjoint from γ.

By the BK inequality the probability of a given tree-skeleton $\mathfrak{T}^{(k)}$ is bounded above as

$$\mathbb{P}_p\left(\ \mathfrak{T}^{(k)}\ \right) \ \leqslant \ \mathbb{P}_p\left(\gamma^{(k)}\right) {\rm e}^{-\xi_{\rm p}(x) - c_7 \mathfrak{l}^{(k)} k}, \tag{2.5.3}$$

where the number of leaves $\mathfrak{l}^{(k)}$ is defined by

$$\mathfrak{l}^{(k)} \ \stackrel{\Delta}{=} \ \#\left(\mathfrak{L}^{(k)}\right) \ = \ \mathfrak{t}^{(k)} - \mathfrak{g}^{(k)}.$$

Lemma 2.3. *There exists a constant* $c_8 = c_8(d,p)$*, such that for every* $\delta > 0$ *fixed,*

$$\mathbb{P}_p\left(\mathfrak{L}^{(k)} \ : \ \mathfrak{l}^{(k)} \ \geqslant \ \frac{\delta\|x\|_d}{k} \ ; \ 0 \longleftrightarrow x\right) \ \leqslant \ \exp\left\{-\xi_p(x) - c_8\delta\|x\|_d\right\}, \tag{2.5.4}$$

uniformly in k *and* x *sufficiently large.*

Proof. Given the skeleton $\gamma^{(k)}$ of a a self-avoiding path $\gamma : 0 \longleftrightarrow x$, let us estimate the number of ways one can attach l leaves to the "trunk" $\gamma = \{y_0, ..., y_m\}$:

To each point $y \in \gamma^{(k)}$ one can attach at most

$$n \ = \ n(k,d) \ \stackrel{\Delta}{=} \ c_9 k^{d-1}$$

new points from $k\partial\mathbf{U}^{\rm p}$. To each of these new points one can attach another point again in, at most, n ways. And so on. Thus, the number of ways to attach l_i leaves to a point $y_i \in \gamma^{(k)}$ is bounded above by a number of connected trees with l_i vertices and branching ratio n. By the well-known estimate of Kesten on the number of lattice animals (see, for example, [10]) the latter is bounded above by

$$\left\{\max_{q\in(0,1)} q^{l_i}(1-q)^{nl_i}\right\}^{-1} \ \leqslant \ \exp\left\{c_{10} l_i \log n\right\} \ \leqslant \ \exp\left\{c_{11} l_i \log k\right\}. \tag{2.5.5}$$

Finally, the number of ways to distribute l leaves to m different branches of the trunk $\gamma^{(k)}$ is bounded above by

$$\left(\begin{array}{c} m+l \\ l \end{array}\right) \ \approx \ \exp\left\{l\log\left(1+\frac{m}{l}\right) + m\log\left(1+\frac{l}{m}\right)\right\}. \tag{2.5.6}$$

By (2.3.5) there is no loss to assume that the the cardinality m of the k-skeleton $\gamma^{(k)}$ satisfies $m \ \leqslant \ c_5\|x\|/k$. Thus,

$$\left(\begin{array}{c} m+l \\ l \end{array}\right) \ \leqslant \ \exp\left\{c_{12} l \log\frac{1}{\delta}\right\},$$

whenever $l \ \geqslant \ \delta\|x\|/k$.

The bounds (2.5.3), (2.5.5),(2.5.6) and the estimate (2.3.4) on the number of different trunks $\gamma^{(k)}$ of the maximal cardinality $c_5\|x\|_d/k$ readily imply that

$$\begin{aligned}\mathbb{P}_p\left(\mathfrak{L}^{(k)} \geqslant \frac{\delta\|x\|_d}{k}\ ;\ 0 \longleftrightarrow x\right)\\ \leqslant \exp\left\{-\xi_{\mathrm{p}}(x) - c_7\delta\|x\|_d + c_{11}\frac{\log(1/\delta)}{k} + c_{13}\frac{\log k}{k}\|x\|_d\right\},\end{aligned}$$

and the conclusion (2.5.4) of the lemma follows. □

3. SEPARATION OF MASSES

As we have seen in Proposition 1.2, any point $\hat{t} \in \operatorname{int}(\mathcal{D}_{\mathbb{H}})$ gives rise to the point $t = (-m_{\mathbb{H}}(\hat{t}), \hat{t})$ on the boundary $\partial\mathbf{K}^{\mathrm{p}}$. In order to prove the mass-gap at $\hat{t}$ we shall fix this t and use it to quantify the surcharge costs of the increments, as defined in the framework of the renormalization results of the preceding section. Such approach happens to be useful, if at some surcharge value $\epsilon > 0$, all ϵ-good increments have strictly positive $\vec{e}_1$-component. We shall then argue that such "forward" structure of typical $\mathfrak{T}^{(k)}$-tree skeletons necessarily decouples the event $\left\{0 \xleftrightarrow{f} x\right\}$ into intersection of many localized independent sub-events, each of the latter having probability strictly less than one.

The appropriate forward condition on the increments is formulated in Subsection 3.1. Geometrically it boils down to certain strict convexity requirements on the connectivity function ξ_{p}. Because of lattice symmetries (1.4.2) of ξ_{p} and, accordingly, of $\mathbf{K}^{\mathrm{p}}$ the proof that such condition always holds for the interior points $\hat{t} \in \operatorname{int}(\mathcal{D}_{\mathbb{H}})$ is essentially trivial, see Lemma 3.1 below.

A more robust approach, which does not rely on lattice symmetries, is explained and worked out in Section 4 in the course of giving a "direct" proof of Theorem A.

3.1. Positive cone property.

Let $\hat{t} \in \operatorname{int}(\mathcal{D}_{\mathbb{H}})$. We say that $\hat{t}$ satisfies the positive cone property if there exists $\epsilon > 0$, such that

$$\alpha(t,\epsilon) \ \stackrel{\Delta}{=} \min_{x\in\mathcal{C}_\epsilon(t)\setminus\{0\}} \frac{x_1}{\|x\|} > 0, \tag{3.1.1}$$

where $t = (-m_{\mathbb{H}}(\hat{t}), \hat{t})$, and $\mathcal{C}_\epsilon\,(t)$ is the surcharge cone defined in (2.3.1).
Informally, the positive cone condition is satisfied if all $\mathfrak{s}_t$-reasonable increments x have a nontrivial forward component in the direction of the axis e_1. By the continuity the positive cone condition is satisfied, iff

$$\min\{x_1 :\ x \in \partial\mathbf{U}^{\mathrm{p}} \text{ and } \mathfrak{s}_t(x) = 0\} > 0. \tag{3.1.2}$$

Lemma 3.1. *The positive cone property is satisfied for every* $t \in$ *int*$(\mathcal{D}_{\mathbb{H}})$.

Proof. By the remark following the proof of Proposition 1.2, the point $\tilde{t} \ \stackrel{\Delta}{=}\ (0,\hat{t})$ belongs to $\operatorname{int}(\mathbf{K}^{\mathrm{p}})$, as soon as $\hat{t} \in \operatorname{int}(\mathcal{D}_{\mathbb{H}})$. With t defined as $t = (-m_{\mathbb{H}}(\hat{t}), \hat{t})$, let us assume that there exists $x = (x_1, ..., x_d) \in \partial\mathbf{U}^{\mathrm{p}}$ such that $(x,t)_d = \xi_{\mathrm{p}}(x)$ and $x_1 = 0$. In this case, however; $(x,\tilde{t})_d = \xi_{\mathrm{p}}(x)$, as well. This, by (1.4.4) , implies that $\tilde{t} \in \partial\mathbf{K}^{\mathrm{p}}$, a contradiction. □

Our main result in this section states:

Lemma 3.2. *If* $\hat{t}$ *satisfies the positive cone property, then* $m_{\mathbb{H}}(\hat{t}) > m_{\mathbb{F}}(\hat{t})$.

Consequently, the mass-gap condition is satisfied at any interiour point $\hat{t} \in \operatorname{int}(\mathcal{D}_{\mathbb{H}})$. This sets up the stage for the local limit analysis of the multi-dimensional renewal relation (1.2.6) along the lines of Theorem B, see Subsection 3.4 below.

3.2. **Reduction to regular tree-skeletons.** We wish to show that if $\hat{t} \in \mathrm{int}\,(\mathcal{D}_{\mathbb{H}})$ satisfies the positive cone condition (3.1.1), then there exists $\nu > 0$, such that

$$\mathrm{e}^{-nm_{\mathbb{H}}(\hat{t})}\sum_k \mathbb{P}_p\left(0 \xleftrightarrow{f} (n,k)\right)\mathrm{e}^{(\hat{t},k)_{d-1}} \ \leqslant\ \mathrm{e}^{-\nu n}, \tag{3.2.3}$$

uniformly in n sufficiently large.

By Proposition 1.2 the point $t \ \stackrel{\Delta}{=}\ (-m_{\mathbb{H}}(\hat{t}), \hat{t}) \in \partial\mathbf{K}^{\mathrm{p}}$. Thus, the left hand side of (3.2.3) could be rewritten as

$$\sum_{x\in\mathcal{H}_n} \mathbb{P}_p\left(0 \xleftrightarrow{f} x\right)\mathrm{e}^{(t,x)_d}, \tag{3.2.4}$$

where, as before, $\mathcal{H}_n$ denotes the lattice hyperplane (1.2.2).

In fact, we shall prove a slightly more general claim:

Definition. given a point $x = (n,k) \in \mathcal{H}_n$, let us say that it is d-connected to the origin; $\left\{0 \xleftrightarrow{d} x\right\}$, if $\{o \longleftrightarrow x\}$, and the common cluster $\mathbf{C}_{\{0,x\}}$ satisfies;

$$\#\left(\mathbf{C}_{\{0,x\}} \cap \mathcal{H}_m\right)\ >\ 1\ \ \forall\, m = 1, ..., m-1.$$

Set

$$d(n,k)\ =\ \mathbb{P}_p\left(0 \xleftrightarrow{d} x\right).$$

☐

Definition Given $x = (n,k)$ and a cluster $\mathbf{C}_{\{0,x\}}$. let us say that a point $y = (m,l)$ is a regeneration point for $\mathbf{C}_{\{0,x\}}$, if $1\ \leqslant\ m\ \leqslant\ n-1$ and $\mathbf{C}_{\{0,x\}} \cap \mathcal{H}_m = \{y\}$. ☐

Notice, that clusters $\mathbf{C}_{\{0,x\}}$ corresponding to d-connections could be defined as those which have no regeneration points. Clearly, d-connectivities dominate the f-ones; $d(n,k) > f(n,k)$. We claim that there exists $\nu > 0$, such that

$$\sum_{x\in\mathcal{H}_n} \mathbb{P}_p\left(0 \xleftrightarrow{d} x\right)\mathrm{e}^{(t,x)_d}\ \leqslant\ \mathrm{e}^{-\nu n}. \tag{3.2.5}$$

By (2.3.5) and (1.2.4)

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{x\in\mathcal{H}_n}\mathbb{P}_p\left(0\longleftrightarrow x\right)\mathrm{e}^{(t,x)_d}\right)\ =\ \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{k\in\mathbb{Z}^{d-1}} h(n,k)\mathrm{e}^{(t,x)_d}\right)\ =\ 0.$$

Consequently, for the purpose of proving (3.2.5), one is entitled to work with full clusters $\mathbf{C}_{\{0,x\}}$ instead of the restricted clusters $\mathbf{C}^n_{\{0,x\}}$ which appear in the definition of $h(n,k)$.

For the rest of the proof we fix $\epsilon = \epsilon(\hat{t})$ and $\alpha = \alpha(\hat{t},\epsilon)$ as in the (3.1.1).

The renormalization estimates of the previous subsections imply that that a non-negligible contribution to the left hand side of (3.2.5) could come only from the clusters $\mathbf{C}_{\{0,x\}}$, which are compatible with sufficiently well behaved tree-skeletons $\mathfrak{T}^{(k)}$. This enables several reductions in the sum (3.2.4). Specifically:

Reduction in $\|x\|_d$: By (1.4.5) it is enough to restrict summation only to those $x \in \mathcal{H}_n$ which satisfy; $\|x\|_n\ \leqslant\ c_1 n$.

Reduction in $\mathfrak{g}^{(k)} = \#\left(\gamma^{(k)}\right)$: In view of the reduction in $\|x\|$ and by by (2.3.5) it is enough to restrict the summation only to the case of k-skeleton (k - large enough) connections, which satisfy $\mathfrak{g}^{(k)}\ \leqslant\ c_2 n/k$.

Reduction in $\#_{t,\epsilon}\left(\gamma^{(k)}\right)$: Fix any $\delta > 0$. By Lemma 2.2;

$$\mathbb{P}_p\left(\gamma^{(k)}\,:\ \#_{t,\epsilon}\left(\gamma^{(k)}\right)\ \geqslant\ \delta\frac{n}{k}\ ;\ 0 \xleftrightarrow{\gamma^{(k)}} x\right)\ \leqslant\ \exp\left\{-c_3\delta\epsilon n - (t,x)_d\right\},$$

uniformly in n and $x \in \mathcal{H}_n$; $\|x\|_d \leqslant c_1 n$, as soon as the scale k is chosen to be sufficiently large, that is $k \geqslant k_0(\delta, \epsilon, \hat{t})$.

Consequently, for all such scales k we obtain;

$$\sum_{x \in \mathcal{H}_n} \mathrm{e}^{(t,x)_d} \mathbb{P}_p \left(\mathfrak{T}^{(k)} : \#_{t,\epsilon} \left(\gamma^{(k)} \right) \geqslant \delta \frac{n}{k} \; ; \; 0 \overset{\mathfrak{T}^{(k)}}{\longleftrightarrow} x \right) \leqslant \mathrm{e}^{-c_4 \delta \epsilon n}. \tag{3.2.6}$$

Reduction in $\mathfrak{l}^{(k)} = \# \left(\mathfrak{L}^{(k)} \right)$: Similarly, Lemma 2.3 implies that for every $\delta > 0$ fixed;

$$\sum_{x \in \mathcal{H}_n} \mathrm{e}^{(t,x)_d} \mathbb{P}_p \left(\mathfrak{T}^{(k)} : \mathfrak{l}^{(k)} \geqslant \delta \frac{n}{k} \; ; \; 0 \overset{\mathfrak{T}^{(k)}}{\longleftrightarrow} x \right) \leqslant \mathrm{e}^{-c_5 \delta n}, \tag{3.2.7}$$

uniformly in n and large enough renormalization scales k.

To summarize all the reductions above: for every $\delta > 0$ one can restrict summation in (3.2.4) to the case of $\|x\|_d \leqslant c_1 n$ and the percolation clusters $\mathbf{C}_{\{0,x\}}$ which are, on sufficiently large renormalization scales k, compatible with tree-skeletons $\mathfrak{T}^{(k)} = \gamma^{(k)} \cup \mathfrak{L}^{(k)}$ satisfying

$$\# \left(\gamma^{(k)} \right) \leqslant c_2 \frac{n}{k}; \quad \#_{t,\epsilon} \left(\gamma^{(k)} \right) \leqslant \delta \frac{n}{k} \quad \text{and} \quad \# \left(\mathfrak{L}^{(k)} \right) \leqslant \delta \frac{n}{k}. \tag{3.2.8}$$

Let us say that $\mathfrak{T}^{(k)}$ is a δ-regular tree-skeleton if it complies with (3.2.8). Similarly, let us say that a cluster $\mathbf{C}_{\{0,x\}}$ is (k, δ)-regular if its tree skeleton $\mathfrak{T}^{(k)}$ on the k-th renormalization scale is δ-regular.

It remains, thus, to show that for an appropriate choice of k and δ;

$$\sum_{x \in \mathcal{H}_n} \mathbb{P}_p \left(0 \overset{d}{\longleftrightarrow} x \; ; \; \mathbf{C}_{\{0,x\}} \text{ is } (k,\delta)\text{-regular} \right) \mathrm{e}^{(t,x)_d} \leqslant \mathrm{e}^{-\nu n}. \tag{3.2.9}$$

3.3. **Proof of the mass-gap.** There is a transparent logic behind the latter estimate: The condition on the cluster $\mathbf{C}_{\{0,x\}}$ to be (k, δ)-regular forces most of $\mathbf{C}_{\{0,x\}}$ to be localized within chunks of $k\mathbf{U}^{\mathrm{p}}$-balls centered around vertices of the trunk $\gamma^{(k)}$ with successive ϵ-good increments. Notice that by the positive cone condition (3.1.1) each ϵ-good increment shifts the e_1-projection of the corresponding endpoints by a fixed fraction of k, which gives rise to decoupling properties along finite sequences of such successive vertices. Small values of δ insure a fixed fraction of n/k of such disjoint sequences which, already, leads to the target bound (3.2.9). Let us proceed with a rigorous implementation of the above idea:

For every $x \in \mathcal{H}_n$; $\|x\|_d \leqslant c_1 n$,

$$\mathbb{P}_p \left(0 \overset{d}{\longleftrightarrow} x \; ; \; \mathbf{C}_{\{0,x\}} \text{ is } (\delta,k)\text{-regular} \right) = \sum_{\mathfrak{T}^{(k)} \text{ is } \delta\text{-regular}} \mathbb{P}_p \left(0 \overset{d}{\longleftrightarrow} x \; ; \; \mathfrak{T}^{(k)} \sim \mathbf{C}_{\{0,x\}} \right).$$

Similarly,

$$\mathbb{P}_p \left(0 \longleftrightarrow x \; ; \; \mathbf{C}_{\{0,x\}} \text{ is } (\delta,k)\text{-regular} \right) = \sum_{\mathfrak{T}^{(k)} \text{ is } \delta\text{-regular}} \mathbb{P}_p \left(0 \longleftrightarrow x \; ; \; \mathfrak{T}^{(k)} \sim \mathbf{C}_{\{0,x\}} \right).$$

We claim that there exists $\nu > 0$, such that

$$\mathbb{P}_p \left(0 \overset{d}{\longleftrightarrow} x \; ; \; \mathfrak{T}^{(k)} \sim \mathbf{C}_{\{0,x\}} \right) \leqslant \mathrm{e}^{-\nu n} \mathbb{P}_p \left(0 \longleftrightarrow x \; ; \; \mathfrak{T}^{(k)} \sim \mathbf{C}_{\{0,x\}} \right), \tag{3.3.1}$$

uniformly in all sufficiently large renormalization scales k and in all δ-regular tree-skeletons $\mathfrak{T}^{(k)}$.

Indeed, let us choose a sufficiently large number $r \in \mathbb{N}$; below we shall specify an appropriate choice, eventually it will depend on the value of α in the positive cone condition (3.1.1), but not on the particular renormalization scale k. Given r we associate, on every coarse graining scale k, a sequence of slabs $\mathcal{S}^j_{k,r}$; $j = 1, 2, ...$,

$$\mathcal{S}^j_{k,r} \triangleq \left\{ x \in \mathbb{Z}^d : \; |x_1 - 4jkr| \leqslant rk \right\}. \tag{3.3.2}$$

In other words, $\mathcal{S}^j_{k,r}$ is the lattice slab of the width $2kr$ centered at the point $4jrk\vec{e}_1 = (4jrk, 0)$. For a given tree-skeleton $\mathfrak{T}^{(k)}$, let us say that a slab $\mathcal{S}^j_{k,r}$ is good, if $4jrk < n$, and

$$\mathcal{S}^j_{k,r} \bigcap \bigcup_{z \in \mathfrak{T}^{(k)}_{\text{bad}}} k\mathbf{U}^{\mathrm{p}}(z) \;=\; \emptyset,$$

where the bad part of $\mathfrak{T}^{(k)}$ is defined via

$$\mathfrak{T}^{(k)}_{\text{bad}} \;\triangleq\; \mathfrak{L}^{(k)} \bigcup \left\{ \gamma^{(k)}(i) \,:\, \gamma^{(k)}(i+1) - \gamma^{(k)}(i) \notin \mathcal{C}_\epsilon(t) \right\}.$$

By (3.2.8) the number of good slabs $\mathcal{S}^j_{k,r}$ is, uniformly in all δ-regular tree skeletons $\mathfrak{T}^{(k)}$, bounded below:

$$\left\{ j \,:\, \mathcal{S}^j_{k,r} \text{ is good} \right\} \;\geqslant\; \frac{n}{8rk}, \tag{3.3.3}$$

as soon as δ is sufficiently small, which, again by (3.2.8), amounts to choosing a sufficiently big coarse-graining scale k.

So let us fix a δ-regular tree-skeleton $\mathfrak{T}^{(k)}$, and let us renumber all good slabs of $\mathfrak{T}^{(k)}$ as $\mathcal{S}^{j_1}_{k,r}, \ldots, \mathcal{S}^{j_m}_{k,r}$; $m \;\geqslant\; n/8rk$. For every good slab $\mathcal{S}^{j_l}_{k,r}$ and every cluster $\mathbf{C}_{\{0,x\}}$ compatible with $\mathfrak{T}^{(k)}$, the intersection $\mathbf{C}_{\{0,x\}} \cap \mathcal{S}^{j_l}_{k,r}$ is confined to the set $\mathbf{R}^{j_l}_{\mathfrak{T}^{(k)}}$:

$$\mathbf{C}_{\{0,x\}} \cap \mathcal{S}^{j_l}_{k,r} \;\subseteq\; \mathbf{R}^{j_l}_{\mathfrak{T}^{(k)}} \;\triangleq\; \bigcup_{\gamma^{(k)}(i) \in \mathcal{S}^{j_l}_{k,r}} 2k\mathbf{U}^{\mathrm{p}}\left(\gamma^{(k)}(i)\right).$$

By the construction, the number of points $\#\left(\mathbf{R}^{j_l}_{\mathfrak{T}^{(k)}}\right) \;\leqslant\; c_3(rk)^d$. Also all the $\gamma^{(k)}$-increments inside $\mathbf{R}^{j_l}_{\mathfrak{T}^{(k)}}$ are ϵ-good. Thus, if $r > c_4/\alpha$, then, in view of (3.1.1), one can locally modify at most $c_5(rk)^d$ bonds inside $\mathbf{R}^{j_l}_{\mathfrak{T}^{(k)}}$ in such a way that:

1. The modified cluster is still compatible with $\mathfrak{T}^{(k)}$.
2. There is at least one regeneration point inside $\mathbf{R}^{j_l}_{\mathfrak{T}^{(k)}}$.

Since these modifications could be performed independently in each of the sets $\mathbf{R}^{j_l}_{\mathfrak{T}^{(k)}}$; $l = 1, \ldots, m$; $m \;\geqslant\; n/2rk$, the inequality (3.3.1) follows. □

3.4. **Asymptotics of h-connectivities.** With the mass-gap condition (1.3.2) verified at all the points $\hat{t} \in \operatorname{int}(\mathcal{D}_{\mathbb{H}})$, we literally proceed as in [11]. In particular, the non-degeneracy condition (1.3.3) follows by the conditional variance argument as on pp.341-342 there.

Similarly , following the proof of Lemma 4.1 in [11], let us describe the analytic function Λ_p; $\Lambda_p : \mathbb{S}^{d-1}_{+} \mapsto \mathbb{R}$ appearing in the right hand side of (1.3.4):

Given $x = (n, k)$, choose the unique point $\hat{t} = \hat{t}(n,k) = \hat{t}(\vec{\mathrm{n}}(x)) \in \mathcal{D}_{\mathbb{H}}$, which satisfies

$$\frac{k}{n} \;=\; \nabla m_{\mathbb{H}}(\hat{t}).$$

The existence and uniqueness of such $\hat{t}$ follows from Theorem D. Set $t = (-m_{\mathbb{H}}(\hat{t}), \hat{t}) \in \partial\mathbf{K}^{\mathrm{p}}$ and

$$\kappa(\hat{t}) = \sum_{n=1}^{\infty} n\mathbb{F}_n(\hat{t})\mathrm{e}^{-nm_{\mathbb{H}}(\hat{t})}.$$

Then

$$\Lambda_p\left(\vec{\mathrm{n}}(x)\right) \;=\; \frac{(1-p)^{d-1}}{\kappa(\hat{t})\sqrt{(1 + \|k/n\|^2_{d-1})\rho_{\partial\mathbf{K}^{\mathrm{p}}}(t)}}, \tag{3.4.1}$$

where, as before, $\rho_{\partial\mathbf{K}^{\mathrm{p}}}(t)$ is the Gaussian curvature of the polar shape $\partial\mathbf{K}^{\mathrm{p}}$ at the point t.

3.5. **Asymptotics of full connectivities.** Because of the exponential bound (3.2.5) we, actually, have all the data to proceed as in [7] and [11] (see the detailed computation on pp.347-349 in the latter paper).

Namely, let $\alpha \in \mathbb{R}_+$ be fixed, and let $x = (n,k) \in \mathcal{H}_n \cap \mathcal{C}_\alpha$. Choose $t = t(x) = (-m_{\mathbb{H}}(\hat{t}), \hat{t})$ as in the preceding subsection. Of course, t is polar to x; $(t,x)_d = \xi_{\mathrm{p}}(x)$.

Decomposing the cluster $\mathbf{C}_{\{0,x\}}$ with respect to the left-most and right-most regeneration points on the interval $[1,...,n-1]$, we obtain that

$$\mathbb{P}_p\left(0 \longleftrightarrow x\right) \;=\; d(n,k) \;+\; \sum_{r=1}^{n} \sum_{l \in \mathbb{Z}^{d-1}} u(r,l) h(n-r, k-l);$$

where u is the connectivity function along the clusters with exactly one regeneration point, that is by the definition;

$$\left\{0 \overset{u}{\longleftrightarrow} x\right\} \iff \exists \text{ unique } r \in [1,...,m-1] : \; \#\left(\mathbf{C}_{\{0,x\}} \cap \mathcal{H}_r\right) = 1.$$

We set $u(r,l) = \mathbb{P}_p\left(0 \overset{u}{\longleftrightarrow} (r,l)\right)$.

As in the case of d-connectivities the results of Section 3 imply that there exists $\nu' > 0$, such that

$$\mathbb{U}_r(\hat{t}) \;\triangleq\; \sum_{l \in \mathbb{Z}^{d-1}} u(r,l) \mathrm{e}^{(\hat{t},l)_{d-1}} \;\leqslant\; \mathrm{e}^{-\nu' r + r m_{\mathbb{H}}(\hat{t})},$$

uniformly in $r \in \mathbb{N}$. Using the local asymptotics (1.3.4) of h-connectivities, we, therefore, arrive to (1.2.5) of Lemma 1.1 with

$$\Phi_p\left(\vec{\mathfrak{n}}(x)\right) \;=\; \sum_{r=1}^{\infty} r \mathbb{U}_r(\hat{t}) \mathrm{e}^{-r m_{\mathbb{H}}(\hat{t})}.$$

4. DIRECT APPROACH

4.1. **Renewal along generic directions.** Let $t \in \partial \mathbf{K}^{\mathrm{p}}$. Given $x, y \in \mathbb{Z}^d$ we define the hyperplane

$$\mathcal{H}_x^t = \{z \in \mathbb{R}^d \mid (t,z)_d = (t,x)\},$$

and the slab

$$\mathcal{S}_{x,y}^t = \{z \in \mathbb{R}^d \mid (t,x)_d \;\leqslant\; (t,z)_d \leq (t,y)_d\}.$$

If $(t,x)_d > (t,y)_d$, we set $\mathcal{S}_{x,y}^t = \emptyset$.

We shall define connectivity functions h_t, f_t associated with t.

Let e be a unit vector in the direction of one of the axes such that the scalar product of e with t is maximal.

Definition For $x, y \in \mathbb{Z}^d$ let $\left\{x \overset{h_t}{\longleftrightarrow} y\right\}$ to denote the (possibly empty) event that

1. x and y are connected in the restriction of the percolation configuration to the slab $\mathcal{S}_{x,y}^t$. Let $\mathbf{C}_{x,y}^t$ to denote the corresponding common cluster. If $x \neq y$, then in addition;
2. $\mathbf{C}_{x,y}^t \cap \mathcal{S}_{x,x+e}^t = \{x, x+e\}$ and $\mathbf{C}_{x,y}^t \cap \mathcal{S}_{y-e,y}^t = \{y-e, y\}$

□

Set

$$h_t(x) \;\triangleq\; \mathbb{P}_p\left(0 \overset{h_t}{\longleftrightarrow} x\right).$$

Notice that $h_t(0) = 1$, and, by the translational invariance, that $h_t(x) = \mathbb{P}_p\left(z \overset{h_t}{\longleftrightarrow} z + x\right)$ for every $x, z \in \mathbb{Z}^d$. Also, as in the case of the parameterized h-connectivities, it is easy to show, that

$$\xi_{\mathrm{p}}(x) \;=\; -\lim_{n\to\infty} \frac{1}{n} \log h_t([nx]) \tag{4.1.1}$$

for any t and x satisfying $(t,x)_d > 0$.

The d-dimensional array of h_t-connectivities possesses a natural renewal structure, which we proceed to describe:

Definition For $x, y \in \mathbb{Z}^d$ let us say that they are f_t-connected; $\left\{x \overset{f_t}{\longleftrightarrow} y\right\}$, if

1. The event $\left\{x \overset{h_t}{\longleftrightarrow} y\right\}$ happens, and $x \neq y$.
2. For no $z \in \mathbb{Z}^d \setminus \{x, y\}$ both
$$\left\{x \overset{h_t}{\longleftrightarrow} z\right\} \qquad \text{and} \qquad \left\{z \overset{h_t}{\longleftrightarrow} y\right\} \tag{4.1.2}$$
take place.

$\square$

Set
$$f_t(x) \ \overset{\Delta}{=}\ \mathbb{P}_p\left(0 \overset{f_t}{\longleftrightarrow} x\right).$$
Notice that $f_t(0) = 0$, and, moreover, the f_t-connectivities are $\mathbb{Z}^d$-shift invariant.

The parameterized construction of the preceding sections corresponds to the choice of t along one of the axis direction. The point we are making here is that it is natural to relate the asymptotics of $\mathbb{P}_p\left(0 \longleftrightarrow x\right)$ to the asymptotics of the h_{t_0}-connectivities, where $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$ is essentially chosen to be polar to x.

From now on we assume that $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$ is not orthogonal to any of the axis directions. We shall adjust the notion of regeneration point to the direction t_0: Let $y \in \mathbb{Z}^d$ and assume that y is connected to the origin. We say that $z \in \mathbb{Z}^d$ is a regeneration point of $\mathbf{C}_{\{0,y\}}$ if

1. $(t_0, e)_d \leq (t_0, z)_d \leq (t_0, y)_d - (t_0, e)_d$
2. $\mathcal{S}^t_{z-e,z+e} \cap \mathbf{C}_{\{0,y\}}$ contains exactly three points $z - e$, z and $z + e$, where e is a unit axis direction, such that the scalar product $(t_0, e)_d$ is maximal.

For any point y and for any realization of the cluster $\mathbf{C}^{t_0}_{0,y}$ there are at most finite number of regeneration points. Notice that if 0 and y are h_{t_0} connected and z is a regeneration point, then (4.1.2) is satisfied (with $x = 0$). If there is no such points at all then, by the definition, 0 is f_t-connected to x. Otherwise, 0 is f_t-connected to the regeneration point z, which has the minimal t-projection. Using the corresponding decomposing the clusters $\mathbf{C}^{t_0}_{0,x}$ one gets the following "renewal type" equation:

$$h_{t_0}(y) = \sum_{z \in \mathbb{Z}^d} f_{t_0}(z) h_{t_0}(y - z) \tag{4.1.3}$$

4.2. **Regeneration points.** By compactness of $\partial\mathbf{K}^{\mathrm{p}}$ for every $\epsilon \in (0, 1/2)$ one can choose $\lambda = \lambda(\epsilon) > 0$, such that; for every $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$,

$$t \in B_\lambda(t_0) \cap \partial\mathbf{K}^{\mathrm{p}} \ \overset{\Delta}{=}\ \left\{s \in \mathbb{R}^d \ : \ \|s - t_0\|_d \ \leqslant\ \lambda\right\} \cap \partial\mathbf{K}^{\mathrm{p}} \qquad \text{implies that} \qquad \mathcal{C}_\epsilon(t) \subset \mathcal{C}_{2\epsilon}(t_0) \tag{4.2.4}$$

uniformly in $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$.

This is the appropriate version of the positive cone condition for the t_0- adjusted renewal structure. Notice that the forthcoming proofs do not rely on the lattice symmetries of $\mathbb{Z}^d$. Let us use $\mathcal{R}^{t_0}_x$ to denote the (random) set of t_0-regeneration points of $\mathbf{C}_{\{0,x\}}$. The following lemma gives a uniform probabilistic estimate on the typical size of $\mathcal{R}^{t_0}_x$ as $\|x\|_d$ increases.

Lemma 4.1. *For every $\epsilon \in (0, \frac{1}{2})$ there exists $\delta > 0$ and $\nu > 0$ such that*

$$\mathbb{P}_p\left(\#(\mathcal{R}^{t_0}_x) < \delta\|x\|_d \ ; \ 0 \overset{h_{t_0}}{\longleftrightarrow} x\right) \leq \exp(-(t, x)_d - \nu\|x\|_d) \tag{4.2.5}$$

uniformly in $t_0 \in \partial\mathbf{K}^p$, $t \in B_\lambda(t_0) \cap \partial\mathbf{K}^p$ and $x \in \mathcal{C}_\epsilon(t)$, where $\lambda = \lambda(\epsilon)$ has been defined in (4.2.4).

Proof. The proof is identical to that of the estimate (3.3.1) in Section 3. The only required modification is to redefine the slabs $\mathcal{S}^j_{k,r}$ as

$$\mathcal{S}^j_{k,r} \triangleq \{y \in \mathbb{Z}^d : |\,(t_0) - 4jkr\,| \leq rk\},$$

instead of the parameterized definition employed there. ☐

For $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$ and $t \in \mathbb{R}^d$ define

$$\mathbb{H}_{t_0}(t) \triangleq \sum_{x \in \mathbb{Z}^d} h_{t_0}(x) e^{(t,x)_d},$$

$$\mathbb{F}_{t_0}(t) \triangleq \sum_{x \in \mathbb{Z}^d} f_{t_0}(x) e^{(t,x)_d}.$$

An almost immediate consequence of Lemma 4.1 is the mass-gap type condition for the t_0-adjusted connectivities.

Lemma 4.2. *For every $\epsilon \in \left(0, \frac{1}{2}\right)$ there exists $\overline{\lambda} = \overline{\lambda}(\epsilon) > 0$ such that, uniformly in $t_0 \in \partial\mathbf{K}^p$*

$$\mathbb{F}_{t_0}(t) < \infty \text{ on } B_{\overline{\lambda}}(t_0). \tag{4.2.6}$$

Furthermore, for every $t_0 \in \partial\mathbf{K}^p$, the implicit description of $\partial\mathbf{K}^p$ in the $\partial\mathbf{K}^p \cap B_{\overline{\lambda}}(t_0)$ neighbourhood of t_0 is given by

$$t \in \partial\mathbf{K}^p \cap B_{\overline{\lambda}}(t_0) \iff \mathbb{F}_{t_0}(t) = 1. \tag{4.2.7}$$

$\partial\mathbf{K}^p$ is a real analytic surface and it is strictly convex with Gaussian curvature uniformly bounded away from 0.

Proof. With $\epsilon \in (0, \frac{1}{2})$ let $\lambda = \lambda(\epsilon)$ as in (4.2.4). For every $t \in \mathbb{R}^d \backslash 0$ define t_p to be the unique point of the boundary $\partial\mathbf{K}^{\mathrm{p}}$ in the direction of p. By the very definition of $\lambda(\epsilon)$ in (4.2.4)

$$\mathbb{P}_p(0 \overset{f_{t_0}}{\longleftrightarrow} x) \leq \mathbb{P}_p(0 \longleftrightarrow x) \leq \exp\left(-(t_p, x)_d - c_1\, \epsilon \|x\|_d \right) \tag{4.2.8}$$

uniformly over $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$, $t \in B_\lambda(t_0)$ and $x \notin \mathcal{C}_\epsilon(t_p)$. Consequently, there exists $\lambda_1 = \lambda_1(\epsilon)$, such that

$$\mathbb{P}_p(0 \overset{f_{t_0}}{\longleftrightarrow} x) \leq \exp\left(-(t, x)_d - c_1\, \epsilon \|x\|_d \right) \tag{4.2.9}$$

also uniformly in $t \in B_{\lambda_1}(t_0)$ and $x \notin \mathcal{C}_\epsilon(t_p)$.

In a similar fashion, using the fact that f_{t_0}-clusters have no regeneration point at all, we infer from (4.2.5) that there exist $\nu' > 0$ and $\lambda_2 > 0$, such that

$$\mathbb{P}_p(0 \overset{f_{t_0}}{\longleftrightarrow} x) \leq \exp\left(-(t, x)_d - \nu' \|x\|_d \right) \tag{4.2.10}$$

uniformly over $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$, $t \in B_{\lambda_2}(t_0)$ and $x \in \mathcal{C}_\epsilon(t_p)$. Thus (4.2.6) follows with $\overline{\lambda}(\epsilon) = \lambda_1(\epsilon) \wedge \lambda_2(\epsilon)$.

Since for $t \in \mathrm{int}(\mathbf{K}^{\mathrm{p}}) \cap B_{\overline{\lambda}}(t_0)$ the moment generating function $\mathbb{H}_{t_0}$ is finite and, moreover,

$$\mathbb{H}_{t_0}(t) = \frac{1}{1 - \mathbb{F}_{t_0}(t)}. \tag{4.2.11}$$

(4.2.7) follows by the continuity of $\mathbb{F}_{t_0}$ and the fact that $\mathbb{H}_{t_0}$ diverges on $B_\lambda(t_0) \setminus \mathbf{K}^{\mathrm{p}}$.

Given $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$, $t \in \mathbb{R}^d$ and $x \in \mathbb{Z}^d$ let us define the measure

$$\mathbb{Q}^t_{t_0}(x) = f_{t_0}(x) e^{(t,x)_d}. \tag{4.2.12}$$

By (4.2.6) and (4.2.7), $\mathbb{Q}^t_{t_0}$ is a probability measure with exponentially decaying tails whenever $t \in B_{\overline{\lambda}}(t_0) \cap \partial\mathbf{K}^{\mathrm{p}}$.

In the latter case let $\mu_{t_0}(t) \in \mathbb{R}^d$ be the expectation of a random variable X under the probability distribution $\mathbb{Q}_{t_0}^t$:

$$\mu_{t_0}(t) = \mathbb{E}_{t_0}^t X = \sum_{x\in\mathbb{Z}^d} x\mathbb{Q}_{t_0}^t(x) = \nabla \log \mathbb{F}_{t_0}(t). \tag{4.2.13}$$

Let $A_{t_0}(t) = \mathrm{Hess}(\log \mathbb{F}_{t_0}(t))$ be the corresponding covariance matrix. It is straightforward to check that $A_{t_0}(t)$ is uniformly (in $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$ and $t \in B_{\overline{\lambda}}(t_0) \cap \partial\mathbf{K}^{\mathrm{p}}$) non-degenerate. Consequently, as the measure $\mathbb{Q}_{t_0}^t$ is concentrated on one side of a hyperplane containing the origin, $\mu_{t_0}^t \neq 0$. By the analytic implicit function theorem we infer that $\partial\mathbf{K}^{\mathrm{p}}$ is a real analytic surface in a neighbourhood of t_0. Similarly, strict convexity of and positive Gaussian curvature of $\partial\mathbf{K}^{\mathrm{p}}$ at t_0 follow from the strict convexity of $\log \mathbb{F}_{t_0}$ and non-degeneracy of A_{t_0} in a neighbourhood of this point. ☐

Using the general theory of convex bodies ([16], [11])) we can obtain a corresponding result for the surface $\partial\mathbf{U}^{\mathrm{p}}$ which is polar to $\partial\mathbf{K}^{\mathrm{p}}$.

Lemma 4.3. *The surface $\partial\mathbf{U}^p$ is an analytic convex surface with Gaussian curvature uniformly bounded away from* 0. *The Gaussian curvatures of $\partial\mathbf{U}^p$ and $\partial\mathbf{K}^p$ at two conjugate points x and t are reciprocal of one other.*

Remark. It should be mentioned that our choice of the renewal relation (4.1.3), and, accordingly of the connectivity type functions h_t and f_t is certainly not the only possible one. This is, however, not that important - whatever notion of regeneration points one employs, the localized structure of percolation clusters (in the sense of renormalization results of Section 2) will lead to an appropriate version of the entropic bound (4.2.5). ☐

4.3. **Tilted measures and strict convexity.**

I is instructive to give a direct proof of the strict convexity of the norm ξ_{p}. Notice that the argument below suggests that this property is, in fact, a purely entropic phenomenon.

Lemma 4.4. *The norm ξ_p is strictly convex.*

Proof. First of all we shall show that for every $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$ and each $t \in B_{\overline{\lambda}}(t_0) \cap \partial\mathbf{K}^{\mathrm{p}}$, the points t and $\mu_{t_0}(t)$ (defined in (4.2.13)) are in polar relation:

$$(t, \mu_{t_0}(t))_d = \xi_{\mathrm{p}}(\mu_{t_0}(t)). \tag{4.3.14}$$

Moreover, we claim that for every $\mu \neq \mu_{t_0}(t)$ with $\|\mu - \mu_{t_0}(t)\|_d \leq 1$ and $(\mu, t)_d = (\mu_{t_0}(t), t)_d$;

$$\xi_{\mathrm{p}}(\mu) \geq (\mu, t)_d + c_4\|\mu - \mu_{t_0}(t)\|_d^2 = \xi_{\mathrm{p}}(\mu_{t_0}(t)) + c_4\|\mu - \mu_{t_0}(t)\|_d^2, \tag{4.3.15}$$

for some strictly positive constant c_4. Strict convexity of ξ_{p} then instantly follows.

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables distributed according to $\mathbb{Q}_{t_0}^t$. One can then rewrite the renewal relation (4.1.3) as

$$h_{t_0}([n\mu]) = \delta_0([n\mu]) + \exp(-(t, [n\mu])_d) \sum \bigotimes_1^k \mathbb{Q}_{t_0}^t(X_1 + \ldots + X_k = [n\mu]). \tag{4.3.16}$$

Since $\mathbb{Q}_{t_0}^t$ is supported by $\{x \in \mathbb{Z}^d \,|\, (t_0, x)_d \geq 0\}$, the expected value

$$(t_0, \mu_{t_0}(t))_d = \mathbb{E}_{t_0}^t(t_0, X_1) > 0.$$

Thus, for $t \in B_\lambda(t_0)$ with λ sufficiently small; $(t, \mu_{t_0}(t))_d > 0$ as well. For these $t \in B_\lambda(t_0)$ there exist $c_5,\ c_6 > 0$, such that

$$\| n\mu - \sum_{i=1}^k \mathbb{E}_{t_0}^t X_i \|_d = \| n\mu - k\mu_{t_0}(t) \|_d \geq c_5 \,|n-k|\, \|\mu_{t_0}\|_d + c_6\, n\, \|\mu - \mu_{t_0}\|_d,$$

for every $k,\ n\ \in\ \mathbb{N}$. By the usual large deviation upper bound,

$$\bigotimes_1^k\ \mathbb{Q}_{t_0}(\sum_{i=1}^k X_i\ =\ [n\mu])\ \leq\ \exp\left(-c_7\ \frac{(n-k)^2}{k}\wedge|n-k|\ -\ c_8\ \frac{n^2}{k}\wedge n\|\mu-\mu_{t_0}(t)\|_d^2\ \right). \tag{4.3.17}$$

Substituting the latter estimate to (4.3.16), we obtain:

$$h_{t_0}([n\mu])\ \leq\ c_9\ \sqrt{n}\ \exp\left(-(t,\mu)_d\ -\ c_{10}\ n\ \|\mu-\mu_{t_0}(t)\|_d^2\ \right).$$

The claim (4.3.15) follows from the asymptotic relation (4.1.1). □

4.4. **Local limit structure of connectivities.** Let us fix $\epsilon > 0$ sufficiently small. We shall give sharp large-$\|x\|_d$ asymptotics of $h_{t_0}(x)$ uniformly over $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$ and $x \in \mathcal{C}_\epsilon(t_0)$. As in the parameterized approach of Subsection 3.5, the passage from the asymptotics of $h_{t_0}(x)$ to the full asymptotics (1.1.6) is, in view of the mass-gap assertion of Lemma 4.1, secured by the decomposition of the cluster $\mathbf{C}_{\{0,x\}}$ with respect to the left-most and right-most t_0-regeneration points. In other words, the claim of Theorem A follows, once we show that:

Lemma 4.5. *Uniformly in $t_0 \in \partial\mathbf{K}^p$ and in $x \in \mathcal{C}_\epsilon(t_0)\cap\mathbb{Z}^d$;*

$$h_{t_0}(x)\ =\ \frac{\Lambda_{t_0}\left(\vec{\mathrm{n}}(x)\right)}{\sqrt{(2\pi\|x\|_d)^{d-1}}}e^{-\xi_{\mathrm{p}}(x)}\left(1\ +\ o(1)\right). \tag{4.4.18}$$

Proof. We use notation and results from the previous subsections. Since $A_{t_0}(t) = \mathrm{Hess}\left(\log\mathbb{F}_{t_0}(t)\right)$ is non-degenerate at $t = t_0$, the cone $\mathcal{C}_\epsilon(t_0)$ lies inside the cone generated by the vectors

$$\left\{\mu_{t_0}(t) = \mathbb{E}_{t_0}^t X_1\ \middle|\ t \in B_{\overline{\lambda}}(t_0)\cap\partial\mathbf{K}^{\mathrm{p}}\right\}.$$

In particular, for every $x \in \mathcal{C}_\epsilon(t_0)$ there exists $t \in B_{\overline{\lambda}}(t_0)$, such that x and t are in the polar relation, which, by the relations (4.3.14) and (4.3.15) of the preceding subsection, means that

$$\vec{\mathrm{n}}(x)\ =\ \frac{\mu_{t_0}(t)}{\|\mu_{t_0}(t)\|_d}. \tag{4.4.19}$$

Furthermore, there exists a number $n = n(x)$, such that

$$\|x - n\mu_{t_0}(t)\|_d \leq c_8, \tag{4.4.20}$$

the latter estimate being uniform in $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$ and $x \in \mathcal{C}_\epsilon(t_0)$.

We now rewrite 4.3.16 as

$$h_{t_0}(x)\ e^{\xi_{\mathrm{p}}(x)}\ =\ \delta_0(x)\ +\ \exp(-(t,x)_d)\ \sum_{k=1}^{\infty}\ \bigotimes_1^k\ \mathbb{Q}_{t_0}^t(X_1+\ldots+X_k = x). \tag{4.4.21}$$

Fix $\alpha \in (0,\frac{1}{2})$. Notice that the support of $Q_{t_0}^t$ spans the whole lattice $\mathbb{Z}^d$. In view of (4.4.20), uniform exponential bounds (4.2.6) on the tails of $\mathbb{Q}_{t_0}^t$ and uniform non-degeneracy of A_{t_0}, we, by the usual d-dimensional local CLT, infer:

$$\bigotimes_1^k\ \mathbb{Q}_{t_0}(X_1+\ldots+X_k = x)\ =\ \frac{\exp\left\{-\frac{(n-k)^2}{2n}\left(A_{t_0}^{-1}\mu_{t_0},\mu_{t_0}\right)_d(t)\right\}}{\sqrt{(2\pi n)^d\det(A_{t_0}(t))}}\times\left(1\ +\ o(1)\right) \tag{4.4.22}$$

uniformly over $t_0 \in \partial\mathbf{K}^{\mathrm{p}}$, $x \in \mathcal{C}_\epsilon(t_0)$ satisfying (4.4.20) and k in the range $|n-k| < n^{1/2+\alpha}$.

In the remaining range of k's one, proceeding as in (4.3.17), we obtain:

$$\sum_{|k-n|\geq n^{1/2+\alpha}}\ \bigotimes_1^k\ \mathbb{Q}_{t_0}(X_1+\ldots+X_k = x)\ \leq \exp(-c_9\ n^{2\alpha}). \tag{4.4.23}$$

Substituting (4.4.22) and (4.4.23) into (4.4.21) we, using (4.4.20), recover (4.4.18) with

$$\Lambda_{t_0}\left(\vec{\mathrm{n}}(x)\right) \;=\; \sqrt{\frac{\|\mu_{t_0}(t)\|_d^{d-1}}{\left(A_{t_0}^{-1}\mu_{t_0},\mu_{t_0}\right)_d(t)\,\det(A_{t_0}(t))}}.$$

Finally, the analyticity of Λ_{t_0} follows from the relation (4.4.19) and the analyticity of $\log \mathbb{F}_{t_0}$, which has been discussed in Subsection 4.2. ☐

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