Escape Probability, Mean Residence Time and Geophysical Fluid Particle Dynamics

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Abstract

Stochastic dynamical systems arise as models for fluid particle motion in geophysical flows with random velocity fields. Escape probability (from a fluid domain) and mean residence time (in a fluid domain) quantify fluid transport between flow regimes of different characteristic motion.

We consider a quasigeostrophic meandering jet model with random perturbations. This jet is parameterized by the parameter $\beta = \frac{2\Omega}{r} \cos(\theta)$, where Ω is the rotation rate of the earth, r the earth's radius and θ the latitude. Note that Ω and r are fixed, so β is a monotonic decreasing function of the latitude. The unperturbed jet (for $0 < \beta < \frac{2}{3}$) consists of a basic flow with attached eddies. With random perturbations, there is fluid exchange between regimes of different characteristic motion. We quantify the exchange by escape probability and mean residence time.

For an eddy, the average escape probability for fluid particles (initially inside the eddy) escape into the exterior retrograde region is smaller than escape into the jet core for $0 < \beta < 0.3333$, while for $0.3333 < \beta < \frac{2}{3}$, the opposite holds.

For a unit jet core near the jet troughs, the average escape probability for fluid particles (initially inside the jet core) escape into the northern recirculating region is greater than escape into the southern recirculating region for $0 < \beta < 0.115$, while for $0.385 < \beta < \frac{2}{3}$, the opposite holds. Moreover, for $0.115 < \beta < 0.385$, fluid particles are about equally likely to escape into either recirculating regions. Furthermore, for a unit jet core near the jet crests, the situation is the opposite as for near the jet troughs.

The maximal mean residence time of fluid particles initially in an eddy increases as β increases from 0 to 0.432 (or as latitude decreases accordingly), then decreases as β increases from 0.432 to $\frac{2}{3}$ (or as latitude decreases accordingly). However, the maximal mean residence time of fluid particles initially in a unit jet core always increases as β increases (or as latitude decreases).

Key Words: stochastic dynamics, transport and mixing, geophysical flows, escape probability, mean residence time

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1 Stochastic dynamics: Escape probability and mean residence time

Stochastic dynamical systems are used as models for various scientific and engineering problems. We consider the following class of stochastic dynamical systems

$$\begin{aligned}\dot{x} &= a_1(x,y) + b_1(x,y)\dot{w}_1, &(1)\\ \dot{y} &= a_2(x,y) + b_2(x,y)\dot{w}_2, &(2)\end{aligned}$$

where $w_1(t), w_2(t)$ are two real independent Wiener processes (white noises) and a_1, a_2, b_1, b_2 are given deterministic functions. More complicated stochastic systems also occur in applications ([1], [25], [10]), [11], [34], [17]).

For a planar bounded domain D, we can consider the exit problem of random solution trajectories of (1)-(2) from D. To this end, let ∂D denote the boundary of D and let Γ be a part of the boundary ∂D. The escape probability $p(x,y)$ is the probability that the trajectory of a particle starting at (x,y) in D first hits ∂D (or escapes from D) at some point in Γ, and $p(x,y)$ is known to satisfy ([19], [32] and references therein)

$$\begin{aligned}\frac{1}{2}b_1^2(x,y)p_{xx} + \frac{1}{2}b_2^2(x,y)p_{yy} + a_1(x,y)p_x + a_2(x,y)p_y &= 0, &(3)\\ p|_{\Gamma} &= 1, &(4)\\ p|_{\partial D-\Gamma} &= 0. &(5)\end{aligned}$$

Suppose that initial conditions (or initial particles) are uniformly distributed over D. The average escape probability P that a trajectory will leave D along the subboundary Γ, before leaving the rest of the boundary, is given by (e.g., [19], [32])

$$P = \frac{1}{|D|}\int\int_D p(x,y)dxdy, \tag{6}$$

where $|D|$ is the area of domain D.

The residence time of a particle initially at (x,y) inside D is the time until the particle first hits ∂D (or escapes from D). The mean residence time $u(x,y)$ is given by (e.g., [32], [24], [28] and references therein)

$$\begin{aligned}\frac{1}{2}b_1^2(x,y)u_{xx} + \frac{1}{2}b_2^2(x,y)u_{yy} + a_1(x,y)u_x + a_2(x,y)u_y &= -1, &(7)\\ u|_{\partial D} &= 0. &(8)\end{aligned}$$

2 A quasigeostrophic jet model

The Lagrangian view of fluid motion is particularly important in geophysical flows since only Lagrangian data can be obtained in many situations. It is essential to understand fluid particle trajectories in many fluid problems. Escape probability (from a fluid domain) and mean residence time (in a fluid domain) quantify fluid transport between flow regimes of different characteristic motion. Deterministic quantities like escape probability and mean residence time can be computed by solving Fokker-Planck type partial differential equations.

We now use the these ideas in the investigation of meandering oceanic jets. Meandering oceanic jets such as the Gulf Stream are strong currents dividing different bodies of water.

Recently, del-castillo-Negrete and Morrison ([6]), and Pratt et al. ([27], [22]) have studied models for oceanic jets. These models are dynamically consistent to within a linear approximation, i.e., the potential vorticity is approximately conserved. Del-castillo-Negrete and Morrison's model consists of the basic flow plus time-periodic linear neutral modes.

In this paper, we consider an oceanic jet consisting of the basic flow as in del-castillo-Negrete and Morrison ([6]), plus random-in-time noise. This model incorporates small-scale oceanic motions such as the molecular diffusion ([16]), which is an important factor in the Gulf Stream ([9], [12]). The irregularity of RAFOS floats ([3], [33], [20]) also suggests the inclusion of random effects in Gulf Stream modeling.

This random jet may also be viewed as satisfying, approximately in the spirit of del-castillo-Negrete and Morrison ([6]), the randomly wind forced quasigeostrophic model. Several authors have considered the randomly wind forced quasigeostrophic model in order to incorporate the impact of uncertain geophysical forces ([30], [13], [15], [23], [7], [4]). They studied statistical issues such as estimating correlation coefficients for the *linearized* quasigeostrophic equation with random forcing. There is also recent work which investigates the impact of the uncertainty of the ocean bottom topography on quasigeostrophic dynamics ([18]).

The randomly forced quasigeostrophic equation takes the form ([23])

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\psi_x = \frac{dW}{dt} \ , \tag{9}$$

where $W(x, y, t)$ is a space-time Wiener process (white noise). The stream function would have a random or noise component ([12], [9]). Note that β is the meridional derivative of the Coriolis parameter [26] [5], i.e., $\beta = \frac{2\Omega}{r}\cos(\theta)$, where Ω is the rotation rate of the earth, r the earth's radius and θ the latitude. Since Ω and r are fixed, β is a monotonic decreasing function of the latitude.

The deterministic meandering jet derived in del-castillo-Negrete and Morrison ([6]) is

$$\Psi(x, y) = -\tanh(y) + a\ sech^2(y)\cos(kx) + cy,$$

where

$$a = 0.01, c = \frac{1}{3}(1 + \sqrt{1 - \frac{3}{2}\beta}), k = \sqrt{2(1 + \sqrt{1 - \frac{3}{2}\beta})}, 0 \le \beta \le \frac{2}{3}.$$

This $\Psi(x, y)$ is an approximate solution of the usual quasigeostrophic model

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\psi_x = 0 \ . \tag{10}$$

With random wind forcing or molecular diffusive forcing in the stochastic quasigeostrophic model (9), the stream function would have a random or noise component. We approximate this noise component by adding a noise term to the above deterministic stream function $\Psi(x, y)$, that is, in the rest of this paper, we consider the following random stream function as a model for a quasigeostrophic meandering jet,

$$\tilde{\Psi}(x, y) = -\tanh(y) + a\ sech^2(y)\cos(kx) + cy + \text{noise}.$$

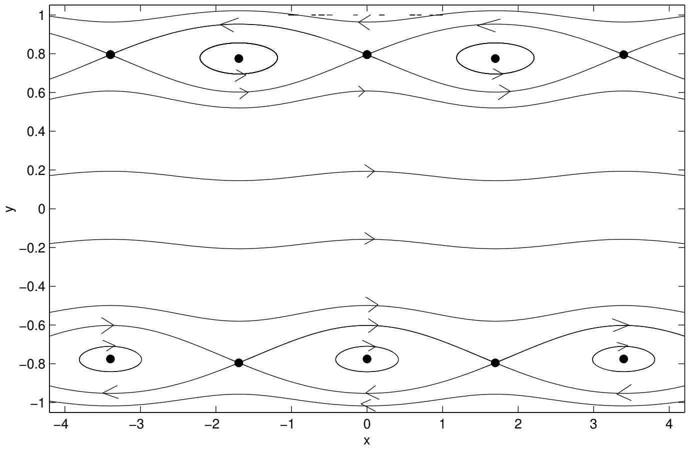

Figure 1: Unperturbed jet: $\epsilon = 0$ and $\beta = 1/3$.

The equations of motion for fluid particles in this jet then have noise terms. We further approximate them as white noises (or Wiener processes)

$$\begin{aligned} dx &= -\Psi_y dt + \sqrt{\epsilon} dw_1, && (11) \\ dy &= \Psi_x dt + \sqrt{\epsilon} dw_2, && (12) \end{aligned}$$

or more specifically,

$$\begin{aligned} dx &= [sech^2(y) + 2a\ sech^2(y)\tanh(y)\cos(kx) - c]dt + \sqrt{\epsilon} dw_1, && (13) \\ dy &= -ak\ sech^2(y)\sin(kx)dt + \sqrt{\epsilon} dw_2, && (14) \end{aligned}$$

where $0 < \epsilon < 1$, and $w_1(t), w_2(t)$ are two real independent Wiener processes (in time only). The calculations below are for $\epsilon = 0.001$. Note that β is now the only parameter 13, 14, as a is given and c, and k depend only on $\beta \in [0, \frac{2}{3}]$.

When $\epsilon = 0$, the deterministic jet consists of the jet core and two rows of recirculating eddies, which are called the northern and southern recirculating regions. Outside the recirculating regions are the exterior retrograde regions; see Figure 1.

3 Escape probability

We take D to be either an eddy or a piece of jet core (see Figures 2, 3). This piece of jet core has the same horizontal length scale as an eddy, and it is one period of the deterministic jet core (note that the deterministic velocity field is periodic in x). Thus we call this piece of jet core a unit jet core.

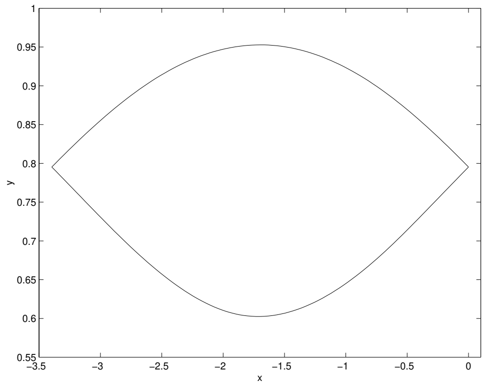

Figure 2: An eddy: $\beta = 1/3$

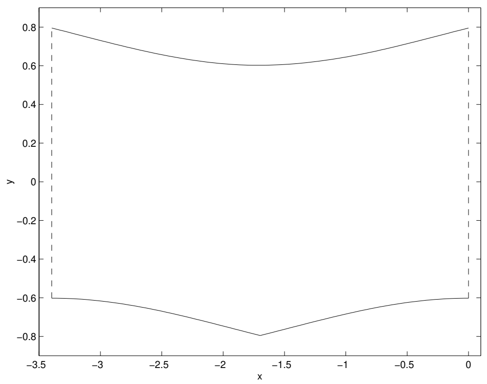

Figure 3: A unit jet core near a trough: $\beta = 1/3$

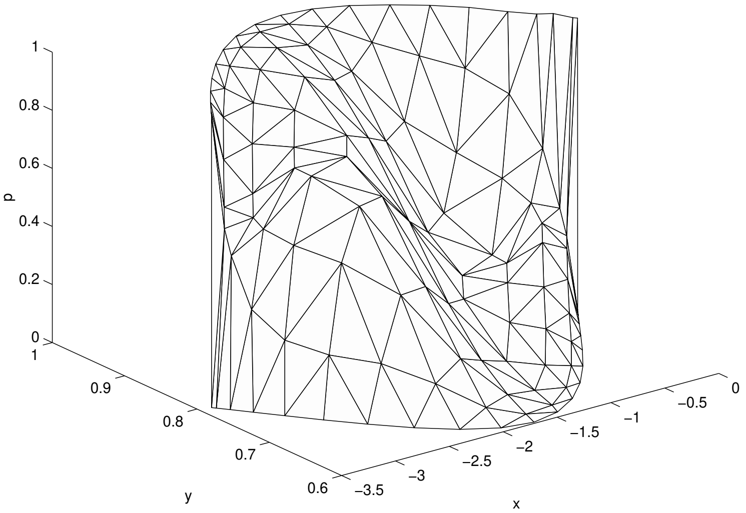

Figure 4: Escape probability of fluid particles (initially in an eddy in Figure 2) exiting into the exterior retrograde region: $\beta = 1/3$

From (3), (4), (5), the escape probability $p(x, y)$ that a fluid particle, initially at (x, y), crosses the subboundary Γ of the domain D satisfies

$$\begin{aligned} \epsilon\Delta p + [sech^2(y) + 2a\ sech^2(y)\tanh(y)\cos(kx) - c]p_x - ak\ sech^2(y)\sin(kx)p_y &= 0, && (15) \\ p|_\Gamma &= 1, && (16) \\ p|_{\partial D - \Gamma} &= 0. && (17) \end{aligned}$$

We take Γ to be either top or bottom boundary of an eddy or a unit jet core (see Figures 2, 3). We numerically solve this elliptic system for various values of β between 0 and $\frac{2}{3}$. In the unit jet core case, we take periodic boundary condition in horizontal (meridional) x direction, with period $\frac{2\pi}{k}$.

A piecewise linear, finite element approximation scheme was used for the numerical solutions of the escape probability $p(x, y)$, and the mean residence time $u(x, y)$, described by the elliptic equations (3), and (7), respectively. Using a collection of points lying on the boundary, piecewise cubic splines were constructed to define the boundary of the eddy, and the top and bottom boundaries of the jet core. Computational (triangular) grids for the eddy and the jet core were then obtained by deforming regular grids constructed for an ellipse and a rectangular region, respectively. The computed escape probability crossing the upper or lower boundary of an eddy or a unit jet core, for the case of $\beta = \frac{1}{3}$, are shown in Figures 4, 5, 6 and 7.

Suppose that the fluid particles are initially uniformly distributed in D (an eddy or a unit jet core). We can also compute the average escape probability P that a particle will leave D along the upper or lower subboundary Γ, using the formula (6); see Figures 8 and 9.

For an eddy, the average escape probability for fluid particles (initially inside the eddy) escape into the exterior retrograde region is smaller than escape into the jet core for $0 < \beta < 0.3333$, while for

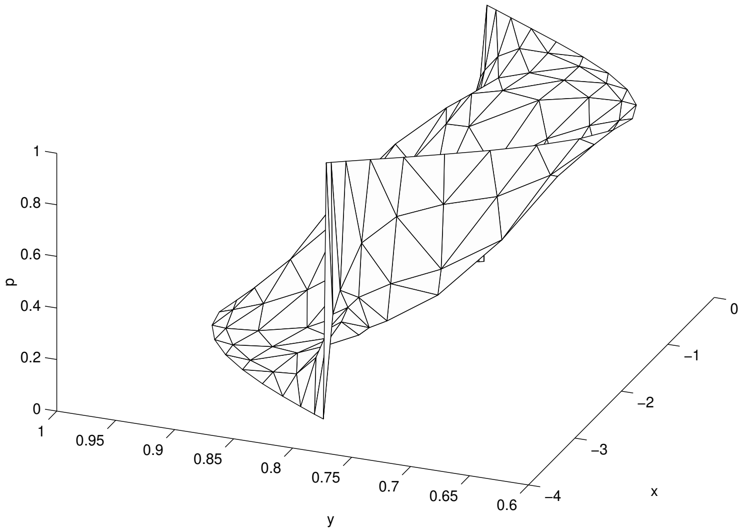

Figure 5: Escape probability of fluid particles (initially in an eddy in Figure 2) exiting into the jet core: $\beta = 1/3$

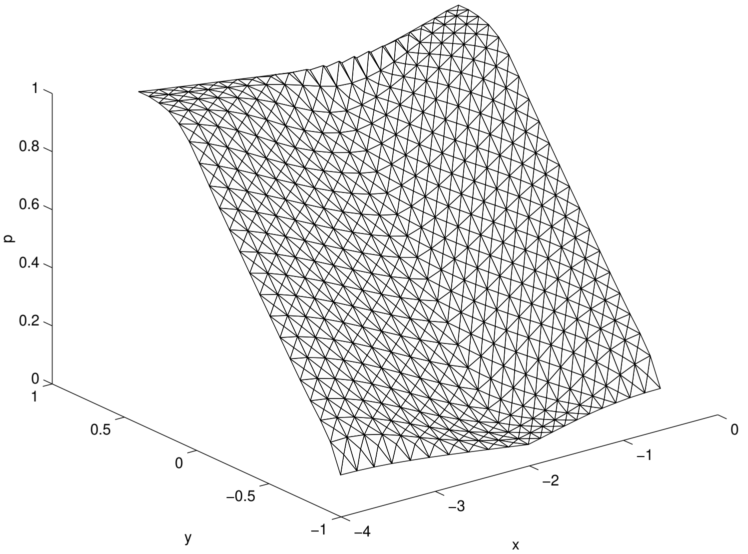

Figure 6: Escape probability of fluid particles (initially in the unit jet core in Figure 3) exiting into the northern recirculating region: $\beta = 1/3$

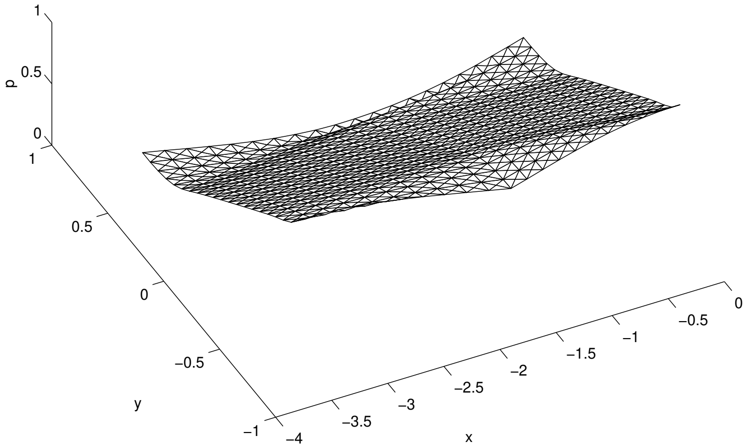

Figure 7: Escape probability of fluid particles (initially in the unit jet core in Figure 3) exiting into the southern recirculating region: $\beta = 1/3$

$0.3333 < \beta < \frac{2}{3}$, the opposite holds (Figure 8). Thus $\beta = 0.3333$ is a bifurcation point. Also, the average escape probability for fluid particles escape into the exterior retrograde region increases as β increases from 0 to 0.54 (or as latitude decreases accordingly), and then decreases as β increases from 0.54 to $\frac{2}{3}$ (or as latitude decreases accordingly). Thus $\beta = 0.54$ is another bifurcation point. The opposite holds for the average escape probability for fluid particles escape into the jet core.

For a unit jet core near the jet troughs, the average escape probability for fluid particles (initially inside the jet core) escape into the northern recirculating region is greater than escape into the southern recirculating region for $0 < \beta < 0.115$, while for $0.385 < \beta < \frac{2}{3}$, the opposite holds. Moreover, for $0.115 < \beta < 0.385$, fluid particles are about equally likely to escape into either recirculating regions (Figure 9).

Furthermore, for a unit jet core near the jet crests, the situation is the opposite as for near the jet troughs.

4 Mean residence time

The mean residence time $u(x, y)$ of a fluid particle, initially at (x, y) in either an eddy or a piece of jet core (see Figures 2, 3), satisfies

$$\begin{aligned} \epsilon\Delta u + [sech^2(y) + 2a\ sech^2(y)\tanh(y)\cos(kx) - c]u_x - ak\ sech^2(y)\sin(kx)u_y &= -1, \qquad (18) \\ u|_{\partial D} &= 0 \qquad (19) \end{aligned}$$

The mean residence times of fluid particles in an eddy or a unit jet core are shown, for the case of $\beta = \frac{1}{3}$, in Figures 10, 11.

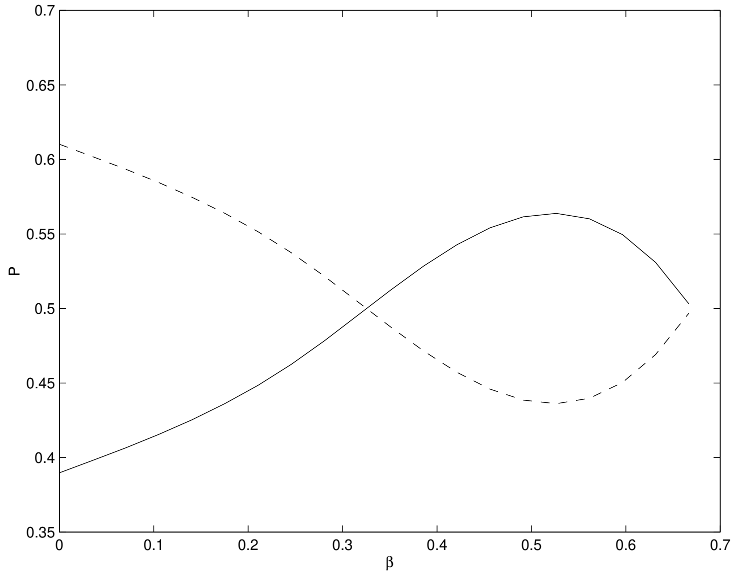

Figure 8: Average escape probability for a fluid particle in an eddy: — for exiting into the exterior retrograde region; ___ for exiting into the jet core region.

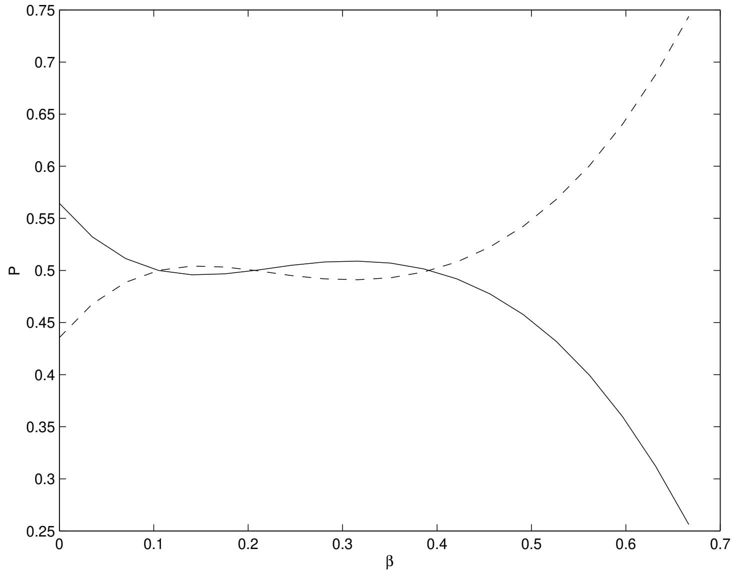

Figure 9: Average escape probability for a fluid particle in the unit jet core near a trough: — for exiting into the northern recirculating region; ___ for exiting into the southern recirculating region.

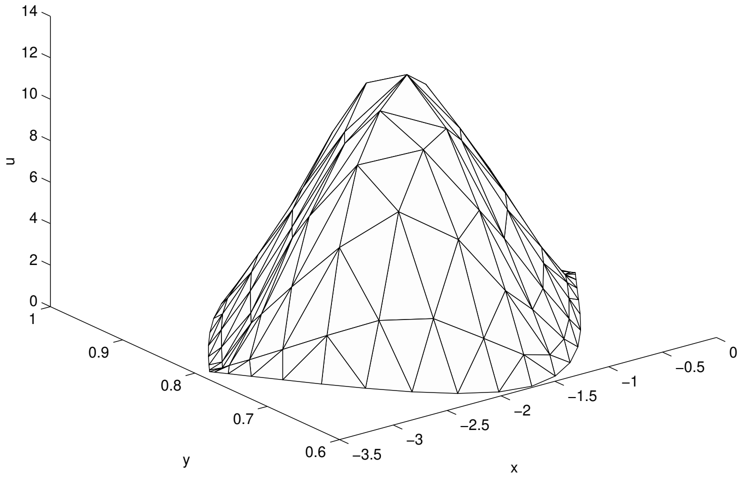

Figure 10: Mean residence time in an eddy: $\beta = \frac{1}{3}$

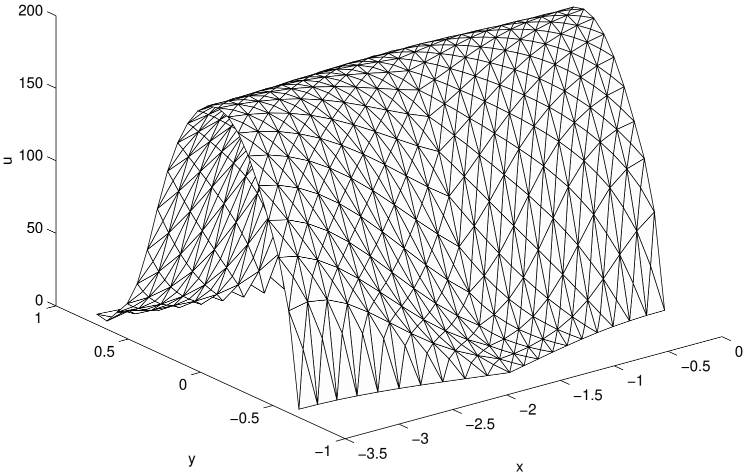

Figure 11: Mean residence time in a unit jet core: $\beta = \frac{1}{3}$

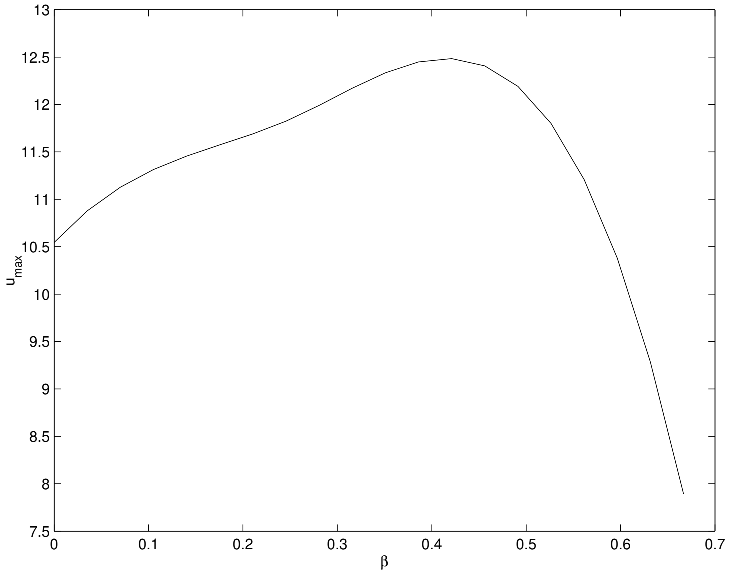

Figure 12: Maximal value of mean residence time, as a function of β, in an eddy.

The maximal mean residence time of fluid particles initially in an eddy increases as β increases from 0 to 0.432 (or as latitude decreases accordingly), then decreases as β increases from 0.432 to $\frac{2}{3}$ (or as latitude decreases accordingly); see Figure 12. However, the maximal mean residence time of fluid particles initially in a unit jet core always increases as β increases (or as latitude decreases accordingly); see Figure 13.

5 Discussions

The present work on fluid particle motion in random flows takes into account of fluid particle *diffusive* as well as advective motion. There has been recent work on fluid particle *advective* motion (molecular diffusion ignored) in time-periodic, quasi-periodic and aperiodic flows (periodic → quasi-periodic → aperiodic → random); see, for example, [29], [35], [31], [2], [8], [21] and [22].

Our work on random particle motion does *not* require that the random part is small. However it does require that the flow can be decomposed into steady or unsteady deterministic (drift) and random (diffusion) parts; Otherwise the Fokker-Planck formalism does not hold.

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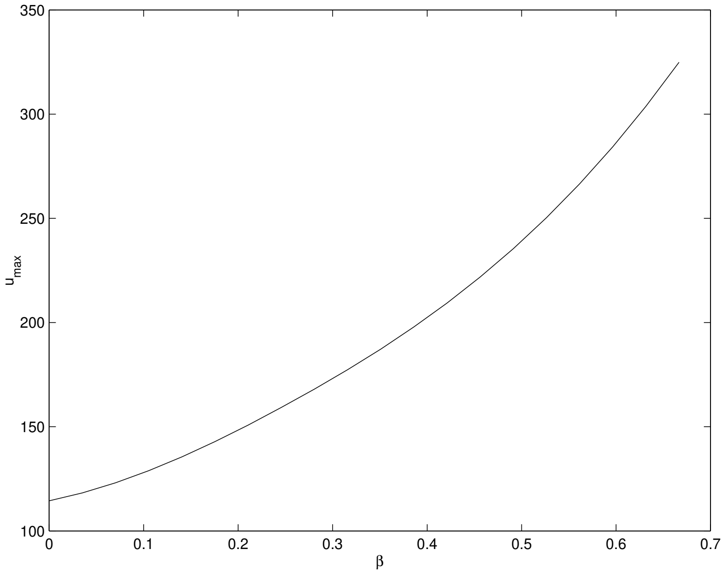

Figure 13: Maximal value of mean residence time, as a function of β, in a unit jet core.

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