Dynamics of a Ring of Diffusively Coupled Lorenz Oscillators

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Abstract

We study the dynamics of a finite chain of diffusively coupled Lorenz oscillators with periodic boundary conditions. Such rings possess infinitely many fixed states, some of which are observed to be stable. It is shown that there exists a stable fixed state in arbitrarily large rings for a fixed coupling strength. This suggests that coherent behavior in networks of diffusively coupled systems may appear at a coupling strength that is independent of the size of the network.

1 Introduction

Lattices of coupled dynamical systems have been studied in many different contexts: as discrete versions of partial differential equations of evolution type [10], [22], models of neuronal networks [6], [18] and phase lock loops [4], and in statistical mechanics [14]. The particular problem of synchronization and emergence of coherent behavior in lattices of diffusively coupled, chaotic, continuous time dynamical systems has received much attention recently due to its applications in neuroscience [1], [18], and chaotic synchronization [19], [16].

Analytical results in the literature suggest that the coupling strength necessary for the appearance of coherent behavior in networks of chaotic systems grow with the size of the network [5],[4],[15]. For instance, estimates of Afraimovich and Lin [5] suggest that the coupling strength necessary to synchronize a lattice of generalized forced Duffing systems grows as a high power of the size of the lattice. Since stable, coherent states in lattices may frequently be viewed as instances of generalized synchronization [19], one might guess that the appearance of dynamically stable states of this type also requires that the coupling grows with the number of oscillators in the system. This would make it unlikely that such networks are of importance in nature, since the coupling strength in realistic systems cannot be varied by orders of magnitude.

In contrast to these theoretical results, the numerical experiments of Huerta, et. al. [18] indicate that partial synchronization may appear in certain networks of chaotic systems at a nearly constant coupling strength, regardless of the size of the network. In particular,

[18] considered a system with about 10^4 coupled neurons. Each neuron was modeled by a system of three nonlinear, ordinary differential equations which, when uncoupled, underwent chaotic motion. They observed a variety of different stable, coherent motions including a number of states that exhibited roughly periodic patterns. It is the goal of this paper to show rigorously that such such stable, coherent states can occur in arbitrarily large systems of coupled chaotic oscillators at a coupling strength which is independent of the size of the system.

Since the Lorenz system is one of the paradigms of chaotic flows, we have chosen a ring of diffusively coupled Lorenz equations for our investigations. The systems in the ring evolve according to the equations

$$\begin{aligned} x_i' &= \sigma(y_i - x_i) + d_x \Delta x_i \\ y_i' &= r x_i - y_i - x_i z_i + d_y \Delta y_i \\ z_i' &= -\beta z_i + x_i y_i + d_z \Delta z_i \end{aligned} \tag{1}$$

where $\Delta x_i = x_{i-1} - 2x_i + x_{i+1}$ with the index i taken modulo n is the discretized Laplacian with periodic boundary conditions. The constants $\sigma = 10, r = 27$, and $\beta = 8/3$ are chosen so that each uncoupled system would be in the chaotic regime. Following the work of [18], we will focus on the case $d_x \neq 0$, $d_y = d_z = 0$. As shown in [19], such a ring is synchronized identically for sufficiently large values of d_x. We will be concerned with the case when d_x is sufficiently large for coherent spatial structures to emerge in the chain, but insufficient to synchronize the ring.

The paper is organized as follows: Typical spatial structures observed in numerical experiments are described in section 2. In particular, we give examples of stable stationary states, traveling waves, and breathers. In section 3 a general framework for the study of fixed states in chains and rings of dynamical systems is discussed. We show that the existence of such states can be reduced to the study of periodic solutions of reversible dynamical systems on $\mathbb{R}^n$. We then apply these results to two different cases, including the chains of coupled Lorenz oscillators (1), to show that a large number of fixed states can be expected in such rings. The conditions under which these states are stable are investigated in section 5. We then show how Floquet theory can be used to reduce the stability question in lattices of arbitrarily many oscillators to the study of the spectrum of a matrix of fixed size (but depending on a parameter – the "quasi-momentum".) Finally, in section 6 we implement this program to rigorously construct an example of a spatially periodic stable fixed state which remains stable for rings of arbitrary size.

2 Numerical Experiments

Numerical experiments were performed on a Sun SuperSparc using the numerical integration program XPP. Several different integration methods were used to check the validity of the results. These included fixed and variable step Runge-Kutta methods of varying order and the Gear method.

At very low values of the coupling constant d_x, the individual systems in the ring behaved as if they were uncoupled. As the coupling strength was increased the behavior of the ring became more coherent. Since the discrete Laplacian acts to synchronize neighboring systems in the ring, these results agree with our expectations. As was emphasized in the introduction, the onset of the coherent behavior occurred at coupling strengths which seemed to be independent of the number of oscillators in the ring. Some of the typical structures observed in these numerical experiments are discussed below.

2.1 Breathers

If some of the systems in the ring are nearly stationary, while others are undergoing oscillations we say that the oscillating systems form a breather. Since we are considering rings of finite size, it is not expected that any single system in the ring is stationary, unless the entire ring is stationary.

An example of a system with two breathers is shown in figure 1. It is interesting to note that in these states each of the individual systems undergoes oscillations that are nearly periodic, and that there is a strong spatial dependence on the motion. Loosely speaking the motion of each individual system is trapped in a region close to one of the 'lobes" of the Lorenz attractor. No stable states in which all systems in the ring are trapped in the vicinity of only *one* "lobe" has been observed. Similar breathers have been studied in coupled map lattices in [8], while occurrences of nearly periodic behavior with pronounced spatial patterns in lattices of chaotically bursting neurons have been observed numerically by M.I. Rabinovich, et. al. [23].

Another interesting feature of breathers, which is also shared with other types of solutions of this network, is that frequently nonadjacent systems synchronize without synchronizing with the oscillators that lie between them. The synchrony is usually not exact, however this may be due to numerical errors in the calculations. Due to the many symmetries of the system this sometimes seems to be due to the stability of an invariant manifold corresponding to a partially synchronized state in the chain as discussed [19]. In other cases the state does not appear to be symmetric, and it is unclear what the mechanism behind this synchronization is.

It is important to note that some of the qualitative features of the dynamics remain unaffected by the size, or the coupling strength in the ring. Figure 2 shows typical timeseries of the x variable at four different coupling values in a ring of 32 oscillators. Note the smooth envelope and regularity of the oscillations, as well as the agreement of the timescales among the different timeseries. These general features remain unchanged as the number of systems in the ring is varied, as long as the coupling is below the synchronization threshold and above the value necessary for the appearance of coherent behavior.

2.2 Stable Stationary States

In numerical experiments with rings of 8 to 52 systems stable steady states were observed for a variety of initial conditions for $d_x > 10$. Typical stable fixed states are shown in figure 3.

For certain values of the coupling it appears that all initial conditions lead to one of the stationary states of the system. The basins of attraction of the stationary states seem to be intertwined in a complicated way in this case, as shown in figure 3. The same is true for other stable states of the system. Small changes in the initial values of the parameters, and changes in the integrating methods can result in very different asymptotic behavior of the system. This situation is similar to that of attractors with riddled basins of attraction [7], and was also observed in numerical simulations of networks of chaotically bursting neurons [18].

2.3 Traveling Pulses

Traveling pulses were observed in the ring only when $d_x \neq 0$ and $d_y \neq 0$. A typical traveling pulse is shown in figure 4. The pulse oscillates as it propagates along the chain, and is thus a periodic, rather than a fixed state in a moving coordinate frame. This can be seen in the timeseries in figure 4.

3 A General Framework for the Study of Steady States

System (1) is a special case of a lattice discrete or continuous time dynamical system

$$(u_j)' = F(\{u_j\}^s) \tag{2}$$

where $j \in \mathbb{Z}^D$, $u_j \in \mathbb{R}^p$ and $\{u_j\}^s = \{u_i : |i-j| \leq s\}$. In the following discussion we will use the convention that in u_i^k the subscript $i \in \mathbb{Z}^D$ denotes the position in the lattice, while the superscript $1 \leq k \leq p$ denotes the component of the vector u_i. Such systems have been studied by many authors [2], [10], [22] as discrete versions of partial differential equations of evolution type.

We will mainly be concerned with systems continuous in time with $D = 1$ and $s = 1$. These are simply chains of systems coupled to their nearest neighbors. The special case of *rings*, that is chains of finite size with periodic boundary condition will be the focus of our attention. A state in a ring of n systems corresponds to a state of period n in the spatial variable in an infinite chain.

A steady state of a chain given in (2) is determined by

$$F(\{u_j\}^s) = F(u_{j-1}, u_j, u_{j+1}) = 0 \tag{3}$$

$$F(\{u_j\}^s) = F(u_{j-1}, u_j, u_{j+1}) = u_j \tag{4}$$

in the continuous, respectively discrete case. We will study systems that satisfy the following condition:

Condition 1 If we write $F : \mathbb{R}^{3p} \to \mathbb{R}^p$ as $F(\chi, \eta, \zeta) = (F^1(\chi, \eta, \zeta),\ F^2(\chi, \eta, \zeta),\ \ldots,\ F^p(\chi, \eta, \zeta))$ then $\det \left[\frac{\partial F^i}{\partial \zeta^j}(\chi, \eta, \zeta)\right]_{i,j} \neq 0$.

By the Implicit Function Theorem in the case of continuous time if $\det \left[\frac{\partial F^i}{\partial \zeta^j}(\chi, \eta, \zeta)\right]_{i,j} \neq 0$ for all values of χ, η, ζ then either there exists a point $(a, b, c) \in \mathbb{R}^{3p}$ such that $F(a, b, c) = 0$ and therefore a function $G(\eta, \chi)$ such that $F(\chi, \eta, G(\eta, \chi)) = 0$, or $F(\chi, \eta, \zeta) = 0$ has no solutions and there are no fixed points of (2). If such a function G exists and we define $u_{j+1} = G(u_j, u_{j-1})$, $x_j = u_j$ and $y_j = u_{j-1}$ this leads to the following dynamical system on $\mathbb{R}^{2p}$:

$$\begin{aligned} x_{j+1} &= G(x_j, y_j) \\ y_{j+1} &= x_j \end{aligned} \tag{5}$$

The steady states of (2) are given by the x-coordinates of the orbits of (5). In general, the function G may not be unique, in which case more than one system of the form (5) is needed to determine all the fixed points of the chain. An equivalent argument holds in the case of discrete time.

Since the function G can assume any form, not much can be said about such systems in general. The subclass of chains of systems with the following type of coupling is easier to analyze.

Definition 3.1 *A nearest neighbor coupling of a chain of systems is said to be* symmetric *if* $F(u_{j-1}, u_j,\ u_{j+1}) = F(u_{j+1}, u_j, u_{j-1})$.

This simply means that a system in the chain is coupled to its left and right neighbor in the same way. For instance the couplings $\Delta u_j = u_{j-1} - 2u_j + u_{j+1}$ and $\Psi u_j = u_{j-1}u_{j+1} + u_j$ are of this type. The stationary states of symmetrically coupled chains are related to the following class of dynamical systems:

Definition 3.2 *Given a diffeomorphism* $\Phi : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ *and an involution* $R : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ *such that the dimension of the fixed point set of* R *is* p *we say that* Φ *is* R-reversible *if*

$$R \circ \Phi = \Phi^{-1} \circ R \tag{6}$$

The dynamical system defined by $x_{i+1} = \Phi(x_i)$ *is also said to be R-reversible.*

The relation between symmetrically coupled chains and reversible systems is given in the following

Theorem 3.3 *The fixed states of a symmetrically coupled chain of systems satisfying condition 1 correspond to the orbits of a dynamical system of the form (5) which is* R*-reversible and volume preserving. For such a system* $R(x, y) = (y, x)$*, with* $x, y \in \mathbb{R}^p$.

Proof: The arguments for the continuous and discrete case are virtually identical, so only the first will be considered. Equation (3) together with the assumption that the coupling is symmetric implies that

$$F(u_{j-1}, u_j, u_{j+1}) = F(u_{j+1}, u_j, u_{j-1}) = 0 \tag{7}$$

Therefore this leads to the recursion equations

$$u_{j+1} = G(u_j, u_{j-1}) \quad u_{j-1} = G(u_j, u_{j+1}) \tag{8}$$

and the two dynamical systems

$$\begin{aligned} x_{j+1} &= G(x_j, y_j) & \qquad z_{j-1} &= G(z_j, w_j) \\ y_{j+1} &= x_j & \qquad w_{j-1} &= z_j \end{aligned} \tag{9}$$

where $x_j = z_j = u_j$, $y_j = u_{j-1}$ and $w_j = u_{j+1}$. The action of these systems is shown schematically in figure 5.

Let $R(x, y) = (y, x)$ and $\Phi(x, y) = (G(x, y), x)$ so that the dynamical systems in (9) is generated by Φ. By the definition of this diffeomorphism

$$\begin{aligned} R \circ \Phi \circ R \circ \Phi(x_i, y_i) &= R \circ \Phi(y_{i+1}, x_{i+1}) = R \circ \Phi(z_i, w_i) \\ &= R(z_{i-1}, w_{i-1}) = R(y_i, x_i) = (x_i, y_i) \end{aligned} \tag{10}$$

Since R is an involution with $\mathrm{Fix}(R) = \{(x,y) : x = y\}$ the diffeomorphism Φ and the dynamical system it induces on the plane are R-reversible.

To prove that the map Φ is volume preserving notice that

$$D\Phi = \begin{bmatrix} D_1G & D_2G \\ I & 0 \end{bmatrix} \tag{11}$$

By the definition of the function G we know that $F(\chi, \eta, G(\eta, \chi)) = 0$ so that differentiating with respect to χ leads to

$$D_1F + D_3FD_2G = 0 \tag{12}$$

D_3F is an invertible matrix by Condition 1, so that $D_2G = -D_1F(D_3F)^{-1}$. By assumption the system is symmetric so that $D_1F = D_3F$ at all points in $\mathbb{R}^{3p}$ and hence $D_2G = -I$. Since

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} D_1G & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -D_1G & I \end{bmatrix}. \tag{13}$$

and

$$\det \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \det \begin{bmatrix} I & 0 \\ -D_1G & I \end{bmatrix} = 1 \tag{14}$$

it follows that $\det D\Phi = 1$ and the diffeomorphism Φ is volume preserving. $\diamondsuit$

R-reversible systems have many properties that facilitate their study. We will make use of several of these summarized in the following proposition adopted from [12]

Proposition 3.4 *Let R and Φ be as in Definition (3.2).*

a. If $p \in$ Fix(R) and $\Phi^k(p) \in$ Fix(R), then $\Phi^{2k}(p) = p$. Such periodic points will be referred to as symmetric periodic points.

b. Let $p \in$ Fix(R) be a fixed point of Φ, then $R(W^u(p)) = W^s(p)$ and $R(W^s(p)) = W^u(p)$ so that if $q \in W^u(p) \cap$ Fix(R) then q is a homoclinic point.

c. If $p \in$ Fix(R) is a fixed point of Φ such that $W^u(p)$ intersects Fix(R) transversally at a point q, then there exist infinitely many symmetric periodic points in any neighborhood of p.

4 Steady States In the Case of Discrete Laplacian Coupling

For a chain of systems of the form $u' = f(u)$ coupled through a discrete Laplacian $\Delta u_j = (u_{j-1} - 2u_j + u_{j+1})$ the equation (2) takes the form

$$(u_j)' = f(u_j) + d(u_{j-1} - 2u_j + u_{j+1}) \tag{15}$$

and the dynamical system (5) takes the form

$$\begin{aligned} x_{i+1} &= -\frac{1}{d}f(x_i) - y_i + (2 + \frac{c}{d})x_i \\ y_{i+1} &= x_i \end{aligned} \tag{16}$$

where $c = 0$ in the case of continuous time and $c = 1$ in the case of discrete time. The following proposition follows immediately from these definitions

Proposition 4.1 *The fixed points of (16) are of the form $x_i = y_i = u_0$ where u_0 is any solution of the equation $f(u) = 0$ in the continuous time and $f(u) = u$ in the discrete time case. Thus all fixed points of (16) lie on Fix$(R) = \{(x, y) : x = y\}$ and are in 1-1 correspondence with the steady states of an uncoupled system in the chain.*

The fixed points of (5) correspond to states of the chain that are constant in space and time and proposition 4.1 shows that the only such states $\{u_j\}_{j\in\mathbb{Z}}$ in the case of a discrete Laplacian coupling are given by $u_j = u_0$ for all $j \in \mathbb{Z}$ where u_0 is a fixed point of an uncoupled system of the chain $u' = f(u)$. Since the fixed points of (16) are $(u_0, u_0) \in \text{Fix}(R)$ this allows us to make use of proposition 3.4.c[1] .

Moreover we can apply Theorem 3.3 directly in this case to conclude that the map defining the fixed points is R-reversible and volume preserving.

Remark: It is interesting to note that if the fixed states of a chain of one dimensional systems are determined by a dynamical system of the form (16) on $\mathbb{R}^2$ and (x_0, y_0) is a fixed point of (16) then the discriminant of the characteristic polynomial of $D\Phi(x_0, y_0)$ is

$$\sqrt{(-\frac{1}{d}f'(x_0) + 2 + c/d)^2 - 4}. \tag{17}$$

Since det $D\Phi(x_0, y_0) = 1$ we can conclude the following:

a. In the case of discrete time the eigenvalues of $D\Phi(x_0, y_0)$ are on the unit circle when $f'(x_0) > 1$ and $d > (f'(x_0) - 1)/4)$ or $f'(x_0) < 1$ and $d < (f'(x_0) - 1)/4)$.

b. In the case of continuous time, the eigenvalues of $D\Phi(x_0, y_0)$ are on the unit circle for sufficiently large *positive* d when $f'(x_0) > 0$ and for sufficiently large *negative* d when $f'(x_0) < 0$,

By a theorem of Birkhoff (see for example [20]) since Φ is area preserving, the dynamical system (16) will generically have infinitely many periodic orbits in any neighborhood of (x_0, y_0). Any such periodic orbit will correspond to a fixed state of (15) which is periodic in space, i.e. in the variable j.

The condition det $D\Phi(x_0, y_0) = 1$ is not sufficient to guarantee that $D\Phi(x_0, y_0)$ will have eigenvalues on the unit circle if (16) is not a dynamical system on the plane. Additional

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[1] The chains under consideration are a special case of chains of the form

$$(u_j)' = f(u_j) + g(u_{j-1}, u_j, u_{j+1})$$

where $g(u_{j-1}, u_j, u_{j+1})$ is a symmetric coupling vanishing on the linear submanifold of $\mathbb{R}^{3p}$ defined by $u_{j-1} = u_j = u_{j+1} = 0$. Such systems can be analyzed using the approach of this section information needs to be considered to conclude the existence of such periodic orbits in this case.

Example 1: Consider a chain of tent maps coupled through a discrete Laplacian described by the equation $u_j' = f(u_j) + \Delta u_j$ with

$$f(u_j) = \begin{cases} 2u_j & \text{if } u_j < 1/2, \\ -2u_j + 2 & \text{if } u_j \geq 1/2. \end{cases} \tag{18}$$

This is a discrete dynamical system with fixed states determined by a homeomorphism of the plane given in (16) with $c = 1$.

The only fixed points of (16) are (0,0) and (2/3, 2/3) by proposition 4.1. Since in this case the dynamical system (16) is linear in each of the two halves of the plane $x < 1/2$ and $x \geq 1/2$ the analysis of the dynamics around the fixed points is straightforward. A direct calculation shows that in the region $x < 1/2$ there is a family of invariant ellipses whose major axis lies on the diagonal $D = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and on which the action of (16) is a rotation. As d is increased, these ellipses become more eccentric.

The fixed states of the chain corresponding to these orbits are not dynamically stable. If $\hat{u} = \{\hat{u}_j\}_{j \in \mathbb{Z}}$ is a fixed state in the chain such that $\hat{u}_j < 1/2$ for all j then around this state the dynamics of the chain are described by:

$$u_j' = 2u_j + d\Delta u_j \tag{19}$$

The spectrum of the operator $2 + d\Delta$ is the interval $[2 - 4d, 2]$ and so $\hat{u}$ cannot be stable, and would not be observed in numerical experiments.

Besides these invariant ellipses, this system will also typically have infinitely many periodic orbits, and will exhibit complicated behavior, as the following argument shows. A simple calculation shows that the point $p = (2/3, 2/3)$ is hyperbolic for all values $d > 0$. Since the dynamics is piecewise linear the stable and unstable manifold can be calculated explicitly. $W^u(p)$ will consist of a ray contained in the half plane $x > 1/2$ and a second more complicated part which is constructed as follows. The first section of $W^u(p)$ is a line A, as shown in figure 6. The part of A contained in the half plane $x < 1/2$ is rotated along the invariant ellipses around the origin to the line A' under forward iteration. The subsequent images of A' are lines that are rotated further. Therefore $f^{n_0}(A')$ intersects the diagonal D for some n_0. Whenever the angle of this intersection is not a right angle $W^u(p)$ will intersect $W^s(p)$ transversely since by proposition 3.4.b the stable manifold is the reflection of the unstable manifold through the diagonal $W^u(p)$. This shows that complicated dynamics of (16) can be expected and implies the existence of infinitely many periodic as well as spatially chaotic states in the chain of tent maps.

Example 2: Next we consider the chain of diffusively coupled Lorenz system (1) with $d_y = d_z = 0$. This system does not satisfy condition 1, however after setting the right–hand side of equations (1) to 0 and using the second and third equation in (1) to eliminate the variables y_i and z_i the following equation for a fixed state is obtained

$$\sigma(\hat{x}_i - \frac{\beta r \hat{x}_i}{\beta + (\hat{x}_i)^2}) - d(\hat{x}_{i-1} - 2\hat{x}_i + \hat{x}_{i+1}) = 0 \tag{20}$$

$$\hat{y}_i = \frac{\beta r \hat{x}_i}{\beta + (\hat{x}_i)^2} \tag{21}$$

$$\hat{z}_i = \frac{r(\hat{x}_i)^2}{\beta + (\hat{x}_i)^2}. \tag{22}$$

The function on the left hand side of equation (20) satisfies condition 1, and we can proceed as in the previous example.

Equation (20) defines the following dynamical system on the plane

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = F\left(\begin{bmatrix} x_i \\ y_i \end{bmatrix}\right) = \begin{bmatrix} \frac{\sigma}{d}(x_i - \frac{\beta r x_i}{\beta + (x_i)^2}) + 2x_i - y_i \\ x_i \end{bmatrix} \tag{23}$$

Orbits of period n of this system are in 1–1 correspondence with the steady states in the ring of n Lorenz systems. The two fixed points of (23) are (0,0) and $(\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}) = (\pm 8.32\bar{6}, \pm 8.32\bar{6})$ which both lie on lie on Fix(R).

The following definition from [12] will be used in the remainder of the argument.

Definition 4.2 *A compact region H in the plane is called* overflowing *in the y-direction for a function F if the image of any point $(x_0, y_0) \in$ int(H) lies strictly above the line $y = y_0$.*

Notice that any point in the interior of H either leaves H or is asymptotic to a periodic point on the boundary of H.

Theorem 4.3 *The map (23) has a homoclinic point and infinitely many periodic points for all* $0 < d \leq 20$.

Proof: Let p denote the fixed point $(-\sqrt{\beta(r-1)}, -\sqrt{\beta(r-1)})$ of (23), G the ray $y = -\sqrt{\beta(r-1)}$ with $x > -\sqrt{\beta(r-1)}$, and $D = \{(x,y) : x = y\}$ the diagonal in $\mathbb{R}^2$. The image of the triangular region W bounded by D and G lies between the two cubics

$$F(D) = \{(x,y) : x = \frac{\sigma}{d}(y - \frac{\beta r y}{\beta + y^2}) + y\} \tag{24}$$

$$F(G) = \{(x,y) : x = \frac{\sigma}{d}(y - \frac{\beta r y}{\beta + y^2}) + 2y + \sqrt{\beta(r-1)}\} \tag{25}$$

Consider the region $F(W) \cap W = H$ depicted in figure 7. Let $(x_0, y_0) \in \text{int}(H)$ and consider the vertical line segment l passing through (x_0, y_0) and connecting G and D. The image of l is a horizontal line segment connecting $F(G)$ and $F(D)$ contained in the line $y = x_0$. Any point of this line segment lies above the line $y = y_0$. Therefore H is overflowing in the y direction.

Since int($F(H)-$ int(H)) and int(H) lie on opposite sides of D, any point in H must have an iterate which either lies on D or crosses D. A direct calculation shows that one branch of $W^u(p)$ enters H. Since H is overflowing in the y direction, $W^u(p)$ must either cross D or be asymptotic to the fixed point $(0,0)$ creating a saddle-saddle connection. The second possibility can be excluded as follows:

The eigenvector of the linearization of F around (0,0) corresponding to the stable direction is $\mathbf{v}_1 = (1, -\frac{d}{d-130+2\sqrt{65^2-d}})$. The vector $\mathbf{v}_2 = (1, \frac{d-260}{d})$ is tangent to $F(D)$ at (0,0). A direct computation shows that for $0 < d < 20$ the vector $\mathbf{v}_1$ points to the left of $\mathbf{v}_2$ and so the tangent to $W^s(0,0)$ at (0,0) does not point into H. This situation is depicted schematically in figure 8 and excludes the possibility that $W^u(p)$ is asymptotic to (0,0).

By proposition 3.4.b the fact that $W^u(p)$ meets D implies the existence of a homoclinic point.

By proposition 3.4.c in the case of R-reversible systems an infinite number of periodic points exists whenever $W^u(p)$ crosses D transversely at a homoclinic point. If this point of intersection is not transverse then a transverse intersection of $W^u(p)$ and D can be produced nearby by an argument given in [12, p. 261]. $\diamondsuit$

Numerical investigations suggest that the theorem remains true for arbitrarily large values of d.

This theorem shows that rings of Lorenz systems coupled through the first variable can be expected to have many periodic stationary states. Numerical computations of $W^u(p)$ suggest that even more complicated stationary states can be expected (see figure 9). However it is unclear whether any of them are stable. Numerical investigations show that not only are they stable, but their basins of attraction occupy a large portion of phase space. We next address the problem of stability of these fixed states.

5 Stability of Steady States in a Ring

The study of stable fixed states in a chain of oscillators is not new. They are described in [13], [6] as instances of oscillator death in a chain of neural oscillators, in [4] as synchronous states of phase lock loops, and conditions for the stability of complex stationary states in general lattices are given in [3]. The situation presented here is different in that an explicit periodic stationary state is analyzed in a case where the methods from the theory of parabolic partial differential equations used in [13] are not applicable, and the conditions proposed in [3] cannot be verified. In addition, since the coupling strength is neither very large nor very small, there is no obvious perturbative approach to the problem.

Even if some of the fixed states in a ring of n systems are stable it can not be concluded that stable equilibria exist in arbitrarily long rings of such systems. If a stable fixed state is viewed as a special case of synchronization, one might guess from results in Afraimovich and Lin [5] that the coupling necessary to have stable steady states in longer rings increases as a power of the size of the ring. It has already been argued that this is not realistic from a physical viewpoint, and we will show below that it is not the case.

In the following sections we will concentrate on the specific example of a ring of Lorenz systems coupled in the x variable discussed in example 2 of the last section. However the ideas presented can be applied to any chain of symmetrically coupled systems for which condition 1 holds.

The linearization of each of the Lorenz equations around a point $\hat{\mathbf{v}}_\mathbf{i} = (\hat{x}_i, \hat{y}_i, \hat{z}_i)$ when

$d = 0$ is

$$D_{\hat{\mathbf{v}}_\mathbf{i}} f = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - \hat{z}_i & -1 & -\hat{x}_i \\ \hat{y}_i & \hat{x}_i & -b \end{bmatrix} \tag{26}$$

and hence linearizing the entire ring around the steady state $\hat{\mathbf{v}} = (\hat{\mathbf{v}}_\mathbf{i})_{i=1}^n = (\hat{x}_1, \hat{y}_1, \hat{z}_1, \ldots, \hat{x}_n, \hat{y}_n, \hat{z}_n)$ leads to the $3n \times 3n$ matrix

$$D_{\hat{\mathbf{v}}} F = \begin{bmatrix} D_{\hat{\mathbf{v}}_\mathbf{1}} f - 2\Gamma & \Gamma & 0 & \ldots & 0 & \Gamma \\ \Gamma & D_{\hat{\mathbf{v}}_\mathbf{2}} f - 2\Gamma & \Gamma & \ldots & 0 & 0 \\ \hdotsfor{6} \\ \Gamma & 0 & 0 & \ldots & \Gamma & D_{\hat{\mathbf{v}}_\mathbf{n}} f - 2\Gamma \end{bmatrix} \tag{27}$$

where the matrix Γ is defined as:

$$\Gamma = \begin{bmatrix} d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{28}$$

It is in general not possible to compute the eigenvalues of this matrix analytically. As mentioned above, we are specifically interested in coherent states in networks with arbitrarily many oscillators. Given a periodic stationary state in a chain of n oscillators, an easy way to obtain a periodic stationary state in arbitrarily long chains is to repeat this state. In particular, if $\hat{\mathbf{v}}$ is a fixed state in a ring of n oscillators then repeating these values K times to obtain $\hat{\mathbf{v}}^{(K)} = \{\hat{\mathbf{v}}, \hat{\mathbf{v}}, \ldots, \hat{\mathbf{v}}\}$ produces a steady state in a chain of Kn oscillators. We refer to such a state as the *K-multiple* of the steady state $\hat{\mathbf{v}}$. An example of a stationary state and its 4-multiple is given in figure 10. We next derive conditions under which all K-multiples of a stable steady state are themselves stable and demonstrate that these conditions hold in a particular case.

The linearization around the steady state $\hat{\mathbf{v}}^{(K)}$ leads to the following stability matrix:

$$D_{\hat{\mathbf{v}}^{(K)}} F = \begin{bmatrix} M & E_d & 0 & \ldots & 0 & E_u \\ E_u & M & E_d & \ldots & 0 & 0 \\ \hdotsfor{6} \\ E_d & 0 & 0 & \ldots & E_u & M \end{bmatrix}$$

where

$$M = \begin{bmatrix} D_{\hat{\mathbf{v}}_\mathbf{1}} f - 2\Gamma & \Gamma & 0 & \ldots & 0 & 0 \\ \Gamma & D_{\hat{\mathbf{v}}_\mathbf{2}} f - 2\Gamma & \Gamma & \ldots & 0 & 0 \\ \hdotsfor{6} \\ 0 & 0 & 0 & \ldots & \Gamma & D_{\hat{\mathbf{v}}_\mathbf{n}} f - 2\Gamma \end{bmatrix}$$

and E is an $3n \times 3n$ matrix of the form

$$E_u = \begin{bmatrix} 0 & \ldots & \Gamma \\ \hdotsfor{3} \\ 0 & \ldots & 0 \end{bmatrix} \qquad E_d = \begin{bmatrix} 0 & \ldots & 0 \\ \hdotsfor{3} \\ \Gamma & \ldots & 0 \end{bmatrix}$$

In general it is impossible to compute the spectrum of this $3Kn \times 3Kn$ matrix. However in the present instance Bloch wave theory, provides a way of expressing the eigenfunctions of a larger system with a periodic dependency as the eigenfunctions of a smaller system. As usual, this reduction is accompanied by the introduction of an additional parameter into the equations. Here the Bloch wave approach will be used to reduce the problem of computing the eigenvalues of the $3Kn \times 3Kn$ matrix $D_{\hat{\mathbf{v}}^{(K)}}F$ for arbitrary K to the computation of the eigenvalues of an $3n \times 3n$ matrix dependent on a parameter t.

Let $\mathbf{e} = (\Psi(j), \Phi(j), \eta(j))_{j=1}^{Kn} = (\Psi(1), \Phi(1), \eta(1), \ldots, \Psi(Kn), \Phi(Kn), \eta(Kn))$ where

$$\begin{aligned} \Psi(j) &= \exp\left(\frac{2\pi iq}{nK} j\right) \tilde{\Psi}_q(j) \\ \Phi(j) &= \exp\left(\frac{2\pi iq}{nK} j\right) \tilde{\Phi}_q(j) \\ \eta(j) &= \exp\left(\frac{2\pi iq}{nK} j\right) \tilde{\eta}_q(j) \end{aligned} \tag{29}$$

for $1 \le q \le Kn$ and $\tilde{\Psi}_q, \tilde{\Phi}_q, \tilde{\eta}_q$ are assumed to be n-periodic. We will show that the vector $\mathbf{e}$ is an eigenvector of $D_{\hat{\mathbf{v}}^{(K)}}$ whenever the vector

$$\mathbf{i} = (\tilde{\Psi}(j), \tilde{\Phi}(j), \tilde{\eta}(j))_{j=1}^{n} \tag{30}$$

is an eigenvector of a particular $3n \times 3n$ matrix derived from $D_{\hat{\mathbf{v}}^{(K)}}F$. A direct calculation leads to:

$$\begin{aligned} [D_{\hat{\mathbf{v}}^{(K)}}F]\,\mathbf{e}\,(3j) =& e^{\frac{2\pi iq}{nK}j}(de^{\frac{2\pi iq}{nK}}\tilde{\Psi}_q(j+1 \bmod n) + de^{-\frac{2\pi iq}{nK}}\tilde{\Psi}_q(j-1 \bmod n) - \\ & 2\tilde{\Psi}_q(j) + v_1(j)\tilde{\Psi}_q(j) + w_1(j)\tilde{\Phi}_q(j) + u_1(j)\tilde{\eta}_q(j)) \\ [D_{\hat{\mathbf{v}}^{(K)}}F]\,\mathbf{e}\,(3j+1) =& e^{\frac{2\pi iq}{nK}j}(v_2(j)\tilde{\Psi}_q(j) + w_2(j)\tilde{\Phi}_q(j) + u_2(j)\tilde{\eta}_q(j)) \\ [D_{\hat{\mathbf{v}}^{(K)}}F]\,\mathbf{e}\,(3j+2) =& e^{\frac{2\pi iq}{nK}j}(v_3(j)\tilde{\Psi}_q(j) + w_3(j)\tilde{\Phi}_q(j) + u_3(j)\tilde{\eta}_q(j)) \end{aligned} \tag{31}$$

for $1 \le j \le Kn$. Here $v_i(j), w_i(j), u_i(j)$ are given by equation (26) as

$$\begin{bmatrix} v_1(j) & w_1(j) & u_1(j) \\ v_2(j) & w_2(j) & u_2(j) \\ v_3(j) & w_3(j) & u_3(j) \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - \hat{z}_{j \bmod n} & -1 & -\hat{x}_{j \bmod n} \\ \hat{y}_{j \bmod n} & \hat{x}_{j \bmod n} & -b \end{bmatrix}$$

and are thus n-periodic.

We want to check when $\mathbf{e}$ is an eigenvector of $D_{\hat{\mathbf{v}}^{(K)}}F$ and so we set $D_{\hat{\mathbf{v}}^{(K)}}F\mathbf{e} = \lambda\mathbf{e}$. Using the expressions obtained in (31) we get

$$\begin{aligned} &\exp\left(\frac{2\pi iq}{nK}j\right)\Big(de^{\frac{2\pi iq}{nK}}\tilde{\Psi}_q(j+1 \bmod n) + de^{-\frac{2\pi iq}{nK}}\tilde{\Psi}_q(j-1 \bmod n) - \\ &2\tilde{\Psi}_q(j) + v_1(j)\tilde{\Psi}_q(j) + w_1(j)\tilde{\Phi}_q(j) + u_1(j)\tilde{\eta}_q(j),\;\; v_2(j)\tilde{\Psi}_q(j) + w_2(j)\tilde{\Phi}_q(j) + u_2(j)\tilde{\eta}_q(j), \\ &v_3(j)\tilde{\Psi}_q(j) + w_3(j)\tilde{\Phi}_q(j) + u_3(j)\tilde{\eta}_q(j)\Big)_{j=1}^{Kn} = \lambda \exp\left(\frac{2\pi iq}{nK}j\right)\left(\tilde{\Psi}_q(j), \tilde{\Phi}_q(j), \tilde{\eta}_q(j)\right)_{j=1}^{Kn}. \end{aligned} \tag{32}$$

Since all the functions on the right hand side of (32) are n-periodic we can express equation (32) as $\mathcal{D}(q)\mathbf{i} = \lambda\mathbf{i}$ where $\mathbf{i}$ is defined in equation (30) and

$$\mathcal{D}(q) = \begin{bmatrix} D_{\hat{\mathbf{v}}_1} f - 2\Gamma & E(q)^+ & 0 & \ldots & 0 & E(q)^- \\ E(q)^- & D_{\hat{\mathbf{v}}_2} f - 2\Gamma & E(q)^+ & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ E(q)^+ & 0 & 0 & \ldots & E(q)^- & D_{\hat{\mathbf{v}}_n} f - 2\Gamma \end{bmatrix} \tag{33}$$

with the matrices $E(q)^\pm$ defined as:

$$E(q)^\pm = \begin{bmatrix} de^{\pm\frac{2\pi i q}{nK}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{34}$$

This implies that the eigenvalues of $D_{\hat{\mathbf{v}}^{(K)}} F$ are the same as the eigenvalues of the $3n \times 3n$ matrices $\mathcal{D}(q)$ for $1 \leq q \leq Kn$. The $3n \times 3n$ matrix $\mathcal{D}(q)$ is easier to handle than the $3Kn \times 3Kn$ matrix $D_{\hat{\mathbf{v}}^{(K)}} F$.

To conclude that any K-multiple of a stable fixed state $\mathbf{v}_0$ of a ring of n oscillator is stable it is sufficient to show that the eigenvalues of the matrix $\mathcal{D}(q)$ have negative real part for all values of q and K. To simplify the calculations we can replace the argument $q \in \mathbb{Z}$ by a continuous parameter $t \in \mathbb{R}$ by replacing the matrices $E(q)^\pm$ with the matrices

$$E_t^\pm = \begin{bmatrix} de^{\pm it} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in the expression for $\mathcal{D}$. If we can show that $\mathcal{D}(t)$ has eigenvalues with negative real part for all $t \in \mathbb{R}$ then the same is true for $\mathcal{D}(q)$ for any values of K and q in $\mathbb{Z}$.

The following lemma about the class of matrices of type (33) will be used in the next section.

Lemma 5.1 *Let $E(t)^\pm$ be as in the previous lemma. The characteristic polynomial of a $Km \times Km$ matrix*

$$B = \begin{bmatrix} M_1 & E(t)^+ & 0 & \ldots & 0 & 0 & E(t)^- \\ E(t)^- & M_2 & E(t)^+ & \ldots 0 & & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \ldots & E(t)^- & M_{K-1} & E(t)^+ \\ E(t)^+ & 0 & 0 & \ldots & 0 & E(t)^- & M_K \end{bmatrix}$$

is of the form

$$p_M(t, \lambda) = \sum_{i=0}^{(K-1)m} (\alpha_i + \beta_i \cos Kt)\lambda^i + \sum_{i=(K-1)m+1}^{Km} \alpha_i \lambda^i. \tag{35}$$

A proof of this lemma is given in the appendix.

6 An Example

This section gives a computer assisted proof that any K-multiple of a particular stable fixed state in a ring of four Lorenz system is stable. The proof involves the use of interval arithmetic and the following theorems about the convergence of the Newton-Raphson-Kantorovich method. The computations were performed using Mathematica's implementation of interval arithmetic.

Theorem 6.1 *Let $f(z)$ be a complex analytic function and assume $f(z_0)f'(z_0) \neq 0$ for some z_0. Define $h_0 = -f(z_0)/f'(z_0)$, the disc $K_0 = \{z : |z - z_0| \leq |h_0|\}$ and $M = \max_{K_0} |f''(z)|$. If $2|h_0|M \leq |f'(z_0)|$ then there is exactly one root of f in the closed disc K_0.*

Theorem 6.2 (Kantorovich's Convergence Theorem) *Given a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ which is nonsingular at a point $x_0 = (x_1^0, x_2^0, \ldots, x_n^0)$ let $[\Gamma_{ik}] = [\frac{\partial f_i}{\partial x_k}(x_0)]^{-1}$. Let A, B, C be positive real numbers such that*

$$\max_i \sum_{k=1}^{n} |\Gamma_{ik}| \leq A \qquad \max_i \sum_{k=1}^{n} |\Gamma_{ik} f_k(x_0)| \leq B \qquad C \leq \frac{1}{2AB}. \tag{36}$$

Define the region $R = \{x \in \mathbb{R}^n : \max_i |x_i - x_i^0| \leq (AC)^{-1}(1 - \sqrt{1 - 2ABC})\}$. If

$$\max_{x \in R} \sum_{j=1}^{n} \sum_{k=1}^{n} \left| \frac{\partial^2 f_i}{\partial x_j \partial x_k} \right| \leq C \qquad i = 1, \ldots, n \tag{37}$$

then the equation $f(x) = 0$ has a solution in R.

The argument will proceed as follows: First a steady state of the ring is found using interval arithmetic. Next it is shown that the characteristic polynomial $p_{\mathcal{D}}(\lambda, t)$ of the matrix $\mathcal{D}(q)$ in (33) corresponding to this steady state has roots with negative real part for $t = \frac{\pi}{4}$. Finally it is proved that $p_{\mathcal{D}}(\lambda, t)$ does not have roots on the imaginary axis for any $t \in \mathbb{R}$. Since the roots of a polynomial depend continuously on its coefficients this implies that the roots of $p_{\mathcal{D}}(\lambda, t)$ cannot cross into right half of $\mathbb{C}$. By the results of section 5 this implies that all K-multiples of the fixed state under consideration must be stable.

In the following intervals of numbers will be denoted with overbars to distinguish them from real numbers, and make the notation less cumbersome. For instance $\bar{\lambda}$ denotes an interval and the interval $[3.16524, 3.16533]$ is denoted $\overline{3.165}$.

Numerical investigations with the program XPP show that a stable state in a chain of four oscillators coupled with $d = 1.5$ occurs close to $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) =$ (-6.4114408, 7.3696656, 8.1984129, 7.3696656). Notice that the orbit of (23) describing this state is not symmetric in the sense of proposition 3.4 although it is mapped into itself under R. This numerically obtained solution is used as an initial guess in Mathematica's implementation of Newton's method to determine the roots x_a^i of the set of equations

$$\sigma(x_i - \frac{\beta r x_i}{\beta + (x_i)^2}) - 1.5(x_{(i-1) \bmod 4+1} - 2x_i + x_{(i+1) \bmod 4+1}) \qquad i = 1, 2, 3, 4. \tag{38}$$

Although Mathematica can be instructed to find roots to an arbitrary precision, so far only floating point arithmetic was used so that none of the results are rigorous yet.

At this point all the quantities in the computations are redefined as intervals rather than floating point numbers and using interval arithmetic and theorem 6.2 we find an interval around each x_a^i in which the roots of equation (38) must lie. These bounds are now rigorous.

In section 5 it was shown that a fixed state and all of its K-multiples are stable if the polynomial $p_{\mathcal{D}}(\lambda, t)$ corresponding to that state has roots with negative real part for all $t \in [0, \pi/2]$. Using interval arithmetic it is shown that

$$\begin{aligned}\bar{p}_{\mathcal{D}}(\lambda, t) =& \overline{3.61524} \times 10^{12} - \overline{5.02753} \times 10^{7} \cos 4t + (\overline{9.49583} \times 10^{11} - \overline{1.33199} \times 10^{7} \cos 4t)\,\lambda + \\ & (\overline{2.67802} \times 10^{11} - \overline{4.94338} \times 10^{6} \cos 4t)\,\lambda^2 + (\overline{4.90382} \times 10^{10} - \overline{771725} \cos 4t)\,\lambda^3 + \\ & (\overline{7.551} \times 10^{9} - \overline{144880} \cos 4t)\,\lambda^4 + (\overline{9.81388} \times 10^{8} - \overline{13673.3} \cos 4t)\lambda^5 + \\ & (\overline{1.03403} \times 10^{8} - \overline{1560.66} \cos 4t)\lambda^6 + (\overline{9.39525} \times 10^{6} - \overline{74.25} \cos 4t)\lambda^7 + \\ & (\overline{703005} - \overline{5.0625} \cos 4t)\lambda^8 + \overline{42026.7}\,\lambda^9 + \overline{2023.92}\,\lambda^{10} + \overline{66.6667}\lambda^{11} + \lambda^{12}\end{aligned} \tag{39}$$

for the fixed state under consideration.

In the next step the roots of the polynomial $\bar{p}_{\mathcal{D}}(\lambda, \frac{\pi}{2})$ are found. To employ Mathematica's implementation of Newton's method we need a polynomial with coefficients that are floating point numbers rather than intervals. Floating point numbers inside the intervals which determine the coefficients of the polynomial $\bar{p}_{\mathcal{D}}(\lambda, \frac{\pi}{2})$ can be chosen to define a polynomial $p^a_{\mathcal{D}}(\lambda, \frac{\pi}{2})$ which approximates $p_{\mathcal{D}}(\lambda, \frac{\pi}{2})$. The roots of $\{r_i^a\}_{i=1}^{12}$ of $p^a_{\mathcal{D}}(\lambda, \frac{\pi}{2})$ are now found using Newton's method.

The complex intervals[2] containing the roots of $\bar{p}_{\mathcal{D}}(\lambda, \frac{\pi}{2})$ are found by using theorem 6.1. We set z_0 equal to complex intervals around r_i^a and use complex interval arithmetic to find the radius K_0 and check the conditions of the theorem for each root r_i^a. The 4 real roots and 4 complex pairs obtained by this procedure are given in the table below.

$\overline{-18.481}$	$\overline{-0.2463} \pm i\overline{9.769}$
$\overline{-17.173}$	$\overline{-0.1961} \pm i\overline{9.317}$
$\overline{-15.437}$	$\overline{-0.1345} \pm i\overline{9.164}$
$\overline{-14.238}$	$\overline{-0.09054} \pm i\overline{8.622}$

Since interval arithmetic was used in these calculations, these estimates are now rigorous. The remainder of the argument shows that these roots will not cross the imaginary axis as t varies.

By lemma 5.1 the characteristic polynomial $p_{\mathcal{D}}(\lambda, t)$ takes the following form when evaluated on the imaginary axis:

$$\begin{aligned}p_{\mathcal{D}}(i\mu, t) &= \sum_{j=0, j \text{ even}}^{12} (\alpha_j + \beta_j \cos 4t)\mu^j (-1)^{\frac{j}{2}} + i \sum_{j=1, j \text{ odd}}^{11} (\alpha_j + \beta_j \cos 4t)\mu^j (-1)^{\frac{j-1}{2}} = \\ &= p^{\mathrm{R}}_{\mathcal{D}}(\mu, t) + i p^{\mathrm{I}}_{\mathcal{D}}(\mu, t)\end{aligned}$$

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[2] There are several ways in which complex intervals can be defined. In this case we use regions of the form $\bar{z} = \bar{x} + i\bar{y}$ where $\bar{x}$ and $\bar{y}$ are real intervals. These are simply rectangular regions in $\mathbb{C}$.

Since the roots of $p_{\mathcal{D}}(\lambda, t)$ depend continuously on the parameter t, if $p_{\mathcal{D}}(\lambda, t)$ has roots with positive real part for some t_1 then there must exist a t_0 such that $p_{\mathcal{D}}(\lambda, t_0)$ has a root on the imaginary axis. In other words there must exist a μ_0 such that

$$p^{\mathrm{R}}_{\mathcal{D}}(\mu_0, t_0) = p^{\mathrm{I}}_{\mathcal{D}}(\mu_0, t_0) = 0. \tag{40}$$

We will use interval arithmetic to show that this cannot happen. The polynomial $p_{\mathcal{D}}(\lambda, t)$ is split into a real and complex part to avoid using complex interval arithmetic in the numerical calculations since complex interval arithmetic leads to much rougher estimates than real interval arithmetic.

By Grešgorin's theorem the eigenvalues of the matrix $\mathcal{D}(t)$ lie in the union of discs $C_i = \{z : |z - \mathcal{D}_{ii}| < R_i\}$ where $\mathcal{D}_{ij}$ are the entries in the matrix $\mathcal{D}(t)$ and $R_i = \sum_{j=1, j\neq i}^{12} |\mathcal{D}_{ij}|$. A direct computation shows that the intersection of $\cup_{i=1}^{12} C_i$ with the imaginary axis is contained in the interval $[-17i, 17i]$ and it is therefore sufficient to show that $p^{\mathrm{R}}_{\mathcal{D}}(\mu, t_0)$ and $p^{\mathrm{I}}_{\mathcal{D}}(\mu, t_0)$ are not zero simultaneously for any value of t and $\mu \in [-17, 17]$.

Since the coefficients of $\bar{p}_{\mathcal{D}}(\lambda, t)$ are $\frac{\pi}{2}$ periodic in t, it is sufficient to show that (40) is not satisfied for any $\lambda \in [-17, 17]$ and $t \in [0, \frac{\pi}{2}]$. This is shown by subdividing these intervals into a sufficient number of subintervals $\bar{\mu}_n$ and $\bar{t}_m$ so that $[-17, 17] \subset \cup_{i=1}^{N} \bar{\mu}_n$ and $[0, \frac{\pi}{2}] \subset \cup_{m=1}^{M} \bar{t}_m$ with the property that the intervals $\bar{p}^{\mathrm{R}}_{\mathcal{D}}(\bar{\mu}_n, \bar{t}_m)$ and $\bar{p}^{\mathrm{I}}_{\mathcal{D}}(\bar{\mu}_n, \bar{t}_m)$ do not contain zero simultaneously for any given pair of subintervals $\bar{\mu}_n$ and $\bar{t}_m$. Therefore the roots of $p_{\mathcal{D}}(\lambda, t)$ stay in the left half plane for all $t \in \mathbb{R}$ and all K-multiples of the fixed state under consideration are also stable.

The paths of the roots of $p_{\mathcal{D}}(\lambda, t)$ as t is varied are shown in figure 11.

The *Mathematica* code used in these calculations is available at *http://math.bu.edu/people/josic/research/code.*

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A Proof of Lemma 5.1

Sublemma A.1 *Assume A is an $(n+m) \times (n+m)$ matrix of the form*

$$A = \begin{bmatrix} B & D^{+} & 0 & \dots & 0 \\ D^{-} & & & & \\ 0 & & C & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \tag{41}$$

where $m > n$, B and C are arbitrary $n \times n$, respectively $m \times m$ matrices and $D^{\pm}$ are $n \times n$ matrices of the form

$$D^{\pm} = \begin{bmatrix} d_{\pm} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}. \tag{42}$$

Writing the determinant of A as

$$\det A = \sum_{\sigma} sgn\,\sigma \prod_{i=1}^{m+n} a_{i,\sigma(i)} \tag{43}$$

then if $a_{1,\sigma(1)} = a_{1,n+1} = d_+$ *then*

$$\prod_{i=2}^{M+n} a_{i,\sigma(i)} \neq 0 \tag{44}$$

only if $a_{n+1,\sigma(n+1)} = a_{n+1,1} = d_-$.

This lemma simply states that the only nonzero products in (43) containing d_+ as a factor necessarily contain d_- as a factor. Notice that since we can interchange rows and columns when calculating determinants the opposite is also true: nonzero products containing d_- as a factor necessarily contain d_+ as a factor.

Proof: Write the matrix A as

$$A = \left[\begin{array}{cccccccc} b_{1,1} & b_{1,2} & \ldots & b_{1,n} & d_+ & 0 & \ldots \\ b_{2,1} & b_{2,2} & \ldots & b_{2,n} & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ d_- & 0 & \ldots & 0 & & & \\ 0 & 0 & \ldots & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & & \mathrm{C} & \\ 0 & 0 & 0 & 0 & & & \end{array}\right]. \tag{45}$$

If $a_{n+1,\sigma(n+1)} = a_{n+1,1} = d_+$ and $\sigma(i) = 1$ for some $2 \leq i \leq n$ another $n-1$ nonzero factors in the product (44) need to be chosen from columns 2 through n. However, barring rows 1 and i which were already used, there are only $n-2$ rows remaining which have nonzero entries in these columns. This shows that factors $a_{2,\sigma(2)}, a_{3,\sigma(3)}, \ldots, a_{n,\sigma(n)}$ must come from columns 2 through n. Since the rows 1 through n are now exhausted, the only nonzero element remaining in column 1 is d_-. This proves the lemma. ♢

Sublemma A.2 *The characteristic polynomial of the $Km \times Km$ matrix*

$$A = \left[\begin{array}{ccccccc} M_1 & E(t)^+ & 0 & \ldots & 0 & 0 & 0 \\ E(t)^- & M_2 & E(t)^+ & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & E(t)^- & M_{K-1} & E(t)^+ \\ 0 & 0 & 0 & \ldots & 0 & E(t)^- & M_K \end{array}\right]$$

where M_i are arbitrary $m \times m$ matrices and $E(t)^{\pm}$ are $m \times m$ matrices defined as

$$E(t)^{\pm} = \left[\begin{array}{cccc} de^{\pm it} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{array}\right]$$

does not depend on t.

Proof: Define the matrix $C = A - \lambda I$. We need to show that the determinant of C is independent of t. The proof is by induction on K. When $K = 1$ the matrix C is itself t independent so the statement is trivial. The cofactors of the entries $c_{1,j}$ for $1 \leq j \leq m$ are of the form

$$\begin{bmatrix} c_{2,1} & \ldots & \hat{c}_{2,j} & \ldots & c_{1,m} & 0 & \ldots & 0 \\ \multicolumn{8}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ c_{m,1} & \ldots & \hat{c}_{m,j} & \ldots & c_{m,m} & 0 & \ldots & 0 \\ de^{-it} & 0 & \ldots & \ldots & 0 & & & \\ 0 & 0 & \ldots & \ldots & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \mathrm{M} & \\ 0 & 0 & 0 & 0 & 0 & & & \end{bmatrix} \tag{46}$$

where the hat signifies the omission of that particular column. Since the columns and rows can be interchanged when the determinant is computed lemma A.1 implies that any product in (43) containing de^{-it} as a factor also contains $c_{1,n+1} = 0$ as a factor and is thus zero. Lemma A.1 also implies that any nonzero product containing $c_{1,n+1} = de^{it}$ as a factor necessarily contains $c_{1,n+1} = de^{-it}$ as a factor and so does not depend on t either. Since this exhausts all possible cofactors of the nonzero elements of the first row the lemma is proved. ♢

Lemma 5.1 can now be proved. We need to show that the determinant of $C = B - \lambda I$ is of the form (35). Definition (43) will again be used for the determinant. We now have three cases:

Case 1: Assume that for σ in the product in (43) we have $1 \leq \sigma(1) \leq m$. Since the next $m - 1$ rows have zero entries in all but the first m columns this particular product will be zero unless it contains $c_{j,1}$ as a factor for some $1 \leq j \leq m$. This excludes the possibility that either $c_{1,(K-1)m+1} = de^{-it}$ or $c_{(K-1)m+1,1} = de^{it}$ appear as factors. If we fix $\sigma(1), \ldots, \sigma(m)$, the part of the sum (43) corresponding to all such permutations σ is equal to

$$c_{1,\sigma(1)} \cdots c_{m,\sigma(m)} \begin{vmatrix} M_2 & E(t)^+ & 0 & \ldots & 0 & 0 & 0 \\ E(t)^- & M_3 & E(t)^+ & \ldots 0 & & 0 & 0 \\ \multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ 0 & 0 & 0 & \ldots & E(t)^- & M_{K-1} & E(t)^+ \\ 0 & 0 & 0 & \ldots & 0 & E(t)^- & M_K \end{vmatrix} \tag{47}$$

by cofactor expansion. By sublemma A.2 these summands will not depend on t.

Case 2: Assume that $\sigma(1) = m + 1$ so that the first factor in the product in (43) is $c_{1,m+1} = de^{it}$. Then the elements $c_{2,\sigma(2)}, \ldots, c_{m,\sigma(m)}$ must come from the matrix

$$\tilde{C} = \begin{bmatrix} c_{2,2} & \cdots & c_{2,m} \\ c_{3,2} & \cdots & c_{3,m} \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots} \\ c_{m,2} & \cdots & c_{m,m} \end{bmatrix}$$

by an argument identical to the one given in the second paragraph of sublemma (A.1). The only nonzero factors that can be chosen from column 1 are therefore $c_{m+1,\sigma(m+1)} =$

$c_{m+1,1} = de^{-it}$ and $c_{(K-1)m+1,\sigma(K-1)m+1} = c_{(K-1)m+1,1} = de^{it}$. We therefore have the following two subcases:

Case 2a: If $c_{m+1,1} = de^{-it}$ is chosen it cancels the term $c_{1,m+1} = de^{it}$ that was chosen as the first factor in the product. In this case we are again in the situation of sublemma A.2 using Laplace expansion as in case 1, and so this product will not depend on t.

The contributions to the characteristic polynomial from Case 1 and 2a are time independent and therefore enter the term $\sum_{i=(K-1)m+1}^{Km} \alpha_i \lambda^i$ in (35). The remainder of the lemma follows from induction.

Case 2b: If the assumptions of Case 2 hold and $c_{(K-1)m+1,\sigma((K-1)m+1)} = c_{(K-1)m+1,1} = de^{it}$ then if a permutation σ leads to a nonzero product in (43) it must satisfy $\sigma(jm+1) = (j+1)m+1$ for all $j < K-1$. This simply means that any nonzero product of this kind contains all the entries $c_{jm+1,(j+1)m+1} = de^{it}$ as factors. This assertion is proved as follows:

The factor $c_{1,m+1}$ is contained in the product by assumption. Assume that $\sigma(lm+1) = (l+1)m+1$ for all $l < j$. It will be shown that in this case $\sigma(jm+1) = (jm+1)m+1$ which will complete the induction argument.

By assumption the $\sigma((j-1)m+1) = jm+1$. The only nonzero elements in columns $jm+2, \ldots (j+1)m$ come from the submatrix

$$\begin{bmatrix} c_{jm+2,jm+2} & \cdots & c_{jm+2,(j+1)m} \\ c_{jm+3,jm+2} & \cdots & c_{jm+3,(j+1)m} \\ \cdots & \cdots & \cdots \\ c_{(j+1)m,jm+2} & \cdots & c_{(j+1)m,(j+1)m} \end{bmatrix}$$

and since an entry from row $jm+1$ has already been chosen, the coefficients $c_{jm+2,\sigma(jm+2)}$, $\ldots$, $c_{(j+1)m,\sigma((j+1)m)}$ must all come from this submatrix. The only nonzero entries remaining on row $jm+1$ are $c_{jm+1,(j-1)m+1} = de^{-it}$ and $c_{jm+1,(j+1)m+1} = de^{it}$. By the induction hypothesis we can exclude the first possibility and we are left to conclude that $c_{jm+1,(j+1)m+1} = de^{it}$ for any $1 < j < K$ is a coefficient in any nonzero product in the case 2b.

This shows that any permutation σ of the type described in Case 2b contributes a factor $\gamma_\sigma e^{Kit} r(\lambda)$ to the characteristic polynomial (35).

Case 3: The only remaining nonzero entry in row 1 of matrix C is $c_{1,(K-1)m+1} = de^{-it}$. An argument identical to the one presented in case 2 shows that any permutation σ leading to a nonzero summand in (43) must satisfy either $c_{(K-1)m+1,\sigma((K-1)m+1)} = c_{(K-1)m+1,1} = de^{it}$ or $c_{(K-1)m+1,\sigma((K-1)m+1)} = c_{(K-1)m+1,(K-2)m+1} = de^{-it}$ which again leads to two subcases:

Case 3a: If $c_{(K-1)m+1,\sigma((K-1)m+1)} = de^{it}$ the situation is virtually identical to case 2a. Laplace expansion and sublemma A.2 show that the contributions to the characteristic polynomial are independent of t.

Case 3b: If $c_{(K-1)m+1,\sigma((K-1)m+1)} = de^{-it}$ the situation is similar to case 2b. A parallel argument shows that all the terms de^{-it} in the matrix must be coefficients in any nonzero product. Moreover due to the properties of the matrix, to each product $\gamma_\sigma e^{Kit} r(\lambda)$ from case 2b corresponds a product $\gamma_\sigma e^{-Kit} r(\lambda)$ from this case. Since these are the only time dependent contributions to the characteristic polynomial the lemma is proved. $\diamondsuit$

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List of Figures

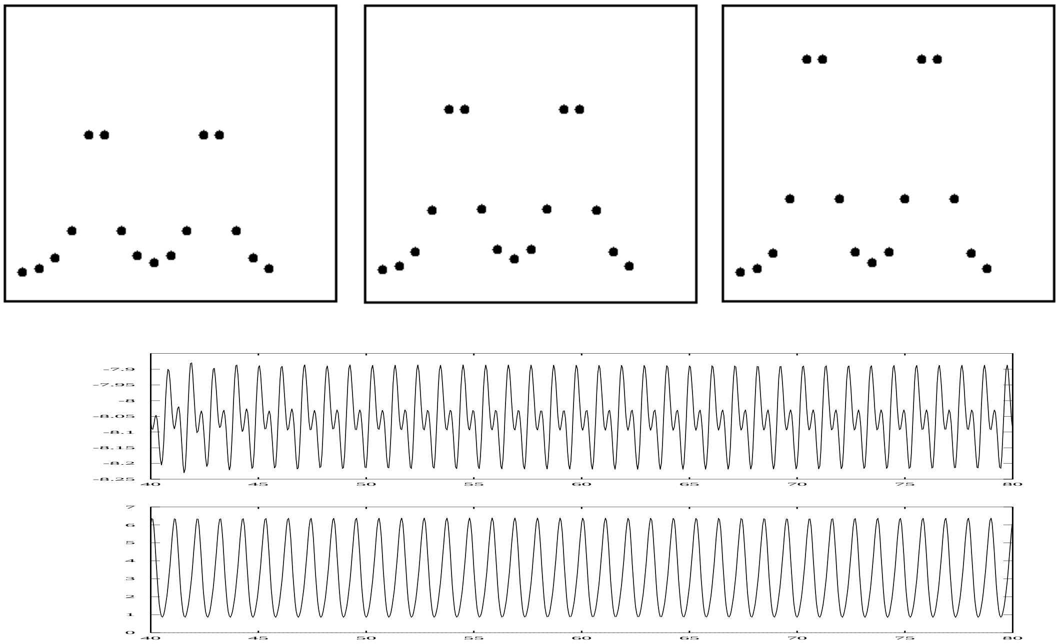

Figure 1: A stable state of two breathers in a ring of 16 Lorenz systems coupled through the x variable at $d_x = 15$. The first series of figures shows a snapshot of the x variables of the different systems at different points of one oscillation. In these figures, the position of the oscillator in the ring is given on the x-axis, while $x_i(t)$ is given on the y-axis. The timeseries of $x_1(t)$ and $x_5(t)$, which are nearly periodic, are given below.

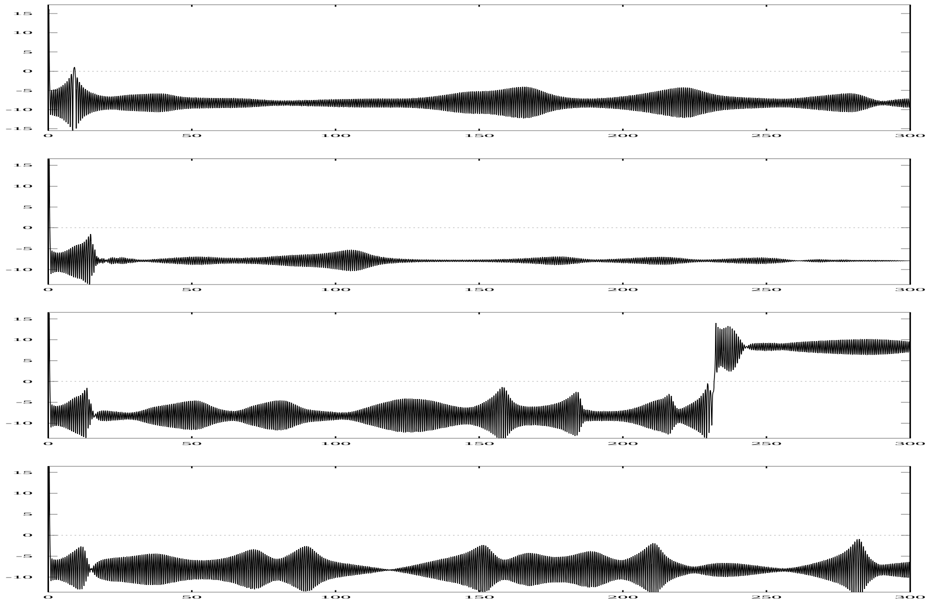

Figure 2: Typical timeseries of $x_{15}(t)$ at coupling strengths $d_x = 5, 15, 20, 25$ from top to bottom. Many features of the system remain similar at different coupling values. The oscillations have a smooth envelope, a feature that is typical to solutions for coupling strengths at which the network behaves coherently, but is not synchronized exactly. Even the transients that precede a stable fixed state have similar shapes.

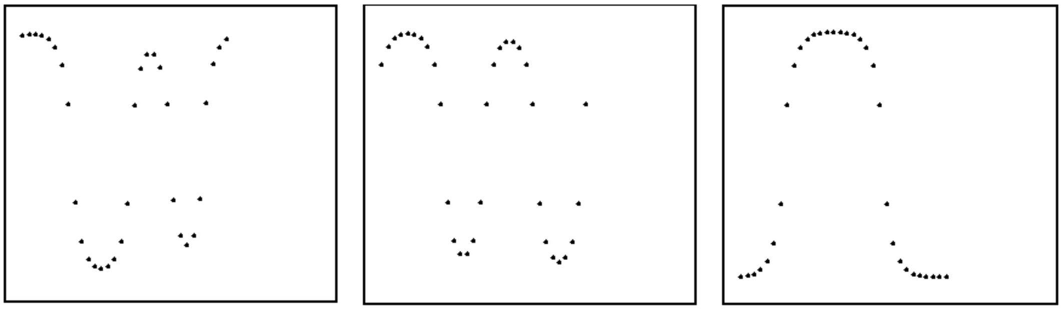

Figure 3: The system is very sensitive to initial conditions. The eventual stable state in a chain of 32 oscillators with $d_x = 40$ is shown. Only one initial condition $x_0(0)$ is different for each figure - it was incremented by 0.01. Very different states are reached. This suggests that the basins of attraction are intertwined in a complicated way. The stable states in these figures are typical, although fixed states with one “hump” are more common than states with more “humps”.

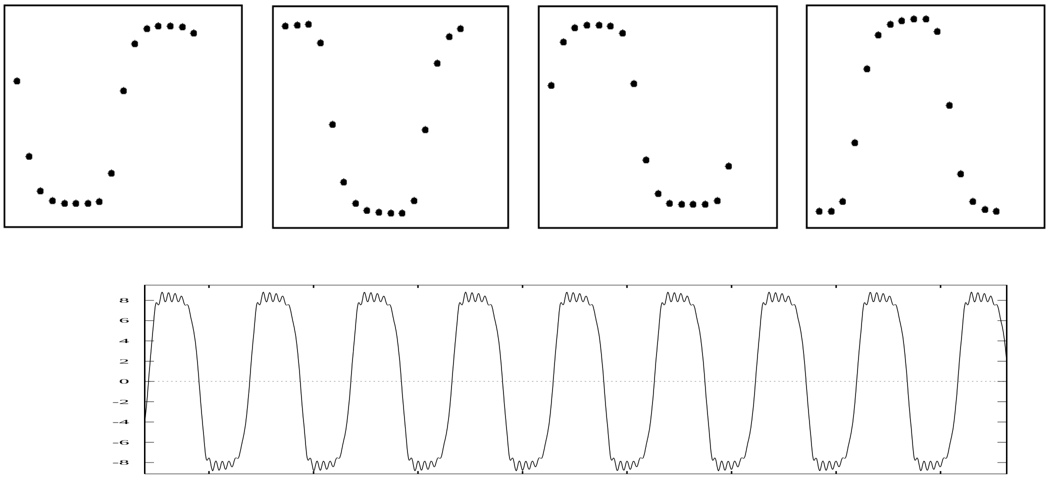

Figure 4: The wave when $d_x = d_y = 6$. This seems to be a globally stable solution of the equations.

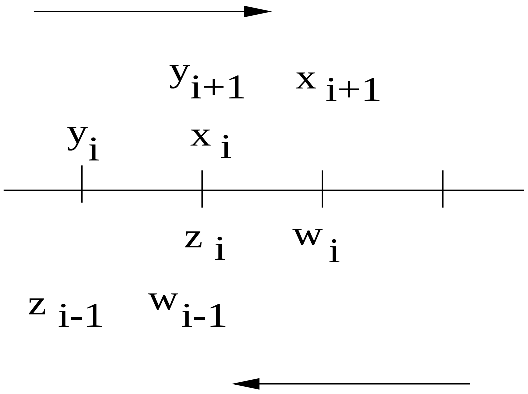

Figure 5: The action of the two systems in equation (9)

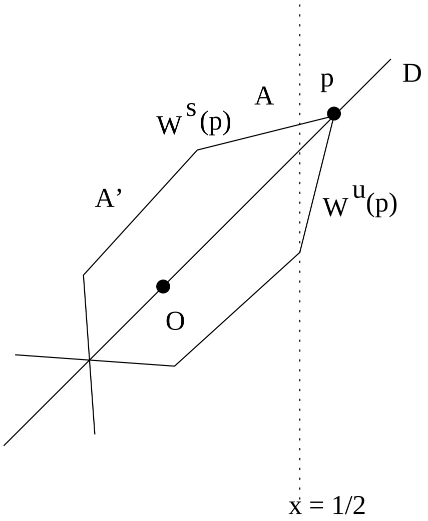

Figure 6: The invariant circles around (0,0) give way to complicated behavior close to (1/2, 1/2) due to the transversal intersection of $W^u(2/3, 2/3)$ and $W^s(2/3, 2/3)$.

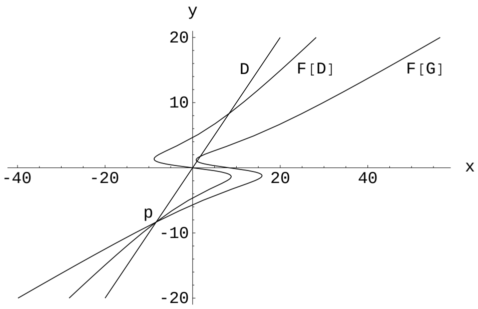

Figure 7: The region H is bounded by the two cubics $F(D)$ and $F(G)$ and D. The figure on the left W corresponds to the case $d = 20$. For $d > 20 + \epsilon$ these curves do not bound a compact region.

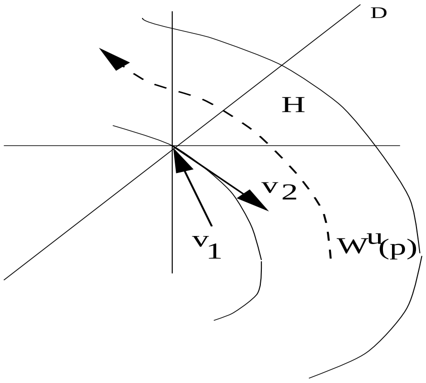

Figure 8: If $W^u(p)$ coincides with the stable manifold of the origin then v_1 must be tangent to $W^u(p)$ at the origin. Since v_1 points outside of H this is impossible.

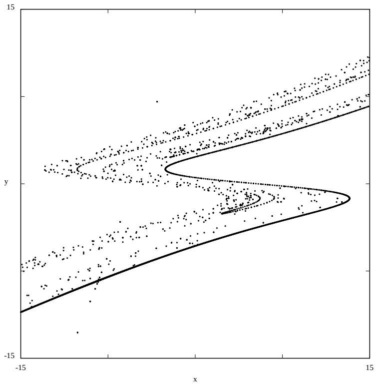

Figure 9: A numerical approximation of the manifold $W^u(p)$ is shown on the right. The complex structure of the manifold suggests complicated behavior on some invariant set of points.

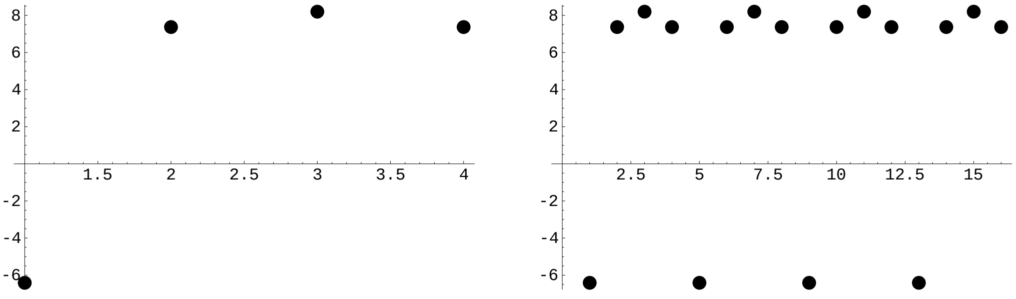

Figure 10: The fixed state that will be used as an example, and its 4-multiple.

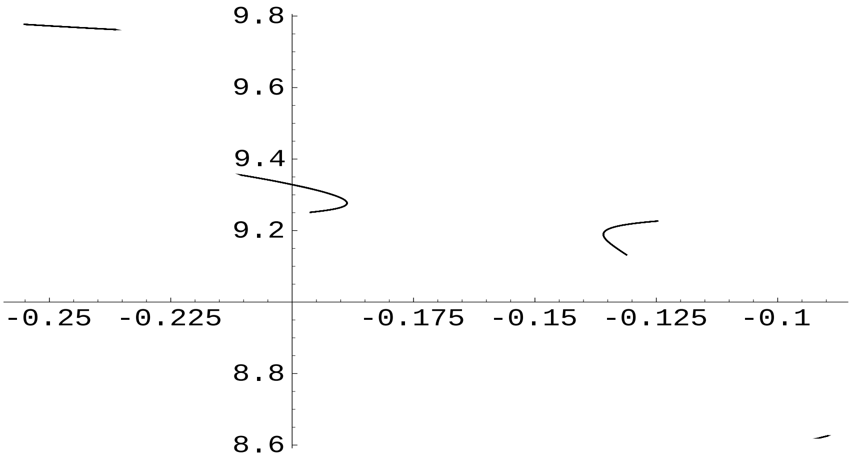

Figure 11: A numerical computation of the paths that the 8 complex eigenvalues with smallest real part trace out as t varies. Only one eigenvalue in the complex conjugate pair is shown. Notice the small trace of the eigenvalue with the least negative real part.