On the Wulff crystal in the Ising model

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Abstract

We study the phase separation phenomenon in the Ising model in dimensions $d \geq 3$. To this end we work in a large box with plus boundary conditions and we condition the system to have an excess amount of negative spins so that the empirical magnetization is smaller than the spontaneous magnetization m^*. We confirm the prediction of the phenomenological theory by proving that with high probability a single droplet of the minus phase emerges surrounded by the plus phase. Moreover, the rescaled droplet is asymptotically close to a definite deterministic shape – the Wulff crystal – which minimizes the surface free energy. In the course of the proof we establish a surface order large deviation principle for the magnetization. Our results are valid for temperatures T below a limit of slab-thresholds $\widehat{T}_c$ conjectured to agree with the critical point T_c. Moreover, T should be such that there exist only two extremal translation invariant Gibbs states at that temperature; a property which can fail for at most countably many values and which is conjectured to be true for every T. The proofs are based on the Fortuin-Kasteleyn representation of the Ising model along with coarse-graining techniques. To handle the emerging macroscopic objects we employ tools from geometric measure theory which provide an adequate framework for the large deviation analysis. Finally, we give a heuristic argument that for subcritical temperatures close enough to T_c, the dominant minus spin cluster of the Wulff droplet *permeates* the entire box and and has a strictly positive local density everywhere.

Key Words: phase separation, Wulff crystal, Ising model, large deviations, FK percolation
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1 Introduction

In order to develop heuristics we begin with an informal discussion of phase separation and phase coexistence phenomena - the main subject of this paper - with a simplistic description of a physical system familiar from everyday experience. The system we propose to consider is the mixture of two substances: oil and water. For our purpose it will be enough to recall the facts that water and oil tend to repel each other, and yet a certain (even if only a very small) amount of oil can be dissolved in water, and vice versa. [3]

As we know, solubility is not unlimited. Depending on the temperature T, there exist threshold densities $d_c^{\mathrm{o/w}}(T)$ and $d_c^{\mathrm{w/o}}(T)$ (both increasing in T) corresponding to *saturated* solutions of oil in water and water in oil, respectively. These two types of saturated solutions are called the pure phases "o/w" and "w/o" and they correspond to a perfect balance between energy and entropy.

In order to see phase separation in this system all we have to do is to take an almost saturated solution of oil in water with density d at temperature T, and let the system cool down to a temperature T' such that $d > d_c^{\mathrm{o/w}}(T')$. The excess amount of oil precipitates and macroscopic droplets emerge. The inside of the droplets does not contain pure oil, rather the droplets are regions where we see the phase w/o. They swim around in the phase o/w (assuming there is no gravitation, otherwise the phase w/o gathers at the top.) What is the law which governs the behavior of the coexisting phases?

The classical phenomenological theory asserts that there is a positive "surface free energy" $\mathcal{I}$ associated with the macroscopic phase boundaries and the system will settle down in a state minimizing this energy. Assuming isotropy, $\mathcal{I}$ is proportional to the surface area. Hence, in perfect equilibrium, there should be one single droplet of the phase w/o with a spherical shape floating in the phase o/w, since, by the isoperimetric inequality, this is the energetically most favorable configuration.

A mathematical challenge is to confirm this theory starting from a microscopic description of the system; to explain the existence and breakdown of solubility, the occurrence of phase separation on a macroscopic scale, and to verify the prediction about the existence of a single droplet with a specific shape.

Let us try to set up a simple model for our system. A convenient choice is a lattice model: each site of the lattice is occupied either by a water particle or an oil particle which we indicate by $+1$ or -1. The interaction between particles is repulsive and occurs when the substances are in immediate contact. Hence a repulsive nearest neighbor interaction is a sensible choice. If we want to focus only on the dominant repulsive interaction between different molecules, we could simplify the model by making the two substances "symmetric" by assuming that their self–interactions are of equal magnitude, or equivalently, equal to zero. Thus the total energy of a configuration should be simply the sum of all nearest neighbor pairs with different sign. Recall that in our experiment the density of oil is fixed, therefore we have a constraint on the possible

[3] The mixture of phenol and water would be a more realistic choice since they are more miscible, however, the potential reader is hardly familiar with that mixture. The mixture of salt and water is also a good example but the phenomenon we want to describe is more complex in that case.

configurations: the proportion of +1-s and −1-s has to be fixed. The reader familiar with the Ising model has already observed that our simple model is equivalent to the Ising model with plus boundary conditions (guaranteeing the water dominance) *conditioned* on the event that the magnetization is fixed and is smaller than the spontaneous magnetization (corresponding to the saturation density $d_c^{\mathrm{o/w}}$) at the given temperature. Note that this event is extremely unlikely when the system is large. In fact, the study of this restricted system leads to questions about large deviations in the (unrestricted) Ising model.

There is an additional difficulty due to the use of the lattice: the system is anisotropic, thus we have to deal with a surface energy depending on the direction which leads to a non-spherical droplet. It is called the *Wulff droplet* or *Wulff crystal* after Wulff [65] who first studied and solved the corresponding isoperimetric problem.

The goal of the present paper is to continue the study of the questions stated above in the context of the Ising model (which is, as we have just seen, a fairly natural choice) with the specific aim to develop methods which work independently from the dimension and temperature. Before we start with the presentation of our results we give a brief summary of the previous work on this subject.

The first efforts were devoted to the study of large deviations of the empirical magnetization, i.e., the average value of the spins in a large box. A *volume order* large deviation principle (LDP) has been established for the Ising model by various authors: Comets, Ellis, Föllmer, Orey, Olla [17, 28, 32, 52]. The corresponding rate function has been found to vanish in $[-m^*, m^*]$ where m^* denotes the spontaneous magnetization. In fact, it was suspected that the correct order of decay is exponential to *surface order*. Indeed, Schonmann [58] found a proof of this conjecture, valid for any dimensions and low enough temperatures and Chayes, Chayes and Schonmann extended the result for the supercritical $\beta > \beta_c$ regime in the two dimensional case. Föllmer and Ort [33] investigated this phenomenon on the level of empirical measures. Finally, inspired by the work of Kesten and Zhang [45] on related questions in percolation, Pisztora [55] established surface order upper bounds for the remaining dimensions $d \geq 3$ above the slab-threshold $\widehat{\beta}_c$ which is conjectured to agree with the critical point β_c.

The monograph of Dobrushin, Kotecký and Shlosman [26] opened the way to the study of the phase separation phenomenon for the two dimensional Ising model, by establishing a surface order LDP for the magnetization and proving the existence of the Wulff droplet. The main tool of their work is the cluster expansion, thus their results are confined to low temperatures. Pfister [53] simplified the proof through duality arguments. Ioffe [42, 43] and then Ioffe and Schonmann [44] extended these results up to the critical temperature. The results around the Wulff construction for the two dimensional Ising model are now fairly precise. They go far beyond large deviations statements. Hryniv [25, 41] even managed to describe the Gaussian fluctuations around the interface. The validity of the Wulff construction for two dimensional Bernoulli percolation has been proved by Alexander, Chayes and Chayes [5]. Alexander [4] subsequently refined the probabilistic estimates. This whole area of research is now very active (see [15, 54, 59, 60, 61]). Nevertheless these works are confined to dimension two. The main reason is that the probabilistic estimates for the presence of an interface separating the two phases rely on the skeleton coarse–graining technique. Through the skeleton, one can approximate a polygonal line drawn on the lattice by a coarser one and then use a combinatorial bound for the number of polygonal lines. Unfortunately, it seems hard to find a higher dimensional analogue of this elegant and efficient technique.

The next challenge is to analyze phase separation and coexistence for microscopic models in higher dimensions. A first step towards this goal was accomplished by Cerf [14]: large deviations statements are proven for the *cluster shapes* in three dimensional Bernoulli percolation, from which a picture of the Wulff crystal of this model emerges. From the very beginning, the problem is embedded in a continuous setting, and adequate tools from geometric measure theory (à la Caccioppoli et De Giorgi) are used with the crucial benefit that the two dimensional combinatorial argument associated with skeletons can be circumvented by a compactness argument. The key ingredients to get the required probabilistic estimates are the coarse–graining results of Pisztora [55], specialized to the case of Bernoulli percolation.

The current paper is a natural successor to [14] and [55]. Our first aim is to provide the essential tools to extend the large deviations results of [14] to FK percolation in a regime where the coarse–graining results of [55] hold. A novel issue is to control the dependence between events occurring in distant regions. The second goal of this paper is to give convincing evidence of the existence of the Wulff crystal in the Ising model. We propose here an Ising specific approach of this question. The reason that our results have an almost optimal range ($\beta > \widehat{\beta}_c$) is that the non-perturbative coarse graining methods hold beyond the contour regime where perturbative techniques are doomed to fail. For reasons of space we give here only one result using the locally averaged magnetization to identify the Wulff crystal.

In an independent line of development, important ideas were already present in a series of works in the context of the Ising model with Kac potentials. Alberti, Bellettini, Cassandro, Presutti [3, 7] introduced the local averaging of the magnetization and the BV setting (the equivalent functional formulation of the sets of finite perimeter) to study surface order large deviations. The subsequent work of Benois, Bodineau, Buttà, Presutti [8, 9] contains useful techniques to handle the renormalized picture. In particular, the idea of transforming adequately the configuration in order to get estimates on certain probabilities, which is an important element of the interface lemma, appears there as well. However, these works lose the microscopic structure of the model: they perform a mean–field limit where the range of interaction tends to infinity and everything becomes isotropic. While preparing the manuscript of this article, we received the preprint [11]. Building upon [8, 9, 14, 55], Bodineau has proved results similar to ours for sufficiently low temperatures.

We next state our results and outline the strategy of the proofs.

Range of validity of the results. We consider Fortuin-Kasteleyn (FK) percolation (also known as the random cluster model) in dimensions $d \geq 3$ in the regime $q \geq 1$, $p > \widehat{p}_c$, $\theta^f(p) = \theta^w(p)$. Here $\widehat{p}_c$ is the slab percolation threshold introduced in [55] and $\theta^f(p)$, $\theta^w(p)$ are the densities of the infinite open cluster for the infinite volume FK measures Φ^f_∞, Φ^w_∞ corresponding to free and wired boundary conditions. The equality $\theta^f(p) = \theta^w(p)$ implies that there exists a unique infinite volume FK measure Φ_∞ on the cubic lattice $\mathbb{L}^d$ corresponding to the parameters (p, q), by results of Lebowitz and Martin-Löf [47, 46] and Grimmett [38]. It is also known that the condition $\theta^f(p) = \theta^w(p)$ holds for values of p close to 1 and might only be violated at countably many values. It is conjectured that $\theta^f(p) = \theta^w(p)$ for every $p \neq p_c$ and that $\widehat{p}_c$ coincides with the critical point p_c. (This is the case at least for Bernoulli percolation

($q=1$) by the result of Grimmett and Marstrand [39].) For the Ising model, we choose $q=2$ and the inverse temperature $\beta=1/T$ is related to p via the relation $p=1-\exp(-\beta)$. The spontaneous magnetization is denoted by $m^*(\beta)$ and it is well known that $m^*(\beta)=\theta(p)$. We set $\widehat{\beta}_c=1/\widehat{T}_c=-\log(1-\widehat{p}_c)$ and $\mathcal{U}(d)=\{\,-\log(1-p) : p \text{ such that } \theta^f(p)=\theta^w(p)\,\}$. The set $\mathcal{U}(d)$ is the domain of inverse temperatures where there exist only two extremal translation invariant Gibbs states. Our results for the Ising model hold in the region: $d\geq 3,\ \beta>\widehat{\beta}_c,\ \beta\in\mathcal{U}(d)$.

Results. We first extract from the FK percolation model a direction dependent surface tension τ. For a unit vector ν, let A be a unit hypersquare orthogonal to ν, let $\operatorname{cyl} A$ be the cylinder $A+\mathbb{R}\nu$, then $\tau(\nu)$ is equal to the limit

$$\lim_{n\to\infty}-\frac{1}{n^{d-1}}\log\Phi_\infty\left(\begin{matrix}\text{inside } n\operatorname{cyl}A \text{ there exists a set of closed edges } E \text{ cutting } n\operatorname{cyl}A \text{ in at}\\ \text{least 2 unbounded components and the edges of } E \text{ at distance less than}\\ 2d \text{ from the boundary of } n\operatorname{cyl}A \text{ are at distance less than } 2d \text{ from } nA\end{matrix}\right)$$

This function τ satisfies the weak simplex inequality, is continuous, positive and invariant under the isometries which leave $\mathbb{Z}^d$ invariant. The Wulff crystal $\mathcal{W}_\tau$ of τ is

$$\mathcal{W}_\tau=\left\{x\in\mathbb{R}^d\ \middle|\ x\cdot\nu\leq\tau(\nu)\text{ for all unit vectors }\right\}$$

The crystal $\mathcal{W}_\tau$ is convex, closed, bounded and contains the origin in its interior. We define the surface energy $\mathcal{I}(\mathcal{W}_\tau)$ of the Wulff crystal by

$$\mathcal{I}(\mathcal{W}_\tau)\ =\ \sup\ \left\{\ \int_{\mathcal{W}_\tau}\operatorname{div} f(x)\,dx : f\in C^1_0(\mathbb{R}^d,\mathcal{W}_\tau)\ \right\}$$

where $C^1_0(\mathbb{R}^d,\mathcal{W}_\tau)$ is the set of the compactly supported C^1 vector fields taking values in the Wulff crystal $\mathcal{W}_\tau$ and div is the usual divergence operator, cf. (6) for an explanation of this formula.

Our first theorem gives the logarithmic asymptotics of the probability of having a defect of magnetization in the cubic box $\Lambda(n)$ of diameter n with plus boundary conditions. For $m\in[-m^*,m^*]$, we define the rescaled Wulff crystal

$$\mathcal{W}(m)=\left(\frac{m^*-m}{2m^*\operatorname{vol}\mathcal{W}_\tau}\right)^{1/d}\mathcal{W}_\tau$$

Let m be such that $\mathcal{W}(m)$ fits completely into the unit box. It is easy to see that this is the case if

$$1-2(\operatorname{diam}\mathcal{W}_\tau)^{-d}\operatorname{vol}\mathcal{W}_\tau<m/m^*<1$$

and we call such m *admissible.*

Theorem 1.1 (logarithmic asymptotics of the magnetization)

Let $d\geq 3,\ \beta>\widehat{\beta}_c,\ \beta\in\mathcal{U}(d)$ and m be admissible. Then

$$\lim_{n\to\infty}\frac{1}{n^{d-1}}\log\mu^+_{\Lambda(n)}\Big[\frac{1}{n^d}\sum_{x\in\Lambda(n)}\sigma(x)\leq m\Big]=-\Big(\frac{m^*-m}{2m^*\operatorname{vol}\mathcal{W}_\tau}\Big)^{(d-1)/d}\mathcal{I}(\mathcal{W}_\tau)=-\mathcal{I}(\mathcal{W}(m))$$

Here we choose to work with finite boxes rather than with the infinite volume measure since usually a finite volume result is more useful in applications. As a benefit, on the technical side, we gain compactness after rescaling. The drawback is that we have to pay attention to the boundary; it is necessary to handle separately the case of an interface sitting on the boundary, for the probabilistic estimates as well as for the geometric approximations.

Our second theorem gives further information on the mechanism creating a defect of magnetization in $\Lambda(n)$. Alternatively, it describes the equilibrium of the system when forced to have an excess amount of negative spins. As indicated earlier, a single Wulff droplet of the minus phase emerges (with a local magnetization close to $-m^*$) which is surrounded by the plus phase and contains all the excess amount of negative spins. A convenient way to localize this droplet is to look at local averages of the magnetization over a (small) intermediate scale.

For $x \in \mathbb{R}^d$ and $r > 0$ we define the box $\Lambda(x,r)$ by

$$\Lambda(x,r) = \Big\{ y = (y_1, \cdots, y_d) \in \mathbb{R}^d \,\Big|\, -r/2 < y_i - x_i \le r/2,\ 1 \le i \le d \Big\}$$

The L^1 distance between two Borel sets E, F is the volume of their symmetric difference, i.e.,

$$\operatorname{dist}_{L^1}(E,F) = \operatorname{vol} E \Delta F$$

The mass center $mc(E)$ of a (bounded) Borel set E is

$$\operatorname{mc}(E) = \frac{1}{\operatorname{vol} E} \int_E x\, dx$$

We make the convention that $\operatorname{mc}(\emptyset) = 0$. Let $f(n)$ be a fixed function from $\mathbb{N}$ to $\mathbb{N}$ such that both $n/f(n)^{d-1}$ and $f(n)/\log n$ tend to ∞ as $n \to \infty$. The *locally averaged magnetization* σ_n is the map from the closed unit cube $\Gamma = [-1/2, -1/2]^d$ to $[-1,1]$ defined by

$$\forall x \in \Gamma \qquad \sigma_n(x) = \frac{1}{f(n)^d} \sum_{y \in \Lambda(nx, f(n)) \cap \Lambda(n)} \sigma(y)$$

We partition Γ into the random sets Γ_n^-, Γ_n^0 and Γ_n^+ according to whether the value of the local magnetization is smaller, equal or larger than zero.

Theorem 1.2 (typical configuration of the locally averaged magnetization)

Let $d \ge 3$, $\beta > \widehat{\beta}_c, \beta \in \mathcal{U}(d)$. For $\delta > 0$,

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mu^+_{\Lambda(n)} \Big[\int_{\Gamma_n^-} |\sigma_n(x) + m^*|\, dx + \operatorname{vol} \Gamma_n^0 + \int_{\Gamma_n^+} |\sigma_n(x) - m^*|\, dx > \delta \Big] = -\infty$$

Let m be admissible. There exist constants $b = b(d,\beta,m,\delta)$, $c = c(d,\beta,m,\delta) > 0$ such that

$$\mu^+_{\Lambda(n)} \Big[\operatorname{dist}_{L^1}\Big(\Gamma_n^-, \operatorname{mc}(\Gamma_n^-) + \mathcal{W}(m)\Big) < \delta \,\Big|\, \frac{1}{n^d} \sum_{x \in \Lambda(n)} \sigma(x) \le m \Big] \ge 1 - b \exp(-cn^{d-1})$$

We finish the presentation of our results with some comments and speculations.

Remarks on the microscopic structure of the Wulff droplet. It is well known [26, 44] that in two dimensions the Wulff droplet can be identified with a (random) region surrounded by a *minus spin cluster* (a set surrounds itself). Its external boundary is therefore a large contour separating plus and minus spins and its shape (when rescaled) follows closely the Wulff shape in the sense of the Hausdorff metric. It has already been realized in [26] that in higher dimensions this picture can not remain valid: large but very thin "hairs" (or spikes) may be attached to the droplet without significantly increasing its surface energy. In fact, it is to expect that such spikes will occur according to the balance between their entropy and penalty in energy. A single hair is depicted in [26], fig. 1.3.

We argue that for $d \geq 3$ the situation is even more cumbersome. We expect that the crossing of the threshold β_p, indicating the onset of percolation of minority spins [4], implies a drastic change in the physical appearance of the Wulff droplet as seen in the spin model. In the low-temperature regime $\beta > \beta_p$, the situation is as predicted by [26]: there is a big minus spin cluster surrounding the Wulff droplet with attached hairs. In the regime (β_c, β_p) however, we expect that the dominant minus spin cluster S^- of the Wulff droplet will percolate all the way to the boundary of the box, in fact, it permeates the entire box with a positive density in the complement of the Wulff region. More precisely, there exist two big spin clusters in the box, S^+ and S^-, which are both omnipresent in the entire box. Moreover there exist densities $0 < \rho_1 < \rho_2 < 1$ such that the (local) density of S^- within the Wulff region and the density of S^+ in the complement are both concentrated around ρ_2 while the density of S^- outside and S^+ inside the Wulff region is close to ρ_1. This means that in this regime the Wulff droplet can not be described directly with contours, and the analysis by contour techniques appears to be horrendous.

At the same time the corresponding object at the FK percolation level do not show this strange behavior: in the entire regime $\beta > \beta_c$ the Wulff region can be identified with a region associated to a single big FK cluster C^- with the right shape. The phenomenon described above occurring at the spin level is simply a consequence of the percolation of negatively colored small FK clusters starting at small FK clusters sitting next to the Wulff cluster C^-. In our view, this phenomenon is one of the numerous examples demonstrating that the FK representation has deep *physical* relevance for the spin model, besides being a superb technical tool. In fact, the FK cluster of a site x can be interpreted as the "region of influence" (which is of course random) of the spin at x. Finally we remark that in this paper we make no attempt to prove the picture described above; it remains a challenging open problem.

The strategy of the proofs. Our basic strategy is to use the FK representation to relate the events occurring at the Ising level to events occurring at the FK level. We try here to provide simple and efficient proofs of Theorems 1.1 and 1.2, hence we do not care to describe accurately the picture at the FK level. A defect of magnetization in a cubic box with positive boundary conditions is caused with overwhelming probability by the presence of one or several large open FK clusters which has been colored negatively (large means here of diameter exceeding $f(n)$).

[4] We recall that in dimensions $d \geq 3$, $\beta_p(d)$ is conjectured to be strictly larger than β_c meaning that minus spins (as well as plus spins) percolate in the plus phase for $\beta_c < T < \beta_p$, but not when $\beta > \beta_p$. It is known [1] that this is the case for dimensions large enough. In the case of $d = 2$ this phenomenon does not occur, i.e., $\beta_p = \beta_c$, cf. [16].

Thus we keep track of the region M^- consisting of the points of $n\Gamma$ whose $f(n)$ neighborhood intersects only large clusters colored negatively. We first show that

$$\frac{1}{n^d}\sum_{x\in\Lambda(n)}\sigma(x) \;=\; m^*(1-2\,\mathrm{vol}\,(M^-/n))+\delta(n) \tag{1}$$

where the (random) correction term $\delta(n)$ goes to 0 in probability faster than exponential to surface order. Therefore everything boils down to getting probabilistic estimates on the law of the random set M^-/n as $n\to\infty$. Obviously the law of M^-/n converges to the Dirac mass at the empty set: the typical state of the system is the pure phase μ^+. We are interested in the large deviations event $\{\ \mathrm{vol}\,(M^-/n)\geq (m^*-m)/(2m^*)\,\}$. We first obtain a lower bound: for λ, x such that $0<\lambda\,\mathrm{diam}\,\mathcal{W}_\tau\leq 1$ and $x+\lambda\mathcal{W}_\tau\subset\Gamma$, for any $\delta>0$,

$$\varliminf_{n\to\infty}\frac{1}{n^{d-1}}\,\log\,\mathbb{P}^+_{\Lambda(n)}\Big[\,\mathrm{dist}_{L^1}(M^-/n, x+\lambda\mathcal{W}_\tau)<\delta\Big]\geq -\lambda^{(d-1)}\,\mathcal{I}(\mathcal{W}_\tau) \tag{2}$$

where $\mathbb{P}^+_{\Lambda(n)}$ is the measure on edge–spin configurations realizing the coupling between the FK measure $\Phi^w_{\Lambda(n)}$ and the Ising Gibbs measure $\mu^+_{\Lambda(n)}$. The difficult part is to get the upper bound. In order to control the combinatorial explosion of all possible discrete shapes for M^-/n, we embed M^-/n in a continuous space and we use a compactness argument. This is the place where Geometric Measure Theory enters the game. Notice that the use of this theory was already suggested in [26] (see remark 1.11.2 c) p14). We consider the random rescaled region M^-/n as an element of the space $\mathcal{B}(\Gamma)$ of the Borel subsets of Γ endowed with the metric dist_{L^1}. We define the surface energy $\mathcal{I}(E)$ of a Borel set E by

$$\mathcal{I}(E)\;=\;\sup\;\Big\{\int_E \mathrm{div}\,f(x)\,dx : f\in C^1_0(\mathbb{R}^d,\mathcal{W}_\tau)\,\Big\} \tag{3}$$

cf (6). Notice that the surface energy might be infinite. First, we prove that the law of M^-/n concentrates fast near sets having a finite surface energy: there exists a positive constant $c=c(d,p,q)$ such that

$$\forall\lambda,\delta>0\qquad \varlimsup_{n\to\infty}\frac{1}{n^{d-1}}\,\log\,\mathbb{P}^+_{\Lambda(n)}\Big[\,\mathrm{dist}_{L^1}\Big(M^-/n,\,\mathcal{I}^{-1}([0,\lambda])\Big)\geq\delta\Big]\;\leq\;-c\lambda \tag{4}$$

Second, we estimate the probability that M^-/n is close to a fixed set having finite surface energy: for E in $\mathcal{B}(\Gamma)$ such that $\mathcal{I}(E)<\infty$ and for $\epsilon>0$, there exists $\delta=\delta(E,\epsilon)>0$ such that,

$$\varlimsup_{n\to\infty}\frac{1}{n^{d-1}}\,\log\,\mathbb{P}^+_{\Lambda(n)}\Big[\,\mathrm{dist}_{L^1}(M^-/n,\,E)<\delta\Big]\;\leq\;-(1-\varepsilon)\,\mathcal{I}(E) \tag{5}$$

The essential tools for the bound (5) are the coarse–graining results of [55] to get the probabilistic estimates and the Vitali covering Theorem for Hausdorff measures to get appropriate geometric approximations.

The surface energy $\mathcal{I}(E)$ is infinite unless E is a set of finite perimeter in the sense of Caccioppoli and De Giorgi. The Gauss–Green Theorem shows that, in case the boundary ∂E of E is smooth, then

$$\mathcal{I}(E)\;=\;\int_{\partial E}\sup_{y\in\mathcal{W}_\tau}(y\cdot\nu_E(x))\,\mathcal{H}^{d-1}(dx)\;=\;\int_{\partial E}\tau(\nu_E(x))\,\mathcal{H}^{d-1}(dx) \tag{6}$$

where $\nu_E(x)$ is the exterior normal vector to E at x and $\mathcal{H}^{d-1}$ is the $(d-1)$ dimensional Hausdorff measure in $\mathbb{R}^d$. The theory of the sets of finite perimeter was historically invented by Caccioppoli [12, 13] and subsequently developed by De Giorgi [18, 19, 20, 21]. The goal of Caccioppoli was to build a general theory of integration for differential forms and to extend the classical Gauss–Green Theorem to sets whose boundary is not C^1. Independently, De Giorgi was seeking to generalize some isoperimetric problems, starting with the Gauss–Green Theorem. This framework is extremely convenient to deal with variational problems. Indeed the surface energy $\mathcal{I}$ is lower semicontinuous (l.s.c.) with respect to the L^1 convergence. In fact, the surface energy defined in formula (3) is the largest l.s.c. extension of the expression (6) to the class of all Borel sets. Moreover the level sets $\{\, E \in \mathcal{B}(\Gamma) : \mathcal{I}(E) \leq \lambda \,\}$, $\lambda \geq 0$, are compact. This crucial compactness property, in conjunction with the local estimate (5) and the exponential tightness given by (4), yield the upper bound

$$\forall v > 0 \qquad \overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{vol}(M^-/n) \geq v\Big] \leq -\inf_{\operatorname{vol} E \geq v} \mathcal{I}(E) \tag{7}$$

It turned out that the class of the sets of locally finite perimeter (also called Caccioppoli sets) is isomorphic to the currents of codimension one of the general Geometric Measure Theory developed by Federer [31]. Taylor [62, 63, 64] proved the Wulff isoperimetric Theorem with the theory of currents. This Theorem (originally due to Wulff [65], followed by Dinghas [24]) states that the Wulff crystal $\mathcal{W}_\tau$ is the only solution to the variational problem

$$\text{minimize } \mathcal{I}(E) \text{ under the constraint } \operatorname{vol} E \geq \operatorname{vol} \mathcal{W}_\tau\,.$$

The Wulff isoperimetric Theorem has been reworked and slightly generalized in the framework of the Caccioppoli sets by Fonseca [34] and Fonseca and Müller [35]. Theorem 1.1 is an easy consequence of (1), (2), (7) and the Wulff isoperimetric Theorem. For Theorem 1.2, one needs in addition some slightly refined coarse–graining estimates from [55].

2 Preliminaries

In this section we introduce first the notation and we give some basic definitions. In the second part, we recall some useful properties of FK (or random cluster) measures and we give a short description of the Ising model and its FK representation.

Notation.

The cardinality of a set A is denoted by $|\,A\,|$. The symmetric difference between two sets A_1, A_2 is denoted by $A_1 \triangle A_2$. If $\mathcal{A}$ is a family of sets then we write $\operatorname{cup}\mathcal{A}$ for $\cup_{A\in\mathcal{A}} A$. For $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the integer part of r and $\lceil r \rceil$ stands for the smallest integer larger or equal to r.

Metric: We denote by d_p the metric associated with the p-norm, i.e., $d_p(x,y) = |x-y|_p$ for any x, y in $\mathbb{R}^d$. We will only use the $1, 2$ and ∞ norms. The d_p distance between two subsets E_1 and E_2 of $\mathbb{R}^d$ is $d_p(E_1, E_2) = \inf\{\, |x_1 - x_2|_p : x_1 \in E_1,\, x_2 \in E_2 \,\}$. The r–neighbourhood of $E \subseteq \mathbb{R}^d$ with respect to the d_p metric is the set $\mathcal{V}_p(E, r) = \{\, x \in \mathbb{R}^d : d_p(x, E) < r \,\}$. The d_p diameter of a subset E of $\mathbb{R}^d$ is $\operatorname{diam}_p E = \sup\{\, |x-y|_p : x, y \in E \,\}$. We will usually work with the Euclidean distance d_2 on the continuous space $\mathbb{R}^d$ and with the distance d_1 or d_∞ on the discrete lattice

$\mathbb{Z}^d$. By default, when we speak of the diameter of a set without any specification, we mean the d_∞ diameter.

Geometric objects: We denote by Γ the closed unit cube $\Gamma = [-1/2, 1/2]^d$. Let $x = (x_1, \cdots, x_d)$ be a point of $\mathbb{R}^d$ and let r be positive. The closed ball of center x and Euclidean radius r is denoted by $B(x,r)$. The sphere of center x and radius r is $\partial B(x,r)$. When $x = 0$ we usually drop x from the notation; for instance $B(r) = B(0,r)$. This convention applies to all of the objects described below. The unit sphere of $\mathbb{R}^d$ is denoted by S^{d-1}. The projective sphere PS^{d-1} is obtained by identifying opposite points on S^{d-1}. Let w be a unit vector. We set

$$\mathrm{hyp}\,(x,w) = \{y \in \mathbb{R}^d \mid (y-x)\cdot w = 0\}$$

The specific hyperplane containing the origin and perpendicular to the d-th axis is denoted by $\mathbb{D}^d$. For r_1, r_2 in $\mathbb{R} \cup \{-\infty, +\infty\}$, we define

$$\mathrm{slab}\,(x, w, r_1, r_2) = \{y \in \mathbb{R}^d \mid r_1 \leq (y-x)\cdot w \leq r_2\}$$

The half spaces $\mathrm{slab}\,(x,w,0,\infty)$ and $\mathrm{slab}\,(x,w,-\infty,0)$ are denoted by $\mathbb{H}_+(x,w)$ and $\mathbb{H}_-(x,w)$ respectively. We set

$$B_-(x,r,w) = B(x,r) \cap \mathbb{H}_-(x,w)\,, \quad B_+(x,r,w) = B(x,r) \cap \mathbb{H}_+(x,w)\,.$$

The "upper" half space $\{x \in \mathbb{Z}^d \mid x_d \geq 0\}$ is denoted by $\mathbb{H}^d$. By $\mathrm{disc}\,(x,r,w)$ we denote the closed disc centered at x of radius r and normal vector w. A *box* is a set of the form

$$\Lambda(x,r) = \Big\{ y = (y_1, \cdots, y_d) \in \mathbb{R}^d \,\Big|\, -r_i/2 < y_i - x_i \leq r_i/2,\ i = 1, \cdots, d \Big\}$$

where $x, r \in \mathbb{R}^d$. Clearly, x is the center and r determines the side lengths of the box. If $r_i = t$ for each $i = 1, ..., d$, where $t \in \mathbb{R}$, then we write simply $\Lambda(x,t)$. Notice that $\Lambda(x,t)$ has diameter t and is neither open nor closed. If $d_\infty(x,y) \geq t$ then $\Lambda(x,t)$ and $\Lambda(y,t)$ are disjoint. Let A be a subset of $\mathbb{R}^d$ of linear dimension $d-1$, that is A spans a hyperplane of $\mathbb{R}^d$, which we denote by $\mathrm{hyp}\,A$. We call such a set an hyper set. By $\mathrm{nor}\,A$ we denote one of the two unit vectors orthogonal to $\mathrm{hyp}\,A$, or equivalently the element of PS^{d-1} orthogonal to $\mathrm{hyp}\,A$. The cylinder of basis A is the set

$$\mathrm{cyl}\,A \;=\; \Big\{ x + t\ \mathrm{nor}\,A \,\Big|\, t \in \mathbb{R},\ x \in A \Big\}$$

We set also $\mathrm{cyl}\,(A,r) = \{x + t\ \mathrm{nor}\,A : |t| \leq r,\ x \in A\} = \mathrm{cyl}\,A \cap \mathrm{slab}\,(x, \mathrm{nor}\,A, -r, r)$. An hyper–rectangle is an hyper set which, up to an orthonormal change of coordinates, is a $d-1$ dimensional box.

Topology: Let E be a subset of $\mathbb{R}^d$. We denote its interior by $\mathrm{int}\,E$, its closure by $\mathrm{clo}\,E$, its boundary by ∂E. Whenever A is an hyper set of $\mathbb{R}^d$, that is A spans a hyperplane of $\mathbb{R}^d$, we use the induced $(d-1)$ dimensional topology of $\mathrm{hyp}\,A$ to define ∂A, $\mathrm{int}\,A$, $\mathrm{clo}\,A$.

Measure: The collection of all the Borel subsets of a set E of $\mathbb{R}^d$ is denoted by $\mathcal{B}(E)$. The volume $\mathrm{vol}\,E$ of a Borel set E is simply its Lebesgue measure. A Borel set is said to be negligible if its volume is zero. We define a (pseudo) metric dist_{L^1} on $\mathcal{B}(\mathbb{R}^d)$ by

$$\forall E, F \in \mathcal{B}(\mathbb{R}^d) \qquad \mathrm{dist}_{L^1}(E,F) \;=\; \mathrm{vol}\,(E \Delta F)$$

When dealing with topological questions on the space $\mathcal{B}(\mathbb{R}^d)$, we consider the equivalence classes of the Borel sets modulo negligible sets. Notice that $\mathcal{B}(\Gamma)$ is a closed subset of $\mathcal{B}(\mathbb{R}^d)$. We denote by $\mathcal{H}^k$ the standard k–dimensional Hausdorff measure, for $k = 1, 2, ..., d$. We recall that for any subset A of $\mathbb{R}^d$, denoting by α_k the volume of the unit ball of $\mathbb{R}^k$,

$$\mathcal{H}^k(A) = \sup_{\delta>0} \inf \Big\{ \alpha_k\, 2^{-k} \sum_{i\in I} (\,\mathrm{diam}\,_2 E_i)^k \,\Big|\, A \subset \bigcup_{i\in I} E_i,\ \sup_{i\in I}\ \mathrm{diam}\,_2 E_i \le \delta \Big\}$$

Let E be a Borel subset of $\mathbb{R}^d$. A collection of sets $\mathcal{U}$ is called a Vitali class for E if for each x in E and δ positive there exists a set U in $\mathcal{U}$ containing x such that $0 < \mathrm{diam}\,_2 U < \delta$.

The Vitali covering Theorem for $\mathcal{H}^{d-1}$ (see for instance [30], Theorem 1.10). Let E be an $\mathcal{H}^{d-1}$–measurable subset of $\mathbb{R}^d$ and let $\mathcal{U}$ be a Vitali class of closed sets for E. Then we may select a (countable) disjoint sequence $(U_i)_{i\in I}$ from $\mathcal{U}$ such that either $\sum_{i\in I}(\,\mathrm{diam}\, U_i)^{d-1} = \infty$ or $\mathcal{H}^{d-1}(E \setminus \bigcup_{i\in I} U_i) = 0$. If $\mathcal{H}^{d-1}(E) < \infty$ then, given $\epsilon > 0$, we may also require that $\mathcal{H}^{d-1}(E) \le \alpha_{d-1} 2^{-d+1} \sum_{i\in I}(\,\mathrm{diam}\,_2 U_i)^{d-1} + \epsilon$.

The lattice $\mathbb{L}^d$: We turn $\mathbb{Z}^d$ into a graph with vertex set $\mathbb{Z}^d$ and edge set

$$\mathbb{E}^d = \Big\{ \{x,y\} \,\Big|\, x \in \mathbb{Z}^d,\, y \in \mathbb{Z}^d,\ \mathrm{d}_1\,(x,y) = 1 \Big\}$$

This graph is called the *d-dimensional cubic lattice* and is denoted by $\mathbb{L}^d$. We often think of this graph as embedded in $\mathbb{R}^d$, the edges $\{x,y\}$ being straight line segments $[x,y]$ between nearest neighbors. If x and y are nearest neighbors, we denote this relation by $x \sim y$.

Let D be a subset of $\mathbb{R}^d$. An edge $\{x,y\}$ of $\mathbb{E}^d$ is said to be included in D if both sites x,y belong to D. We denote by $\mathbb{E}^d(D)$ the set of the edges of $\mathbb{E}^d$ included in D. For D a subset of $\mathbb{Z}^d$, the graph $(D, \mathbb{E}^d(D))$ will be often identified with its vertex set D. For E a subset of $\mathbb{E}^d$, a formula like $E \subset \mathbb{E}^d(D)$ will be abbreviated into $E \subset D$.

To simplify notation, we will sometimes **identify** subsets of $\mathbb{R}^d$ with their traces on the lattice, i.e., we identify $A \subseteq \mathbb{R}^d$ with $A \cap \mathbb{Z}^d$. For example, $\Lambda(n)$ denotes a box both in the continuum and in the lattice. On the other hand if $A \subseteq \mathbb{Z}^d$, we define an associated thickened region, regarded as a continuum object in $\mathbb{R}^d$, by

$$\mathrm{cube}\, A = \bigcup_{x\in A} \Lambda(x,1)$$

Discrete topology: Let A be a subset of $\mathbb{Z}^d$. We define its

- edge boundary: $\partial^{edge} A = \{\{x,y\} \in \mathbb{E}^d \,|\, x \in A, y \in A^c\}$
- inner vertex boundary: $\partial^{in} A = \{x \in A \,|\, \exists\, y \in A^c$ such that $y \sim x\}$
- outer vertex boundary: $\partial^{out} A = \{x \in A^c \,|\, \exists\, y \in A$ such that $y \sim x\}$

These definitions are extended to the subsets of $\mathbb{R}^d$ by setting, for any $E \subset \mathbb{R}^d$, $\partial^* E = \partial^*(\mathbb{Z}^d \cap E)$, where $*$ stands for *edge*, *in* or *out*.

A *path* γ in $(\mathbb{Z}^d, \mathbb{E}^d)$ (or in any graph) is an alternating sequence $x_0, e_0, x_1, e_1, \cdots, e_{n-1}, x_n, \cdots$ of distinct vertices x_i and edges e_i, where e_i is the edge between x_i and x_{i+1}. The path is said to connect every pair of its vertices. If the path terminates at some vertex x_n it is said to have length n, otherwise it is infinite. Two paths are disjoint if they have no edges in common. The set A is said to be connected if the graph $(A, \mathbb{E}^d(A))$ is connected. Let A, B, D be subsets of $\mathbb{R}^d$. A set of edges $E \subseteq \mathbb{E}^d$ is said to *separate* A and B in D if there is no path in the graph $(\mathbb{Z}^d(D), \mathbb{E}^d(D) \setminus E)$ connecting A and B. The set E *separates* ∞ *in* D if the graph $(\mathbb{Z}^d(D), \mathbb{E}^d(D) \setminus E)$ has at least two infinite components.

The lattice $\mathbb{L}^{d,\infty}$: We introduce another graph structure on $\mathbb{Z}^d$. First we define the edge set $\mathbb{E}^{d,\infty} = \{\{x,y\} \mid \mathrm{d}_\infty(x,y) = 1\}$. The lattice $\mathbb{L}^{d,\infty}$ is defined to be the graph $(\mathbb{Z}^d, \mathbb{E}^{d,\infty})$. The relevance of this lattice stems from the fact that the *exterior* boundary of any connected finite set A in $\mathbb{Z}^d$ is itself connected when regarded as a subgraph of $\mathbb{L}^{d,\infty}$ (but not of $\mathbb{L}^d$). To be more precise, let us define the residual components of A as the connected components of the graph $(A^c, \mathbb{E}^d(A^c))$. A subset A of $\mathbb{Z}^d$ is $\mathbb{L}^{d,\infty}$-connected if the graph $(A, \mathbb{E}^{d,\infty}(A))$ is connected, where $\mathbb{E}^{d,\infty}(A)$ is the set of the edges of $\mathbb{E}^{d,\infty}$ whose both endpoints belong to A. Note that connectedness in the usual ($\mathbb{L}^d$) sense implies $\mathbb{L}^{d,\infty}$-connectedness. Let A be a $\mathbb{L}^{d,\infty}$-connected subset of $\mathbb{Z}^d$. If R is a residual component of A, then its inner and outer vertex boundaries are also $\mathbb{L}^{d,\infty}$-connected (cf. [22]). Suppose in addition that A is finite. Then exactly one of these residual components, say R_1, is infinite. The boundary between R_1 and A is called the exterior (edge, inner vertex, outer vertex) boundary. For future reference, we prove next a little geometrical Lemma.

Lemma 2.1 *For any finite $\mathbb{L}^{d,\infty}$–connected subset A of $\mathbb{Z}^d$, for $r \geq 4$,*

$$\operatorname{vol} \mathcal{V}_\infty(A,r) \;\leq\; 4^{d+1}\, r^{d-1}(|\,A\,| \vee r)$$

Proof: If $\operatorname{diam} A \leq r$, then $\operatorname{vol} \mathcal{V}_\infty(A,r) \;\leq\; (3r)^d$. Suppose now that $r < \operatorname{diam} A < \infty$. Let $\{\, x_1, \cdots, x_l \,\}$ be a collection of vertices of A of maximal cardinality such that

$$\forall i,j \in \{1\cdots l\}, \quad i \neq j, \quad \Lambda(x_i,r) \cap \Lambda(x_j,r) = \emptyset\,.$$

The maximality of the collection implies that $A \subset \Lambda(x_1, 2r) \cup \cdots \cup \Lambda(x_l, 2r)$. Because l is necessarily larger or equal than 2 and A is $\mathbb{L}^{d,\infty}$–connected, for each i in $\{1\cdots l\}$,

$$|\,A \cap \Lambda(x_i,r)\,| \;\geq\; d_\infty(x_i, \partial^{in}\Lambda(x_i,r)) \;\geq\; r/2 - 1$$

so that $|\,A\,| \geq l(r/2-1)$. Since $\mathcal{V}_\infty(A,r)$ is included in l boxes of diameter $4r$, we obtain

$$\operatorname{vol} \mathcal{V}_\infty(A,r) \leq l(4r)^d \leq (4r)^d (r/2-1)^{-1}\,|\,A\,| \leq 4^{d+1} r^{d-1}\,|\,A\,| \qquad \square$$

Ising Gibbs measure: Let $\Lambda \subset \mathbb{Z}^d$ be a box. A *spin configuration* in Λ is an element σ of $\{-,+\}^\Lambda$. We denote by $\sigma(x)$ the spin at site x in the configuration σ. We define the Hamiltonian or energy $H_\Lambda(\sigma)$ of a spin configuration σ in Λ by

$$H_\Lambda(\sigma) \;=\; -\frac{1}{2} \sum_{\{x,y\} \in \mathbb{E}^d(\Lambda)} \sigma(x)\sigma(y)$$

Let $\beta > 0$. The Ising Gibbs measure $\mu^+_{\Lambda,\beta}$ in Λ with $+$ boundary conditions at inverse temperature β is the probability measure on $\{-,+\}^\Lambda$ defined by: $\mu^+_{\Lambda,\beta}(\sigma) = 0$ if $\sigma(x) = -$ for some $x \in \partial^{in}\Lambda$ and

$$\mu^+_{\Lambda,\beta}(\sigma) \;=\; (Z^+_{\Lambda,\beta})^{-1} \exp\big(-\beta H_\Lambda(\sigma)\big)$$

otherwise, where $Z^+_{\Lambda,\beta}$ is a normalizing factor called the partition function. The following is known. The limit $\lim_{n\to\infty}\mu^+_{\Lambda(n),\beta}(\sigma(0))$ exists for any $\beta > 0$. We denote it by m^*. For $d \geq 2$, there exists a critical value $\beta_c = \beta_c(d)$ in $(0, \infty)$ such that $m^* > 0$ for $\beta > \beta_c$ and $m^* = 0$ for $\beta < \beta_c$.

FK percolation.

Edge configurations: The basic probability space for edge processes is given by $\Omega = \{0,1\}^{\mathbb{E}^d}$; its elements are called *edge configurations in* $\mathbb{Z}^d$. The natural projections are given by $\mathrm{pr}_e : \omega \in \Omega \mapsto \omega(e) \in \{0,1\}$, where $e \in \mathbb{E}^d$. An edge e is called open in the configuration ω if $\mathrm{pr}_e(\omega) = 1$, and closed otherwise.

For $E \subseteq \mathbb{E}^d$ with $E \neq \emptyset$, we write $\Omega(E)$ for the set $\{0,1\}^E$; its elements are called *configurations in* E. Note that there is a one-to-one correspondence between cylinder sets and configurations, which is given by $\eta \in \Omega(E) \mapsto \{\eta\} := \{\omega \in \Omega \,|\, \omega(e) = \eta(e) \text{ for every } e \in E\}$. We will use the following convention: the set Ω is regarded to be a cylinder (set) corresponding to the 'empty configuration' (with the choice $E = \emptyset$.) We will sometimes identify cylinders with the corresponding configuration. For $A \subseteq \mathbb{Z}^d$, let Ω_A stand for the set of the configurations in A: $\{0,1\}^{\mathbb{E}^d(A)}$, and Ω^A for the set of the configurations *outside* A: $\{0,1\}^{\mathbb{E}^d\setminus\mathbb{E}^d(A)}$. In general, for $A \subseteq B \subseteq \mathbb{Z}^d$, we set $\Omega^A_B = \{0,1\}^{\mathbb{E}^d(B)\setminus\mathbb{E}^d(A)}$. Given $\omega \in \Omega$ and $E \subseteq \mathbb{E}^d$, we denote by $\omega(E)$ the restriction of ω to $\Omega(E)$. Analogously, ω^A_B stands for the restriction of ω to the set $\mathbb{E}^d(B)\setminus\mathbb{E}^d(A)$.

Given $\eta \in \Omega$, we denote by $\mathcal{O}(\eta)$ the set of the edges of E which are open in the configuration η. The connected components of the graph $(\mathbb{Z}^d, \mathcal{O}(\eta))$ are called *η-clusters*. The path $\gamma = (x_1, e_1, x_2, ...)$ is said to be η-open if all the edges e_i belong to $\mathcal{O}(\eta)$. We write $\{A \leftrightarrow B\}$ for the event that there exists an open path joining some site in A with some site in B. Similarly, we denote by $\{A \leftrightarrow \infty\}$ the event that there exists $x \in A$ lying in an infinite component.

If $V \subseteq \mathbb{Z}^d$ and E consists of all the edges between vertices in V, the graph $G = (V, E) \subseteq \mathbb{L}^d$ is called the maximal subgraph of $\mathbb{L}^d$ on the vertices V. Let ω be an edge configuration in $\mathbb{Z}^d$ (or in a subgraph of $\mathbb{L}^d$). We can look at the *open clusters in* V or alternatively the open *V-clusters.* These clusters are simply the connected components of the random graph $(V, \mathcal{O}(\omega(E)))$, where $\omega(E)$ is the restriction of ω to E.

Given $E \subseteq \mathbb{E}^d$, we write $\mathcal{F}(E)$ for the σ-field generated by the finite-dimensional cylinders associated with configurations in $\Omega(E)$. Similarly, for $A \subseteq B \subseteq \mathbb{Z}^d$, we use the notation $\mathcal{F}^A_B$ for the σ-field generated by finite-dimensional cylinders associated with configurations in Ω^A_B. If $A = \emptyset$ or $B = \mathbb{Z}^d$, then we omit them from the notation.

Stochastic domination: There is a partial order $\preceq$ in Ω given by $\omega \preceq \omega'$ iff $\omega(e) \leq \omega'(e)$ for every $e \in \mathbb{E}^d$. A function $f : \Omega \to \mathbb{R}$ is called *increasing* if $f(\omega) \leq f(\omega')$, whenever $\omega \preceq \omega'$. An event is called increasing if its characteristic function is increasing. Let $\mathcal{F}$ be a σ-field of subsets of Ω. For a pair of probability measures μ and ν on $(\Omega, \mathcal{F})$, we say that μ *(stochastically) dominates* ν if for any $\mathcal{F}$-measurable increasing function f the expectations satisfy $\mu(f) \geq \nu(f)$. If, in addition, for each $\mathcal{F}$-measurable cylinder Z with $\mu(Z)\wedge\nu(Z) > 0$, we have $\mu(f|Z) \geq \nu(f|Z)$, then we say that μ *strongly dominates* ν, and we denote this relation by $\mu \succeq \nu$.

FK measures: Let $V \subseteq \mathbb{Z}^d$ be finite and $E = \mathbb{E}^d(V)$. We first introduce *(partially wired) boundary conditions* as follows. Consider a partition π of the set $\partial^{in}V$, say $\pi = \{B_1, ..., B_n\}$.

(The sets B_i are disjoint non-empty subsets of $\partial^{in}V$ with $\cup_{i=1,..,n}B_i = \partial^{in}V$). We say that $x, y \in \partial^{in}V$ are *π-wired*, if $x, y \in B_i$ for an $i \in \{1, ..., n\}$. Fix a configuration $\eta \in \Omega_V$. We want to count the η-clusters in V in such a way that π-wired sites are considered to be connected. This can be done in the following formal way. We introduce an equivalence relation on V: x and y are said to be *$\pi\cdot\eta$-wired* if they are both joined by η-open paths to (or identical with) sites $x', y' \in \partial^{in}V$ which are themselves π-wired. The new equivalence classes are called $\pi\cdot\eta$-clusters, or η-clusters in V with respect to the boundary condition π. The number of clusters with respect to the boundary condition π (i.e., the number of $\pi\cdot\eta$-clusters) is denoted by $cl^{\pi}(\eta)$. (Note that cl^{π} is simply a random variable.)

For fixed $p \in [0, 1]$ and $q \geq 1$, the *FK measure with parameters (p, q) and boundary conditions π* is a probability measure on the σ-field $\mathcal{F}_V$, defined by the formula

$$\forall \eta \in \Omega_V \qquad \Phi_V^{\pi,p,q}[\{\eta\}] = \frac{1}{Z_V^{\pi,p,q}} \, (\prod_{e \in E} p^{\eta(e)}(1-p)^{1-\eta(e)}) \, q^{cl^{\pi}(\eta)} \tag{8}$$

where $Z_V^{\pi,p,q}$ is an appropriate normalization factor. Since $\mathcal{F}_V$ is an atomic σ-field with atoms $\{\eta\}$, $\eta \in \Omega_V$, (8) determines a unique measure on $\mathcal{F}_V$. Note that every cylinder has non-zero probability. There are two extremal b.c.s: the free boundary condition corresponds to the partition f defined to have exactly $|\partial^{in}V|$ classes, and the wired b.c. corresponds to the partition w with only one class. The set of all such measures called FK (or random cluster) measures corresponding to different b.c.s will be denoted by $\mathcal{FK}(p, q, V)$, and we write $c\mathcal{FK}(p, q, V)$ for its convex hull. The stochastic process $(\mathrm{pr}_e)_{e \in \Omega_V} : \Omega \to \Omega_V$ given on the probability space $(\Omega, \mathcal{F}, \Phi_V^{\pi,p,q})$ is called *FK percolation with boundary conditions π*.

We will list some useful properties of FK measures. A property of crucial importance is that for $q \geq 1$, every $\Phi \in \mathcal{FK}(p, q, V)$ is *strong FKG*. This means that for every $\mathcal{F}_V$-measurable cylinder Z, and for all $\mathcal{F}_V$-measurable increasing functions f, g, we have

$$\Phi\,[fg \mid Z] \geq \Phi\,[f \mid Z]\,\Phi\,[g \mid Z]\;. \tag{9}$$

In some cases it is possible to compare FK measures with different b.c.s. There is a partial order on the set of partitions of $\partial^{in}V$. We say that π *dominates* π', $\pi \geq \pi'$, if x, y π'-wired implies that they are π-wired. We then have $\Phi_V^{\pi',p,q} \preceq \Phi_V^{\pi,p,q}$. This implies immediately that for each $\Phi \in \mathcal{FK}(p, q, V)$, $\Phi_V^{f,p,q} \preceq \Phi \preceq \Phi_V^{w,p,q}$. Next we discuss properties of conditional FK measures. For given $U \subseteq V$ and $\omega \in \Omega$, we define a partition $W_V^U(\omega)$ of $\partial^{in}U$ by declaring $x, y \in \partial^{in}U$ to be $W_V^U(\omega)$-wired if they are joined by an ω_V^U-open path. Fix a partition π of $\partial^{in}V$. We define a new partition of $\partial^{in}U$, denoted by $\pi\cdot W_V^U(\omega)$, by considering $x, y \in \partial^{in}U$ to be $\pi\cdot W_V^U(\omega)$-wired if they are both joined by ω_V^U-open paths to (or identical with) sites x', y', which are themselves π-wired. Then, for every $\mathcal{F}_U$–measurable function f,

$$\Phi_V^{\pi,p,q}[f \mid \mathcal{F}_V^U](\omega) = \Phi_U^{\pi\cdot W_V^U(\omega),p,q}[f], \qquad \Phi_V^{\pi,p,q}\text{-a.s.} \tag{10}$$

Note that (10) can be interpreted as a kind of Markov property. A direct consequence of this formula is that the restriction of every FK measure Φ in $\mathcal{FK}(p, q, V)$ to $\mathcal{F}_U$ is contained in the convex hull $c\mathcal{FK}(p, q, U)$.

FK Ising coupling: We describe the fundamental coupling between the FK measures and the Ising Gibbs measure cf. [51, 40] for more details. Let $\Lambda \subset \mathbb{Z}^d$ be a box. An *edge–spin* *configuration* in Λ is an element (ω,σ) of $\{0,1\}^{\mathbb{E}^d(\Lambda)}\times\{-,+\}^\Lambda$. Let p belong to $(0,1)$. Let $\mathbb{P}_\Lambda^+$ be the probability measure on the space of edge–spin configurations in Λ obtained through the following procedure.

The edges in Λ are opened with probability p and closed with probability $1-p$. The spin value of the sites in $\partial^{in}\Lambda$ is set to $+$. The spin value of the sites in $\Lambda\setminus\partial^{in}\Lambda$ is drawn randomly with the uniform distribution on $\{-,+\}$. The previous operations are performed independently. Finally the measure is conditioned on the event that there is no open edge in Λ between two sites with different spin values.

The support of $\mathbb{P}_\Lambda^+$ consists of the edge–spin configurations (ω,σ) in Λ such that: $\sigma(x)=+$ for any $x\in\partial^{in}\Lambda$, and all the sites belonging to one ω–cluster C have the same spin value, which we denote by $\sigma(C)$. The first marginal of $\mathbb{P}_\Lambda^+$ on $\{0,1\}^{\mathbb{E}^d(\Lambda)}$ is the FK measure $\Phi_\Lambda^{w,p,2}$, its second marginal on $\{-,+\}^\Lambda$ is the Ising Gibbs measure $\mu_{\Lambda,\beta}^+$ where $\beta=-\log(1-p)$. Therefore to draw a spin configuration in Λ according to $\mu_{\Lambda,\beta}^+$ we can proceed as follows. First we draw an edge configuration in Λ according to $\Phi_\Lambda^{w,p,2}$. Second we color each open cluster independently, with $+$ for clusters intersecting $\partial^{in}\Lambda$ and with the uniform distribution on $\{-,+\}$ for the other clusters.

Coarse graining of FK processes.

The blocks and the rescaled lattice: Let K be a fixed positive integer. We divide $\mathbb{Z}^d$ into small boxes called *blocks* of size K in the following way. For $\underline{x}\in\mathbb{Z}^d$, we define the block indexed by $\underline{x}$ as $B(\underline{x})=\Lambda(K\underline{x},K)$. Note that the blocks partition $\mathbb{R}^d$ (or $\mathbb{Z}^d$). Let A be a region in $\mathbb{R}^d$ (or $\mathbb{Z}^d$). Depending on the context, we define the rescaled region $\underline{A}$ as either $\underline{A}=\{\underline{x}\,|\,B(\underline{x})\subseteq A\}$ or $\underline{A}=\{\underline{x}\,|\,B(\underline{x})\cap A\neq\emptyset\}$. In general, we use underline in the notation to emphasize that we are dealing with rescaled objects. For instance, we denote by $\underline{\Lambda}(n)=\{\underline{x}\,|\,B(\underline{x})\subseteq\Lambda(n)\}$ the box $\Lambda(n)$ rescaled by a factor K. Clearly, the rescaled lattice inherits the structure of $\mathbb{Z}^d$. In particular we can equip it with the graph structures corresponding to $\mathbb{L}^d$ or $\mathbb{L}^{d,\infty}$.

With a block we often associate events which can be observed in or in a certain neighborhood of the block. Let α be a positive integer, called the *event–block size*. For $\underline{x}\in\underline{\mathbb{Z}}^d$, we introduce a larger block $B'(\underline{x})$ around $K\underline{x}$, called the *event–block*, by setting

$$B'(\underline{x})=\bigcup_{\underline{z};\ \mathrm{d}_\infty(\underline{z},\underline{x})<\alpha}B(\underline{z})\tag{11}$$

When rescaling a finite object, like a large box Λ, some care is needed in the definition of the blocks sitting close to the boundary if a partition of Λ is desired. For $\underline{x}\in\partial^{in}\underline{\Lambda}$, we modify the definition of the block associated with $\underline{x}$ as follows. First we set

$$\mathcal{M}(\underline{x})=\{\underline{x}\}\cup\Big\{\underline{z}\in\mathbb{Z}^d\,\Big|\,\mathrm{d}_1(\underline{z},\underline{x})=1,\ \Lambda(K\underline{z},K)\cap\Lambda\neq\emptyset,\ \Lambda(K\underline{z},K)\cap\Lambda^c\neq\emptyset\Big\}$$

The block $B(\underline{x})$ is then defined as the smallest box containing the set $\Lambda\cap\Big(\bigcup_{\underline{z}\in\mathcal{M}(\underline{x})}\Lambda(K\underline{z},K)\Big)$, see figure 1. Note that $B(\underline{x})$ is a box with (in general unequal) side-lengths between K and $2K$, and the blocks $B(\underline{x})$, $\underline{x}\in\underline{\Lambda}$, partition Λ. For $\underline{x}\in\underline{\Lambda}$, the event-blocks are still defined with formula (11) but using the enlarged blocks.

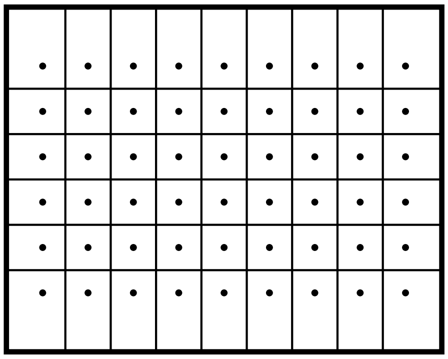

figure 1: a rescaled box

Block events: Let Λ be a box with side-lengths between n and $2n$. An open cluster within Λ is called crossing for Λ if it intersects each of the $2d$ faces of $\partial^{in}\Lambda$. Let g be an increasing function from $\mathbb{N}$ to $\mathbb{R}^+$ such that $g(n) \leq n$ for all n and let $\delta > 0$. We consider the following events:

$$\begin{aligned}
&U(\Lambda) = \{\,\text{there exists a unique open crossing cluster } C^* \text{ in } \Lambda\,\}\\
&R(\Lambda, g) = U(\Lambda) \,\cap\, \{\,\exists!\text{ open cluster with diameter } \geq g(n)\,\}\\
&O(\Lambda, g) = R(\Lambda, g) \,\cap\, \{\,C^* \text{ intersects every sub-box of } \Lambda \text{ of diameter } \geq g(n)\,\}\\
&V(\Lambda, \delta) = U(\Lambda) \,\cap\, \{\,(\theta - \delta)\,|\Lambda| < |C^*| < (\theta + \delta)\,|\Lambda|\,\}\\
&T(\Lambda, g, \delta) = O(\Lambda, g) \,\cap\, \{\,\text{for any box } \Lambda' \subset \Lambda \text{ of diameter } \geq g(n),\\
&\qquad\qquad\qquad (\theta - \delta)\,|\Lambda'| < |C^* \cap \Lambda'| < (\theta + \delta)\,|\Lambda'|\,\}
\end{aligned} \tag{12}$$

Theorem 3.1 in [55] implies that for $d \geq 3$, $q \geq 1$, $p > \widehat{p}_c$, $\theta^f(p) = \theta^w(p)$, there exist positive constants $b = b(p, q, d)$, $c = c(p, q, d)$ and $\kappa = \kappa(p, q, d)$, such that for each $n \geq 1$, each box with side-lengths between n and $2n$, and each measure $\Phi \in c\mathcal{FK}(p, q, \Lambda)$

$$\Phi\,[U(\Lambda)^c] \;\leq\; \Phi\,[V(\Lambda, \delta)^c] \;\leq\; b\exp(-cn) \tag{13}$$

Moreover, if $\kappa \log n \leq g(n) \leq n$ for all n in $\mathbb{N}$,

$$\Phi\,[R(\Lambda, g)^c] \;\leq\; \Phi\,[O(\Lambda, g)^c] \;\leq\; \Phi\,[T(\Lambda, g, \delta)^c] \;\leq\; b\exp(-cg(n)) \tag{14}$$

Notice that we have introduced a new type of event named $T(\Lambda, g, \delta)$. The corresponding estimate follows from Theorems 3.1 and 1.2 in [55].

Block variables: In the course of the proofs we will often use coarse graining in $\Lambda(n)$ by looking at a block process $(X(\underline{x}))_{\underline{x}\in\underline{\Lambda}(n)}$, indicating the non-occurrence of one of the typical events listed in (12) in the corresponding event-block. By controlling the coarse grained process X we can extract useful information about the underlying FK process; in fact this is our main tool to analyze the microscopic behavior of the model. The estimates (13) and (14) guarantee that the block process satisfies the following properties:

– the variable $X(\underline{x})$ depends only on the edges in $B'(\underline{x})$

– $\displaystyle\max_{\Phi \in c\mathcal{FK}(p,q,B'(\underline{x}))} \Phi[\,X(\underline{x}) = 1\,] \leq \varepsilon$ (15)

These two properties imply furthermore

$$- \max_{\Phi \in c\mathcal{FK}(p,q,\Lambda(n))} \Phi\left[X(\underline{x})=1 \;\Big|\; \sigma\Big(X(\underline{z}),\ \mathrm{d}_\infty(\underline{x},\underline{z}) \geq 2\alpha-1\Big)\right] \leq \varepsilon \tag{16}$$

The properties (15), (16) alone imply some simple estimates for the block process. For future reference we formulate them in the subsequent lemmas.

Lemma 2.2 *Consider a 0-1 valued random field $(X_z)_{z\in\Lambda(m)}$ with the property that there exist a positive integer D and $\varepsilon \in [0,1]$ such that for each $z \in \Lambda(m)$,*

$$P\left[X_z = 1 \,\Big|\, \sigma(X_y;\ \mathrm{d}_\infty(z,y) \geq D)\right] \leq \varepsilon \tag{17}$$

Then, for every $\delta \in (\varepsilon, 1]$,

$$P\left[\frac{1}{m^d}\sum_{z\in\Lambda(m)} X_z \geq \delta\right] \leq D^d \exp\Big(-\Lambda_\varepsilon^*(\delta)\,\Big\lfloor\frac{m}{D}\Big\rfloor^d\Big)$$

where $\Lambda_\varepsilon^*(\delta) = \delta\log\frac{\delta}{\varepsilon} + (1-\delta)\log\frac{1-\delta}{1-\varepsilon}$ is the Legendre transform of the logarithmic moment generating function of a Bernoulli variable with parameter ε. (We remark that if $\varepsilon < \delta \leq 1/2$, then $\Lambda_\varepsilon^*(\delta) \geq \delta\log(\delta/\varepsilon) - \log 2$).

Proof: We introduce an equivalence relation on $\Lambda(m)$: $z \approx y$ iff D divides each component of $z-y$. The corresponding equivalence classes $V_1, ..., V_{D^d}$ partition $\Lambda(m)$. Condition (17) guarantees that each of the the fields $(X_z)_{z\in V_i}$, $i=1,...,D^d$, is stochastically dominated by i.i.d. Bernoulli variables with parameter ε. By applying optimized exponential Chebyshev estimates (cf. the section on Cramer's Theorem in [23]), we obtain

$$P\left[\frac{1}{m^d}\sum_{z\in\Lambda(m)} X_z \geq \delta\right] \leq \sum_{i=1,...,D^d} P\left[\frac{1}{|V_i|}\sum_{z\in V_i} X_z \geq \delta\right] \leq D^d\exp\Big(-\Lambda_\varepsilon^*(\delta)\min_{i=1,...,D^d}|V_i|\Big)$$

and the claim follows from the fact that $|V_i| \geq \lfloor m/D\rfloor^d$ for $i=1,...,D^d$. □

Applying the previous lemma to a block process satisfying (16), we obtain for every $\Phi \in \mathcal{FK}(p,q,\Lambda(n))$ and $\delta > \varepsilon$,

$$\Phi\left[\frac{1}{|\underline{\Lambda}(n)|}\sum_{\underline{x}\in\underline{\Lambda}(n)} X(\underline{x}) \geq \delta\right] \leq (2\alpha)^d\exp\Big(-\Lambda_\varepsilon^*(\delta)\,\frac{1}{(2\alpha)^d}\,\Big\lfloor\frac{n}{K}\Big\rfloor^d\Big) \tag{18}$$

Note that this estimate is valid for each fixed $n, \varepsilon, \alpha, K$ (with $\alpha K < n$). In particular ε, α, K may depend on n.

The block process can be viewed as a (dependent) site percolation process where a site $\underline{x}$ is occupied iff $X(\underline{x}) = 1$. The occupied ∞-cluster of the site $\underline{x}$, i.e., the ∞-connected component of the occupied sites containing $\underline{x}$, is then denoted by $\underline{C}(\underline{x})$.

Lemma 2.3 *Assume (15) holds. There exists a dimension dependent constant $b(d) > 0$ such that, for any $\Phi \in \mathcal{FK}(p,q,\Lambda(n))$, any $s, t > 0$,*

$$\Phi\left[\Big|\Big\{\underline{x}\in\underline{\Lambda}(n)\,\Big|\,|\underline{C}(\underline{x})| \geq t\Big\} \geq s\Big|\right] \leq \sum_{j\geq s} 2\exp j\Big(2d\log(n/K)/t + \log b + (2\alpha)^{-d}\log\varepsilon\Big) \tag{19}$$

Proof: We first prove the following estimate: for any $\Phi \in \mathcal{FK}(p,q,\Lambda(n))$, any subset $\underline{A} \subseteq \underline{\Lambda}(n)$,

$$\Phi[\forall\ \underline{x} \in \underline{A} : X(\underline{x}) = 1\,] \le \exp\Big(\ (2\alpha)^{-d}|\underline{A}|\ \log\varepsilon\Big) \tag{20}$$

We use again the equivalence relation on $\underline{\Lambda}(n)$: $\underline{x} \approx \underline{y}$ iff $D = 2\alpha - 1$ divides each component of $\underline{x} - \underline{y}$. Since there are at most $D^d \le (2\alpha)^d$ distinct classes in $\underline{\Lambda}(n)$, there certainly exists $\underline{x}^* \in \underline{\Lambda}(n)$ such that the intersection of $\underline{A}$ and the equivalence class of $\underline{x}^*$ has cardinality at least $(2\alpha)^{-d}|\underline{A}|$. By (16), the field $X(\underline{x})$, $\underline{x} \in \underline{\Lambda}(n)$, $\underline{x} \approx \underline{x}^*$ is stochastically dominated by a Bernoulli product field with parameter ε. Finally, for any $\Phi \in \mathcal{FK}(p,q,\Lambda(n))$,

$$\Phi[\,\forall\ \underline{x} \in \underline{A},\ X(\underline{x}) = 1\,] \le \Phi\Big[\,\forall\ \underline{x} \in \{\underline{y} \in \underline{A}\,|\,\underline{y} \approx \underline{x}^*\} :\ X(\underline{x}) = 1\Big] \le \exp\Big(\ (2\alpha)^{-d}|\underline{A}|\ \log\varepsilon\Big)$$

as claimed. Now we turn to the statement of the lemma. By decomposing the event in question, we can estimate the left hand side of (19) by

$$\sum_{j\ge s}\ \sum_{1\le i\le j/t}\ \sum_{\substack{m_1,m_2,\dots,m_i\ge t\\ m_1+m_2+\dots+m_i=j}}\ \sum_{\underline{A}_1,\underline{A}_2,\dots,\underline{A}_i} \Phi\Big[\forall\,\underline{x} \in \underline{A}_1 \cup \underline{A}_2 \cup \ldots \cup \underline{A}_i :\ X(\underline{x}) = 1\Big]$$

The ultimate summation extends over the pairwise disjoint sets $\underline{A}_1, ..., \underline{A}_i$ such that $\underline{A}_l$ is $\mathbb{L}^{d,\infty}$-connected and $|\underline{A}_l| = m_l$ for $1 \le l \le i$. By (20), the probability appearing in the summation is less than $\exp((2\alpha)^{-d} j \log\varepsilon)$. For fixed j and i, there are at most 2^j ways to choose the values $m_1, ..., m_i$. Recall that there exists a constant $b = b(d) > 0$ such that the number of $\mathbb{L}^{d,\infty}$-connected sets of size m containing the origin is bounded by $(b/2)^m$. The number of possibilities for choosing the set $\underline{A}_l$ is bounded by $(n/K)^d b^{m_l}$ for $1 \le l \le i$. Thus the number of terms involved in the last three summations is less than $2\exp\Big((2jd/t)\log(n/K) + j\log b\Big)$. □

3 Decay of boundary effects and decoupling

The main results of this section are Proposition 3.1 which gives sufficient control of boundary effects and leads directly to Lemma 3.2, which allows to decouple rare events in distant regions. This lemma is one of the key results which allow to extend the large deviations results for percolation of [14] to FK percolation. We begin with the statement of the main result and a useful consequence of it. The proof will be given at the end of the section after a series of preparatory lemmas.

Let $\Theta \subseteq \mathbb{R}^d$ be a box building, i.e., the union of finitely many d-dimensional boxes with non empty interior. Fix a monotone increasing function $\phi : \mathbb{N} \to \mathbb{N}$ satisfying $\lim_{n\to\infty}\phi(n) = \infty$ and $\lim_{n\to\infty}\phi(n)/n = 0$. We will consider the $\phi(n)$-interior of the building $n\Theta$ which is defined as

$$\mathrm{int}\,(n\Theta, \phi(n)) = \{x \in n\Theta \mid \mathrm{d}_\infty\,(x, \partial^{in} n\Theta) > \phi(n)\}$$

Proposition 3.1 *Assume $d \ge 3$, $q \ge 1$, $p > \widehat{p}_c$ with $\theta^f(p) = \theta^w(p)$. For $n \in \mathbb{N}$, let S_n be an event depending only on the edges in* $\mathrm{int}\,(n\Theta,\phi(n))$ *and let $\pi(n)$ be a partially wired b.c.s on $n\Theta$. We have*

$$\overline{\lim_{n\to\infty}}\ \frac{1}{n^{d-1}}\ \log\ \Phi^{\pi(n),p,q}_{n\Theta}[S_n] \ =\ \overline{\lim_{n\to\infty}}\ \frac{1}{n^{d-1}}\ \log\ \Phi^{f,p,q}_{n\Theta}[S_n] \tag{21}$$

The same equality is valid when $\overline{\lim}$ is replaced by $\underline{\lim}$.

Remark: 1) Note that free b.c.s on the left hand side could be replaced by wired b.c.s as well. A direct consequence of equation (21) is

$$\begin{aligned}\overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log\Phi^{f,p,q}_{n\Theta}[S_n] &= \overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log\sup_{\Phi\in c\mathcal{FK}(n\Theta,p,q)}\Phi\left[S_n\right]\\ &= \overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log\inf_{\Phi\in c\mathcal{FK}(n\Theta,p,q)}\Phi\left[S_n\right] \qquad (22)\end{aligned}$$

2) We will work in the box $\Lambda(n)$ with wired boundary conditions, hence we will need the following slight generalization of Proposition 3.1, whose proof is similar. Assume $d\geq 3,\ q\geq 1,\ p>\widehat{p}_c$ with $\theta^f(p)=\theta^w(p)$. Let Θ be a box building included in $\Lambda(1)$. We define the $\phi(n)$-interior of the building $n\Theta$ relative to $\Lambda(n)$ by

$$\operatorname{int}_{\Lambda(n)}(n\Theta,\phi(n))=\{x\in n\Theta\,|\ \mathrm{d}_\infty\left(x,\partial^{in}n\Theta\setminus\partial^{in}n\Lambda(n)\right)>\phi(n)\}$$

For $n\in\mathbb{N}$, let S_n be an event depending only on the edges in $\operatorname{int}_{\Lambda(n)}(n\Theta,\phi(n))$ and let $\pi(n)$ be a partially wired b.c.s on $n\Theta$ such that the sites in $\partial^{in}n\Theta\cap\partial^{in}\Lambda(n)$ are wired together. We have

$$\overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log\Phi^{\pi(n),p,q}_{n\Theta}[S_n] = \overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log\Phi^{w,p,q}_{\Lambda(n)}[S_n] \qquad (23)$$

Lemma 3.2 (Decoupling lemma) *Let $d\geq 3,\ q\geq 1,\ p>\widehat{p}_c$ with $\theta^f(p)=\theta^w(p)$. Let D_i, $i\in I$, be a finite collection of disjoint compact subsets of the unit cube $\Gamma=[-1/2,1/2]^d$. Assume that these sets have non-empty connected interiors. For $i\in I$ and $n\geq 1$ let S^i_n be an event depending only on the edges in $nD_i\cap\Lambda(n)$. Then*

$$\overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log\Phi^{w}_{\Lambda(n)}\Big[\bigcap_{i\in I}S^i_n\Big] \leq \sum_{i\in I}\overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log\Phi^{w}_{\Lambda(n)}\Big[S^i_n\Big]$$

Remark: The same result is valid for $\underline{\lim}$ with the opposite inequality. Again, w could be replaced by f. The statements are valid also for the infinite volume measure or for free b.c.s.

Proof: It is sufficient to prove the statement for two sets D_1 and D_2 only. We suppose that D_1 and D_2 are closed subsets of the interior of Γ. Let Θ_1 be a box building such that $D_1\subset\operatorname{int}\Theta_1$, $D_2\cap\operatorname{clo}\Theta_1=\emptyset$. Let $\phi:\mathbb{N}\to\mathbb{N}$ be an increasing function such that $\lim_{n\to\infty}\phi(n)=\infty$ and $\lim_{n\to\infty}\phi(n)/n=0$. For n large enough, the event S^1_n depends only on the edges in $\operatorname{int}(n\Theta_1,\phi(n))$, while S^2_n is in the σ–field $\mathcal{F}^{n\Theta_1}_{\Lambda(n)}$. Therefore

$$\Phi^w_{n\Gamma}[S^1_n\cap S^2_n] = \Phi^w_{n\Gamma}[1_{S^2_n}\Phi^w_{n\Gamma}[S^1_n\,|\,\mathcal{F}^{n\Theta_1}_{\Lambda(n)}]]\leq\Big(\max_{\Phi\in c\mathcal{FK}(n\Theta_1)}\Phi[S^1_n]\Big)\,\Phi^w_{n\Gamma}[S^2_n] \qquad (24)$$

Hence taking the limsups and using (22) we obtain the claim of Lemma 3.2. For sets intersecting the boundary of Γ, the argument is analogous and it relies on (23). $\square$

We next present the lemmas involved in the proof of Proposition 3.1. The first lemma is a simple observation which, nonetheless, plays a crucial role in the derivation of the results mentioned above.

Lemma 3.3 (Monotone perturbation of boundary conditions in FK percolation) *Let $q \geq 1$ and $\pi' \preceq \pi$ be two (partially wired) b.c.s on the region $V \subseteq \mathbb{Z}^d$. We denote by $|\pi|$ the number of equivalence classes in π (note $|\pi| \leq |\pi'|$.) Assume $|\pi'| - |\pi| \leq C$ for a certain constant C. Then for any event S depending only on the edges in V*

$$q^{-C} \leq \frac{\Phi_V^{\pi'}[S]}{\Phi_V^{\pi}[S]} \leq q^C$$

Proof: Note that $|\pi'| - |\pi| \leq C$ together with $\pi \succeq \pi'$ imply that for each ω,

$$0 \leq cl^{\pi'}(\omega) - cl^{\pi}(\omega) \leq C$$

For any event A we define the partition sum $Z^\pi[A]$ by

$$Z^\pi[A] = \sum_{\omega \in A} \Big(\prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \Big)\, q^{cl^\pi(\omega)}$$

We have the inequalities: $Z^\pi[A] \leq Z^{\pi'}[A] \leq q^C\, Z^\pi[A]$. The first inequality is obvious and the second follows from

$$Z^{\pi'}[A] = \sum_{\omega \in A} \Big(\prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \Big)\, q^{cl^\pi(\omega)}\, q^{cl^{\pi'}(\omega) - cl^\pi(\omega)} \leq q^C\, Z^\pi[A]$$

The inequalities of the Lemma can be proved as follows:

$$\begin{aligned}
\Phi_V^\pi[S] &= \frac{Z^\pi[S]}{Z^\pi} \leq \frac{Z^{\pi'}[S]}{q^{-C}\, Z^{\pi'}} = q^C\, \Phi_V^{\pi'}[S] \\
\Phi_V^{\pi'}[S] &= \frac{Z^{\pi'}[S]}{Z^{\pi'}} \leq \frac{q^C\, Z^\pi[S]}{Z^\pi} = q^C\, \Phi_V^{\pi'}[S] \qquad \square
\end{aligned}$$

The other ingredient of the proof of Proposition 3.1 is the control of (random) boundary conditions on $\operatorname{int}(n\Theta, \phi(n))$ *induced* by the measures $\Phi_{n\Theta}^{\pi(n)}$. Denote by $A(n)$ the "boundary region" $n\Theta \setminus \operatorname{int}(n\Theta, \phi(n))$. We will show that the average number of $A(n)$-clusters per site is concentrated around κ^+, which is the half-space analogue of the expected number of clusters per site κ. We begin the lengthy but rather simple proof with the introduction of *half-clusters*. Given an edge configuration in $\mathbb{Z}^d$, the half-clusters are simply the $\mathbb{H}^d$-clusters, i.e., the open clusters of the configuration restricted to $\mathbb{H}^d$. For $A \subseteq \mathbb{D}^d = \{x \in \mathbb{Z}^d \,|\, x_d = 0\}$ we denote by K'_A the number of half-clusters intersecting A. Finally, we set $\kappa'_n(p,q) = \Phi_\infty^{p,q}[K'_{D(n)}/n^{d-1}]$, where $D(n)$ is the hypersquare $\mathbb{D}^d \cap \Lambda(n)$.

Lemma 3.4 *Assume $d \geq 3$, $q \geq 1$, $0 \leq p \leq 1$, $\theta^f(p) = \theta^w(p)$. The limit $\lim_{n\to\infty} \kappa'_n = \kappa^+$ exists and is equal to $\inf_n \kappa'_n$.*

Remark: If $0 < p < 1$, the limit κ^+ is easily seen to be confined to $(0,1)$.

Proof: The proof is based on the following subadditive property: If $A, B \subseteq \mathbb{D}^d$ and ω is a configuration then

$$K'_{A\cup B}(\omega) \leq K'_A(\omega) + K'_B(\omega)$$

and the same is true for the expectations. The claim of the lemma follows from a well-known d-dimensional generalization of the subadditive inequality. For the readers convenience we sketch the argument. It is enough to show that for each fixed m, $\overline{\lim}_{n\to\infty} \kappa'_n \leq \kappa'_m$. Let $n \geq m$ and divide the hypersquare $D(n) \subseteq \mathbb{D}^d$ into disjoint hypersquares congruent to $D(m)$. There are $\lfloor n/m \rfloor^{d-1}$ boxes which fit into $D(n)$ and it remains an uncovered region of size not exceeding $2(d-1)mn^{d-2}$. By subadditivity and translation invariance of $\Phi_\infty^{p,q}$,

$$\Phi_\infty[K'_n/n^{d-1}] \leq \lfloor n/m \rfloor^{d-1}\ (m/n)^{d-1}\Phi_\infty[K'_m/m^{d-1}] + \frac{1}{n^{d-1}}2(d-1)mn^{d-2}$$

Taking $\overline{\lim}_{n\to\infty}$ we arrive at the claim. $\square$

We next introduce the number K_n of $\Lambda(n)\cap\mathbb{H}^d$-clusters intersecting the hypersquare $D(n-\sqrt{n})$ and set $\kappa_n = \Phi_\infty^{p,q}[K_n/n^{d-1}]$.

Lemma 3.5 *Assume $d \geq 3,\ q \geq 1,\ p > \widehat{p}_c,\ \theta^f(p) = \theta^w(p)$. Then $\lim_{n\to\infty} \kappa_n = \kappa^+$.*

Proof: We define the event

$$R_n = \Big\{ \exists !\ \Lambda(n)\cap\mathbb{H}^d\text{-cluster } C^* \text{ with diameter} \geq \sqrt{n} \text{ and } C^*\cap D(n-\sqrt{n}) \neq \emptyset \Big\} \tag{25}$$

On R_n, if two sites in $D(n-\sqrt{n})$ are connected in $\mathbb{H}^d$ then they are already connected in $\Lambda(n)\cap\mathbb{H}^d$. Hence, on R_n,

$$\big| K_n - K'_n \big| \leq \big|\, D(n)\setminus D(n-\sqrt{n})\,\big| \leq 2(d-1)n^{d-2}\sqrt{n}$$

By Theorem 3.1 in [55] we know that $\lim_{n\to\infty} \Phi_\infty^{p,q}[R_n^c] = 0$. Thus

$$\overline{\lim_{n\to\infty}}\, \big|\kappa_n - \kappa'_n\big| \leq \overline{\lim_{n\to\infty}}\ \Phi_\infty^{p,q}\Big[n^{1-d}\,\big|K_n - K'_n\big|\,;\, R_n\Big] + \overline{\lim_{n\to\infty}}\ \Phi_\infty^{p,q}[R_n^c] = 0 \qquad \square$$

Lemma 3.6 *Assume $d \geq 3,\ q \geq 1,\ p > \widehat{p}_c$ with $\theta^f(p) = \theta^w(p)$. The quantity K_n/n^{d-1} concentrates around κ^+ as n goes to ∞, i.e., for each $\varepsilon > 0$,*

$$\lim_{n\to\infty}\ \max_{\Phi\in\, c\mathcal{FK}(\Lambda(n))} \Phi\left[\Big|K_n/n^{d-1} - \kappa^+\Big| > \varepsilon\right] = 0$$

Proof: Let $\varepsilon > 0$. Since K_n is decreasing, we need only to show

$$\lim_{n\to\infty} \Phi^w_{\Lambda(n)}[K_n/n^{d-1} < \kappa^+ - 5\varepsilon] \;=\; 0 \tag{26}$$

$$\lim_{n\to\infty} \Phi^f_{\Lambda(n)}[K_n/n^{d-1} > \kappa^+ + 5\varepsilon] \;=\; 0 \tag{27}$$

By Theorem 3.1 in [55] and Lemma 3.5, there exists $N = N(\varepsilon)$ such that

$$\max_{\Phi\in\, c\mathcal{FK}(\Lambda(N)\cap\mathbb{H}^d)} \Phi\,[R_N^c] \leq \varepsilon \tag{28}$$

$$\kappa^+ - \varepsilon/2 \leq \kappa_N = \Phi_\infty[K_N/N^{d-1}] \leq \kappa^+ + \varepsilon/2 \tag{29}$$

(the event R_N was defined in (25)). We require in addition that

$$2(d-1)/\sqrt{N} < \varepsilon \tag{30}$$

Note that K_n is also a local variable. Since according to our assumptions $\Phi^f_\infty = \Phi^w_\infty (= \Phi_\infty)$, then the measures $\Phi^*_{\Lambda(m)}$, $* = f$ or w, converge weakly towards Φ_∞ as $m \to \infty$, and there exists $M_0 = M_0(\varepsilon, N)$ such that for every $m \geq M_0$,

$$\Phi^f_{\Lambda(m)}[K_N/N^{d-1}] - \varepsilon/2 \leq \Phi_\infty[K_N/N^{d-1}] \leq \Phi^w_{\Lambda(m)}[K_N/N^{d-1}] + \varepsilon/2 \tag{31}$$

Therefore, by (29) and (31), for $M \geq M_0(\varepsilon, N)$,

$$\kappa^+ - \varepsilon \quad \leq \quad \Phi^w_{\Lambda(M)}[K_N/N^{d-1}] \leq \Phi^f_{\Lambda(M)}[K_N/N^{d-1}] \leq \kappa^+ + \varepsilon \tag{32}$$

Let $M \geq M_0(\varepsilon, N)$ be such that M/N is an odd integer ≥ 3. Thus for these values N, M the conditions (28) and (32) hold true. Let $n > M^2$ and $l = \lfloor (n - \sqrt{n})/N \rfloor$. We use the blocks $B(\underline{i}) = \Lambda(\underline{i}N, N)$, $\underline{i} \in D(l)$, to rescale $D(n - \sqrt{n})$. For given $\underline{i}$, we introduce the corresponding "middle square" $D(\underline{i}) = \underline{i}N + D(N - \sqrt{N})$ and we denote by $K(\underline{i})$ the number of clusters in $B(\underline{i}) \cap \mathbb{H}^d$ intersecting $D(\underline{i})$. We call the block $B(\underline{i})$ regular if the event described in (25) occurs in $B(\underline{i}) \cap \mathbb{H}^d$ (instead of $\Lambda(n)$); in this case we set $X(\underline{i}) = 0$ and $X(\underline{i}) = 1$ otherwise. Condition (28) guarantees that the block process $(X(\underline{i}))_{\underline{i} \in D(l)}$ is stochastically dominated by i.i.d. Bernoulli variables with parameter ε.

Let T_n denote the event that the proportion of irregular blocks $B(\underline{i})$, $\underline{i} \in D(l)$, exceeds ε. By Cramer's theorem there exists $c = c(\varepsilon) > 0$ such that

$$\Phi^w_{\Lambda(n)}[T_n] \leq \exp(-c\,|D(l)|) = \exp(-c\, \left\lfloor (n - \sqrt{n})/N \right\rfloor^{d-1}) \tag{33}$$

We next observe that $K_n > \sum_{\underline{i} \in D(l)} K(\underline{i})(1 - X(\underline{i})) - |D(l)|$, since the number of clusters in $\mathbb{H}^d \cap \Lambda(n)$ intersecting $D(\underline{i})$, when $\underline{i}$ is regular, can be determined by looking at the configuration in $\mathbb{H}^d \cap B(\underline{i})$. Moreover, in each such block there exists at most one half-cluster which can intersect another block. Hence the maximal overcount is bounded by the number of regular blocks (which is bounded by $|D(l)|$.) On T_n we can use the following estimate

$$K_n > \sum_{\underline{i} \in D(l)} K(\underline{i}) - |\, D(l)\,| - \varepsilon |D(l)|\, N^{d-1} \geq \sum_{\underline{i} \in D(l)} K(\underline{i}) - 2\varepsilon n^{d-1} \tag{34}$$

The term $\varepsilon |D(l)|\, N^{d-1}$ is a bound on the number of sites in $D(n - \sqrt{n})$ not covered by regular blocks, which itself is a crude upper bound on the (extra) overcount. The second inequality follows from (30).

It will be useful to subdivide all blocks in the collection $B(\underline{i})$, $\underline{i} \in D(l)$, into further classes containing blocks which are sufficiently separated in space. First we set $B'(\underline{i}) = \Lambda(\underline{i}N, M)$. Recall that M/N is an odd integer ≥ 3. Thus the blocks $B(\underline{k})$, $\underline{k} \in D(M/N)$, partition $B'(\underline{0})$. For $\underline{k} \in D(M/N)$, we look at the collection of (indices of) blocks

$$J(\underline{k}) = \left\{ \underline{j} \in D(l) \;\middle|\; \underline{j} = \underline{k} + \underline{i}(M/N) \text{ for some } \underline{i} \in \mathbb{D}^d \right\}$$

Note that $J(\underline{k})$, $\underline{k} \in D(M/N)$ is a partition of $D(l)$ and the d-dimensional blocks $B'(\underline{j})$, $\underline{j} \in J(\underline{k})$ are all disjoint and contained in $\Lambda(n)$ (since $\sqrt{n} > M$.) Returning to the proof of (26),

$$
\begin{aligned}
&\overline{\lim_{n\to\infty}}\, \Phi^w_{\Lambda(n)}\Big[K_n/n^{d-1} < \kappa^+ - 5\varepsilon\Big] \\
&\quad \le \overline{\lim_{n\to\infty}}\, \Phi^w_{\Lambda(n)}[T_n] + \overline{\lim_{n\to\infty}}\, \Phi^w_{\Lambda(n)}\Big[\frac{1}{n^{d-1}} \sum_{\underline{k}\in D(M/N)} \sum_{\underline{j}\in J(\underline{k})} K(\underline{j}) < \kappa^+ - 3\varepsilon;\ T_n\Big] \\
&\quad \le \overline{\lim_{n\to\infty}}\, \Phi^w_{\Lambda(n)}\Big[\exists\, \underline{k} \in D(M/N) \quad \frac{|\, D(M/N)\,|}{n^{d-1}} \sum_{\underline{j}\in J(\underline{k})} K(\underline{j}) < \kappa^+ - 3\varepsilon\Big] \\
&\quad \le \overline{\lim_{n\to\infty}}\, |D(M/N)| \max_{\underline{k}\in D(M/N)} \Phi^w_{\Lambda(n)}\Big[\frac{1}{|\, J(\underline{k})|} \sum_{\underline{j}\in J(\underline{k})} K(\underline{j})/N^{d-1} < \kappa^+ - 2\varepsilon\Big]
\end{aligned}
\tag{35}
$$

(Note that $|\, D(M/N)\,|/n^{d-1} \to 1/(|\, J(\underline{k})\,|\, N^{d-1})$ as $n \to \infty$). In order to estimate the probabilities above we use the decoupling event

$$
F(\underline{k}) = \bigcap_{\underline{j}\in J(\underline{k})} \{\text{each edge connecting two sites in } \partial^{in} B'(\underline{j}) \text{ is open}\}
$$

Under the conditional measure $\Phi^w_{\Lambda(n)}[\,\cdot\,|\, F(\underline{k})]$, the variables $K(\underline{j})/N^{d-1}$, $\underline{j} \in J(\underline{k})$, become i.i.d., and by (32) we know that their expected values are at least $\kappa^+ - \varepsilon$. Hence, for any $\underline{k} \in D(M/N)$, by the FKG inequality and Cramer's theorem we have

$$
\begin{aligned}
&\Phi^w_{\Lambda(n)}\Big[\frac{1}{|\, J(\underline{k})|} \sum_{\underline{j}\in J(\underline{k})} K(\underline{j})/N^{d-1} < \kappa^+ - 2\varepsilon\Big] \\
&\qquad \le \Phi^w_{\Lambda(n)}\Big[\frac{1}{|\, J(\underline{k})|} \sum_{\underline{j}\in J(\underline{k})} K(\underline{j})/N^{d-1} < \kappa^+ - 2\varepsilon \;\Big|\; F(\underline{k})\Big] \le \exp(-c'\,|\, J(\underline{k})|)
\end{aligned}
\tag{36}
$$

for some $c' = c'(\varepsilon) > 0$. Note finally that $|\, J(\underline{k})| \ge (n/M)^{d-1}/2$ for n large enough which together with (35) and (36) implies (26).

The proof of (27) is very similar, in fact it is easier since subadditivity can be used instead of the regularity argument. The block construction is the same as before except that we set $l = \lceil (n - \sqrt{n})/N \rceil$ to have a complete cover of $D(n - \sqrt{n})$ and we choose $N = N(\varepsilon)$ and $M = M(\varepsilon, N)$ such that (28) and (32) are satisfied and M/N is an odd integer ≥ 3. In this case

$$
K_n \le |\, D(n) \setminus D(n - \sqrt{n})\,| + \Big| \bigcup_{\underline{i}\in D(l)} (B(\underline{i}) \cap \mathbb{D}^d) \setminus D(\underline{i}) \Big| + \sum_{\underline{i}\in D(l)} K(\underline{i})
$$

By (30), both $|\, D(n) \setminus D(n - \sqrt{n})\,|$ and $|\bigcup_{\underline{i}\in D(l)} (B(\underline{i}) \cap \mathbb{D}^d) \setminus D(\underline{i})\,|$ are bounded by εn^{d-1}. Hence,

$$
\begin{aligned}
&\overline{\lim_{n\to\infty}}\, \Phi^f_{\Lambda(n)}\Big[K_n/n^{d-1} > \kappa^+ + 5\varepsilon\Big] \\
&\quad \le \overline{\lim_{n\to\infty}}\, \Phi^f_{\Lambda(n)}\Big[n^{1-d} \sum_{\underline{i}\in D(l)} K(\underline{i}) > \kappa^+ + 3\varepsilon\Big]
\end{aligned}
$$

$$\leq \varlimsup_{n\to\infty} \Phi^f_{\Lambda(n)}\Big[\exists \underline{k} \in D(M/N) \quad \frac{|D(M/N)|}{n^{d-1}} \sum_{\underline{j}\in J(\underline{k})} K(\underline{j}) > \kappa^+ + 3\varepsilon\Big]$$

$$\leq |D(M/N)| \varlimsup_{n\to\infty} \max_{\underline{k}\in D(M/N)} \Phi^f_{\Lambda(n)}\Big[\frac{1}{|J(\underline{k})|} \sum_{\underline{j}\in J(\underline{k})} K(\underline{j})/N^{d-1} > \kappa^+ + 2\varepsilon\Big]$$

To estimate the probabilities above, we use the decoupling event

$$F(\underline{k}) = \bigcap_{\underline{j}\in J(\underline{k})} \{\text{each edge in } \partial^{edge} B'(\underline{j}) \text{ is closed}\}$$

Note that under the conditional measure $\Phi^f_{\Lambda(n)}[\,\cdot \mid F(\underline{k})]$, the variables $K(\underline{j})/N^{d-1}$, $\underline{j} \in J(\underline{k})$, are i.i.d. with expected value smaller than $\kappa^+ + \varepsilon$. The proof can be finished as in the previous case by employing the FKG inequality and Cramer's theorem. $\square$

Now we turn to the **proof of Proposition** 3.1. To preserve transparency, in particular to simplify notation, we will give the proof for the choice $\Theta = \Lambda(1)$; the generalization to arbitrary buildings is straightforward. Note that for $\Theta = \Lambda(1)$, $n\Theta = \Lambda(n)$. We begin with the description of the idea behind the proof. Let us first recall Lemma 3.3 which says that whenever two b.c.s π, π' given on a region are not too different, more precisely when $\pi' \preceq \pi$ and the difference of the number of equivalence classes is bounded by some number C, then the corresponding probabilities of any event S given in that region are comparable in the following sense:

$$q^{-C} \leq \Phi^{\pi'}[S]/\Phi^{\pi}[S] \leq q^C$$

Consider a monotone coupling of the two measures $\Phi^f_{\Lambda(n)}$ and $\Phi^\pi_{\Lambda(n)}$ where π is a partially wired boundary condition on $\Lambda(n)$. Our first goal is to show that for any ε, with exceedingly high probability (i.e. up to LD-s of *higher* than surface order) we are able to find a (random) box B in $\Lambda(n)$ (only a little smaller than $\Lambda(n)$ itself) such that the b.c.s W^f_B and W^π_B on B, induced by the configuration in $\Lambda(n) \setminus B$, satisfy $0 \leq |W^f_B| - |W^\pi_B| \leq \varepsilon n^{d-1}$. It is then easy to show, by using Lemma 3.3, that

$$q^{-\varepsilon n^{d-1}} \leq \Phi^\pi_{\Lambda(n)}[S]/\Phi^f_{\Lambda(n)}[S] \leq q^{\varepsilon n^{d-1}}$$

for any event S which is measurable in a region not too close to the boundary of $\Lambda(n)$.

Recall that $\phi : \mathbb{N} \to \mathbb{N}^+$ is a fixed monotone increasing function such that $\lim_{n\to\infty} \phi(n) = \infty$ and $\lim_{n\to\infty} \phi(n)/n = 0$. Set $L = L(n) = \lfloor \phi(n)^{1/(d+1)} \rfloor$ if this number is odd and let $L = \lfloor \phi(n)^{1/(d+1)} \rfloor - 1$ otherwise. We assume that n is large enough such that $L \geq 3$. We divide $\mathbb{Z}^d$ into blocks of side length L in the usual way: for $\underline{i} \in \mathbb{Z}^d$ we set $B(\underline{i}) = \Lambda(L\underline{i}, L)$. Set $\Lambda(n,\phi) = \mathrm{int}\,(\Lambda(n), \phi(n))$ and let $\underline{\Lambda}(n,\phi)$ be the set of indices $\underline{i}$ such that $\Lambda(n,\phi) \cap B(\underline{i}) \neq \emptyset$. Note that $\underline{\Lambda}(n,\phi)$ is itself a box of the form $\Lambda(j_0')$ for some $j_0' \in \mathbb{N}$. We set $j_0 = j_0'$ if this is an odd number and $j_0 = j_0'+1$ otherwise. For $k \geq 1$, the set $\underline{S}(k) = \Lambda(j_0+2k) \setminus \Lambda(j_0+2(k-1))$ is called the *k-th rescaled shell* around $\Lambda(n,\phi)$ and $S(k) = \cup_{\underline{i}\in\underline{S}(k)} B(\underline{i})$ is the *k–th shell* around $\Lambda(n,\phi)$. Note that $S(k) \subseteq \Lambda(n) \setminus \Lambda(n,\phi)$ for $k = 1, 2, \ldots, s$, where $s = s(n) = \lfloor \phi(n)/(2L(n)) \rfloor - 1$. We divide the k-th rescaled shell into *corner* and *non-corner* sites (blocks) according to whether their d_1 distance to $\underline{\Lambda}(j_0+2(k-1))$ is equal to or strictly larger than one. In the subsequent arguments the middle hypersquares of these blocks will play an important role. They are contained in the inner vertex boundary of the box

$$\Gamma(k) = \Lambda((j_0 + 2k)L)$$

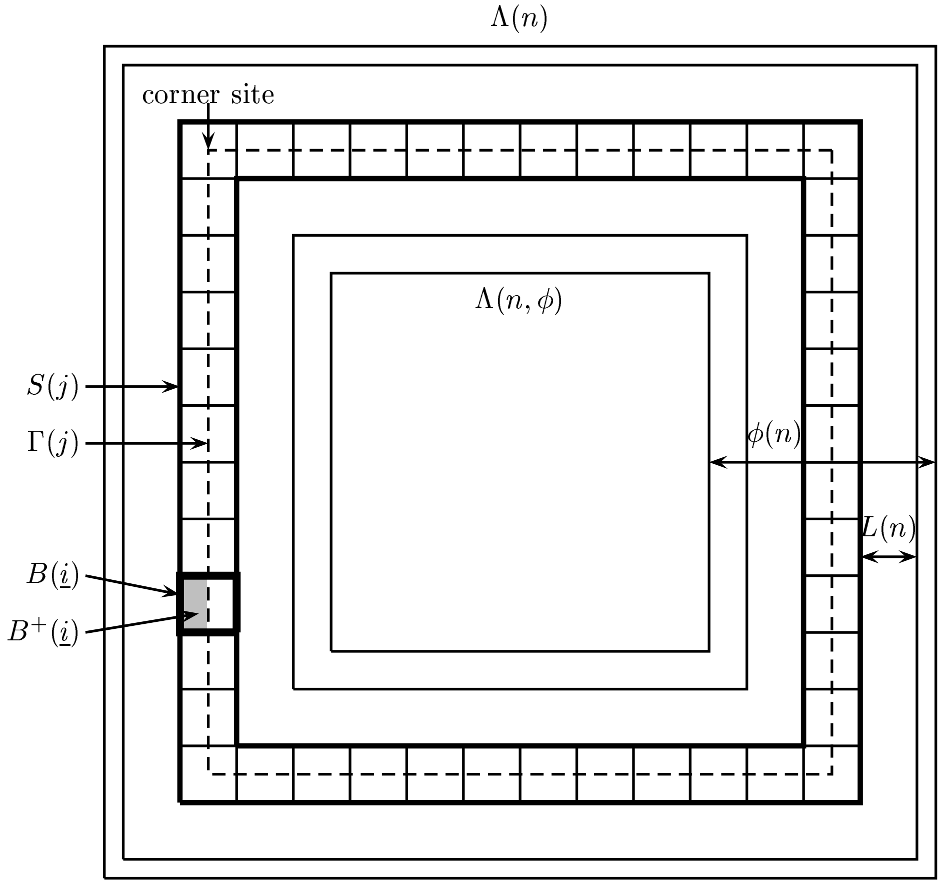

figure 2: the blocks and the shells

Note that $\partial^{in}\Gamma(k)$ intersects every non-corner block of the k-th shell in its middle hypersquare. We define for $k = 1, ..., s$, $\underline{i} \in S(k)$, $\underline{i}$ non-corner,

$$\begin{aligned}
B^+(\underline{i}) &= B(\underline{i}) \cap \Big(\partial^{in}\Gamma(k) \cup (\Lambda(n) \setminus \Gamma(k)\Big) \\
D(\underline{i}) &= \Big\{x \in B(\underline{i}) \cap \partial^{in}\Gamma(k) \,\Big|\, \mathrm{d}_\infty\,(x, \partial^{in}B(\underline{i})) \geq \sqrt{L}\Big\} \\
K(\underline{i}) &= \text{ number of } B^+(\underline{i})\text{-clusters intersecting } D(\underline{i})
\end{aligned}$$

Finally, for $\varepsilon > 0$, we define the block event

$$R_{\underline{i}}(\varepsilon) = \Big\{\exists\,!\ B^+(\underline{i})\text{-cluster } C^* \text{ with diameter } \geq \sqrt{L} \text{ intersecting } D(\underline{i}),\ |\,K(\underline{i})/L^{d-1} - \kappa^+\,| < \varepsilon\Big\}$$

Recall that we wish to understand the difference between the measures $\Phi^f_{\Lambda(n)}$ and $\Phi^\pi_{\Lambda(n)}$ in $\Lambda(n,\phi)$. It is natural to look at a monotone coupling $\mathbb{P}_{\Lambda(n)}$ of these measures governing two layers of configurations $\omega = (\omega^f, \omega^\pi)$ with $\omega^f \leq \omega^\pi$. We will work with this coupled process in the sequel. A non-corner block $\underline{i}$ is called *regular* if $R_{\underline{i}}(\varepsilon)$ occurs in *both* layers. We denote the corresponding block process by $X_{\underline{i}}$. Hence $X_{\underline{i}} = 0$ if $\underline{i}$ is regular and $X_{\underline{i}} = 1$ otherwise. For $k = 1, ..., s$, we set

$$G_k(\varepsilon) = \Big\{\text{the proportion of regular non-corner blocks in the } k\text{-th shell is larger than } 1 - 2\varepsilon\Big\}$$

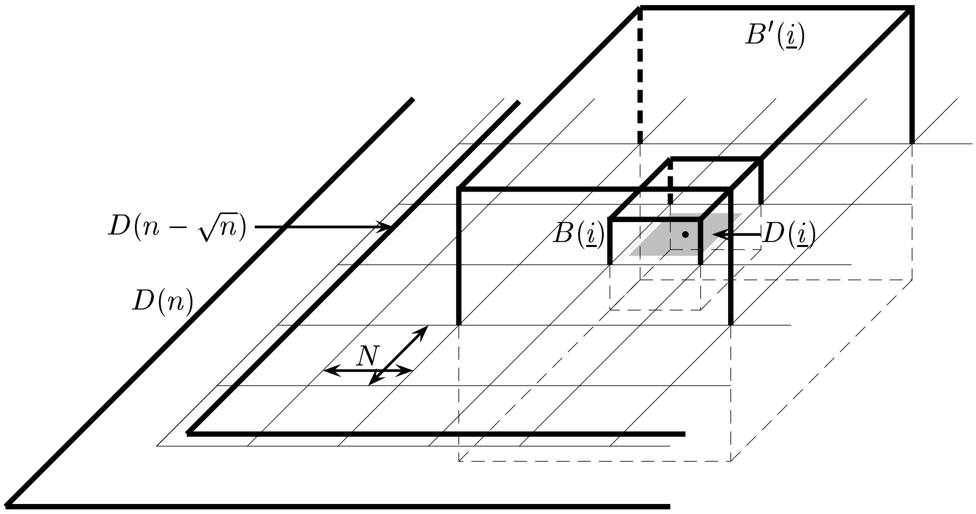

figure 3: the blocks $B(\underline{i})$, $B'(\underline{i})$ and the middleplane $D(\underline{i})$

The uniform estimates (14) and Lemma 3.6 guarantee that for every $n \geq n_0(\varepsilon, d, p, q)$, the process $X_{\underline{i}}$, indexed by non-corner blocks in the k-th shell, is stochastically dominated by i.i.d. Bernoulli variables with parameter ε. Hence, there exists $c = c(\varepsilon) > 0$ such that for every $n \geq n_0$,

$$\begin{aligned}
\mathbb{P}_{\Lambda(n)}\Big[\bigcap_{k=1,\dots,s} G_k^c\Big] &\leq \mathbb{P}_{\Lambda(n)}\Big[\Big(\text{the proportion of regular blocks in} \bigcup_{k=1,\dots,s} \underline{S}(k)\Big) \leq 1-2\varepsilon\Big] \\
&\leq \exp\Big(-c'\Big|\bigcup_{k=1,\dots,s} \underline{S}(k)\Big|\Big) \leq \exp\big(-c' n^{d-1} L(n)\big) \qquad (37)
\end{aligned}$$

where we used Cramer's theorem. In words: we know that up to negligible events, we can find a shell in the $\phi(n)$-boundary of $\Lambda(n)$ where most of the blocks are regular. As we will show, this implies that the induced boundary conditions in each layer are close to each other. For $* = f$ or π, we denote by $W^*_{\Gamma(k)}(\omega)$ the induced b.c.s $*\cdot W^{\Gamma(k)}_{\Lambda(n)}(\omega^*)$ (see section 2 p.13 for the notation). Note that these are random b.c.s on $\Gamma(k)$ which arise as a combination of the b.c.s on $\Lambda(n)$ and of the configurations occurring in the annulus

$$A(k) = \Lambda(n) \setminus \Gamma(k)$$

We claim that there exists $n_0 = n_0(\varepsilon, d, \phi)$ such that for every $n \geq n_0$ and $k = 1, \dots, s$, we have on the event G_k

$$0 \leq |W^f_{\Gamma(k)}| - |W^\pi_{\Gamma(k)}| \leq 9\varepsilon\, 2d\, n^{d-1} \qquad (38)$$

Note that the first inequality is a simple consequence of the monotonicity of the coupling. To prove the other bound we first observe that if $B(\underline{i})$ is a regular non-corner block in the k-th shell then for $* = f$ and $* = \pi$, $K(\underline{i})(\omega^*)$ is equal to the number of $A(k)$-clusters on the $*$-layer which intersect $D(\underline{i})$. The next step is to derive an appropriate upper bound on the number $|W^f_{\Gamma(k)}(\omega)|$ of $A(k)$-clusters intersecting $\partial^{in}\Gamma(k)$ in the f layer, when $\omega \in G_k$. We estimate it (using subadditivity) by the sum of the numbers of $A(k)$-clusters intersecting $\partial^{in}\Gamma(k)$

in corner blocks, $= N_1$
in irregular non-corner blocks, $= N_2$
in regular non-corner blocks $B(\underline{i})$ "outside" $D(\underline{i})$, i.e., in $(\partial^{in}\Gamma(k) \cap B(\underline{i})) \setminus D(\underline{i})$, $= N_3$
in $D(\underline{i})$ when $B(\underline{i})$ is a regular non-corner block, $= N_4$

The corresponding estimates are as follows:

$$
\begin{aligned}
N_1 &\leq \varepsilon n^{d-1}; \quad \text{for } n \text{ large enough depending on } \varepsilon, d, \phi \\
N_2 &\leq 2d\, 2\varepsilon (n/L(n))^{d-1} L(n)^{d-1} = 4d\varepsilon n^{d-1}; \;\; \text{on } G_k \\
N_3 &\leq 2d(n/L(n))^{d-1} \times L(n)^{d-2}\sqrt{L(n)} < \varepsilon n^{d-1}; \quad \text{for } n \text{ large enough depending on } \varepsilon, d, \phi \\
N_4 &\leq \sum_{\substack{\underline{i}\in \underline{S}(k),\ B(\underline{i}) \text{ is} \\ \text{regular, non-corner}}} K(\underline{i}) \quad \leq L(n)^{d-1}(\kappa^+ + \varepsilon)\, 2d\, (n/L(n))^{d-1} = 2d(\kappa^+ + \varepsilon) n^{d-1}
\end{aligned}
$$

This implies

$$
|W^f_{\Gamma(k)}| \leq N_1 + N_2 + N_3 + N_4 \leq 2d(\kappa^+ + 4\varepsilon) n^{d-1} \tag{39}
$$

To get a lower bound on $|W^\pi_{\Gamma(k)}(\omega)|$ when $\omega \in G_k$ we first observe that in a regular non-corner block $B(\underline{i})$, $\underline{i} \in \underline{S}(k)$ there are $K(\underline{i}) - 1 \geq (\kappa^+ - \varepsilon)L^{d-1} - 1$ $A(k)$-clusters with diameter strictly smaller than $\sqrt{L}$ intersecting $D(\underline{i})$. Note that these clusters can not intersect any other block $B(\underline{j})$, $\underline{j} \in \underline{S}(k) \setminus \{\underline{i}\}$. Thus the total number of $A(k)$-clusters intersecting $\partial^{in}\Gamma(k)$ is certainly not smaller than the sum of the $A(k)$-clusters with diameter strictly smaller than $\sqrt{L}$ intersecting $D(\underline{i})$ for some $\underline{i} \in \underline{S}(k)$, $B(\underline{i})$ non-corner and regular. The number of such blocks is bounded from below by

$$
\Big(\frac{n - 2\phi(n)}{L(n)}\Big)^{d-1} 2d\,(1 - 2\varepsilon) > (1 - 3\varepsilon) 2d\, (n/L(n))^{d-1}
$$

where n is large enough (depending on ε, d, ϕ only.) Hence for $\omega \in G_k$ and n large

$$
|W^\pi_{\Gamma(k)}(\omega)| \geq (1 - 3\varepsilon) 2d (n/L)^{d-1}((\kappa^+ - \varepsilon)L^{d-1} - 1) \geq 2d(\kappa^+ - 5\varepsilon) n^{d-1}
$$

This, together with (39) gives (38).

The remainder of the proof of Proposition 3.1 is based on Lemma 3.3. For an integer $m < n$ we set

$$
J_m = J_m(\varepsilon) = \Big\{ \omega \,\Big|\, |W^f_{\Lambda(m)}| - |W^\pi_{\Lambda(m)}| < \varepsilon n^{d-1} \Big\}
$$

By (37) and (38) we know that

$$
\varlimsup_{n\to\infty} n^{1-d} \log \mathbb{P}_{\Lambda(n)} \Big[\big(\bigcup_{n - \phi(n) < m < n} J_m \big)^c \Big] = -\infty \tag{40}
$$

Let S_n be an event which depends only on the edges in $\Lambda(n,\phi)$. In the coupled model denote by S_n^*, $*=f$ or π, the event that S_n occurs on the $*$-layer. Let $\eta=(\eta^f,\eta^\pi)$ be a configuration in $\Lambda(n)\setminus\Lambda(m)$ and assume $\eta\in J_m(\varepsilon)$. Note that, by (10),

$$\mathbb{P}_{\Lambda(n)}[S_n^*\,|\,\eta]=\Phi^*_{\Lambda(n)}[S_n\,|\,\eta^*]=\Phi_{\Lambda(m)}^{W^*_{\Lambda(m)}(\eta^*)}[S_n]$$

By Lemma 3.3 we have

$$q^{-\varepsilon n^{d-1}}\mathbb{P}_{\Lambda(n)}[S_n^\pi\,|\,\eta]\le\mathbb{P}_{\Lambda(n)}[S_n^f\,|\,\eta]\le q^{\varepsilon n^{d-1}}\mathbb{P}_{\Lambda(n)}[S_n^\pi\,|\,\eta] \tag{41}$$

Hence

$$\begin{aligned}\mathbb{P}_{\Lambda(n)}[S_n^f\cap J_m] &= \sum_{\substack{\eta:\ \text{config. on }\Lambda(n)\setminus\Lambda(m)\\ \{\eta\}\subseteq J_m}}\mathbb{P}_{\Lambda(n)}[S_n^f\,|\,\eta]\,\mathbb{P}_{\Lambda(n)}[\eta]\\ &\le \sum_{\{\eta\}\subseteq J_m} q^{\varepsilon n^{d-1}}\mathbb{P}_{\Lambda(n)}[S_n^\pi\,|\,\eta]\,\mathbb{P}_{\Lambda(n)}[\eta]\\ &= q^{\varepsilon n^{d-1}}\mathbb{P}_{\Lambda(n)}[S_n^\pi\cap J_m]\le q^{\varepsilon n^{d-1}}\mathbb{P}_{\Lambda(n)}[S_n^\pi]\end{aligned} \tag{42}$$

Therefore, by using (40) and (42)

$$\begin{aligned}\overline{\lim_{n\to\infty}}\ n^{1-d}\log\Phi^f_{\Lambda(n)}[S_n] &= \overline{\lim_{n\to\infty}}\ n^{1-d}\log\mathbb{P}_{\Lambda(n)}\Big[S_n^f\cap\Big(\bigcup_{n-\phi(n)<m<n}J_m\Big)\Big]\\ &\le \overline{\lim_{n\to\infty}}\ n^{1-d}\log\Big(\phi(n)\max_{n-\phi(n)<m<n}\mathbb{P}_{\Lambda(n)}\Big[S_n^f\cap J_m\Big]\Big)\\ &\le \overline{\lim_{n\to\infty}}\ n^{1-d}\log q^{\varepsilon n^{d-1}}\,\mathbb{P}_{\Lambda(n)}\Big[S_n^{\pi(n)}\Big]\\ &\le \varepsilon\log q+\overline{\lim_{n\to\infty}}\ n^{1-d}\log\Phi^{\pi(n)}_{\Lambda(n)}[S_n]\end{aligned}$$

Since $\varepsilon>0$ can be chosen arbitrarily, we have

$$\overline{\lim_{n\to\infty}}\ n^{1-d}\log\Phi^f_{\Lambda(n)}[S_n]\le\overline{\lim_{n\to\infty}}\ n^{1-d}\log\Phi^{\pi(n)}_{\Lambda(n)}[S_n] \tag{43}$$

By interchanging the roles of f and $\pi(n)$ and using the first inequality in (41) we derive similarly the opposite inequality which gives the claim (21). The derivation of the inequality involving $\underline{\lim}$ is analogous. □

4 Surface tension

We will have to work with enlargements of continuous subsets of $\mathbb{R}^d$ so that they have a significant trace on the discrete lattice $\mathbb{Z}^d$. We fix a real number $\zeta>2d$ and we enlarge a subset A of $\mathbb{R}^d$ by considering its ζ–neighbourhood $\mathcal{V}_2(A,\zeta)$. A minimal requirement to choose ζ is that, whenever A is an arcwise connected subset of $\mathbb{R}^d$, the graph $(\mathbb{Z}^d(\mathcal{V}_2(A,\zeta)),\mathbb{E}^d(\mathcal{V}_2(A,\zeta)))$ is also connected. Some of the constants appearing in the statements and the proofs depend on ζ. However the direction dependent surface tension and the probabilistic estimates are independent of the particular choice of $\zeta>2d$.

We next identify the surface tension of the model, whose existence is guaranteed by subadditivity and the FKG inequality. We work here in the following regime: $d \geq 3$, $q \geq 1$, $p > \widehat{p}_c$, $\theta^f(p) = \theta^w(p)$. We study the surface tension of the infinite volume FK measure Φ_∞.

Let A be a closed hyper-rectangle and let s be positive or infinite. Recall that $\operatorname{cyl} \partial A = \{x + t \operatorname{nor} A \,|\, x \in \partial A,\, t \in \mathbb{R}\}$. We denote by $W(\partial A, s, \zeta)$ the event that there exists a set of closed edges $E \subseteq \mathcal{V}_2(\operatorname{hyp} A, s)$ such that

- E separates ∞ in $\operatorname{cyl} A$
- the edges in $E \cap \mathcal{V}_2(\operatorname{cyl} \partial A, \zeta)$ are close to $\operatorname{hyp} A$ so that they are contained in $\mathcal{V}_2(\operatorname{hyp} A, \zeta)$.

Loosely speaking, the second condition means that the "boundary" of E is "pinned down" at ∂A within a distance ζ. Note that the event $W(\partial A, s, \zeta)$ is decreasing and it depends only on edges inside $\operatorname{cyl} A \cap \mathcal{V}_2(\operatorname{hyp} A, s)$.

Proposition 4.1 *Let A be a hyper–rectangle. Let $\phi(n)$ be a function from $\mathbb{N}$ to $\mathbb{R}^+ \cup \{\infty\}$ such that $\lim_{n\to\infty} \phi(n) = \infty$. The limit*

$$\lim_{n\to\infty} -\frac{1}{\mathcal{H}^{d-1}(nA)} \log \Phi_\infty \Big[W(\partial nA, \phi(n), \zeta) \Big]$$

exists $\in [0, \infty]$ and depends only on $\operatorname{nor} A$. *We denote it by $\tau(\operatorname{nor} A)$ and call it the surface tension in the direction* $\operatorname{nor} A$.

Proof: This result is proved with the help of the same subadditivity argument used in [2], Proposition 2.4. The only additional problem is that we work with curves whose position with respect to the discrete lattice $\mathbb{Z}^d$ is arbitrary. Let w be a unit vector of $\mathbb{R}^d$ and let A, A' be two hyper–rectangles such that $\operatorname{nor} A = \operatorname{nor} A' = w$. Let $\phi(n), \phi'(n)$ be two functions from $\mathbb{N}$ to $\mathbb{R}^+ \cup \{\infty\}$ such that $\lim_{n\to\infty} \phi(n) = \infty$, $\lim_{n\to\infty} \phi'(n) = \infty$. Let ζ, ζ' be two real numbers larger than $2d$. Let n, m in $\mathbb{N}$ be such that $n \operatorname{diam}_2 A > m \operatorname{diam}_2 A' > \max(\zeta, \zeta')$. Because we deal with hyper–rectangles, certainly there exists a collection of sets $(T(i), i \in I)$ such that: each set $T(i)$ is a translate of mA' intersecting the set

$$D(m,n) \;=\; \{\, x \in nA : d_2(x, n\partial A) > 2m \ \operatorname{diam}_2 A' \,\} \,;$$

the sets $(T(i), i \in I)$ have pairwise disjoint interiors; their union $\bigcup_{i\in I} T(i)$ contains the set $D(m,n)$. Since A is a hyper–rectangle, then

$$\begin{aligned} \mathcal{H}^{d-1}(nA \setminus D(m,n)) \;&\leq\; \mathcal{H}^{d-1}(nA) - 2m(\operatorname{diam}_2 A')\mathcal{H}^{d-2}(n\partial A) \\ &\leq\; |I|\mathcal{H}^{d-1}(mA') \;\leq\; \mathcal{H}^{d-1}(nA) \end{aligned} \tag{44}$$

For each i in I, let $t(i)$ be a vector in $\mathbb{R}^d$ such that $|t(i)|_\infty \leq 1$ and $t(i) + T(i)$ is the image of mA' by an integer translation (a translation that leaves $\mathbb{Z}^d$ globally invariant). Let $T'(i) = t(i) + T(i)$. Suppose that all the events $W(\partial T'(i), \phi(n), \zeta')$, $i \in I$ occur, and let $E(i)$, $i \in I$, be sets of closed edges realizing these events. Let E_0 be the set of edges included in

$$\Big(\operatorname{cyl}(nA \setminus D(m,n)) \cap \mathcal{V}_2(\operatorname{hyp} nA, \zeta) \Big) \,\cup\, \bigcup_{i\in I} \Big(\mathcal{V}_2(\operatorname{cyl} \partial T'(i), c(d,\zeta')) \cap \mathcal{V}_2(\operatorname{hyp} nA, c(d,\zeta')) \Big)$$

Let $E = E_0 \cup \bigcup_{i\in I} E(i)$. The constant $c(d,\zeta')$ can be chosen large enough to guarantee that the edges of E separate ∞ inside $\mathrm{cyl}\, nA$. Then the set of edges E realizes the event $W(\partial nA, \phi(n), \zeta)$. An attempt of proof is done in [14], Proposition 5.2. Therefore

$$\Big\{\omega \,\Big|\, \forall e \in E_0 \quad \omega(e) = 0\Big\} \cap \Big(\bigcap_{i\in I} W(\partial T'(i), \phi(n), \zeta')\Big) \subseteq W(\partial nA, \phi(n), \zeta)$$

Since all these events are decreasing, by the FKG inequality,

$$\Phi_\infty\Big[W(\partial nA, \phi(n), \zeta)\Big] \geq (1-p)^{|E_0|} \prod_{i\in I} \Phi_\infty\Big[W(\partial T_i', \phi(n), \zeta')\Big] \tag{45}$$

Since the model is invariant under the integer translations, for any i in I,

$$\Phi_\infty\Big[W(\partial T_i', \phi(n), \zeta')\Big] = \Phi_\infty\Big[W(\partial mA', \phi(n), \zeta')\Big] \tag{46}$$

Because $\phi(n)$ goes to ∞ as n goes to ∞,

$$\lim_{n\to\infty} \Phi_\infty\Big[W(\partial mA', \phi(n), \zeta')\Big] = \Phi_\infty\Big[W(\partial mA', \infty, \zeta')\Big]$$

whence, for n sufficiently large,

$$\Phi_\infty\Big[W(\partial mA', \phi(n), \zeta')\Big] \geq (1/2)\Phi_\infty\Big[W(\partial mA', \infty, \zeta')\Big] \tag{47}$$

For such integers n, combining (45), (46), (47) and passing to the logarithm,

$$\log \Phi_\infty\Big[W(\partial nA, \phi(n), \zeta)\Big] \geq |I| \log \Phi_\infty\Big[W(\partial mA', \phi'(m), \zeta')\Big] + |E_0| \log(1-p) - |I| \log 2 \tag{48}$$

There exists a further constant $c(d,\zeta,\zeta',A,A')$ such that

$$|E_0| \leq c(d,\zeta,\zeta',A,A')(n^{d-2}m + n^{d-1}/m + 1) \tag{49}$$

Using the inequalities (44), (48) and (49), we obtain

$$\begin{aligned}\mathcal{H}^{d-1}(nA)^{-1} \log \Phi_\infty\Big[W(\partial nA, \phi(n), \zeta)\Big] &\geq \mathcal{H}^{d-1}(mA')^{-1} \log \Phi_\infty\Big[W(\partial mA', \phi'(m), \zeta')\Big] + \\ &c(d,\zeta,\zeta',A,A')\mathcal{H}^{d-1}(A)^{-1}(m/n + 1/m + 1/n^{d-1})\log(1-p) - \mathcal{H}^{d-1}(mA')^{-1}\log 2\end{aligned}$$

Sending successively n to ∞ and then m to ∞ yields

$$\varliminf_{n\to\infty} \mathcal{H}^{d-1}(nA)^{-1} \log \Phi_\infty\Big[W(\partial nA, \phi(n), \zeta)\Big] \geq \varlimsup_{m\to\infty} \mathcal{H}^{d-1}(mA')^{-1} \log \Phi_\infty\Big[W(\partial mA', \phi'(m), \zeta')\Big]$$

which implies the result of the Proposition. □

Properties. The surface tension τ inherits automatically some symmetry properties from the model. For instance, if f is a linear isometry of $\mathbb{R}^d$ such that $f(0) = 0$ and $f(\mathbb{Z}^d) = \mathbb{Z}^d$ then $\tau \circ f = \tau$. Besides, the surface tension τ satisfies another important inequality called the weak triangle inequality. For details and results concerning this kind of inequalities, see [27, 49, 50].

Proposition 4.2 (weak triangle inequality) *Let (ABC) be a non–degenerate triangle in $\mathbb{R}^d$ and let ν_A, ν_B, ν_C be the exterior normal unit vectors to the sides $[BC]$, $[AC]$, $[AB]$ in the plane spanned by A, B, C. Then*

$$\mathcal{H}^1([BC])\tau(\nu_A) \;\leq\; \mathcal{H}^1([AC])\tau(\nu_B) + \mathcal{H}^1([AB])\tau(\nu_C)$$

Proof: We consider first the case where $BA \cdot BC \geq 0$ and $CA \cdot CB \geq 0$. Let ϵ, h be positive with than $\epsilon \leq 1 \leq h$. Let $(e_1, \cdots, e_d)$ be an orthonormal basis of $\mathbb{R}^d$ such that e_1, e_2 belong to the two dimensional space spanned by A, B, C. Let K be the compact convex set defined by

$$K \;=\; \Big\{ x + \sum_{3\leq i\leq d} u_i e_i \,\Big|\, x \in (ABC),\, (u_3, \cdots, u_d) \in [0,h]^{d-2} \Big\}$$

The boundary of K consists of the three hyper–rectangles R_A, R_B, R_C defined by

$$R_A \;=\; \Big\{ x + \sum_{3\leq i\leq d} u_i e_i \,\Big|\, x \in [BC],\, (u_3, \cdots, u_d) \in [0,h]^{d-2} \Big\}$$
$$R_B \;=\; \Big\{ x + \sum_{3\leq i\leq d} u_i e_i \,\Big|\, x \in [AC],\, (u_3, \cdots, u_d) \in [0,h]^{d-2} \Big\}$$
$$R_C \;=\; \Big\{ x + \sum_{3\leq i\leq d} u_i e_i \,\Big|\, x \in [AB],\, (u_3, \cdots, u_d) \in [0,h]^{d-2} \Big\}$$

and the set

$$T \;=\; \bigcup_{3\leq j\leq d} \Big\{ x + \sum_{3\leq i\leq d} u_i e_i \,\Big|\, x \in (ABC),\, u_j \in \{0,h\},\, (u_3, ..., u_{j-1}, u_{j+1}, ..., u_d) \in [0,h]^{d-3} \Big\}$$

Notice that the set T is connected in dimension $d \geq 4$ and consists of two disjoint triangles in dimension $d = 3$. The intersection of the hyper–rectangles R_A and R_B is a $d-2$ dimensional rectangle and it is denoted by $R_{A,B}$. Similar notation is used for the other intersections. Let E_0 be the set of the edges included in

$$\Big(\mathrm{cyl} \Big(\mathrm{hyp}\, nR_A \cap \mathcal{V}_2(\partial nR_A, 4\epsilon n) \Big) \cap \mathcal{V}_2(\mathrm{hyp}\, nR_A, \zeta) \Big)$$
$$\cup\, \mathcal{V}_2(nR_{B,C}, 2\zeta) \,\cup\, \mathcal{V}_2(nR_{A,C}, 2\zeta) \,\cup\, \mathcal{V}_2(nR_{A,B}, 2\zeta) \cup \mathcal{V}_2(nT, 2\zeta)$$

There exists a constant $c = c(d, \zeta)$ such that

$$|E_0| \;\leq\; c\Big(\varepsilon n^{d-1} h^{d-2} + (nh)^{d-2} + 2(d-2) n^{d-1} h^{d-3} \Big) \tag{50}$$

Let R^ϵ be a hyper-rectangle in $\mathrm{hyp}\, R_A$ such that $R_A \subset R^\epsilon \subset \mathcal{V}_2(R_A, 4\epsilon)$ and $d_2(\partial R^\epsilon, R_A) > 2\epsilon$. For n large enough, so that $\epsilon n > \zeta$, if the events

$$W(\partial nR_C, \epsilon n, \zeta)\,, \quad W(\partial nR_B, \epsilon n, \zeta)\,, \quad \{\text{all the edges of } E_0 \text{ are closed}\}$$

occur simultaneously, then the event $W(\partial nR^\epsilon, \infty, \zeta)$ occurs as well; by the assumptions $BA \cdot BC \geq 0$ and $CA \cdot CB \geq 0$, the set $\mathcal{V}_2(nR_C \cup nR_B, \epsilon n)$ is included in $\mathcal{V}_2(\mathrm{cyl}\; nR_A, \epsilon n)$ and does not intersect $\mathcal{V}_2(\operatorname{cyl}\partial nR^\epsilon,\zeta)$, so that the separating sets will be correctly localized. By the FKG inequality, this inclusion implies

$$(1-p)^{|E_0|}\Phi_\infty\Big[W(\partial nR_C,\epsilon n,\zeta)\Big]\,\Phi_\infty\Big[W(\partial nR_B,\epsilon n,\zeta)\Big]\,\leq\,\Phi_\infty\Big[W(\partial nR^\epsilon,\infty,\zeta)\Big] \tag{51}$$

The inequalities (50), (51) and Proposition 4.1 yield

$$\mathcal{H}^{d-1}(R^\epsilon)\tau(\nu_A)\leq\mathcal{H}^{d-1}(R_C)\tau(\nu_C)+\mathcal{H}^{d-1}(R_B)\tau(\nu_B)-c(\varepsilon h^{d-2}+2(d-2)h^{d-3})\log(1-p) \tag{52}$$

We observe

$$\mathcal{H}^{d-1}(R_B)=h^{d-2}\mathcal{H}^1([AC]),\ \mathcal{H}^{d-1}(R_C)=h^{d-2}\mathcal{H}^1([AB]),\ h^{d-2}\mathcal{H}^1([BC])\,\leq\,\mathcal{H}^{d-1}(R^\epsilon)$$

By substituting this into the inequality (52) and dividing by h^{d-2},

$$\mathcal{H}^{d-1}([BC])\tau(\nu_A)\leq\mathcal{H}^{d-1}([AC])\tau(\nu_B)+\mathcal{H}^{d-1}([AB])\tau(\nu_C)-c\,(\varepsilon+2(d-2)/h)\log(1-p)$$

By letting h go to ∞ and ϵ go to 0, we obtain the weak triangle inequality for the triangle (ABC). Let now A,B,C be three points such that $BA\cdot BC<0$, $CA\cdot CB\geq 0$. Let D be the orthogonal projection of B on $[AC]$. Then $BC\cdot BD>0$, $DB\cdot DA=0$, $BA\cdot BD>0$. We apply the weak triangle inequality to the triangles (BCD) and (BDA):

$$\begin{aligned}\mathcal{H}^1([BC])\tau(\nu_A)\,&\leq\,\mathcal{H}^1([BD])\tau(\nu_{BD})+\mathcal{H}^1([DC])\tau(\nu_B)\,,\\ \mathcal{H}^1([BD])\tau(\nu_{BD})\,&\leq\,\mathcal{H}^1([AB])\tau(\nu_C)+\mathcal{H}^1([AD])\tau(\nu_B)\,,\end{aligned}$$

where ν_{BD} is a unit vector orthogonal to $[BD]$. Combining the two inequalities, we get the weak triangle inequality for the triangle (ABC). The case $BA\cdot BC\geq 0$, $CA\cdot CB<0$ is similar. □

The weak triangle inequality implies a lot of nice properties for the surface tension.

Corollary 4.3 *The homogeneous extension τ_0 of τ to $\mathbb{R}^d$ defined by $\tau_0(0)=0$ and*

$$\forall w\in\mathbb{R}^d\setminus\{0\}\qquad\tau_0(w)\;=\;|w|_2\tau(w/|w|_2)$$

is a convex function.

The convexity of τ_0 is in fact equivalent to the weak triangle inequality.

Corollary 4.4 *The surface tension $\tau:S^{d-1}\mapsto\mathbb{R}^+$ is bounded and continuous.*

Proof: Let $w\in S^{d-1}$ and let A be a hyper–rectangle orthogonal to w such that $\mathcal{H}^{d-1}(A)=1$. Let $E(n)$ be the set of the edges included in $\operatorname{cyl}nA\cap\mathcal{V}_2(\operatorname{hyp}nA,2d)$. Then $|E(n)|\leq c(d)n^{d-1}$ and

$$\Phi_\infty\Big[W(\partial nA,\infty,2d)\Big]\,\geq\,\Phi_\infty\Big[\text{the edges of }E(n)\text{ are closed}\Big]\,\geq\,(1-p)^{|E(n)|}$$

Passing to the limit, we get $\tau(w)\leq -c(d)\log(1-p)$. Since τ_0 is homogeneous, convex (by Corollary 4.3) and bounded on S^{d-1}, it is finite everywhere. By a standard result of convex analysis [56], Corollary 10.1.1, it follows that τ_0 is continuous, as well as τ. □

The next Corollary is a consequence of [27], Theorem 3.1: the weak triangle inequality automatically implies the weak simplex inequality.

Corollary 4.5 (weak simplex inequality) *Let $A_0, \cdots, A_d$ be $d+1$ points in general position in $\mathbb{R}^d$. For i in $\{0 \cdots d\}$ let $\Delta(i)$ be the hyper–simplex defined by the points $\{A_0, \cdots, A_d\} \setminus \{A_i\}$. Up to the sign, there exists a unique family of unit vectors $\nu_0, \cdots, \nu_d$ such that for i in $\{0 \cdots d\}$, the vector ν_i is orthogonal to the vector space spanned by the hyper–simplex $\Delta(i)$. Then*

$$\mathcal{H}^{d-1}(\Delta(0))\tau(\nu_0) \leq \mathcal{H}^{d-1}(\Delta(1))\tau(\nu_1) + \cdots + \mathcal{H}^{d-1}(\Delta(d))\tau(\nu_d)$$

Proposition 4.6 *The surface tension $\tau : S^{d-1} \mapsto \mathbb{R}^+$ is bounded away from 0.*

Proof: Recall that we work here in the region: $d \geq 3$, $q \geq 1$, $p > \widehat{p}_c$, $\theta^f(p) = \theta^w(p)$. We first show that $\tau(1, 0, \cdots, 0)$ is positive. Let δ be positive and let $F_1, \cdots, F_{2d}$ be the $2d$ faces of the cubic box $\Lambda(1 - 2\delta)$. Let E_0 be the set of edges included in $\mathcal{V}_2(n\partial F_1 \cup \cdots \cup n\partial F_{2d}, 2\zeta)$. There exists a constant $c(d, \zeta)$ such that $|E_0| \leq c(d, \zeta)n^{d-2}$. If the events $W(n\partial F_i, \delta n, \zeta)$, $1 \leq i \leq 2d$, $\{$ all the edges of E_0 are closed $\}$, occur simultaneously, then there exists a set of closed edges inside the box $\Lambda(n)$ which separates the box $n\Lambda(1 - 4\delta)$ from $\partial^{in}\Lambda(n)$. By the FKG inequality,

$$(1-p)^{|E_0|} \prod_{1 \leq i \leq 2d} \Phi_\infty \Big[W(n\partial F_i, \delta n, \zeta) \Big] \leq \Phi_\infty \Big[\{ n\Lambda(1 - 4\delta) \nleftrightarrow \partial^{in}\Lambda(n) \}^c \Big]$$

The bound on $|E_0|$, Proposition 4.1 and the symmetry of τ imply that

$$\lim_{n \to \infty} \frac{1}{n^{d-1}} \log \left((1-p)^{|E_0|} \prod_{1 \leq i \leq 2d} \Phi_\infty \Big[W(n\partial F_i, \delta n, \zeta) \Big] \right) = -2d(1 - 2\delta)^{d-1}\tau(1, 0, \cdots, 0)$$

Next, the event $\{ n\Lambda(1 - 4\delta) \leftrightarrow \partial^{in}\Lambda(n) \}$ depends only on the status of the edges inside $\Lambda(n)$. By conditioning on the configuration outside $\Lambda(n)$, we get

$$\begin{aligned} &\Phi_\infty \Big[\{ n\Lambda(1 - 4\delta) \leftrightarrow \partial^{in}\Lambda(n) \}^c \Big] \\ &\quad = \Phi_\infty \Big[\Phi_\infty \Big[\{ n\Lambda(1 - 4\delta) \leftrightarrow \partial^{in}\Lambda(n) \}^c | \mathcal{F}^{\Lambda(n)} \Big] \Big] \\ &\quad \leq \sup_{\Phi \in c\mathcal{FK}(\Lambda(n))} \Phi \Big[\{ n\Lambda(1 - 4\delta) \leftrightarrow \partial^{in}\Lambda(n) \}^c \Big] \end{aligned}$$

Yet the event $\{ n\Lambda(1 - 4\delta) \leftrightarrow \partial^{in}\Lambda(n) \}^c$ implies that any cluster in $\Lambda(n)$ intersecting $\partial^{in}\Lambda(n)$ is included in $\Lambda(n) \setminus n\Lambda(1 - 4\delta)$ and has therefore cardinality less than $n^d(1 - (1 - 4\delta)^d)$. We choose δ small enough so that $1 - (1 - 4\delta)^d \leq \theta/2$. By the result of Pisztora [55], Theorem 1.2, we have then

$$\overline{\lim_{n \to \infty}} \frac{1}{n^{d-1}} \log \sup_{\Phi \in c\mathcal{FK}(\Lambda(n))} \Phi \Big[\{ \Lambda(n(1 - 4\delta)) \leftrightarrow \partial^{in}\Lambda(n) \}^c \Big] < 0$$

so that $\tau(1, 0, \cdots, 0)$ is positive.

Suppose that $\tau(\nu) = 0$ for some ν in S^{d-1}. Let $f_1, \cdots, f_{d-1}$ be linear isometries of $\mathbb{R}^d$ such that $f_1(0) = \cdots = f_{d-1}(0) = 0$, $f_1(\mathbb{Z}^d) = \cdots = f_{d-1}(\mathbb{Z}^d) = \mathbb{Z}^d$ and $(f_1(\nu), \cdots, f_{d-1}(\nu), \nu)$ is an orthonormal basis of $\mathbb{R}^d$. Then $\tau(f_1(\nu)) = \cdots = \tau(f_{d-1}(\nu)) = 0$. Applying the weak simplex inequality to a pyramid having for basis a hyper–simplex orthogonal to $(1, 0, \cdots, 0)$ and whose d other faces are orthogonal to $f_1(\nu),\cdots,f_{d-1}(\nu),\nu$, we obtain that $\tau(1,0,\cdots,0)=0$, a contradiction. Thus the surface tension τ does not vanish on S^{d-1}. Since τ is continuous on S^{d-1} (by Corollary 4.4), it is bounded away from 0 on S^{d-1}. $\square$

The previous properties of τ can equivalently be described through its Wulff crystal

$$\mathcal{W}_\tau = \Big\{ x\in\mathbb{R}^d \,\Big|\, x\cdot w\leq \tau(w) \text{ for all } w \text{ in } S^{d-1}\Big\}$$

Corollary 4.7 *The Wulff crystal $\mathcal{W}_\tau$ associated with τ is bounded, closed, convex and contains 0 in its interior. If f is a linear isometry of $\mathbb{R}^d$ such that $f(0)=0$ and $f(\mathbb{Z}^d)=\mathbb{Z}^d$ then $f(\mathcal{W}_\tau)=\mathcal{W}_\tau$. The surface tension τ is the support function of its Wulff crystal, i.e.,*

$$\forall\nu\in S^{d-1}\qquad \tau(\nu)\;=\;\sup\,\{x\cdot\nu\,|\,x\in\mathcal{W}_\tau\}$$

These properties are equivalent to the symmetry properties of τ and Corollaries 4.3, 4.4, Proposition 4.6. The function τ is the support function of $\mathcal{W}_\tau$ because τ_0 is convex and coincides with its bipolar, see for instance [56], Corollary 13.2.1, [34], Proposition 3.5, or [27], Theorem 2.1, Corollary 3.6.

Separating sets. With the help of the surface tension, we next estimate the probability of the occurrence of a separating set of closed edges near an hyperplane. Let A be a hyperset in $\mathbb{R}^d$ and let r be positive. We denote by $S(A,r)$ the event that there exists a set of closed edges in $\operatorname{cyl} A\cap\mathcal{V}_2(\operatorname{hyp} A,r)$ which separates ∞ in $\operatorname{cyl} A$, that is,

$$S(A,r)=\Big\{\exists\, E\subset \operatorname{cyl} A\cap\mathcal{V}_2(\operatorname{hyp} A,r),\ \forall e\in E\quad \omega(e)=0,\, E \text{ separates } \infty \text{ in } \operatorname{cyl} A\Big\}$$

From now on, we work with a fixed value of ζ larger than $2d$ and we drop ζ in the notation $W(\partial A,s,\zeta)$, thus writing simply $W(\partial A,s)$.

Lemma 4.8 *Let O be an open hyperset in $\mathbb{R}^d$ and let $\phi(n)$ be a function from $\mathbb{N}$ to $\mathbb{R}^+\cup\{\infty\}$ such that $\lim_{n\to\infty}\phi(n)=\infty$. We have*

$$\varliminf_{n\to\infty}\frac{1}{n^{d-1}}\log\Phi_\infty\Big[S(nO,\phi(n))\Big]\;\geq\;-\mathcal{H}^{d-1}(O)\,\tau(\operatorname{nor} O)$$

Proof: Let $(A_i, i\in I)$ be a finite family of hyper–rectangles in $\operatorname{hyp} O$ having disjoint relative interiors and covering O. Let $c=c(d,\zeta)$ be a large constant. Let E_0 be the set of edges included in the union

$$\bigcup_{i\in I}\mathcal{V}_2(\operatorname{cyl}\partial nA_i,c)\cap\mathcal{V}_2(\operatorname{hyp} nO,c)$$

There exists a further constant $c'=c'(d,\zeta,c)$ such that $|E_0|\leq c'|I|n^{d-2}$. If all the events $W(\partial nA_i,\phi(n))$, $i\in I$, occur and all the edges of E_0 are closed, then $S(nO,\phi(n))$ occurs as well, provided the constant c is large enough. By the FKG inequality,

$$\Phi_\infty\Big[S(nO,\phi(n))\Big]\;\geq\;(1-p)^{|E_0|}\prod_{i\in I}\Phi_\infty\Big[W(\partial nA_i,\phi(n))\Big]$$

whence, by Proposition 4.1 and the bound on $|E_0|$,

$$\underline{\lim}_{n\to\infty} \frac{1}{n^{d-1}} \log \Phi_\infty\Big[S(nO,\phi(n))\Big] \;\geq\; -\tau(\,\mathrm{nor}\,O)\sum_{i\in I}\mathcal{H}^{d-1}(A_i)$$

By taking the supremum of the righthand side over all possible coverings of O, we obtain the claim of the Lemma. $\square$

Lemma 4.9 *There exists a positive constant $c=c(p,d,\zeta)$ such that, for any hyper–rectangle A of $\mathbb{R}^d$, for any positive r,*

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \Phi_\infty\Big[S(nA,nr)\Big] \;\leq\; -\tau(w)\mathcal{H}^{d-1}(A) + c\,r\mathcal{H}^{d-2}(\partial A)$$

Proof: Let $\epsilon>0$. Let A^ϵ be a hyper–rectangle such that

$$\mathcal{V}_\infty(A,2\epsilon)\cap \mathrm{hyp}\,A \;\subset\; A^\epsilon \;\subset\; \mathcal{V}_\infty(A,3\epsilon)\cap \mathrm{hyp}\,A$$

Let E_0 be the set of the edges included in

$$\Big(\mathcal{V}_2(\,\mathrm{cyl}\,n\partial A,\zeta)\cap \mathcal{V}_2(\,\mathrm{hyp}\,nA,nr)\Big)\cup\Big(\,\mathrm{cyl}\,(nA^\epsilon\setminus nA)\cap\mathcal{V}_2(\,\mathrm{hyp}\,nA,\zeta)\Big)$$

Suppose that the event $S(nA,nr)$ occurs, and let E_S be a set of closed edges realizing it. Suppose also that all the edges of E_0 are closed. Then the set of closed edges $E_0\cup E_S$ realizes the event $W(\partial nA^\epsilon,nr)$. Therefore

$$\Big\{\omega\,\Big|\,\forall e\in E_0\quad \omega(e)=0\Big\}\cap S(nA,nr)\;\subset\; W(\partial nA^\epsilon,nr)$$

Since all these events are decreasing, by the FKG inequality,

$$\Phi_\infty\Big[S(nA,nr)\Big]\,(1-p)^{|E_0|}\;\leq\;\Phi_\infty\Big[W(\partial nA^\epsilon,nr)\Big]$$

There exists a constant $c=c(d,\zeta)$ such that

$$|E_0|\leq cn^{d-1}(r\mathcal{H}^{d-2}(\partial A)+\mathcal{H}^{d-1}(A^\varepsilon\setminus A))$$

whence, passing to the logarithm,

$$\log\Phi_\infty\Big[S(nA,nr)\Big]\leq \log\Phi_\infty\Big[W(\partial nA^\epsilon,nr)\Big]-cn^{d-1}(r\mathcal{H}^{d-2}(\partial A)+\mathcal{H}^{d-1}(A^\varepsilon\setminus A))\log(1-p)$$

Letting n go to ∞, applying Proposition 4.1, and sending ϵ to 0, we get the desired inequality. $\square$

Lemma 4.10 *There exists a positive constant $c=c(p,d,\zeta)$ such that: for any open hyperset O in $\mathbb{R}^d$, for any finite family $(A_i, i\in I)$ of disjoint hyper–rectangles included in O, for any positive r,*

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \Phi_\infty\Big[S(nO,nr)\Big] \;\leq\; -\tau(\,\mathrm{nor}\,O)\sum_{i\in I}\mathcal{H}^{d-1}(A_i)+c\,r\sum_{i\in I}\mathcal{H}^{d-2}(\partial A_i)$$

Proof: The very definition of the event $S(nO, nr)$ implies that

$$S(nO, nr) \subset \bigcap_{i \in I} S(nA_i, nr)$$

For i in I, the event $S(nA_i, nr)$ depends only on the status of the edges inside $n\,\mathrm{cyl}\,(A_i, r)$. Since the hyper–rectangles $(A_i, i \in I)$, are pairwise disjoint and compact, so are the sets $(\,\mathrm{cyl}\,(A_i, r), i \in I)$. Thus the decoupling Lemma 3.2 is in force and

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \Phi_\infty \Big[S(nO, nr) \Big] \;\leq\; \sum_{i \in I} \overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \Phi_\infty \Big[S(nA_i, nr) \Big]$$

The conclusion follows from Lemma 4.9. $\square$

Corollary 4.11 *There exists a positive constant $c = c(p, d, \zeta)$ such that, for any x in $\mathbb{R}^d$, any positive ρ, η with $\eta \leq \rho$, any w in S^{d-1},*

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \Phi_\infty \Big[S(n\,\mathrm{disc}\,(x, \rho, w), n\eta) \Big] \;\leq\; -\alpha_{d-1} \rho^{d-1} \tau(w) + c\,\eta \rho^{d-2}$$

Proof: We apply Lemma 4.10 with $O = \mathrm{disc}\,(x, \rho, w)$, $r = \eta$. There exists a constant $c' = c'(d)$ such that, for any ϵ positive, there exists a finite family $(A_i, i \in I)$ of disjoint hyper–rectangles included in O such that

$$\sum_{i \in I} \mathcal{H}^{d-1}(A_i) \;\geq\; \alpha_{d-1} \rho^{d-1} - \epsilon\,, \qquad \sum_{i \in I} \mathcal{H}^{d-2}(\partial A_i) \;\leq\; c' \rho^{d-2}$$

The result follows by taking the infimum over all possible families in the inequality stated in Lemma 4.10. $\square$

5 The surface energy

We work here within the region: $d \geq 3$, $q \geq 1$, $p > \widehat{p}_c$, $\theta^f(p) = \theta^w(p)$. With the help of the surface tension τ defined in Proposition 4.1, or equivalently its Wulff crystal $\mathcal{W}_\tau$, we build a surface energy functional defined on the collection of the Borel sets of $\mathbb{R}^d$. The surface energy $\mathcal{I}(A)$ of a Borel set A is defined as

$$\mathcal{I}(A) = \sup \Big\{ \int_A \mathrm{div}\, f(x)\, dx \,\Big|\, f \in C_0^1(\mathbb{R}^d, \mathcal{W}_\tau) \Big\}$$

where $C_0^1(\mathbb{R}^d, \mathcal{W}_\tau)$ is the set of the compactly supported C^1 vector fields taking values in the Wulff crystal $\mathcal{W}_\tau$ and div is the usual divergence operator. The definition readily implies that the map $\mathcal{I} : (\mathcal{B}(\mathbb{R}^d), \mathrm{dist}_{L^1}) \to [0, \infty]$ is lower semicontinuous. We denote by $\mathcal{P}(A)$ the isotropic perimeter of a set A, introduced by Caccioppoli [12, 13]. Some references and basic results on the theory of sets of finite perimeter are recalled in the appendix. In case A is a set of finite perimeter, we denote by $\partial^* A$ its *reduced boundary*. Let $\tau_{\min}$ and $\tau_{\max}$ be the infimum and supremum of τ over S^{d-1}. Then $B(\tau_{\min}) \subseteq \mathcal{W}_\tau \subseteq B(\tau_{\max})$ so that for every $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\tau_{\min} \mathcal{P}(A) \leq \mathcal{I}(A) \leq \tau_{\max} \mathcal{P}(A) \tag{53}$$

By Corollary 4.4 and Proposition 4.6, we have $0 < \tau_{\min} \leq \tau_{\max} < \infty$. Thus a set A has finite surface energy if and only if it has finite perimeter. In this case

$$\mathcal{I}(A) = \int_{\partial^* A} \tau(\nu_A(x))\, \mathcal{H}^{d-1}(dx)$$

(see Proposition 6.3 in [14] for a detailed proof). Of crucial importance is the following compactness result.

Compactness of the level sets of $\mathcal{I}$ in Γ. The sets $\{E \in \mathcal{B}(\Gamma) \,|\, \mathcal{I}(E) \leq \lambda\}$, $\lambda > 0$, are compact.

Since $\mathcal{I}$ is lowersemicontinuous, its level sets are closed. By inequality (53), the level set of $\mathcal{I}$ in Γ associated to λ is included in $\{\, E \in \mathcal{B}(\Gamma) : \mathcal{P}(E) \leq \lambda\tau_{\max} \}$ which is compact for the metric dist_{L^1} (see the appendix).

We state next the geometric approximation results needed to prove the lower bound (2) and the local upper bound (5).

The lower bound (2) relies on the possibility to approximate the Wulff crystal by a polyhedral set. A Borel subset of $\mathbb{R}^d$ is polyhedral if its boundary is included in the union of a finite number of hyperplanes.

Lemma 5.1 *For any $\varepsilon > 0$, there exists a polyhedral convex set E such that $\mathcal{W}_\tau \subset E \subset \mathcal{V}_2(\mathcal{W}_\tau, \varepsilon)$ and $\mathcal{I}(E) < \mathcal{I}(\mathcal{W}_\tau) + \varepsilon$.*

Remark: This result is stronger than the general polyhedral approximation result for Caccioppoli sets (see [14], Proposition 6.7). Indeed, we are able to approximate the Wulff crystal from the outside and with respect to the Hausdorff metric, thanks to its convexity.

Proof: We first recall some results from the theory of convex sets (for details, see [57]). Let A be a convex compact set with non–empty interior. A point x of ∂A is called smooth if the supporting hyperplane of A at x is unique. By a classical result due to Reidemeister (see [57], Theorem 2.2.4), the boundary of a convex compact set admits $\mathcal{H}^{d-1}$-almost everywhere a unique supporting hyperplane. In particular, $\mathcal{H}^{d-1}(\partial A \setminus \partial^* A) = 0$. The area measure $S_{d-1}(A, \cdot)$ associated to A is the Borel measure on the sphere S^{d-1} defined by

$$\forall E \in \mathcal{B}(S^{d-1}) \qquad S_{d-1}(A, E) \;=\; \mathcal{H}^{d-1}\Big(\{x \in \partial A \,|\, \nu_A(x) \in E\}\Big)$$

We can express the surface energy $\mathcal{I}(A)$ of A with the help of $S_{d-1}(A, \cdot)$:

$$\mathcal{I}(A) \;=\; \int_{S^{d-1}} \tau(\nu)\, S_{d-1}(A, d\nu)\,.$$

Moreover, if $(A_n)_{n\in\mathbb{N}}$ is a sequence of convex compact sets converging for the Hausdorff metric to a convex compact set A having non–empty interior, then the sequence of the area measures $S_{d-1}(A_n, \cdot)_{n\in\mathbb{N}}$ converges weakly to $S_{d-1}(A, \cdot)$ (see for instance the proof of Theorem 4.2.5 in [57]).

Let $(\nu_n)_{n\in\mathbb{N}}$ be a dense subset of S^{d-1} and let for n in $\mathbb{N}$

$$A_n \;=\; \bigcap_{1\leq i\leq n} \Big\{x \in \mathbb{R}^d \,\Big|\, x \cdot \nu_i \leq \tau(\nu_i)\Big\}$$

Since τ is bounded (by Corollary 4.4), then for n large enough, the set A_n is convex polyhedral and compact. The sequence $(A_n)_{n\in\mathbb{N}}$ is decreasing and converges to $\mathcal{W}_\tau$. Therefore $S_{d-1}(A_n,\cdot)_{n\in\mathbb{N}}$ converges weakly to $S_{d-1}(\mathcal{W}_\tau,\cdot)$. Since the surface tension τ is continuous (Corollary 4.4), the surface energy $\mathcal{I}(A_n)$ converges towards $\mathcal{I}(\mathcal{W}_\tau)$. □

The proof of the local upper bound (5) relies on the following covering lemma.

Lemma 5.2 *Let E be a Borel subset of Γ having finite perimeter. For ε, $\delta > 0$ there exists a finite collection of disjoint balls $B(x_i, r_i)$, $i \in I$, such that for any $i \in I$,*

- *either $x_i \in \partial^* E \cap \operatorname{int}\Gamma$ and $B(x_i, r_i) \subseteq \operatorname{int}\Gamma$*
- *or $x_i \in \partial^* E \cap \partial\Gamma$, $B_-(x_i, r_i, \nu_E(x_i)) \subseteq \Gamma$*

In both cases $\operatorname{dist}_{L^1}\Big(E \cap B(x_i, r_i),\ B_-(x_i, r_i, \nu_E(x_i))\Big) \leq \delta r_i^d$

Moreover

$$\Big|\,\mathcal{I}(E) - \sum_{i\in I} \alpha_{d-1} r_i^{d-1} \tau(\nu_E(x_i))\,\Big| \leq \varepsilon$$

Proof: Let ε, δ be positive with $\varepsilon < 1/2$. By definition of the measure theoretic normal, for any $x \in \partial^* E$ there exists a positive $r_1(x,\delta)$ such that for any $r < r_1(x,\delta)$

$$\operatorname{dist}_{L^1}\Big(E \cap B(x,r),\ B_-(x,r,\nu_E(x))\Big) \leq \delta r^d$$

Since $E \subset \Gamma$, then $\partial^* E \subset \operatorname{int}\Gamma \cup \partial^*\Gamma$. If x belongs to $\partial^* E \cap \partial^*\Gamma$ then $\nu_E(x) = \nu_\Gamma(x)$. The map $x \in \partial^* E \mapsto \nu_E(x)$ is $||\nabla\chi_E||$ measurable and $\tau : S^{d-1} \to \mathbb{R}^+$ is continuous and bounded, thus the map $x \in \partial^* E \mapsto \tau(\nu_E(x))$ is $||\nabla\chi_E||$ measurable and bounded. By the Besicovitch differentiation Theorem (see the appendix), for $\mathcal{H}^{d-1}$ almost every x in $\partial^* E$,

$$\lim_{r\to\infty} \left(\alpha_{d-1} r^{d-1}\right)^{-1} \mathcal{H}^{d-1}(\partial^* E \cap B(x,r)) = 1 \tag{54}$$

$$\lim_{r\to\infty} \left(\alpha_{d-1} r^{d-1}\right)^{-1} \int_{\partial^* E\cap B(x,r)} \tau(\nu_E(y))\mathcal{H}^{d-1}(dy) = \tau(\nu_E(x)) \tag{55}$$

Let $\partial^{**}E$ be the set of the points of $\partial^* E$ where (54) and (55) hold simultaneously. Clearly $\mathcal{H}^{d-1}(\partial^* E \setminus \partial^{**} E) = 0$. For any $x \in \partial^{**} E$, there exists a positive $r_2(x,\varepsilon)$ such that for any $r < r_2(x,\varepsilon)$,

$$\begin{aligned}\Big|\,\mathcal{H}^{d-1}(\partial^* E \cap B(x,r)) - \alpha_{d-1} r^{d-1}\,\Big| &\leq \varepsilon\alpha_{d-1} r^{d-1}\\ \Big|\int_{\partial^* E\cap B(x,r)} \tau(\nu_E(y))\mathcal{H}^{d-1}(dy) - \alpha_{d-1} r^{d-1}\tau(\nu_E(x))\,\Big| &\leq \varepsilon\alpha_{d-1} r^{d-1}\end{aligned}$$

The family of the balls $B(x,r)$, $x \in \partial^{**}E$ and $0 < r < \min\Big(r_1(x,\delta),\ r_2(x,\varepsilon), \mathrm{d}_\infty(x, \partial\Gamma \setminus \partial^*\Gamma)\Big)$ is a Vitali class for $\partial^{**}E$ (where $\partial\Gamma \setminus \partial^*\Gamma$ is the union of the boundaries of the faces of Γ). By the Vitali covering Theorem for $\mathcal{H}^{d-1}$, we can select a countable collection of disjoint balls

$B(x_i,r_i)$, $i\in I$, belonging to this family such that either $\mathcal{H}^{d-1}(\partial^{**}E\setminus\cup_{i\in I}B(x_i,r_i))=0$ or $\sum_{i\in I}r_i^{d-1}=\infty$. By our assumption, $\mathcal{P}(E)=\mathcal{H}^{d-1}(\partial^*E)$ is finite. For each $i\in I$, r_i is smaller than $r_2(x_i,\varepsilon)$, thus

$$(1-\varepsilon)\sum_{i\in I}\alpha_{d-1}r_i^{d-1}\le\mathcal{H}^{d-1}(\partial^{**}E)<\infty$$

Therefore, the first case must occur and we may select a finite subset $J\subseteq I$ such that

$$\mathcal{H}^{d-1}\Big(\partial^{**}E\setminus\bigcup_{i\in J}B(x_i,r_i)\Big)<\varepsilon\mathcal{H}^{d-1}(\partial^{**}E)$$

We claim that the collection of balls indexed by J enjoys the desired properties. Indeed, there is only the last condition to be checked:

$$\begin{aligned}\Big|\mathcal{I}(E)-\sum_{i\in I}\alpha_{d-1}r_i^{d-1}\tau(\nu_E(x_i))\Big| \;&\le\; \int_{\partial^{**}E\setminus\bigcup_{i\in J}B(x_i,r_i)}\tau(\nu_E(y))\mathcal{H}^{d-1}(dy)\\ &\quad+\sum_{i\in J}\Big|\int_{\partial^{**}E\cap B(x_i,r_i)}\tau(\nu_E(y))\mathcal{H}^{d-1}(dy)-\alpha_{d-1}r_i^{d-1}\tau(\nu_E(x_i))\Big|\\ &\le\;\varepsilon\mathcal{H}^{d-1}(\partial^*E)(\tau_{\max}+2)\;\le\;\varepsilon\mathcal{I}(E)(\tau_{\max}+2)/\tau_{\min}\end{aligned}$$

where we have used the inequality (53) in the last step. □

6 The interface lemma

This section is devoted to the proof of a lemma which is crucial for linking the surface tension to the desired large deviation upper bounds. The interface lemma gives a probabilistic estimate for the local presence of a collection of open clusters creating a small flat interface near a middle hyperplane of a ball. Near is understood with respect to the L^1 topology. The estimate is uniform with respect to the location, the size and the direction of the interface. Moreover this lemma suggests an alternative way for defining the surface tension, which is physically more natural.

Notation. Let $B(x,r)$ be a ball in $\Gamma=[-1/2,1/2]^d$, let w belong to S^{d-1}, n to $\mathbb{N}$ and let r,δ be positive. Recall that for a collection $\mathcal{C}$ of sets we denote by $\operatorname{cup}\mathcal{C}$ the union of all the sets belonging to the collection $\mathcal{C}$. If $A\subseteq\mathbb{Z}^d$ we denote its "thickened" version, regarded as a continuum object of $\mathbb{R}^d$, by $\operatorname{cube}A=\cup_{x\in A}\Lambda(x,1)$. Recall that the open $B(nx,nr)$–clusters are the open clusters in the configuration restricted to the ball $B(nx,nr)$. Let $\operatorname{Sep}(n,x,r,w,\delta)$ be the event: there exists a collection $\mathcal{C}$ of open $B(nx,nr)$–clusters such that

$$\operatorname{dist}_{L^1}\Big(\operatorname{cube}\operatorname{cup}\mathcal{C},\,B_-(nx,nr,w)\Big)\le\delta r^dn^d \tag{56}$$

We will work in the unit cube Γ, hence we have to examine the possibility of having an interface intersecting $\partial\Gamma$. Let $B(x,r)$ be a ball such that x belongs to $\partial^*\Gamma$ and $B_-(x,r,\nu_\Gamma(x))$ is included in Γ. Let $n\in\mathbb{N}$ and $\delta>0$. Let $\operatorname{Sep}^{\mathrm{bd}}(n,x,r,\delta)$ be the event: there exists a collection $\mathcal{C}$ of open $B_-(nx,nr,\nu_\Gamma(x))$–clusters such that none of them intersects $\partial^{in}\Lambda(n)$ and the inequality (56) is satisfied with w replaced by $\nu_\Gamma(x)$.

Lemma 6.1 (interface lemma) *Let $d \geq 3$, $q \geq 1$, $p > \widehat{p}_c$, $\theta^f(p) = \theta^w(p)$. There exists a constant $c = c(p,q,d,\zeta)$ such that for every r in $(0,1)$, every ball $B(x,r) \subseteq \Gamma$, every unit vector w in S^{d-1}, and every δ in $(0,\theta/2)$,*

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \max_{\Phi\in c\mathcal{FK}(\Lambda(n),p,q)} \Phi\left[\mathrm{Sep}(n,x,r,w,\delta)\right] \leq -\alpha_{d-1} r^{d-1}\tau(w)\left(1 - c\,\delta^{1/2}\right)$$

Remarks: 1) Note that the event $\mathrm{Sep}(n,x,r,w,\delta)$ depends only on the status of the edges inside $B(nx,nr)$ and it can be seen as an event guaranteeing the existence of a certain set of closed edges inside $B(nx,nr)$, hence it is decreasing.
2) For η sufficiently small (depending on δ), we have $S(n\,\mathrm{disc}\,(x,r,w,\eta)) \subset \mathrm{Sep}(n,x,r,w,\delta)$. Proposition 3.1 and Lemma 4.8 together imply the corresponding lower bound

$$\underline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \min_{\Phi\in c\mathcal{FK}(\Lambda(n),p,q)} \Phi\left[\mathrm{Sep}(n,x,r,w,\delta)\right] \geq -\alpha_{d-1} r^{d-1}\tau(w)$$

3) The event $\mathrm{Sep}^{\mathrm{bd}}(n,x,r,\delta)$ is also decreasing. By the monotonicity of FK measures with respect to boundary conditions,

$$\Phi^w_{\Lambda(n)}\Big[\mathrm{Sep}^{\mathrm{bd}}(n,x,r,\delta)\Big] \leq \Phi_\infty\Big[\mathrm{Sep}^{\mathrm{bd}}(n,x,r,\delta)\Big] \leq \Phi_\infty\Big[\mathrm{Sep}(n,x,r,\nu_\Gamma(x),\delta)\Big]$$

so that the interface lemma yields also an upper bound for $\Phi^w_{\Lambda(n)}[\mathrm{Sep}^{\mathrm{bd}}(n,x,r,\delta)]$. This upper bound turns out to be of the correct order. Indeed, the L^1 constraint is weak and the interface might be created at a positive distance from the boundary: for η small enough (depending on δ),

$$S\Big(n\,\mathrm{disc}\,(x-\eta\nu_\Gamma(x),r,\nu_\Gamma(x)),n\eta\Big) \subset \mathrm{Sep}^{\mathrm{bd}}(n,x,r,\delta)$$

and we get the lower bound with the help of Proposition 3.1 and Lemma 4.8 as in remark 2) above. However, the situation could be radically different with another kind of boundary condition.

Lemma 6.1 and remark 2) suggest the following alternative definition of surface tension.

Corollary 6.2 (alternative definition of surface tension)

Let $d \geq 3$, $q \geq 1$, $p > \widehat{p}_c$, $\theta^f(p) = \theta^w(p)$. Let $\delta(n) : \mathbb{N} \to (0,1)$ be such that $\lim_{n\to\infty}\delta(n) = 0$, $\lim_{n\to\infty} n\delta(n) = \infty$. For $x \in \mathbb{R}^d$ and $w \in S^{d-1}$, the limit

$$\lim_{n\to\infty} -\left(\alpha_{d-1}n^{d-1}\right)^{-1}\log\Phi_\infty[\mathrm{Sep}(n,x,1,w,\delta(n))]$$

exists and depends only on w. It is equal to the surface tension $\tau(w)$.

Remark: The above limit does not depend on x because our model is translation invariant. Yet in a more general model, we might obtain a surface tension depending simultaneously on the direction and a space variable.

Proof: Since the event $\mathrm{Sep}(n,x,1,w,\delta)$ is non-decreasing with respect to δ, then, for any $\delta > 0$ and n large enough so that $\delta(n) < \delta$, we have $\mathrm{Sep}(n,x,1,w,\delta(n)) \subset \mathrm{Sep}(n,x,1,w,\delta)$ whence by lemma 6.1,

$$\overline{\lim_{n\to\infty}} \left(\alpha_{d-1}n^{d-1}\right)^{-1}\log\Phi_\infty[\mathrm{Sep}(n,x,1,w,\delta(n))] \leq -\tau(w)$$

Next, we have

$$S\Big(n\,\mathrm{disc}\,(x,1,w), n\delta(n)/(2\alpha_{d-1})\Big) \subset \mathrm{Sep}(n,x,r,w,\delta(n))$$

Since $\lim_{n\to\infty} n\delta(n)=\infty$, Lemma 4.8 yields

$$\varliminf_{n\to\infty}\,(\alpha_{d-1}n^{d-1})^{-1}\log\Phi_\infty[\mathrm{Sep}(n,x,1,w,\delta(n))]\;\geq\;-\tau(w)\qquad\square$$

An essential ingredient of the proof is Lemma 6.3 which might be of independent interest. This lemma relates the probabilities of events which can be transformed into each other by closing (or opening) certain configuration-dependent edges "by force" − a procedure which we refer to as perturbation or surgery. It can be used for estimating the probability of an event S about which we know that by "changing it a little bit" we get (a subset of) another event whose probability is known or can be controlled. Of course, the change (surgery) will cost some penalty; the corresponding factors can be seen in (57).

Let E be a finite set of edges in a graph and let S be an event defined on these edges. Let $\psi : S \to \Omega_E$ be a map changing certain open edges into closed ones. Assume that the number of changed edges is uniformly bounded by $N(\psi)$. Note that the change in the number of clusters is non negative since we destroy connections, and it is bounded by $N(\psi)$. Let $d(\psi)=\max_{\psi(S)}\left|\psi^{-1}\right|$ be the "degree of degeneracy" (non-injectiveness) of the map ψ.

Lemma 6.3 (monotone perturbation of an event in FK percolation)

Let $q>0$, $p\in(0,1)$ and $\Phi\in\mathcal{FK}(E,p,q)$. Then

$$\Phi\,[S]\leq d(\psi)\,(1\vee\frac{1}{q})^{N(\psi)}\left(1\vee\frac{p}{1-p}\right)^{N(\psi)}\Phi\,[\psi(S)] \tag{57}$$

Let S_j, $j\in J$, be a finite partition of S and for $j\in J$, let $\psi_j=\psi_{\lceil S_j}$ be the restriction of ψ to S_j. Assume that the number of changed edges in the mapping ψ_j is bounded by $N(\psi_j)$. Then

$$\Phi\,[S]\leq |J|\,(\max_{j\in J}\,d(\psi_j))\left((1\vee\frac{1}{q})\,(1\vee\frac{p}{1-p})\right)^{\max_{j\in J}N(\psi_j)}\Phi\,[\psi(S)] \tag{58}$$

Remark: Analogous estimates can be derived for opening edges (instead of closing) as well as for non-monotonic perturbations.

Proof: For $\omega\in\Omega_E$ and $\Phi\in\mathcal{FK}(E,p,q)$, we denote by $cl(\omega)$ the number of clusters in ω counted with respect to the partially wired boundary conditions associated with Φ. Then

$$\begin{aligned}
\Phi\,[S] \;&=\; Z^{-1}\sum_{\omega\in S} q^{cl(\omega)}\Big(\prod_{e\in E}p^{\omega(e)}(1-p)^{1-\omega(e)}\Big)\\
&= Z^{-1}\sum_{\omega'\in\psi(S)}\sum_{\omega\in\psi^{-1}(\omega')} q^{cl(\omega')}\Big(\prod_{e\in E}p^{\omega'(e)}(1-p)^{1-\omega'(e)}\Big)\,(\frac{1}{q})^{cl(\omega')-cl(\omega)}\Big(\prod_{e\in E}(\frac{p}{1-p})^{\omega(e)-\omega'(e)}\Big)\\
&\leq Z^{-1}\sum_{\omega'\in\psi(S)}\sum_{\omega\in\psi^{-1}(\omega')} q^{cl(\omega')}\Big(\prod_{e\in E}p^{\omega'(e)}(1-p)^{1-\omega'(e)}\Big)\,(1\vee\frac{1}{q})^{cl(\omega')-cl(\omega)}\prod_{e\in E}(1\vee\frac{p}{1-p})^{\omega(e)-\omega'(e)}
\end{aligned}$$

$$\leq \sum_{\omega' \in \psi(S)} \sum_{\omega \in \psi^{-1}(\omega')} \Phi[\omega'] (1 \vee \frac{1}{q})^N (1 \vee \frac{p}{1-p})^N$$
$$= (1 \vee \frac{1}{q})^N (1 \vee \frac{p}{1-p})^N \sum_{\omega' \in \psi(S)} \Phi[\omega'] \left|\psi^{-1}(\omega')\right|$$
$$\leq \left(\max_{\psi(S)} \left|\psi^{-1}\right|\right) (1 \vee \frac{1}{q})^N (1 \vee \frac{p}{1-p})^N \Phi[\psi(S)]$$

Inequality (58) is a direct consequence of (57). □

Proof of the interface lemma: The basic idea of the proof is as follows. We will show that whenever the event "Sep" occurs, it is always possible to perform a surgery in the spirit of Lemma 6.3 by closing not too many edges so that in the modified configuration a separating set of closed edges appears near the middle hyperplane $\mathrm{hyp}\,(nx, w)$ of the ball $B(nx, nr)$. Recall that the occurrence of such a separating set of closed edges is directly related to our definition of surface tension. By controlling the penalty factors in (58) we will be able to give an estimate for $\Phi[\mathrm{Sep}]$ in terms of the surface tension $\tau(w)$.

Throughout the proof, we fix x, r, w, δ as in the statement of the interface lemma and we note $\mathrm{Sep}(n) = \mathrm{Sep}(n, x, r, w, \delta)$. By Proposition 3.1, the proof can be carried out for the measure $\Phi^w_{\Lambda(n)}$. Let $f(n)$ be a function from $\mathbb{N}$ to $\mathbb{N}$ such that $n/f(n)^{d-1}$ goes to ∞ as n goes to ∞ and $f(n) \geq \kappa \log n$ for all n, where κ is a sufficiently large constant, so that the estimate (14) holds. We work in the box $\Lambda(n)$ rescaled by a factor $f(n)$ with event-block size 1. For $\underline{x}$ in $\underline{\Lambda}(n)$, let $X(\underline{x})$ be the indicator function of $R(B'(\underline{x}), f(n)-1)^c \cup V(B(\underline{x}), \theta/2)^c$. Suppose that the event $\mathrm{Sep}(n, x, r, w, \delta)$ occurs and let $\mathcal{C}$ be a collection of open $B(nx, nr)$–clusters realizing it. Let ρ, η be such that $0 < \eta < \rho < r$, $0 < 2\eta < \sqrt{r^2 - \rho^2}$ (ρ will be chosen later to be close to r). Set $Z = \mathrm{cyl}\,(n\,\mathrm{disc}\,(x, \rho, w))$. We suppose that n is large enough so that every asymptotic inequality coming up in the proof is fulfilled (like $4\eta n > \zeta > f(n)^{d-1}/n$, where $\zeta > 2d$ is an arbitrary but fixed constant which we used to define the surface tension). It will be useful to define the following subsets of $B(nx, nr)$:

$$\begin{aligned}
D = D(n, x, w, \eta, \zeta, \rho) &= Z \cap \mathrm{slab}\,(nx, w, -n\eta - \zeta, n\eta + \zeta) \\
D^+ &= Z \cap \mathrm{slab}\,(nx, w, 1, n\eta + \zeta) \\
D^- &= Z \cap \mathrm{slab}\,(nx, w, -n\eta - \zeta, 0) \\
\partial^+ D &= Z \cap \mathrm{slab}\,(nx + n\eta w, w, -\zeta, \zeta) \\
\partial^- D &= Z \cap \mathrm{slab}\,(nx - n\eta w, w, -\zeta, \zeta) \\
\partial^- D^+ &= Z \cap \mathrm{slab}\,(nx, w, 1, 1 + \zeta) \\
\partial^+ D^- &= Z \cap \mathrm{slab}\,(nx, w, -\zeta, 0)
\end{aligned}$$

Let C be an open D–cluster joining the sets $\partial^+ D$ and $\partial^- D$. There are two possibilities: Either C is contained in some cluster of $\mathcal{C}$ or it is contained in its complement $D \setminus \mathcal{C}$. In the first case there exists an open D^+-cluster C' connecting $\partial^- D^+$ and $\partial^+ D$ such that $C' \subset C$ and therefore

$$C' \subset (\mathrm{cube\ cup}\,\mathcal{C}) \setminus B_-(nx, nr, w)$$

In the second case there exists an open D^--cluster C' connecting $\partial^- D$ and $\partial^+ D^-$ such that $C' \subset C$ and therefore

$$C' \subset B_-(nx, nr, w) \setminus \mathrm{cube\ cup}\,\mathcal{C}$$

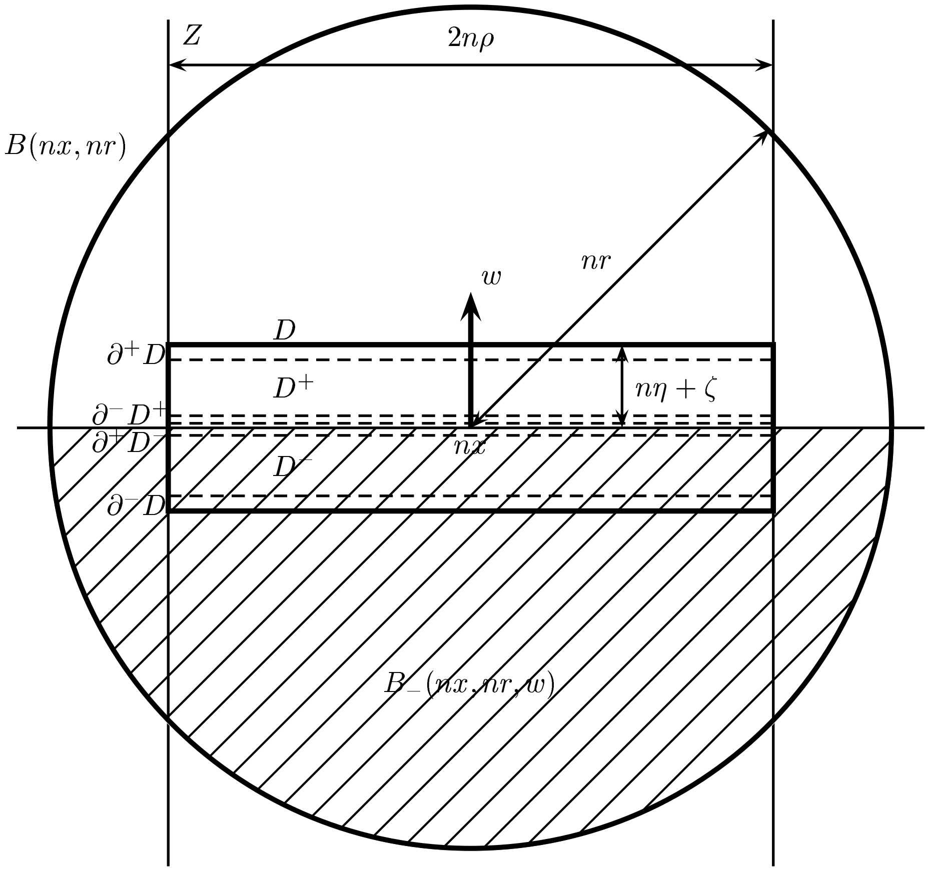

figure 4

Let F be the union of all such clusters C'. Then, on $\mathrm{Sep}(n)$,

$$|F| \ \leq\ \mathrm{dist}_{L^1}\Big(\mathrm{cube\ cup}\,\mathcal{C}, B_-(nx,nr,w)\Big) \ \leq\ \delta r^d n^d \tag{59}$$

We define

$$\underline{F} = \Big\{ \underline{x} \in \underline{\Lambda}(n) \,\Big|\, B(\underline{x}) \cap F \neq \emptyset,\, B'(\underline{x}) \subset D^- \cup D^+ \Big\}$$

Let $\underline{x}$ be a regular block in $\underline{F}$, i.e., $X(\underline{x}) = 0$. Since F intersects $B(\underline{x})$ and the diameter of any open cluster contained in F is larger than $f(n)$, the box $B'(\underline{x})$ contains an open path of diameter larger than $f(n)-1$ included in F. The occurrence of the event $R(B'(\underline{x}), f(n)-1)$ implies that this open path is contained in the crossing cluster $C^*(B'(\underline{x}))$, which in turn implies that the latter cluster is contained in F. Similarly, the crossing cluster $C^*(B(\underline{x}))$ associated with the event $U(B(\underline{x}))$ has diameter $f(n)-1$ and it is thus contained in $C^*(B'(\underline{x}))$. Since the event $V(B(\underline{x}), \theta/2)$ occurs as well, then

$$|F \cap B(\underline{x})| \ \geq\ |C^*(B(\underline{x}))| \ \geq\ \theta f(n)^d/2\,.$$

Summing over $\underline{x}$ yields

$$|F| \ \geq \sum_{\underline{x}\in\underline{F},\, X(\underline{x})=0} |F \cap B(\underline{x})| \ \geq\ (\theta/2) f(n)^d \,\Big|\Big\{ \underline{x} \in \underline{F} \,\Big|\, X(\underline{x}) = 0 \Big\}\Big|$$

By (59), on $\mathrm{Sep}(n)$, we can bound the number of the regular blocks in $\underline{F}$

$$\Big|\Big\{\underline{x}\in\underline{F}\,\Big|\,X(\underline{x})=0\Big\}\Big|<\frac{2\delta}{\theta}\Big(\frac{rn}{f(n)}\Big)^d$$

On the other hand, by the estimate (14), the block process $X(\underline{x})$ satisfies (15) with $\varepsilon = b\exp(-cf(n))$. Applying (18) with $\delta = an^{d-1}(|\underline{\Lambda}(n)|f(n))^{-1}$, where $a>0$, we obtain

$$\begin{aligned}\Phi^w_{\Lambda(n)}\Big[\sum_{\underline{x}\in\underline{F}}X(\underline{x})\geq a\frac{n^{d-1}}{f(n)}\Big] &\leq \Phi^w_{\Lambda(n)}\Big[\frac{1}{|\underline{\Lambda}(n)|}\sum_{\underline{x}\in\underline{\Lambda}(n)}X(\underline{x})\geq\frac{an^{d-1}}{|\underline{\Lambda}(n)|f(n)}\Big]\\ &\leq 2^d\exp\Big(-2^{-d}\Lambda^*_\varepsilon\Big(\frac{an^{d-1}}{|\underline{\Lambda}(n)|f(n)}\Big)\Big\lfloor\frac{n}{f(n)}\Big\rfloor^d\Big)\leq b\exp(-can^{d-1})\end{aligned}$$

where b,c are two positive constants. By setting

$$\mathrm{Sep}^*(n)=\mathrm{Sep}(n)\cap\Big\{|\underline{F}|\leq a\frac{n^{d-1}}{f(n)}+\frac{2\delta}{\theta}\Big(\frac{rn}{f(n)}\Big)^d\Big\}$$

and using the previous estimates, we see that

$$\Phi^w_{\Lambda(n)}[\mathrm{Sep}(n)]\leq b\exp(-can^{d-1})\;+\;\Phi^w_{\Lambda(n)}[\mathrm{Sep}^*(n)] \tag{60}$$

For i in $\mathbb{Z}$, we define the slab-like sets $\underline{H}(i)=\{\underline{y}\in\underline{D}\,|\,if(n)\leq w\cdot(f(n)\underline{y}-nx)<(i+1)f(n)\}$. The sets $\underline{H}(i)$, $i\in\mathbb{Z}$, are pairwise disjoint. Hence for any subset I of $\mathbb{Z}$, the sum $\sum_{i\in I}|\underline{F}\cap\underline{H}(i)|$ is less than $|\underline{F}|$ and there exists i in I such that $|\underline{F}\cap\underline{H}(i)|\leq|\underline{F}|/|I|$. Applying the preceding remark to the sets $(-\eta n/f(n)+3,\,-3)\cap\mathbb{Z}$ and $(3,\,\eta n/f(n)-3)\cap\mathbb{Z}$, whose cardinalities are larger than $\eta n/f(n)-7>\eta n/(2f(n))$, we find that there exist two random indices I_- and I_+ in $\mathbb{Z}$ such that $-\eta n/f(n)+3<I_-<-3$, $3<I_+<\eta n/f(n)-3$ and both $\underline{F}\cap\underline{H}(I_-)$, $\underline{F}\cap\underline{H}(I_+)$ have a cardinality less than $2f(n)(\eta n)^{-1}|\underline{F}|$. We choose the indices I_- and I_+ with the smallest possible absolute value. We decompose the event $\mathrm{Sep}^*(n)$ according to the values of I_-, I_+ and the sets $\underline{F}\cap\underline{H}(I_-)$, $\underline{F}\cap\underline{H}(I_+)$. For

$$-\eta n/f(n)+3<i_-<-3,\quad 3<i_+<\eta n/f(n)-3,\quad T_\pm\subseteq\underline{H}(i_\pm)\text{ with }|T_\pm|\leq\frac{2f(n)}{\eta n}|\underline{F}|$$

we define the event

$$G(n,i_\pm,T_\pm)=\mathrm{Sep}^*(n)\cap\{I_\pm=i_\pm,\,\underline{F}\cap\underline{H}(I_\pm)=T_\pm\}$$

Notice that the events $G(n,i_\pm,T_\pm)$ corresponding to different values of i_-,i_+,T_-,T_+ are disjoint. Hence these events partition $\mathrm{Sep}^*(n)$. Using Lemma 6.3, we now derive an estimate for $\Phi^w_{\Lambda(n)}[\mathrm{Sep}^*(n)]$. Let $\psi:\mathrm{Sep}^*(n)\to\Omega_{\Lambda(n)}$ be the map defined on $G(n,i_\pm,T_\pm)$ by

$$\psi(\omega)[e]=\begin{cases}0 & ;\;\text{ if } e\in\bigcup_{\underline{x}\in T_+\cup T_-}\partial^{edge}B(\underline{x})\\ \omega(e) & ;\;\text{ otherwise}\end{cases}$$

Assume $\omega\in G(n,i_\pm,T_\pm)$. By the definition of F, each open path in D connecting ∂^+D and ∂^-D intersects either $\bigcup_{\underline{x}\in T_+}B(\underline{x})$ or $\bigcup_{\underline{x}\in T_-}B(\underline{x})$ depending on whether it belongs to $\mathcal{C}$ or not. Therefore, when we close all the edges in $\bigcup_{\underline{x}\in T_+\cup T_-}\partial^{edge}B(\underline{x})$, we destroy every open connection between $\partial^+ D$ and $\partial^- D$. Hence the map ψ has values in the set of configurations $S(n\,\mathrm{disc}\,(x,\rho,w),\, n\eta)$, cf. section 4 p. 33 for its definition. To apply (58), we have to estimate the penalty factors. We first observe that there exists a dimension dependent constant $c' = c'(d)$ such that for all $i \in \mathbb{Z}$

$$|\underline{H}(i)| \;\le\; c'\Big(\frac{nr}{f(n)}\Big)^{d-1}$$

The first penalty factor in (58), $|J|$, is easily seen to be bounded by

$$|J| \le \Big(\frac{\eta n}{f(n)}\Big)^2 \exp\Big((\max_{i_+}|\underline{H}(i_+)| + \max_{i_-}|\underline{H}(i_-)|)\log 2\Big) \le \Big(\frac{\eta n}{f(n)}\Big)^2 \exp\Big(2c'\Big(\frac{nr}{f(n)}\Big)^{d-1}\Big) \quad (61)$$

To get a bound on the maximal possible number N of edges to close in the surgery, we observe that

$$\begin{aligned}\Big(\max_{\underline{x}\in T_+\cup T_-}|\partial^{edge}B(\underline{x})|\Big)\times|T_+\cup T_-| \;&\le\; 2d^2\, f(n)^{d-1}\, 2\,\frac{2f(n)}{\eta n}\Big(a\frac{n^{d-1}}{f(n)} + \frac{2\delta}{\theta}\Big(\frac{rn}{f(n)}\Big)^d\Big)\\ &=\; \frac{8d^2}{\eta}\Big(a\frac{f(n)}{n}^{d-1} + \frac{2\delta r^d}{\theta}\Big)n^{d-1} = N \qquad (62)\end{aligned}$$

The degree of degeneracy $d(\psi_j)$ in our case is simply bounded by 2^N (since on $G(n, i_\pm, T_\pm)$ the location of all the changes is known). Note that all these estimates are uniform in i_+, i_-, T_+, T_-. Therefore, by (58),

$$\Phi^w_{\Lambda(n)}[\mathrm{Sep}^*(n)] \le |J|\Big(\frac{2p}{1-p}\Big)^N \Phi^w_{\Lambda(n)}[\psi(\mathrm{Sep}^*(n))] \le |J|\Big(\frac{2p}{1-p}\Big)^N \Phi^w_{\Lambda(n)}\Big[S(n\,\mathrm{disc}\,(x,\rho,w), n\eta)\Big]$$

Coming back to (60), we get

$$\Phi^w_{\Lambda(n)}[\mathrm{Sep}(n,x,r,w,\delta)] \;\le\; b\exp(-can^{d-1}) + |J|\Big(\frac{2p}{1-p}\Big)^N \Phi^w_{\Lambda(n)}\Big[S(n\,\mathrm{disc}\,(x,\rho,w), n\eta)\Big]$$

Letting n go to ∞ and using (61), (62) and Corollary 4.11, we get

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\,\frac{1}{n^{d-1}}\log\Phi^w_{\Lambda(n)}[\mathrm{Sep}(n,x,r,w,\delta)] \le\\ &\qquad -\min\Big\{\, ca,\ \alpha_{d-1}\rho^{d-1}\tau(w) - c\,\eta\rho^{d-2} + \frac{16d^2r^d}{\theta\eta}\,\delta\log\frac{2p}{1-p}\,\Big\}\end{aligned}$$

where $c = c(p,d,\zeta)$. We choose now $\eta = \sqrt{\delta}r/3$, $\rho = r\sqrt{1-\delta}$ and we let a go to ∞ in the preceding inequality. Because τ is bounded away from 0 (Proposition 4.6), there exists a constant $c'' = c''(p,q,d,\zeta)$ such that

$$\overline{\lim_{n\to\infty}}\,\frac{1}{n^{d-1}}\log\Phi^w_{\Lambda(n)}[\mathrm{Sep}(n,x,r,w,\delta)] \le -\alpha_{d-1}r^{d-1}\tau(w)(1-c''\delta^{1/2})$$

This inequality holds for every ball $B(x,r)\subseteq\Gamma$, δ in $(0,\theta/2)$ and w in S^{d-1}. $\square$

7 Proofs of the main theorems.

7.1 Asymptotics of the magnetization

We consider an edge–spin configuration (ω, σ) in $\Lambda(n)$ drawn under the FK–Ising coupling measure $\mathbb{P}^+_{\Lambda(n)}$. We analyze the behavior of the magnetization of the Ising model in the box $\Lambda(n)$ via the FK representation. The magnetization $n^{-d}\sum_{x\in\Lambda(n)}\sigma(x)$ of the spin configuration σ is related to the open clusters of the FK configuration ω through the formula

$$\frac{1}{n^d}\sum_{x\in\Lambda(n)}\sigma(x) = \frac{1}{n^d}\sum_{C\in\mathcal{C}}\sigma(C)\,|C|$$

where $\mathcal{C}$ is the collection of the open clusters in $\Lambda(n)$ and $\sigma(C)$ is the color associated to the cluster C. We first show that the relevant information is carried by the large (macroscopic) clusters of $\mathcal{C}$. Throughout the proof, we use a fixed function $f:\mathbb{N}\to\mathbb{N}$ such that both $n/f(n)^{d-1}$ and $f(n)/\log n$ tend to ∞ as $n\to\infty$. A cluster in $\Lambda(n)$ is called *small* if its diameter is strictly smaller than $f(n)$ and *large* otherwise. The next lemma controls the coloring of the small clusters.

Lemma 7.1 *For any $\delta > 0$,*

$$\overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log\mathbb{P}^+_{\Lambda(n)}\Big[\,\Big|\frac{1}{n^d}\sum_{\substack{C\in\mathcal{C}\\ C\ \text{small}}}\sigma(C)|C|\Big|\geq\delta\Big] = -\infty$$

Proof: For later use, we will prove the statement with respect to a measure $\mathbb{P}_{\Lambda(n)}$ describing a coupling between an arbitrary FK measure $\Phi\in c\mathcal{FK}(\Lambda(n))$ and a coloring measure with the property that the colors of the clusters not touching the boundary are i.i.d. ± 1 with probability $1/2$ each. (Clearly, $\mathbb{P}^+_{\Lambda(n)}$ is such a measure). An analogous statement has already been proved in the proof of Theorem 1.1 [55], hence we will merely sketch the argument. Set

$$\mathcal{C}' = \{C\in\mathcal{C}\,|\,C \text{ is small and } C\cap\Lambda(n-2f(n))\neq\emptyset\}$$

Let n be so big that $2dn^{d-1}\,2f(n) < \delta n^d$. Then

$$\Big\{\Big|\sum_{\substack{C\in\mathcal{C}\\ C\,\text{small}}}\sigma(C)|C|\Big|\geq 2\delta n^d\Big\}\subseteq\Big\{\Big|\sum_{C\in\mathcal{C}'}\sigma(C)|C|\Big|\geq\delta n^d\Big\}$$

There exists $\alpha=\alpha(p,d)>0$ and $c_i=c_i(p,d)>0$, $i=1,2$, such that

$$\mathbb{P}_{\Lambda(n)}[|\mathcal{C}'|<\alpha n^d]\leq c_1\exp(-c_2 n^d)$$

Note that already the number of the isolated clusters $C\in\mathcal{C}'$, $|C|=1$, is proportional to n^d, up to volume order large deviations: to prove this, one can use for instance the fact that the FK measure is stochastically dominated by the Bernoulli percolation. We next condition on the edge configuration ω observed on the FK level and assume that $\omega\in\{\,|\mathcal{C}'|\geq\alpha n^d\}$. The coloring variables of the clusters of $\mathcal{C}'$ are i.i.d. ± 1 valued variables with zero mean (since the clusters of $\mathcal{C}'$ can not touch the boundary of $\Lambda(n)$). Let X denote a ± 1 valued variable with zero mean. By Lemma 5.3 in [55],

$$\begin{aligned}
\mathbb{P}_{\Lambda(n)}\Big[\Big|\sum_{C\in\mathcal{C}'}\sigma(C)|C|\Big|\geq \delta n^d\Big|\omega\Big] &\leq \mathbb{P}_{\Lambda(n)}\Big[\Big|\frac{1}{|\mathcal{C}'|}\sum_{C\in\mathcal{C}'}\sigma(C)|C|\Big|\geq \delta/\alpha\Big|\omega\Big]\\
&\leq 2\exp\Big(-|\mathcal{C}'|\,\Lambda^*_{f(n)X}(\delta/\alpha)\Big)\leq 2\exp\Big(-\frac{c_3\delta^2 n^d}{\alpha f(n)^2}\Big)
\end{aligned}$$

where we have used the fact that the Legendre-transform $\Lambda^*_X(x)$ of the log-moment generating function of X satisfies $\Lambda^*_X(x)\geq c_3x^2$ for an appropriate $c_3>0$. Since the previous estimate is uniform on the set $\{|\mathcal{C}'|\geq \alpha n^d\}$, the claim follows. Let us finally remark that the estimate is not optimal; the correct order of decay is $\sim\exp(-cn^d)$ and it could be proved with slightly more effort. $\square$

We next treat the large clusters. For $*=-$ or $+$, let $\mathcal{C}^*$ be the collection of the *large* clusters colored with $*$ and $\mathcal{M}^*=\mathrm{cup}\,\mathcal{C}^*$. We divide the continuum box $n\Gamma$ into three disjoint random sets

$$\begin{aligned}
M^- &= \Big\{x\in n\Gamma\,\Big|\,\mathrm{d}_\infty(x,\mathcal{M}^-)\leq f(n),\ \mathrm{d}_\infty(x,\mathcal{M}^+)>f(n)\Big\}\\
M^+ &= \Big\{x\in n\Gamma\,\Big|\,\mathrm{d}_\infty(x,\mathcal{M}^-)> f(n),\ \mathrm{d}_\infty(x,\mathcal{M}^+)\leq f(n)\Big\}
\end{aligned}$$

and $M^0=n\Gamma\setminus(M^-\cup M^+)$. Note that the region M^- consists of the points of $n\Gamma$ whose $f(n)$ neighborhood intersects only large clusters with negative color, and at least one such cluster.

Lemma 7.2 *For any $\delta>0$,*

$$\overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log\mathbb{P}^+_{\Lambda(n)}\Big[\Big||\mathcal{M}^-|-\theta\,\mathrm{vol}\,M^-\Big|+\mathrm{vol}\,M^0+\Big||\mathcal{M}^+|-\theta\,\mathrm{vol}\,M^+\Big|>\delta n^d\Big]=-\infty$$

Proof: We work with the box $\Lambda(n)$ rescaled by a factor $f(n)$ with event-block size 2. For $\underline{x}\in\underline{\Lambda}(n)$, the block variable $X(\underline{x})$ is the indicator function of the event $T(B'(\underline{x}),f(n)-1,\delta/4)^c$. Suppose that $X(\underline{x})=0$ for some $\underline{x}\in\underline{\Lambda}(n)$, i.e., the block is regular. Then there exists a unique large cluster C intersecting the block $B(\underline{x})$ and the block $B(\underline{x})$ is included in $M^{\sigma(C)}$. Moreover, for n large enough,

$$\Big||C\cap B(\underline{x})|-|C^*(B(\underline{x}))|\Big|\leq\Big||C\cap B(\underline{x})|-\theta|B(\underline{x})|\Big|+(\delta/4)|B(\underline{x})|\leq(\delta/2)f(n)^d$$

Therefore,

$$\Big||\mathcal{M}^-|-\theta\,\mathrm{vol}\,M^-\Big|+\mathrm{vol}\,M^0+\Big||\mathcal{M}^+|-\theta\,\mathrm{vol}\,M^+\Big|\leq\delta n^d/2+f(n)^d\sum_{\underline{x}\in\underline{\Lambda}(n)}X(\underline{x})$$

By the estimate (14), the block process $X(\underline{x})$ satisfies (15) with $\varepsilon=b\exp(-cf(n))$. Applying (18), we obtain

$$\overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log\mathbb{P}^+_{\Lambda(n)}\Big[f(n)^d\sum_{\underline{x}\in\underline{\Lambda}(n)}X(\underline{x})>\delta n^d/2\Big]=-\infty$$

and the proof is completed. $\square$

Lemmas 7.1, 7.2 yield the following result.

Corollary 7.3 *For $\delta > 0$,*

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mathbb{P}^+_{\Lambda(n)}\Big[\Big| \frac{1}{n^d} \sum_{x\in\Lambda(n)} \sigma(x) - \theta\Big(1 - 2\,\mathrm{vol}\,(M^-/n)\Big) \Big| > \delta \Big] = -\infty$$

Thus the magnetization $n^{-d}\sum_{x\in\Lambda(n)}\sigma(x)$ and $\theta(1-2\,\mathrm{vol}\,(M^-/n))$ are exponentially contiguous which implies (1). To derive the large deviation statement for the magnetization we need only to consider the random variable $\mathrm{vol}\,(M^-/n)$. We prove successively the lower bound (2) and the upper bound (5).

Proof of the lower bound. We first prove (2).

Proposition 7.4 *Let $\lambda > 0$ be such that $\lambda\,\mathrm{diam}\,\mathcal{W}_\tau < 1$ and let x be such that $x + \lambda\mathcal{W}_\tau$ is included in* $\mathrm{int}\,\Gamma$. *For any $\delta > 0$,*

$$\underline{\lim}_{n\to\infty} \frac{1}{n^{d-1}} \log \mathbb{P}^+_{\Lambda(n)}\Big[\mathrm{dist}_{L^1}(M^-/n, x + \lambda\mathcal{W}_\tau) < \delta \Big] \geq -\lambda^{(d-1)}\,\mathcal{I}(\mathcal{W}_\tau)$$

Proof: By Lemma 5.1, for any $\delta, \delta' > 0$, there exists a polyhedral set E such that

$$\mathrm{dist}_{L^1}(E, x+\lambda\mathcal{W}_\tau) < \delta/3, \quad x+\lambda\mathcal{W}_\tau \subset E \subset \mathcal{V}_2(x+\lambda\mathcal{W}_\tau, \delta), \quad \mathcal{I}(E) < \lambda^{(d-1)}\mathcal{I}(\mathcal{W}_\tau) + \delta'$$

In particular, since $x+\lambda\mathcal{W}_\tau$ is included in $\mathrm{int}\,\Gamma$, then for δ sufficiently small, E is also included in the interior of Γ. Let F_i, $i \in I$, be the relative interiors (w.r.t. the $d-1$-dimensional topology) of the faces of E. These are polyhedral open hypersets of $\mathbb{R}^d$. Let U be an open connected subset of E such that $\mathrm{dist}_{L^1}(U, E) < \delta/3$ and $\partial U \cap \partial E = \emptyset$. Let V be an open connected subset of $\Gamma\setminus E$ such that $\mathrm{dist}_{L^1}(V, \Gamma\setminus E) < \delta/3$ and $\partial V \cap \partial(\Gamma\setminus E) = \emptyset$. Pick $\varepsilon > 0$ such that $4\varepsilon < \mathrm{d}_\infty(U, \partial E) \wedge \mathrm{d}_\infty(V, \partial(\Gamma\setminus E))$. We cover the compact set $\mathrm{clo}\,(U\cup V)$ by a finite collection of boxes $\Lambda(x_j, \varepsilon)$, $j \in J$, included in $\mathrm{int}\,\Gamma$ and centered in $U \cup V$. Let n be large enough so that $f(n) < \varepsilon n$. Let $\zeta > 2d$ be fixed and consider the following events:

$$\begin{aligned} W(n) &= \bigcap_{i\in I} S(nF_i, f(n)) \cap \Big\{ \text{all the edges in } \mathcal{V}_2(\,\mathrm{cyl}\, n\partial F_i, \zeta) \cap \mathcal{V}_2(\,\mathrm{hyp}\, nF_i, f(n)) \text{ are closed} \Big\} \\ R(n,\varepsilon) &= \bigcap_{j\in J} O(n\Lambda(x_j, 2\varepsilon), f(n)) \end{aligned}$$

In words, we have surfaces of closed edges near the faces of E, and typical configurations inside the regions U and V. Whenever $W(n)$ and $R(n,\varepsilon)$ occur, there exist two distinct large clusters C_1 and C_2 such that

$$\begin{gathered} nU \subseteq \mathcal{V}_\infty(C_1, f(n)) \subseteq \mathcal{V}_\infty(nE, f(n)) \\ nV \subseteq \mathcal{V}_\infty(C_2, f(n)) \subseteq \mathcal{V}_\infty(n(\Gamma\setminus E), f(n)) \end{gathered}$$

and no other large cluster intersects the region $n\mathcal{V}_\infty(U,\varepsilon) \cup n\mathcal{V}_\infty(V,\varepsilon)$. If C_1 is colored with $-$ and C_2 with $+$ then the region M^- will satisfy $nU \subseteq M^- \subseteq n(\Gamma\setminus V)$. Therefore,

$$\begin{aligned} (1/4)\mathbb{P}^+_{\Lambda(n)}\Big[W(n)\cap R(n,\varepsilon)\Big] &= \mathbb{P}^+_{\Lambda(n)}\Big[W(n)\cap R(n,\varepsilon)\cap\{\sigma(C_1) = -,\, \sigma(C_2) = +\}\Big] \\ &\leq \mathbb{P}^+_{\Lambda(n)}\Big[\mathrm{dist}_{L^1}(M^-/n, x+\lambda\mathcal{W}_\tau) < \delta\Big] \end{aligned}$$

The event $R(n,\varepsilon)$ depends only on the edges inside $n\mathcal{V}_\infty(U\cup V,2\varepsilon)$, while the event $W(n)$ depends on the edges in $n\mathcal{V}_\infty(\partial E,\varepsilon)$. By the estimate (14), for j in J, the probability of each event $O(n\Lambda(x_j,2\varepsilon),f(n))$ goes to 1, uniformly over the b.c.s on $n\Lambda(x_j,2\varepsilon)$. Therefore, $\lim_{n\to\infty}\mathbb{P}^+_{\Lambda(n)}[R(n,\varepsilon)\,|\,W(n)]=1$ and by the FKG inequality and Lemma 4.8,

$$\varliminf_{n\to\infty}\frac{1}{n^{d-1}}\log\mathbb{P}^+_{\Lambda(n)}\Big[W(n)\Big]\geq-\sum_{i\in I}\mathcal{H}^{d-1}(F_i)\tau(\operatorname{nor}F_i)=-\mathcal{I}(E)\geq-\lambda^{(d-1)}\mathcal{I}(\mathcal{W}_\tau)-\delta'$$

Thus, for any $\delta,\delta'>0$ sufficiently small,

$$\varliminf_{n\to\infty}\frac{1}{n^{d-1}}\log\mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{dist}_{L^1}(M^-/n,x+\lambda\mathcal{W}_\tau)<\delta\Big]\geq-\lambda^{(d-1)}\,\mathcal{I}(\mathcal{W}_\tau)-\delta'$$

The result of the Proposition follows by letting δ' go to 0. □

We complete now the proof of the lower bound for Theorem 1.1. Let m as in Theorem 1.1 and let $\delta>0$ such that $1-2(\operatorname{diam}\mathcal{W}_\tau)^{-d}\operatorname{vol}\mathcal{W}_\tau<(m-2\delta)/m^*$. By applying successively Corollary 7.3 and Proposition 7.4, we get

$$\begin{aligned}
&\varliminf_{n\to\infty}\frac{1}{n^{d-1}}\log\mathbb{P}^+_{\Lambda(n)}\Big[\,\frac{1}{n^d}\sum_{x\in\Lambda(n)}\sigma(x)\leq m\,\Big]\\
&\qquad\geq\varliminf_{n\to\infty}\frac{1}{n^{d-1}}\log\mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{vol}(M^-/n)\geq\frac{m^*+\delta-m}{2m^*}\Big]\\
&\qquad\geq\varliminf_{n\to\infty}\frac{1}{n^{d-1}}\log\mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{dist}_{L^1}\Big(M^-/n,\Big(\frac{m^*+2\delta-m}{2m^*\operatorname{vol}\mathcal{W}_\tau}\Big)^{1/d}\mathcal{W}_\tau\Big)<\frac{\delta}{m^*}\Big]\\
&\qquad\geq-\Big(\frac{m^*+2\delta-m}{2m^*\operatorname{vol}\mathcal{W}_\tau}\Big)^{\frac{d-1}{d}}\mathcal{I}(\mathcal{W}_\tau)
\end{aligned}$$

Letting δ go to 0, we obtain the correct lower bound for Theorem 1.1. This lower bound is certainly not correct for m such that $1-2(\operatorname{diam}\mathcal{W}_\tau)^{-d}\operatorname{vol}\mathcal{W}_\tau>m/m^*$ (see [59] for the corresponding question in $d=2$).

Proof of the upper bound. We build two auxiliary sets fifa M^- and aglu M^-. Both fifa M^- and aglu M^- are exponentially contiguous to M^-. Fifa M^- helps to prove the exponential tightness (4), while aglu M^- is used to prove the local estimate (5).

The set fifa M^-. This set is a coarser version of M^- obtained through the succession of a filling and a fattening operation, hence the name fifa. Moreover France has won the World Cup '98.[5] We work with the box $\Lambda(n)$ rescaled by a factor k with event-block size 1 (k is a fixed integer larger than $2d$). The block variable $X(\underline{x})$ is the indicator function of the event $O(B'(\underline{x}),k-1)^c$. We introduce a coarse grained image of the set $\mathcal{M}^-$ as follows:

$$\underline{\mathcal{M}}^-=\Big\{\underline{x}\in\underline{\Lambda}(n)\,\Big|\,B(\underline{x})\cap\ \mathcal{M}^-\neq\emptyset\Big\}$$

This set, in general, contains lots of large and small holes. To get rid of the small ones we first fill out these holes by the operation fill which we now describe. We look at the residual components of $\underline{\mathcal{M}}^-$, more precisely the $\mathbb{L}^{d,\infty}$-connected components of $\mathbb{Z}^d\setminus\underline{\mathcal{M}}^-$. We define

$$\operatorname{fill}\underline{\mathcal{M}}^-=\underline{\mathcal{M}}^-\cup\Big\{\underline{R}\,\Big|\,\underline{R}\text{ is a finite residual component of }\underline{\mathcal{M}}^-,\ |\partial^{edge}\underline{R}|\leq f(n)/k\Big\}$$

[5] But Brazil was very weak in the final.

Notice that any $\mathbb{L}^{d,\infty}$-connected component of $\partial^{in}\mathrm{fill}\,\underline{\mathcal{M}}^-$ has cardinality strictly larger than $f(n)/(2dk)$. This is clearly true, by construction, for the parts of $\partial^{in}\mathrm{fill}\,\underline{\mathcal{M}}^-$ associated with finite residual components (holes). Let $\underline{A}$ be an $\mathbb{L}^d$-connected component of $\underline{\mathcal{M}}^-$. Since we deal with large clusters, $\mathrm{diam}\,\underline{A} \geq f(n)/k - 1$ and $|\partial^{ext,\,edge}\underline{A}| \geq \mathrm{diam}\,\underline{A} + 2 > f(n)/k$. This in turn implies that $|\partial^{ext,\,in}\underline{A}| > f(n)/(2dk)$. Note finally that every $\mathbb{L}^{d,\infty}$-connected piece of $\partial^{in}\mathrm{fill}\,\underline{\mathcal{M}}^-$ consists of such boundary pieces, which implies the claim.

We next claim that if $\underline{x}$ belongs to $\partial^{in}\mathrm{fill}\,\underline{\mathcal{M}}^- \subseteq \partial^{in}\underline{\mathcal{M}}^-$ then $X(\underline{x}) = 1$. To see this we first observe that $\underline{x} \in \partial^{in}\underline{\Lambda}(n) \cap \partial^{in}\underline{\mathcal{M}}^-$ implies that the event $O(B'(\underline{x}),\, k-1)$ does not happen, otherwise the crossing cluster of $B'(\underline{x})$ would be colored $+$ and the block of $\underline{x}$ would not intersect $\mathcal{M}^-$. If, on the other hand, $\underline{x} \in \partial^{in}\underline{\mathcal{M}}^- \setminus \partial^{in}\underline{\Lambda}(n)$ then it has a nearest neighbor $\underline{y}$ in $\underline{\Lambda}(n)$ such that $B(\underline{y}) \cap \mathcal{M}^- = \emptyset$, hence the event $O(B'(\underline{x}), k-1)$ does not happen either.

Finally, we define the fattened region

$$\mathrm{fifa}\,M^- = \bigcup_{\underline{x}\in \mathrm{fill}\,\underline{\mathcal{M}}^-} B(\underline{x})$$

Lemma 7.5 *For k large enough, depending on p and d, there exist $b, c > 0$ such that*

$$\forall s > 0 \qquad \mathbb{P}^+_{\Lambda(n)}\Big[\mathcal{P}(\mathrm{fifa}\,M^-) > s\Big] \leq b\,\exp(-cs)$$

Proof: We have the following bound on the perimeter of fifa M^-:

$$\mathcal{P}(\mathrm{fifa}\,M^-) \quad\leq\quad 2dk^{d-1}|\partial^{in}\mathrm{fill}\,\underline{\mathcal{M}}^-| \leq 2dk^{d-1}\Big|\Big\{\underline{x} \in \underline{\Lambda}(n)\,\Big|\, |\underline{C}(\underline{x})| \geq f(n)/(2dk)\Big\}\Big|$$

By the estimate (14), the block process $X(\underline{x})$ satisfies (15) with $\varepsilon = b\exp(-ck)$. The claim of the Lemma follows from Lemma 2.3 applied with $t = f(n)/(2dk)$. $\square$

Lemma 7.6 *For n large enough,*

$$\mathrm{dist}_{L^1}(M^-,\ \mathrm{fifa}\,M^-) \leq\ \mathrm{vol}\,M^0 + 5^d f(n)^{d-1}\mathcal{P}(\mathrm{fifa}\,M^-)$$

Proof: We estimate the distance between M^- and fifa M^-. In the filling operation performed to build fifa M^-, all the holes which are filled on the rescaled lattice have a diameter not exceeding $f(n)/((2d-2)k)$. Therefore, for n large enough, so that $10k < f(n)$,

$$\mathrm{fifa}\,M^- \subseteq \Big\{x \in n\Gamma \,\Big|\, \mathrm{d}_\infty(x,\ \mathcal{M}^-) \leq f(n)\Big\} \tag{63}$$

whence $\mathrm{fifa}\,M^- \setminus M^- \subseteq M^0$. Moreover

$$M^- \setminus \mathrm{fifa}\,M^- \subseteq \Big\{x \in n\Gamma \,\Big|\, \mathrm{d}_\infty(x, \partial\,\mathrm{fifa}\,M^-) \leq f(n)\Big\} \tag{64}$$

Notice that either $\mathcal{M}^- = \emptyset$ or $\mathrm{diam}\,\mathrm{fifa}\,M^- \geq f(n)$. If $\mathcal{M}^- = \emptyset$, then $M^- = \mathrm{fifa}\,M^- = \emptyset$. Suppose that $\mathrm{diam}\,\mathrm{fifa}\,M^- \geq f(n)$. Let $A_1, \cdots, A_l$ be the $\mathbb{L}^{d,\infty}$ components of $\partial^{in}\mathrm{fifa}\,M^-$. Each of these components has cardinality larger than $f(n)k^{d-2}/(2d) \geq f(n)$ (recall that $k \geq 2d$). By Lemma 2.1,

$$\mathrm{vol}\Big\{x \in n\Gamma \,\Big|\, \mathrm{d}_\infty(x,\,\partial\,\mathrm{fifa}\,M^-) \leq f(n)\Big\} \;\leq\; \sum_{1\leq i\leq l} \mathrm{vol}\,\mathcal{V}_\infty(A_i, f(n)+1)$$
$$\leq \sum_{1\leq i\leq l} 4^{d+1}\,(f(n)+1)^{d-1}\,|\,A_i\,| \;\leq\; 5^d\,f(n)^{d-1}\mathcal{P}(\mathrm{fifa}\,M^-)$$

where the last inequality holds for n large enough. □

We now prove the exponential tightness (4).

Lemma 7.7 *There exists a positive constant $c = c(p,d)$ such that for any $\lambda, \delta > 0$,*

$$\overline{\lim_{n\to\infty}}\,\frac{1}{n^{d-1}}\,\log\,\mathbb{P}^+_{\Lambda(n)}\Big[\,\mathrm{dist}_{L^1}\Big(M^-/n,\,\mathcal{I}^{-1}([0,\lambda])\Big) \geq \delta\Big] \;\leq\; -c\lambda$$

Proof: Let $\lambda, \delta > 0$ be given. Using Lemma 7.6 and the inequality (53),

$$\mathbb{P}^+_{\Lambda(n)}\Big[\,\mathrm{dist}_{L^1}\Big(M^-/n,\,\mathcal{I}^{-1}([0,\lambda])\Big) \geq \delta\Big]$$
$$\leq \mathbb{P}^+_{\Lambda(n)}\Big[\,\mathrm{dist}_{L^1}(n^{-1}\mathrm{fifa}\,M^-,\; n^{-1}M^-) \geq \delta/2\Big] + \mathbb{P}^+_{\Lambda(n)}\Big[\,\mathrm{dist}_{L^1}(n^{-1}\mathrm{fifa}\,M^-,\; \mathcal{I}^{-1}([0,\lambda])) \geq \delta/2\Big]$$
$$\leq \mathbb{P}^+_{\Lambda(n)}\Big[\,\mathrm{vol}\,M^0 + 5^d f(n)^{d-1}\,\mathcal{P}(\mathrm{fifa}\,M^-) \geq n^d\delta/2\Big] + \mathbb{P}^+_{\Lambda(n)}\Big[\mathcal{I}(\mathrm{fifa}\,M^-) \geq \lambda n^{d-1}\Big]$$
$$\leq \mathbb{P}^+_{\Lambda(n)}\Big[\,\mathrm{vol}\,M^0 \geq \frac{\delta n^d}{4}\Big] + \mathbb{P}^+_{\Lambda(n)}\Big[\mathcal{P}(\mathrm{fifa}\,M^-) \geq \frac{\delta n^d}{4\,5^d f(n)^{d-1}}\Big] + \mathbb{P}^+_{\Lambda(n)}\Big[\mathcal{P}(\mathrm{fifa}\,M^-) \geq \frac{\lambda n^{d-1}}{\tau_{\max}}\Big]$$

By setting $c = 1/\tau_{\max}$, Lemmas 7.2 and 7.5 imply the desired conclusion. □

The set aglu M^-. We begin with the definition of a certain enlargement A of $\mathcal{C}^-$. The set A consists of the sites belonging either to $\mathcal{M}^-$ or to small clusters not touching the boundary of $\Lambda(n)$ and whose distance to $\mathcal{M}^-$ is bounded by $3f(n)$, i.e.,

$$A = \mathcal{M}^- \bigcup \mathrm{cup}\Big\{C \text{ is a small cluster}\,\Big|\,\mathrm{d}_\infty(C, \mathcal{M}^-) \leq 3f(n),\; C \cap \partial^{in}\Lambda(n) = \emptyset\Big\}$$

The set aglu M^- (the word aglu stands for agglutination) is defined as the union of all the unit continuum boxes $\Lambda(x,1) \subseteq \mathbb{R}^d$ centered at a vertex x in A, that is, aglu M^- = cube A. We first show that M^- and aglu M^- are exponentially contiguous.

Lemma 7.8 *For any $\delta > 0$,*

$$\overline{\lim_{n\to\infty}}\,\frac{1}{n^{d-1}}\,\log\,\mathbb{P}^+_{\Lambda(n)}\Big[\,\mathrm{dist}_{L^1}(M^-,\,\mathrm{aglu}\,M^-) > \delta n^d\Big] = -\infty$$

Proof: We work with the box $\Lambda(n)$ rescaled by a factor $f(n)$ and with event-block size 4. The block variable $X(\underline{x})$ is the indicator function of the event $R(B'(\underline{x}), f(n)-1)^c$. We introduce an intermediate block $B''(\underline{x})$ "around" $\underline{x}$ corresponding to the event–block size 2: hence, if $\mathrm{d}_\infty(\underline{x},\,\partial^{in}\underline{\Lambda}(n)) \geq 3$, then $B''(\underline{x}) = \Lambda(f(n)\underline{x},\,5f(n))$. Let $\underline{x} \in \underline{\Lambda}(n)$ be such that

$d_\infty(\underline{x},\partial^{in}\underline{\Lambda}(n))\geq 3$ and suppose that $B(\underline{x})\cap\mathcal{M}^-\neq\emptyset$ and $X(\underline{x})=0$. Then there is a unique crossing cluster in $B'(\underline{x})$, and all the other clusters intersecting the intermediate block $B''(\underline{x})$ have a diameter less than $f(n)-1$. Therefore $B''(\underline{x})\subseteq \mathrm{aglu}\,M^-$. Since

$$M^-\subseteq\Big\{x\in n\Gamma\,\Big|\,d_\infty(x,\mathcal{M}^-)\leq f(n)\Big\}\subseteq\bigcup_{\underline{x}:\ B(\underline{x})\cap\mathcal{M}^-\neq\emptyset}B''(\underline{x})$$

we have

$$\begin{aligned}&M^-\setminus\mathrm{aglu}\,M^-\subseteq\\&\quad\Big\{x\in\Lambda(n)\,\Big|\,d_\infty(x,\partial n\Gamma)\leq 4f(n)\Big\}\cup\Big(\bigcup_{\underline{x}:\ B(\underline{x})\cap\mathcal{M}^-\neq\emptyset}B''(\underline{x})\setminus\bigcup_{\substack{\underline{x}:\ B(\underline{x})\cap\mathcal{M}^-\neq\emptyset\\X(\underline{x})=0}}B''(\underline{x})\Big)\\&\qquad\subseteq\Big\{x\in\Lambda(n)\,\Big|\,d_\infty(x,\partial n\Gamma)\leq 4f(n)\Big\}\cup\bigcup_{\underline{x}\in\underline{\Lambda}(n),\,X(\underline{x})=1}B''(\underline{x})\end{aligned}$$

so that

$$\mathrm{vol}\,(M^-\setminus\mathrm{aglu}\,M^-)\leq 8df(n)n^{d-1}\,+\,5^df(n)^d\sum_{\underline{x}\in\underline{\Lambda}(n)}X(\underline{x})$$

On the other hand, using (63) and (64),

$$\begin{aligned}\mathrm{aglu}\,M^-\setminus M^-&\subseteq(\mathrm{aglu}\,M^-\setminus\mathrm{fifa}\,M^-)\cup(\mathrm{fifa}\,M^-\setminus M^-)\\&\subseteq\Big\{x\in\Lambda(n)\,\Big|\,d_\infty(x,\partial\,\mathrm{fifa}\,M^-)\leq 4f(n)\Big\}\cup M^0\end{aligned}$$

Using Lemma 2.1 and proceeding as in Lemma 7.6, we obtain

$$\mathrm{dist}_{L^1}(M^-,\mathrm{aglu}\,M^-)\leq 8df(n)n^{d-1}+5^df(n)^d\sum_{\underline{x}\in\underline{\Lambda}(n)}X(\underline{x})+\,\mathrm{vol}\,M^0+5^d(4f(n))^{d-1}\mathcal{P}(\mathrm{fifa}\,M^-)$$

By the estimate (14), the block process $X(\underline{x})$ satisfies (15) with $\varepsilon=b\exp(-cf(n))$. Applying (18) and Lemma 7.5, we get the exponential contiguity between M^- and $\mathrm{aglu}\,M^-$. □

We now prove the upper bound (5) for a set having finite surface energy.

Lemma 7.9 *Let E be a Borel set of Γ such that $\mathcal{I}(E)<\infty$. For every $\varepsilon>0$, there exists $\delta=\delta(E,\varepsilon)>0$ such that*

$$\overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log\mathbb{P}^+_{\Lambda(n)}\Big[\mathrm{dist}_{L^1}(M^-/n,\,E)<\delta\Big]\leq-(1-\varepsilon)\,\mathcal{I}(E)$$

Proof: By the triangle inequality for dist_{L^1} and Lemma 7.8, we need only to prove the above statement for the set $\mathrm{aglu}\,M^-$. Let E be a Borel set of Γ such that $0<\mathcal{I}(E)<\infty$. For $\varepsilon>0$, set $\varepsilon'=\varepsilon(1+1/\mathcal{I}(E))^{-1}$. Pick $\delta_0\in(0,\theta/2)$ such that $c\sqrt{\delta_0}<\varepsilon'$ where $c=c(p,q=2,d,\zeta)$ is the constant appearing in the interface Lemma 6.1. Let $B(x_i,r_i)$, $i\in I$, be a finite collection of disjoint balls associated with E,ε' and $\delta_0/3$, as given in the covering Lemma 5.2. Let $\delta>0$ be smaller than $\min\{\delta_0r_i^d/3\,|\,i\in I\}$. Suppose that $\mathrm{dist}_{L^1}(\mathrm{aglu}\,M^-/n,E)<\delta$.

For i in I let $\mathcal{C}(i)$ be the collection of the open clusters of the configuration restricted to $\operatorname{aglu} M^- \cap B(nx_i, nr_i)$. Note that these clusters are open clusters of the configuration restricted to $B(nx_i, nr_i)$. Setting $B_-(n,i) = B_-(nx_i, nr_i, \nu_E(x_i))$, we have

$$\begin{aligned}\operatorname{dist}_{L^1}\Big(\operatorname{cube\,cup}\mathcal{C}(i),\, B_-(n,i)\Big) \quad &\leq \quad \operatorname{dist}_{L^1}\Big(\operatorname{cube\,cup}\mathcal{C}(i),\, \operatorname{aglu} M^- \cap B_-(n,i)\Big) + \\ + \operatorname{dist}_{L^1}\Big(\operatorname{aglu} M^- \cap B_-(n,i),\, nE \cap B_-(n,i)\Big) &+ \operatorname{dist}_{L^1}\Big(nE \cap B_-(n,i),\, B_-(n,i)\Big) \\ &\leq \quad c'(d) n^{d-1} + \delta n^d + \delta_0 (nr_i)^d/3 \leq \delta_0 (nr_i)^d \qquad (65)\end{aligned}$$

where $c'(d)$ is an appropriate constant depending only on the dimension. If x_i belongs to $\operatorname{int}\Gamma$ then the collection $\mathcal{C}(i)$ realizes the event $\operatorname{Sep}(n, x_i, r_i, \nu_E(x_i), \delta_0)$. If x_i belongs to $\partial^*\Gamma$ then the collection $\mathcal{C}(i)$ realizes the event $\operatorname{Sep}^{\mathrm{bd}}(n, x_i, r_i, \delta_0)$. Indeed, none of the clusters inside $\operatorname{aglu} M^-$ intersects $\partial^{in}\Lambda(n)$: the large clusters in $\operatorname{aglu} M^-$ are colored with minus, and the small clusters attached to $\mathcal{C}^-$ to build $\operatorname{aglu} M^-$ do not intersect $\partial^{in}\Lambda(n)$. We conclude that

$$\Big\{ \operatorname{dist}_{L^1}(M^-/n, E) < \delta \Big\} \subseteq \bigcap_{\substack{i \in I \\ x_i \in \operatorname{int}\Gamma}} \operatorname{Sep}(n, x_i, r_i, \nu_E(x_i), \delta_0) \cap \bigcap_{\substack{i \in I \\ x_i \in \partial^*\Gamma}} \operatorname{Sep}^{\mathrm{bd}}(n, x_i, r_i, \delta_0)$$

Note that the sets $\Gamma \cap B(x_i, r_i)$ are compact and disjoint. The decoupling Lemma 3.2 and the interface Lemma 6.1 together imply

$$\begin{aligned}\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{dist}_{L^1}(\operatorname{aglu} M^-/n, E) < \delta\Big] \quad &\leq \quad -\sum_{i \in I} \alpha_{d-1} r_i^{d-1} \tau(\nu_E(x_i)) \Big(1 - c\sqrt{\delta_0}\Big) \\ &\leq \quad -\mathcal{I}(E)\,(1 - \varepsilon') + \varepsilon' = -\mathcal{I}(E)(1-\varepsilon) \qquad \square\end{aligned}$$

We complete now the proof of the upper bound for Theorem 1.1.

Let $v \in [0,1]$ be fixed and $\lambda > 0$. We define the sets $S(v) = \{E \in \mathcal{B}(\Gamma) \,|\, \operatorname{vol} E \geq v\}$ and $S(v,\lambda) = \{E \in S(v) \,|\, \mathcal{I}(E) \leq \lambda\}$. The sets $S(v,\lambda)$, $\lambda < \infty$, are compact. Let $\varepsilon > 0$. For each E in $S(v-\varepsilon, \lambda)$ we choose $\delta(E,\varepsilon) > 0$ according to Lemma 7.9. The family

$$\Big\{ \{A \in \mathcal{B}(\Gamma) \,|\, \operatorname{dist}_{L^1}(A, E) < \delta(E,\varepsilon)\},\ E \in S(v-\varepsilon,\lambda) \Big\}$$

is an open covering of $S(v-\varepsilon,\lambda)$ from which we can extract a finite subcover associated with a finite collection of sets E_i, $i \in I$. Since $S(v-\varepsilon,\lambda)$ is compact, we can choose $\delta \in (0,\varepsilon)$ such that the δ-neighborhood of $S(v-\varepsilon,\lambda)$ is still contained in the subcover, i.e.,

$$\Big\{ A \in \mathcal{B}(\Gamma) \,\Big|\, \operatorname{dist}_{L^1}\Big(A, S(v-\varepsilon,\lambda)\Big) < \delta \Big\} \subset \bigcup_{i \in I} \Big\{ A \in \mathcal{B}(\Gamma) \,\Big|\, \operatorname{dist}_{L^1}(A, E_i) < \delta(E_i, \varepsilon) \Big\}$$

Since $\operatorname{dist}_{L^1}(S(v),\, S(v-\varepsilon)^c) \geq \varepsilon > \delta$,

$$\begin{aligned}\mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{vol} M^-/n \geq v\Big] &\leq \mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{dist}_{L^1}\Big(M^-/n,\, S(v-\varepsilon,\lambda)\Big) < \delta\Big] + \\ &\qquad + \mathbb{P}^+_{\Lambda(n)}\Big[M^-/n \in S(v),\ \operatorname{dist}_{L^1}\Big(M^-/n,\, S(v,\lambda)\Big) \geq \delta\Big] \\ &\leq \sum_{i \in I} \mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{dist}_{L^1}(M^-/n, E_i) < \delta(E_i,\varepsilon)\Big] + \mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{dist}_{L^1}\Big(M^-/n,\, \mathcal{I}^{-1}([0,\lambda])\Big) \geq \delta\Big]\end{aligned}$$

Passing to the $\overline{\lim}$, the choice of the $\delta(E_i,\varepsilon)$-s and Lemma 7.7 yield

$$\begin{aligned}\overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log \mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{vol}(M^-/n)\geq v\Big] &\leq -\Big(c\lambda\wedge(1-\varepsilon)\min_{i\in I}\mathcal{I}(E_i)\Big)\\ &\leq -\Big(c\lambda\wedge(1-\varepsilon)\inf_{S(v-\varepsilon,\lambda)}\mathcal{I}\Big)\end{aligned}$$

Since the sets $S(v-\varepsilon,\lambda)$ are compact and $\mathcal{I}$ is l.s.c.,

$$\lim_{\varepsilon\to 0}\inf_{S(v-\varepsilon,\lambda)}\mathcal{I}=\inf_{S(v,\lambda)}\mathcal{I}\geq\inf_{S(v)}\mathcal{I}$$

By letting $\varepsilon\to 0$, $\lambda\to\infty$ and applying the Wulff isoperimetric theorem [34], we obtain (7)

$$\overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log \mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{vol}(M^-/n)\geq v\Big]\leq-\inf_{S(v)}\mathcal{I}=-(v/\operatorname{vol}\mathcal{W}_\tau)^{\frac{d-1}{d}}\mathcal{I}(\mathcal{W}_\tau)$$

Let m belong to $[-m^*,+m^*)$. Let $\delta>0$ such that $m+\delta<m^*$. By Corollary 7.3,

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log \mathbb{P}^+_{\Lambda(n)}\Big[\frac{1}{n^d}\sum_{x\in\Lambda(n)}\sigma(x)\leq m\Big]\\ &\leq\ \overline{\lim_{n\to\infty}}\frac{1}{n^{d-1}}\log \mathbb{P}^+_{\Lambda(n)}\Big[\operatorname{vol}(M^-/n)\geq\frac{m^*-\delta-m}{2m^*}\Big]\leq-\Big(\frac{m^*-\delta-m}{2m^*\operatorname{vol}\mathcal{W}_\tau}\Big)^{\frac{d-1}{d}}\mathcal{I}(\mathcal{W}_\tau)\end{aligned}$$

Letting δ go to 0, we obtain the correct upper bound for Theorem 1.1. Notice that this upper bound is valid for any m in $[-m^*,+m^*]$. However we have the corresponding lower bound only for $1-2(\operatorname{diam}\mathcal{W}_\tau)^{-d}\operatorname{vol}\mathcal{W}_\tau<m/m^*<1$. For the remaining values of m in $(-m^*,m^*)$, the correct value should be

$$\inf\Big\{\mathcal{I}(E)\,\Big|\,E\in\mathcal{B}(\Gamma),\ \operatorname{vol}E\geq(m^*-m)/(2m^*)\Big\}$$

See [59] for the corresponding question in dimension two.

7.2 The locally averaged magnetization

We express σ_n with the help of the FK representation:

$$\forall x\in\Gamma\qquad\sigma_n(x)=\frac{1}{f(n)^d}\sum_{C\in\mathcal{C}}\sigma(C)\,|C\cap\Lambda(nx,f(n))|$$

We separate the contribution of the small and the large clusters by setting, for $x\in\Gamma$,

$$\sigma_n^{\text{small}}(x)=\frac{1}{f(n)^d}\sum_{\substack{C\in\mathcal{C}\\ C\text{ small}}}\sigma(C)\,|C\cap\Lambda(nx,f(n))|$$

and $\sigma_n^{\text{large}}(x)$ as in the previous case except that the sum is running over the large clusters.

Lemma 7.10 *For any $\delta > 0$,*

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mathbb{P}^+_{\Lambda(n)}\Big[\int_\Gamma |\sigma_n^{\text{small}}(x)|\,dx > \delta\Big] = -\infty$$

Proof: Let $\delta \in (0,1/2)$ and n so big that $2dn^{d-1}\,2f(n) < \delta n^d$. Set $\Lambda(n)' = \Lambda(n-2f(n)) \cap \mathbb{Z}^d$. We denote by $B(z)$ the open unit cube centered at $z \in \mathbb{Z}^d$. Note that $\sigma_n^{\text{small}}(x)$ is constant in $B(z)$ with common value $\sigma_n^{\text{small}}(z)$. Thus

$$\begin{aligned}\mathbb{P}^+_{\Lambda(n)}\Big[\int_\Gamma |\sigma_n^{\text{small}}(x)|\,dx > 3\delta\Big] &\leq \mathbb{P}^+_{\Lambda(n)}\Big[\sum_{z\in\Lambda(n)'} \int_{B(z)} |\sigma_n^{\text{small}}(x/n)|\,dx > 2\delta n^d\Big]\\ &= \mathbb{P}^+_{\Lambda(n)}\Big[\sum_{z\in\Lambda(n)'} |\sigma_n^{\text{small}}(z/n)| > 2\delta n^d\Big]\end{aligned}$$

Let $z \in \Lambda(n)'$. If C is a cluster intersecting $\Lambda(z, f(n))$ then $C \cap \partial^{in}\Lambda(n) = \emptyset$, hence $\sigma(C)$ is ± 1 with probability $1/2$ each and is independent from everything. Arguing as in the proof of Lemma 7.1 (no need for $\mathcal{C}'$ here) we have

$$\mathbb{P}^+_{\Lambda(n)}\Big[|\sigma_n^{\text{small}}(z)| > \delta\Big] \leq 2\exp\Big(-\frac{c_3\delta^2 f(n)^{d-2}}{\alpha}\Big)$$

We call z regular if $|\sigma_n^{\text{small}}(z)| \leq \delta$. We set $X_z = 0$ if z is regular and $X_z = 1$ otherwise. There exists $c_4 = c_4(p,d) > 0$, such that

$$\mathbb{P}^+_{\Lambda(n)}\Big[X_z = 1 \,\Big|\, \sigma\Big(X_y;\ \mathrm{d}_\infty(z,y) \geq 2f(n)\Big)\Big] \leq 2\exp(-c_4\delta^2 f(n)^{d-2}) \tag{66}$$

Then, for n large enough

$$\begin{aligned}\mathbb{P}^+_{\Lambda(n)}\Big[\sum_{z\in\Lambda(n)'} |\sigma_n^{\text{small}}(z)| > 2\delta n^d\Big] &= \mathbb{P}^+_{\Lambda(n)}\Big[\sum_{\substack{z\in\Lambda(n)'\\ X_z=0}} |\sigma_n^{\text{small}}(z)| + \sum_{\substack{z\in\Lambda(n)'\\ X_z=1}} |\sigma_n^{\text{small}}(z)| > 2\delta n^d\Big]\\ &\leq \mathbb{P}^+_{\Lambda(n)}\Big[\sum_{z\in\Lambda(n)'} X_z > \delta n^d\Big]\end{aligned}$$

In order to apply Lemma 2.2, by using (66) we first estimate $\Lambda^*_\varepsilon(\delta)$:

$$\Lambda^*_{\varepsilon(n)}(\delta) \geq \delta\log(\delta/\varepsilon(n)) - \log 2 \geq \delta\log\Big((\delta/2)\exp(c_4\delta^2 f(n)^{d-2}) - \log 2\Big) \geq c_4\delta^3 f(n)^{d-2}/2 - 2\log 2$$

Thus, for n large,

$$\mathbb{P}^+_{\Lambda(n)}\Big[\sum_{z\in\Lambda(n)'} X_z > \delta n^d\Big] \leq (2f(n))^d \exp\Big(-\frac{cn^d}{2f(n)^2}\Big)$$

for a certain constant $c > 0$ and the claim follows. □

Lemma 7.11 *For $\delta > 0$,*

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mathbb{P}^+_{\Lambda(n)}\Big[\int_{M^-/n} |\sigma_n^{\text{large}}(x) + m^*|\,dx + \int_{M^+/n} |\sigma_n^{\text{large}}(x) - m^*|\,dx > \delta\Big] = -\infty$$

Proof: We work with the box $\Lambda(n)$ rescaled by a factor $f(n)$ and blocksize 2. The block variable $X(\underline{x})$ is the indicator function of the event $T(B'(\underline{x}), f(n)-1, \delta/2)^c$. Suppose that $X(\underline{x}) = 0$ and let x be such that $y = xn \in B(\underline{x})$. Then there exists exactly one large open cluster C such that $\mathrm{d}_\infty(y, C) \leq f(n)$. This cluster C satisfies in addition

$$|C \cap \Lambda(y, f(n)) - \theta f(n)^d| < (\delta/2) f(n)^d$$

Therefore, y belongs to $M^- \cup M^+$ and $|\sigma_n^{\text{large}}(x) - \theta\sigma(C)| < \delta/2$. We split the region of integration into the blocks $B(\underline{x})/n$, $\underline{x} \in \underline{\Lambda}(n)$ and we use the previous inequality:

$$\int_{M^-/n} |\sigma_n^{\text{large}}(x) + m^*|\, dx + \int_{M^+/n} |\sigma_n^{\text{large}}(x) - m^*|\, dx \leq 2^d (f(n)/n)^d \sum_{\underline{x} \in \underline{\Lambda}(n)} X(\underline{x}) + \delta/2$$

By the estimate (14), the block process $X(\underline{x})$ satisfies (15) with $\varepsilon = b\exp(-cf(n))$. The result follows from (18). $\square$

We next compare the two random partitions $(M^-/n, M^0/n, M^+/n)$ and $(\Gamma_n^-, \Gamma_n^0, \Gamma_n^+)$.

Corollary 7.12 *For any $\delta > 0$,*

$$\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mathbb{P}^+_{\Lambda(n)}\Big[\mathrm{dist}_{L^1}(M^-/n, \Gamma_n^-) + \mathrm{vol}\,\Gamma_n^0 + \mathrm{dist}_{L^1}(M^+/n, \Gamma_n^+) > \delta\Big] = -\infty$$

Proof: With the help of some set algebra, we have

$$\begin{aligned}
&\mathrm{dist}_{L^1}(M^-/n, \Gamma_n^-) + \mathrm{dist}_{L^1}(M^0/n, \Gamma_n^0) + \mathrm{dist}_{L^1}(M^+/n, \Gamma_n^+) \leq \\
&\qquad \leq\ 2\,\mathrm{vol}\,(M^-/n \cap (\Gamma_n^0 \cup \Gamma_n^+)) + 2\,\mathrm{vol}\,(M^0/n) + 2\,\mathrm{vol}\,(M^+/n \cap (\Gamma_n^0 \cup \Gamma_n^-)) \\
&\qquad \leq\ \frac{2}{m^*}\Big(\int_{M^-/n} |\sigma_n(x) + m^*|\, dx + \mathrm{vol}\,(M^0/n) + \int_{M^+/n} |\sigma_n(x) - m^*|\, dx\Big)
\end{aligned}$$

The claim of the corollary is a consequence of lemmas 7.2, 7.10, 7.11. $\square$

We now complete the proof of Theorem 1.2. Lemmas 7.10, 7.11 and Corollary 7.12 imply the first claim of Theorem 1.2. Let m be such that $1 - 2(\mathrm{diam}\,\mathcal{W}_\tau)^{-d}\,\mathrm{vol}\,\mathcal{W}_\tau < m/m^* < 1$. Let $\delta > 0$. By Theorem 1.1,

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mu^+_{\Lambda(n)}\Big[\mathrm{dist}_{L^1}(\Gamma_n^-, \mathrm{mc}\,(\Gamma_n^-) + \mathcal{W}(m)) \geq \delta \,\Big|\, \frac{1}{n^d} \sum_{x\in\Lambda(n)} \sigma(x) \leq m\Big] \leq \\
&\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mathbb{P}^+_{\Lambda(n)}\Big[\mathrm{dist}_{L^1}(\Gamma_n^-, \mathrm{mc}\,(\Gamma_n^-) + \mathcal{W}(m)) \geq \delta\,,\ \frac{1}{n^d} \sum_{x\in\Lambda(n)} \sigma(x) \leq m\Big] + \mathcal{I}(\mathcal{W}(m))
\end{aligned}$$

Next, by Corollaries 7.3, 7.12 and Lemma 7.7, for any $\lambda > 0$, $\varepsilon > 0$,

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mathbb{P}^+_{\Lambda(n)}\Big[\mathrm{dist}_{L^1}(\Gamma_n^-, \mathrm{mc}\,(\Gamma_n^-) + \mathcal{W}(m)) \geq \delta\,,\ \frac{1}{n^d} \sum_{x\in\Lambda(n)} \sigma(x) \leq m\Big] \leq \\
&-c\lambda \wedge \overline{\lim_{n\to\infty}} \frac{1}{n^{d-1}} \log \mathbb{P}^+_{\Lambda(n)}\left[\begin{array}{c} \mathrm{dist}_{L^1}(\Gamma_n^-, \mathrm{mc}\,(\Gamma_n^-) + \mathcal{W}(m)) \geq \delta\,,\ \mathrm{dist}_{L^1}(\Gamma_n^-, M^-/n) < \varepsilon\,, \\ \mathrm{dist}_{L^1}(M^-/n, \mathcal{I}^{-1}([0,\lambda])) < \varepsilon,\ \mathrm{vol}\,(M^-/n) \geq \mathrm{vol}\,(\mathcal{W}(m)) - \varepsilon \end{array}\right]
\end{aligned}$$

Let η such that $0 < 2\eta < \mathrm{vol}\,(\mathcal{W}(m))$ and let us denote by $\mathcal{E}(\delta,\lambda,\eta)$ the subset of $\mathcal{B}(\Gamma)$ defined by

$$\mathcal{E}(\delta,\lambda,\eta) = \Big\{ E \in \mathcal{B}(\Gamma) \,\Big|\, \mathrm{dist}_{L^1}(E,\, \mathrm{mc}\,(E)+\mathcal{W}(m)) \geq \delta-\eta,\ \mathrm{vol}\,E \geq \mathrm{vol}\,(\mathcal{W}(m))-\eta,\, \mathcal{I}(E) \leq \lambda \Big\}$$

The map $E \in \mathcal{B}(\Gamma) \mapsto \mathrm{mc}\,(E) \in \Gamma$ is continuous at each set having positive volume. Therefore for ε small enough, the inequalities involved in the above event imply the further inequality $\mathrm{dist}_{L^1}(M^-/n,\, \mathcal{E}(\delta,\lambda,\eta)) < \varepsilon$. Moreover the set $\mathcal{E}(\delta,\lambda,\eta)$ is compact. Proceeding as in the proof of the upper bound of Theorem 1.1, with the help of lemmas 7.7, 7.9, we get the bound

$$\lim_{\varepsilon\to 0}\ \overline{\lim_{n\to\infty}}\ \frac{1}{n^{d-1}} \log \mathbb{P}^+_{\Lambda(n)}\Big[\mathrm{dist}_{L^1}\Big(M^-/n, \mathcal{E}(\delta,\lambda,\eta)\Big) < \varepsilon \Big] \leq -\Big(c\lambda \wedge \inf_{E\in\mathcal{E}(\delta,\lambda,\eta)} \mathcal{I}(E) \Big)$$

Combining the previous inequalities, sending λ to ∞ and η to 0, we arrive at

$$\begin{aligned}\overline{\lim_{n\to\infty}}\ \frac{1}{n^{d-1}} \log \mu^+_{\Lambda(n)}\Big[\mathrm{dist}_{L^1}(\Gamma^-_n,\, \mathrm{mc}\,(\Gamma^-_n) + \mathcal{W}(m)) \geq \delta \,\Big|\, \frac{1}{n^d} \sum_{x\in\Lambda(n)} \sigma(x) \leq m \Big] \\ \leq -\inf_{E\in\mathcal{E}(\delta,\infty,0)} \mathcal{I}(E) + \mathcal{I}(\mathcal{W}(m))\end{aligned}$$

Since $\mathcal{I}$ restricted to Γ has compact level sets it attains its minimum over the set $\mathcal{E}(\delta,\infty,0)$. However this set contains only sets of volume larger than or equal to $\mathrm{vol}\,\mathcal{W}(m)$ but no translate of $\mathcal{W}(m)$. The uniqueness statement in the Wulff isoperimetric Theorem [35] implies that

$$\inf_{E\in\mathcal{E}(\delta,\infty,0)} \mathcal{I}(E) > \mathcal{I}(\mathcal{W}(m))$$

concluding thereby the proof of the second claim of Theorem 1.2.

A Appendix

We recall here some facts concerning the class of the sets of finite perimeter, introduced initially by Caccioppoli [12, 13] and subsequently developed by De Giorgi [18, 19, 20, 21] (see also [29, 31, 36, 48, 66]). The perimeter of a Borel set E of $\mathbb{R}^d$ is defined as

$$\mathcal{P}(E) = \sup \Big\{ \int_E \mathrm{div}\, f(x)\, dx : f \in C_0^\infty(\mathbb{R}^d, B(1)) \Big\}$$

where $C_0^\infty(\mathbb{R}^d, B(1))$ is the set of the compactly supported C^∞ vector functions from $\mathbb{R}^d$ to the unit ball $B(1)$ and div is the usual divergence operator. The set E is of finite perimeter if $\mathcal{P}(E)$ is finite. A set E is a Caccioppoli set if it is locally of finite perimeter. In this paper, we deal with bounded sets, hence we need only to consider sets of finite perimeter. A set E has finite perimeter if and only if its characteristic function χ_E is a function of bounded variation. The distributional derivative $\nabla\chi_E$ of χ_E is then a vector Radon measure and $\mathcal{P}(E) = ||\nabla\chi_E||(\mathbb{R}^d)$, where $||\nabla\chi_E||$ is the total variation measure of $\nabla\chi_E$. The perimeter $\mathcal{P}$ is l.s.c. on the space $(\mathcal{B}(\mathbb{R}^d),\, \mathrm{dist}_{L^1})$.

Compactness property of sets of finite perimeter in Γ. For every bounded domain U and every $\lambda > 0$, the set $\{ E \in \mathcal{B}(U) : \mathcal{P}(E) \leq \lambda \}$ is compact for the metric dist_{L^1}.

This result is stated in this precise form in [21], Teorema 2.4, or [19], Teorema I. It is also an immediate consequence of the compactness theorem stated in [48], chapter 2, p.70. Modern presentations are formulated through functions of bounded variations: if O is an open bounded domain with sufficiently regular boundary (say C^1), then a set of functions in $L^1(O)$ uniformly bounded in BV–norm is relatively compact in $L^1(O)$ (see any of the following references: [29], Section 5.2.3, [36], Theorem 1.19, [66], Corollary 5.3.4). To deduce the compactness result on sets of finite perimeter, we choose an open bounded domain O with regular boundary containing U in its interior. We embed $\mathcal{B}(U)$ in $L^1(O)$ by associating to a Borel set E its characteristic function χ_E and we simply remark that the set $\{\chi_E \,|\, E \in \mathcal{B}(U)\}$ is a closed subset of $L^1(O)$.

Let E be a set of finite perimeter. Its reduced boundary $\partial^* E$ consists of the points x such that
• $||\nabla\chi_E||(B(x,r)) > 0$ for any $r > 0$,
• if $\nu_r(x) = -\nabla\chi_E(B(x,r))/||\nabla\chi_E||(B(x,r))$ then, as r goes to 0, $\nu_r(x)$ converges towards a limit $\nu_E(x)$ such that $|\nu_E(x)|_2 = 1$.
The reduced boundary $\partial^* E$ is countably $(d-1)$–rectifiable, that is $\partial^* E \subset N \cup \bigcup_{i\in\mathbb{N}} M_i$ where $\mathcal{H}^{d-1}(N) = 0$ and each M_i is a $d-1$-dimensional embedded C^1 submanifold of $\mathbb{R}^d$. For a point x belonging to $\partial^* E$, the vector $\nu_E(x)$ is called the generalized exterior normal to E at x. A unit vector ν is called the measure theoretic exterior normal to E at x if

$$\lim_{r\to 0} r^{-d}\,\mathrm{vol}\,(B_-(x,r,\nu)\setminus E) \;=\; 0\,, \quad \lim_{r\to 0} r^{-d}\,\mathrm{vol}\,(B_+(x,r,\nu)\cap E) \;=\; 0\,.$$

At each point x of the reduced boundary $\partial^* E$ of E, the generalized exterior normal $\nu_E(x)$ is also the measure theoretic exterior normal to E at x. The map $x \in \partial^* E \mapsto \nu_E(x) \in S^{d-1}$ is $||\nabla\chi_E||$ measurable. For any Borel set A of $\mathbb{R}^d$,

$$||\nabla\chi_E||(A) = \mathcal{H}^{d-1}(A\cap\partial^* E)\,, \qquad \nabla\chi_E(A) = \int_{A\cap\partial^* E} -\nu_E(x)\,\mathcal{H}^{d-1}(dx)\,.$$

Let $f : \partial^* E \mapsto \mathbb{R}$ be a $||\nabla\chi_E||$ measurable bounded function. By the Besicovitch derivation Theorem [6, 10] applied to the measure $||\nabla\chi_E||$, for $\mathcal{H}^{d-1}$ almost all x in $\partial^* E$,

$$\lim_{r\to 0} \quad (\alpha_{d-1} r^{d-1})^{-1} \int_{B(x,r)\cap\partial^* E} f(y)\,\mathcal{H}^{d-1}(dy) \;=\; f(x)\,.$$

For any vector function f in $C^1_0(\mathbb{R}^d,\mathbb{R}^d)$, any Caccioppoli set E, by the generalized Gauss–Green Theorem,

$$\int_E \mathrm{div}\, f(x)\,dx \;=\; \int_{\partial^* E} f(x)\cdot\nu_E(x)\,\mathcal{H}^{d-1}(dx)\,.$$

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