Stability of Bose dynamical systems and branching theory

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Abstract:
Stability under small nonlinear perturbations is proved for a class of linear quantum dynamical systems, including the harmonic oscillator coupled to a free Bose field and the infinite harmonic chrystal. The main tool is an estimate of Dyson's time-dependent perturbation series, based on a labeling of its terms by rooted trees.

1. Introduction.

In recent years a renewed interest is noticeable in the ergodic theory of quantum systems ([BFS1, BFS2, JaP1, JaP2, Spo, ArH, GrM]). One possible approach proceeds by perturbation of easily solvable models. Here we push this approach the an extreme for the case of linear Bose systems, applying a technique which may well allow wider usage: summation of the Dyson series by summing over rooted trees.

We consider a harmonic oscillator which together with its Bosonic environment forms a mixing linear quantum dynamical system at temperatur $T \geq 0$. We perturb the oscillator's harmonic potential by a bounded anharmonic term and study the dynamics of the perturbed system. For $T > 0$ the non-commutative Radon-Nikodym theorem of A. Connes [Con] says that the perturbed dynamics also possesses a thermal equilibrium state, which is given by a vector in the same Hilbert space.

We give a sufficient condition for the unitary equivalence of the perturbed and the unperturbed dynamics, thereby considerably improving an earlier result by one of the authors [Maa], which was later applied to Rayleigh scattering by H. Spohn [Spo].

At zero temperature the existence of a perturbed equilibrium state (i.e. a ground state) in the same Hilbert space is difficult to establish. We give a few concrete sufficient conditions for this case.

Our starting point is a finite positive measure ν on $[0, \infty)$ which characterises the linear quantum dynamical system of oscillator and environment together. A sum of $n \geq 1$ point masses would correspond to an assembly of $n-1$ (possibly coupled) harmonic oscillators interacting with the central oscillator. We shall be interested in the case of an absolutely continuous measure ν, as is needed for mixing behaviour of the system as a whole. This corresponds to a Bose field as in [Maa1, Spo] or to an infinite assembly of harmonic oscillators as in [FKM, FiL]. A few examples are mentioned in the appendix. For further examples we refer to [LeM].

The measure ν canonically determines a triple $(\mathcal{K}, K, q)$, where $\mathcal{K}$ is a complex separable Hilbert space, K a positive self-adjoint operator on $\mathcal{K}$ and q a vector in $\mathcal{K}$:

$$\mathcal{K} = L^2([0,\infty),\nu); \quad Kf(x) = xf(x), \quad q(x) = 1. \tag{1}$$

We may regard $\mathcal{K}$ as the phase space of a classical mechanical system with linear dynamics $S_t := e^{itK}$. The imaginary part of the inner product represents the symplectic form or Poisson bracket. The vector q stands for the position of the central harmonic oscillator. The triple $(\mathcal{K}, K, q)$ is alternatively characterised by the function

$$g : [0,\infty) \to \mathbb{R} : \quad t \mapsto \int_0^\infty \sin xt \; \nu(dx) = \operatorname{Im} \langle q, e^{itK} q \rangle \; . \tag{2}$$

which is expressed completely in classical mechanical terms.

The linear dynamical system $(\mathcal{K}, K, q)$ is quantised by associating in a real-linear way to a vector $f \in \mathcal{K}$ a self-adjoint operator $\Phi(f)$ on some Hilbert space $\mathcal{H}$ with cyclic vector Ω such that

$$\langle \Omega, \Phi(f)\Phi(h)\Omega \rangle = \begin{cases} \frac{1}{2}\langle f, h \rangle & \text{for } T = 0; \\ \frac{1}{2}\langle f, \coth{(K/T)} h \rangle & \text{for } T > 0. \end{cases}$$

The position operator of our oscillator at time t then becomes

$$Q_t := \Phi(S_t q) \ ,$$

so that $g(t) = [Q_0, Q_t]$. We shall denote Q_0 by Q. If $q \in \mathrm{Dom}\,(K)$ we also have a momentum operator

$$P := m\frac{d}{dt}Q_t|_{t=0} = m\frac{d}{dt}\Phi(S_t q)|_{t=0} = m\Phi(iKq) \ , \tag{3}$$

where m denotes the mass of the oscillator.

We perturb the Hamiltonian of the quantised linear system by a term $v(Q)$, where $v : \mathbb{R} \to \mathbb{R}$ is a smooth bounded function, and we call the dynamical system *stable* for this perturbation, if the perturbed and the unperturbed sytem are unitarily equivalent.

In Section 2 we recall a pair of 'abstract' sufficient conditions for stability: convergence of the Dyson series (I) and existence of an invariant state (II). In Section 3, Theorems 4 and 5, these are translated into concrete sufficient conditions. Our main result is Theorem 4 on stability.

In previous papers on the stability problem as stated above ([Maa, Spo]) it was required that $g(t) = [Q_0, Q_t]$ decays exponentially. This is not typically the case, however: it requires the measure ν to have a density whose odd extension to $\mathbb{R}$ is the imaginary part of an analytic function on an infinite strip around the real axis. It can never hold for measures of compact support, such as the spectral measure of a harmonic crystal. The present improvement was inspired by work of one of us [BFM] which suggested that only integrability of g would be needed. A closer investigation of the natural upper bound for Dyson's perturbation series revealed that its terms can be labeled by 'indexed' rooted trees, and then 'packed' together into a sum over trees without indexation, thus reducing the number of terms of order n from $n!$ to a power of n. The recursive structure of the space of all rooted trees allowed to identify the sum as the fixed point of a generating function, as occurs in the classical Galton-Watson theory of branching processes [Har]. In examples this fixed point can often be found explicitly. This summation over trees is set forth in Section 4.

Our condition on v implies that the total binding potential $\frac{1}{2}\alpha Q^2 + v(Q)$ is a convex function of Q, but we do not believe that this is a necessary condition for stability. (Cf. Lemma 8). In the simplest case, $v(Q) = a\cos\lambda Q$ with $a, \lambda > 0$, our condition reads:

$$a\lambda^2 \leq \frac{1}{e\,\|g\|} \ , \tag{4}$$

and since (Lemma 7) $\alpha\,\|g\| \leq 1$, this implies that

$$\frac{d^2}{dx^2}\left(\tfrac{1}{2}\alpha x^2 + v(x)\right) \geq \alpha - a\lambda^2\cos\lambda x > 0 \ .$$

Shortly before this work was finished, we became aware of comparable results of Fidaleo and Liverani [FiL], who could also prove stability on the basis of integrability of g. In the simplest case mentioned above they give the following sufficient condition:

$$\sup_{n\in\mathbb{N}} \frac{a\lambda^n}{(n-1)!} < \frac{1}{2\,\|g\|}.$$

This condition is considerably more restrictive than (4) in particular for large λ, since

$$\sup_{n\in\mathbb{N}} \frac{\lambda^n}{(n-1)!} \sim \sqrt{\frac{\lambda}{2\pi}}\, e^{\lambda}, \qquad (\lambda \to \infty).$$

A technical point which had to be addressed in Section 3 is the proper choice of the algebra of operators for which the mixing property is to be formulated. The C*-algebra generated by the Weyl operators is too small, since it is not stable for the perturbed dynamics. (This point was overlooked in [Spo].) The von Neumann algebra on the other hand is in general too large: at temperature 0 the mixing property fails. We chose here to use the C*-algebra generated by all the integrals over the Weyl operators with respect to finite complex measures on a linear subspace of $\mathcal{K}$, and call it the *integral Weyl algebra.* The need to employ this algebra was also recognised by Fidaleo and Liverani [FiL].

The crucial estimate for Theorem 4 is proved in Section 4. In Section 5 we prove Theorem 5. In Section 6 we give the measure ν for several simple linear dynamical systems and indicate which of our conditions hold for them.

2. Stability of Quantum Dynamical Systems.

In this Section we provide the framework for the main results in Section 3. In a concise form we present basically known facts from [Spo, Maa, FiL].

By a *quantum dynamical system* we shall mean a triple $(\mathcal{A}, \omega, \alpha_t)$, where $\mathcal{A}$ is a C*-algebra with unit, ω a state on $\mathcal{A}$, and $(\alpha_t)_{t\in\mathbb{R}}$ a one-parameter group of *-automorphisms of $\mathcal{A}$ leaving ω invariant. By the Gel'fand-Naimark-Segal construction the pair $(\mathcal{A}, \omega)$ determines a Hilbert space $\mathcal{H}$, a unit vector $\Omega \in \mathcal{H}$, and a representation of $\mathcal{A}$ as an algebra of bounded operators on $\mathcal{H}$ such that $\mathcal{A}\Omega$ is dense in $\mathcal{H}$ and $\langle \Omega, A\Omega \rangle = \omega(A)$ for all $A \in \mathcal{A}$. In this paper we shall identify the algebra $\mathcal{A}$ with its representation and require that $t \mapsto \alpha_t(A)$ is strongly continuous for all $A \in \mathcal{A}$.
The quantum dynamical system $(\mathcal{A}, \omega, \alpha_t)$ is called *mixing* if for all unit vectors $\Psi \in \mathcal{H}$ and all $A \in \mathcal{A}$

$$\lim_{t\to\pm\infty} \langle \Psi, \alpha_t(A)\Psi \rangle = \omega(A) . \tag{5}$$

The dynamics $(\alpha_t)_{t\in\mathbb{R}}$ determines a one-parameter group $(U_t)_{t\in\mathbb{R}}$ of unitary operators on $\mathcal{H}$ by the relation

$$U_t A\Omega = \alpha_t(A)\Omega .$$

The generator H of this group, given by

$$U_t = e^{itH} ,$$

is called the *Hamiltonian* of the quantum dynamical system.

Proposition 1. *Let $(\mathcal{A}, \omega, \alpha_t)$ be a mixing quantum dynamical system represented on a Hilbert space $\mathcal{H}$ with cyclic vector Ω and Hamiltonian H. Let $V = V^* \in \mathcal{A}$ and let for all $t \in \mathbb{R}$*

$$\widetilde{\alpha}_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) : A \mapsto e^{it(H+V)} A e^{-it(H+V)} .$$

Suppose that $\widetilde{\alpha}_t(\mathcal{A}) \subset \mathcal{A}$ for all $t \in \mathbb{R}$ and that the following two conditions are satisfied.
(I) For all $A \in \mathcal{A}$,

$$\sum_{n=0}^{\infty} \int \cdots \int_{0 \le t_1 \le \cdots \le t_n} \left\| [\alpha_{-t_n}(V), [\cdots [\alpha_{-t_1}(V), A] \cdots]] \right\| dt_1 dt_2 \cdots dt_n < \infty \; ; \tag{6}$$

(II) $H + V$ has an eigenvector $\widetilde{\Omega}$ (of unit length), which is cyclic for $\mathcal{A}$.

Then there exists a unitary operator $\Gamma : \mathcal{H} \to \mathcal{H}$ such that $\Gamma\Omega = \widetilde{\Omega}$ and

$$H + V = \Gamma H \Gamma^* \, .$$

In particular putting $\widetilde{\omega}(A) := \langle \widetilde{\Omega}, A\widetilde{\Omega} \rangle$ we obtain a mixing quantum dynamical system $(\mathcal{A}, \widetilde{\omega}, \widetilde{\alpha}_t)$ unitarily equivalent to $(\mathcal{A}, \omega, \alpha_t)$.

Proof. For all $A \in \mathcal{A}$ and $t \ge 0$, $\widetilde{\alpha}_t(A)$ is given by the Dyson series [BrR]

$$\widetilde{\alpha}_t(A) = \alpha_t(A) + \sum_{n=1}^{\infty} i^n \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \alpha_t\big([\alpha_{-t_n}(V), [\alpha_{-t_{n-1}}(V), [\cdots [\alpha_{-t_1}(V), A] \cdots]]]\big) dt_1 dt_2 \cdots dt_n \, , \tag{7}$$

and for $t \le 0$ by a similar expression.
Condition (I) implies that for all $A \in \mathcal{A}$ the following limits exist in the norm topology:

$$\gamma(A) := \lim_{t \to \infty} \widetilde{\alpha}_{-t} \circ \alpha_t(A) \quad \text{and} \quad \widetilde{\gamma}(A) := \lim_{t \to \infty} \alpha_{-t} \circ \widetilde{\alpha}_t(A) \, .$$

(In fact, for the existence of the first limit only the convergence of the $n = 1$ term in (7) is required.) Since $\mathcal{A}$ is invariant for $\widetilde{\alpha}_t$ and norm-closed, the above limits define *-homomorphisms $\gamma, \widetilde{\gamma} : \mathcal{A} \to \mathcal{A}$. Then we have for all $A, B \in \mathcal{A}$, in the topology of pointwise weak convergence on $\mathcal{A}$:

$$\lim_{t \to \infty} \alpha_t(A) \widetilde{\alpha}_t(B) = \omega(A \widetilde{\gamma}(B)) \cdot \mathbf{1} \, . \tag{8}$$

Indeed, for any unit vector $\Psi \in \mathcal{H}$ and any $\varepsilon > 0$ we can choose t so large that

$$\| \alpha_{-t} \circ \widetilde{\alpha}_t(B) - \widetilde{\gamma}(B) \| < \varepsilon \, ,$$

and, by the mixing property also

$$| \langle \Psi, \alpha_{-t}(A \widetilde{\gamma}(B)) \Psi \rangle - \omega(A \widetilde{\gamma}(B)) | < \varepsilon \, .$$

It follows that

$$\begin{aligned} | \langle \Psi, \alpha_t(A) \widetilde{\alpha}_t(B) \Psi \rangle - \omega(A \widetilde{\gamma}(B)) | &\le | \langle \Psi, \alpha_t(A \alpha_{-t} \circ \widetilde{\alpha}_t(B)) \Psi \rangle - \langle \Psi, \alpha_t(A \widetilde{\gamma}(B)) \Psi \rangle | \\ &\quad + | \langle \Psi, \alpha_t(A \widetilde{\gamma}(B)) \Psi \rangle - \omega(A \widetilde{\gamma}(B)) | \\ &< \varepsilon \, \|A\| + \varepsilon \, , \end{aligned}$$

which proves (8). Putting $A = \mathbf{1}$ we find that the dynamics $\widetilde{\alpha}_t$ eventually sends every observable $B \in \mathcal{A}$ to its value in the state $\omega \circ \widetilde{\gamma}$. By condition (II) this state is given by a vector $\widetilde{\Omega} \in \mathcal{H}$: we have $\widetilde{\omega} = \omega \circ \widetilde{\gamma}$ and $(\mathcal{A}, \widetilde{\omega}, \widetilde{\alpha}_t)$ is mixing. Now put

$$\Gamma_0 : \mathcal{A}\Omega \to \mathcal{A}\widetilde{\Omega} : A\Omega \mapsto \gamma(A)\widetilde{\Omega} \, .$$

Then Γ_0 is well-defined and isometric since, by the mixing property of $(\mathcal{A}, \alpha_t, \omega)$:

$$\begin{aligned} \|\gamma(A)\widetilde{\Omega}\|^2 &= \widetilde{\omega}(\gamma(A^*A)) = \lim_{t\to\infty} \widetilde{\omega}(\widetilde{\alpha}_{-t} \circ \alpha_t(A^*A)) \\ &= \lim_{t\to\infty} \widetilde{\omega}(\alpha_t(A^*A)) = \omega(A^*A) = \|A\Omega\|^2 . \end{aligned}$$

Let $\Gamma : \mathcal{H} \to \mathcal{H}$ denote the isometric extension of Γ_0 to $\overline{\mathcal{A}\Omega} = \mathcal{H}$. Clearly we have:

$$\Gamma\Omega = \Gamma \cdot \mathbf{1}\Omega = \gamma(\mathbf{1})\widetilde{\Omega} = \widetilde{\Omega} .$$

Similarly we consider

$$\widetilde{\Gamma}_0 : \mathcal{A}\widetilde{\Omega} \to \mathcal{A}\Omega : A\widetilde{\Omega} \mapsto \widetilde{\gamma}(A)\Omega .$$

Then $\widetilde{\Gamma}_0$ extends to an isometry $\widetilde{\Gamma} : \mathcal{H} \to \mathcal{H}$ by the mixing property of $(\mathcal{A}, \widetilde{\omega}, \widetilde{\alpha}_t)$. Moreover, we have for all $A, B \in \mathcal{A}$:

$$\begin{aligned} \langle A\Omega, \widetilde{\Gamma}B\widetilde{\Omega}\rangle &= \omega(A^*\widetilde{\gamma}(B)) = \lim_{t\to\infty} \widetilde{\omega}(\alpha_t(A^*)\widetilde{\alpha}_t(B)) \\ &= \lim_{t\to\infty} \widetilde{\omega}(\widetilde{\alpha}_{-t} \circ \alpha_t(A^*)B) = \widetilde{\omega}(\gamma(A^*)B) \\ &= \langle \gamma(A)\widetilde{\Omega}, B\widetilde{\Omega}\rangle = \langle \Gamma A\Omega, B\widetilde{\Omega}\rangle . \end{aligned}$$

Since $\mathcal{A}\Omega$ and $\mathcal{A}\widetilde{\Omega}$ are dense in $\mathcal{H}$, it follows that $\widetilde{\Gamma} = \Gamma^*$, and since both Γ and $\widetilde{\Gamma}$ are isometric $\mathcal{H} \to \mathcal{H}$, they must be unitary and each other's inverse. Finally, from the intertwining relation

$$\gamma \circ \alpha_t(A) = \lim_{s\to\infty} \widetilde{\alpha}_{-s} \circ \alpha_{s+t}(A) = \lim_{u\to\infty} \widetilde{\alpha}_{t-u} \circ \alpha_u(A) = \widetilde{\alpha}_t \circ \gamma(A)$$

we may conclude that

$$e^{it(H+V)}\Gamma = \Gamma e^{itH} .$$

and the result follows. □

3. Stability of Linear Bose Dynamical Systems.

In this Section we shall apply Proposition 1 to the case of certain linear quantum dynamical systems obeying canonical commutation relations. In Theorems 4 and 5 below we shall give concrete sufficient conditions for the convergence of the Dyson series (condition (I) of Prop. 1). and for the existence of an invariant state (condition (II)).

3.1. The linear dynamical system.

Our starting point is a complex separable Hilbert space $\mathcal{K}$, on which a positive self-adjoint operator K is given with absolutely continuous spectrum, and from which a nonzero element q is singled out. The operator K determines a one-parameter group of unitary operators

$$S_t := e^{itK} .$$

By $\mathcal{S} = \mathcal{S}(\mathcal{K}, K, q)$ we shall denote the linear span of the vectors $(S_t q)_{t\in\mathbb{R}}$. If $\mathcal{S}$ is dense in $\mathcal{K}$, the triple $(\mathcal{K}, K, q)$ is completely characterised by the measure ν on $[0, \infty)$ given by

$$\langle q, S_t q\rangle = \int_0^\infty e^{ixt}\nu(dx) , \qquad (t \in \mathbb{R}) . \tag{9}$$

Let a 'temperature' $T \geq 0$ be given. By $D_T = D_T(\mathcal{K}, K)$ we shall mean the whole space $\mathcal{K}$ for $T = 0$, and for $T > 0$ the domain of $K^{-1/2}$.

Definition. By the **linear Bose dynamical system** over $(\mathcal{K}, K)$ at **temperature** $T \geq 0$ we mean a quadruple $(\mathcal{H}, W, \Omega; H)$, where $\mathcal{H}$ is a complex separable Hilbert space, W a strongly continuous map from D_T to the unitary operators on $\mathcal{H}$, Ω a unit vector in $\mathcal{H}$ and H a self-adjoint operator on $\mathcal{H}$ such that the following conditions hold.

(i) For all $f, g \in D_T(\mathcal{K}, K)$:

$$W(f)W(g) = e^{-\frac{i}{2}\mathrm{Im}\,\langle f,g\rangle} W(f+g) \; ;$$

(ii) for all $f \in D_T(\mathcal{K}, K)$:

$$\langle \Omega, W(f)\Omega\rangle = e^{-\frac{1}{4}\|f\|_T^2} \; , \quad \text{where} \quad \|f\|_T^2 := \begin{cases} \|f\|^2 & \text{if } T = 0, \\ \langle f, (\coth(K/T))f\rangle & \text{if } T > 0; \end{cases}$$

(iii) for all $f \in D_T(\mathcal{K}, K)$:

$$e^{itH} W(f)\Omega = W\left(e^{itK} f\right)\Omega \; ;$$

(iv) The linear span of the vectors $W(f)\Omega$, $(f \in D_T(\mathcal{K}, K))$ is dense in $\mathcal{H}$.

The Bose dynamical system is determined up to unitary equivalence by the pair $(\mathcal{K}, K)$ and the number T. For $T > 0$ the state $\omega_T : A \mapsto \langle \Omega, A\Omega\rangle$ satisfies the KMS condition with respect to the evolution $\alpha_t : A \mapsto e^{itH} A e^{-itH}$ on the von Neumann algebra $\mathcal{N}_T = \mathcal{N}_T(\mathcal{K}, K)$ generated by the operators $W(f)$, $f \in D_T$, and for $T = 0$ the Hamiltonian H is positive with ground state vector Ω. The temperature zero algebra $\mathcal{N}_0(\mathcal{K}, K)$ consists of all bounded operators on $\mathcal{H}$ ([BrR]).

Let $\mathcal{A}_T = \mathcal{A}_T(\mathcal{K}, K, q)$ denote the C*-algebra generated by operators $A(\kappa)$ of the form

$$A(\kappa) := \int_{\mathcal{S}} W(f)\kappa(df) \; ,$$

where κ is a finite complex measure on $\mathcal{S}(\mathcal{K}, K, q)$. We call $\mathcal{A}_T(\mathcal{K}, K, q)$ the *integral Weyl algebra* at temperature T over $\mathcal{S}(\mathcal{K}, K, q)$.

Lemma 2. *The quantum dynamical system* $(\mathcal{A}_T(\mathcal{K}, K, q), \omega_T, (\alpha_t)_{t\in\mathbb{R}})$ *is mixing for all* $T \geq 0$. *For* $T > 0$ *the mixing property extends to the von Neumann algebra* $\mathcal{N}_T(\mathcal{K}, K)$.

Remark. The temperature zero quantum dynamical system $(\mathcal{N}_0, \omega_0, \alpha_t)$ is not mixing. Indeed, since $\mathcal{N}_0 = \mathcal{B}(\mathcal{H})$, it contains the projection operator $P_\Omega := |\Omega\rangle\langle\Omega|$, which does not move under α_t:

$$\alpha_t(P_\Omega) = e^{itH}|\Omega\rangle\langle\Omega|e^{-itH} = |\Omega\rangle\langle\Omega| = P_\Omega \; .$$

For the same reason, Lemma 2 can only be true if $P_\Omega \notin \mathcal{A}_0$ and $P_\Omega \notin \mathcal{N}_T$ for $T > 0$. Indeed, $P_\Omega \in \mathcal{A}_0$ if and only if $\dim(\mathcal{K}) < \infty$ ([vNe]). For $T > 0$ the vector Ω is separating for $\mathcal{N}_T$, and since $P_\Omega\Omega = \Omega = \mathbf{1}\Omega$, the projection P_Ω does not lie in $\mathcal{N}_T$.

Proof of Lemma 2. As K has absolutely continuous spectrum, we have by the Riemann-Lebesgue lemma for all $f, h \in \mathcal{K}$:

$$\lim_{t\to\pm\infty} \langle f, e^{itK} h\rangle = 0 \; .$$

It follows that for all $f, g, h \in D_T$:

$$\omega_T\big(W(f)W(S_t g)W(h)\big) = \exp\big(-\tfrac{1}{4}(\|f\|_T^2 + \|g\|_T^2 + \|h\|_T^2) - \tfrac{1}{2}(\langle f, S_t g\rangle_T + \langle S_t g, h\rangle_T + \langle f, h\rangle_T)\big) \xrightarrow[t\to\pm\infty]{} \omega_T(W(f)W(h))\omega_T(W(g)) \; .$$

So we have

$$\lim_{t\to\pm\infty} \langle \Psi_1, \alpha_t(A)\Psi_2\rangle = \langle \Psi_1, \Psi_2\rangle \omega_T(A) \tag{10}$$

for all Ψ_1, Ψ_2 in the linear span of the vectors $W(f)\Omega$, $f \in D_T$, and for A of the form $W(g)$, $g \in D_T$. By cyclicity of Ω, (10) extends to all $\Psi_1, \Psi_2 \in \mathcal{H}$, and by dominated convergence to all A in the integral Weyl algebra. Putting $\Psi_2 = \Omega$ we obtain

$$\text{weak-}\lim_{t\to\pm\infty} e^{itH} = P_\Omega \; .$$

Now, choosing A_1', A_2' in the commutant $\mathcal{N}_T'$, we find that (10) holds for $\Psi_i := A_i'\Omega$ and $A \in \mathcal{N}_T$:

$$\langle A_1'\Omega, \alpha_t(A)A_2'\Omega\rangle = \langle (A_2')^* A_1'\Omega, U_t A\Omega\rangle \xrightarrow[t\to\pm\infty]{} \langle A_1'\Omega, A_2'\Omega\rangle\langle\Omega, A\Omega\rangle \; .$$

For $T > 0$, Ω is cyclic for $\mathcal{N}_T'$ as well as $\mathcal{N}_T$, therefore $(\mathcal{N}_T, \omega, \alpha_t)$ is mixing. □

3.2. The nonlinear dynamical system.

We shall perturb the dynamics α_t on $\mathcal{B}(\mathcal{H})$ by the bounded self-adjoint operator $V \in \mathcal{A}_T$ given by

$$V := v(Q) := \int_{-\infty}^{\infty} W(\lambda q)\mu(d\lambda) \; ,$$

where μ is a finite complex measure on $\mathbb{R}$ satisfying $\mu(-E) = \overline{\mu(E)}$ for all Borel subsets E of $\mathbb{R}$. By μ_+ we denote the total variation measure of μ:

$$\mu_+(E) := \sup\Big\{ \int_E e^{i\vartheta(x)}\mu(dx) \;\Big|\; \vartheta : E \to \mathbb{R} \text{ Borel measurable } \Big\} \; .$$

We now proceed to check the conditions of Prop. 1 one by one.

Lemma 3. *$\mathcal{A}_T(\mathcal{K}, K, q)$ is invariant for* $\widetilde{\alpha}_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) : A \mapsto e^{it(H+V)}Ae^{-it(H+V)}$.

Proof. Let κ be a finite complex Borel measure on $\mathcal{S}(\mathcal{K}, K, q)$. Using (7), $\widetilde{\alpha}_t(A(\kappa))$ can be written in the form

$$\widetilde{\alpha}_t(A(\kappa)) = \text{norm-}\lim_{N\to\infty} \sum_{n=0}^{N} \int_{\mathcal{S}} W(f)\rho^{n,t}(df) \; ,$$

where $(\rho^{n,t})_{n\in\mathbb{N}, t\in\mathbb{R}}$ is a family of measures on $\mathcal{S}$ satisfying

$$\rho_+^{n,t}(\mathcal{S}) \le \frac{|t|^n}{n!}\mu_+(\mathbb{R})^n\kappa_+(\mathcal{S}) \; .$$

Since for each t this is a summable sequence in n, $\widetilde{\alpha}_t(A(\kappa))$ is a norm limit of Weyl integrals, hence $\widetilde{\alpha}_t(A(\kappa)) \in \mathcal{A}_T(\mathcal{K}, K, q)$. If A is any norm limit of Weyl integrals $A(\kappa_n)$, $n \to \infty$, then again $\widetilde{\alpha}_t(A) \in \mathcal{A}_T$ since $\widetilde{\alpha}_t$ is a *-automorphism, hence norm-continuous. $\square$

Our main result is the following. Define

$$M : [0, \infty) \to [0, \infty] : x \mapsto \int_{-\infty}^{\infty} |\lambda| e^{|\lambda| x} \mu_+(d\lambda) \ .$$

Theorem 4. *Suppose that* $\|g\| := \int_0^\infty |\mathrm{Im}\,\langle q, S_t q\rangle| \ dt < \infty$, *and that the equation*

$$y = M(\|g\|\, y) \tag{11}$$

admits a solution $y \geq 0$. *Then condition (I) of Prop. 1 (convergence of the Dyson series) is satisfied for the quantum dynamical system* $(\mathcal{A}_T(\mathcal{K}, K, q), \omega_T, \alpha_t)$.

Proof. It suffices to prove convergence of the Dyson series (6) for

$$A := W\left(\lambda_0 S_{t_0} q\right) \ ,$$

with some fixed $t_0 \in \mathbb{R}$ and $\lambda_0 \in \mathbb{C}$. Now let $g : [0, \infty) \to \mathbb{R}$ be as in (2), and define

$$\widetilde{g}(t) := \langle S_t q, S_{t_0} q\rangle \ .$$

Let $\widetilde{m}_k := |\lambda_0|^k$ and $m_k := \int_0^\infty |\lambda|^{k+1} \mu_+(d\lambda)$. By repeated use of the equality

$$[W(f), W(h)] = -2i \sin(\tfrac{1}{2} \mathrm{Im}\,\langle f, h\rangle) W(f+h) \ ,$$

we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \int_{0 \leq t_1 \leq \cdots \leq t_n} \!\!\!\cdots\!\! \int \left\| [\alpha_{-t_n}(V), [\cdots[\alpha_{-t_1}(V), A]\cdots]] \right\| dt_1 dt_2 \cdots dt_n \\
&\quad \leq \sum_{n=0}^{\infty} \int_{0 \leq t_1 \leq \cdots \leq t_n} \!\!\!\cdots\!\! \int \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \prod_{j=1}^{n} 2 \sin \tfrac{1}{2} \left(\sum_{c=0}^{j-1} \mathrm{Im}\,\langle \lambda_j S_{t_j} q, \lambda_c S_{t_c} q\rangle \right) \right| \\
&\qquad\qquad \times \mu_+(d\lambda_1) \cdots \mu_+(d\lambda_n) dt_1 dt_2 \cdots dt_n \\
&\quad \leq \sum_{n=0}^{\infty} \sum_{c_1=0}^{0} \sum_{c_2=0}^{1} \sum_{c_3=0}^{2} \cdots \sum_{c_n=0}^{n-1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\prod_{j=1}^{n} |\lambda_j \lambda_{c_j}| \right) \mu_+(d\lambda_1) \cdots \mu_+(d\lambda_n) \\
&\qquad\qquad \times \int_{0 \leq t_1 \leq \cdots \leq t_n} \!\!\!\cdots\!\! \int \left(\prod_{j=1}^{n} g_{c_j}(t_j - t_{c_j}) \right) dt_1 dt_2 \cdots dt_n \ , \\
&\quad = \sum_{n=0}^{\infty} \sum_{c_1=0}^{0} \cdots \sum_{c_n=0}^{n-1} \widetilde{m}_{d_c(0)} \prod_{j=1}^{n} m_{d_c(j)} \int_{0 \leq t_1 \leq \cdots \leq t_n} \!\!\!\cdots\!\! \int \left(\prod_{j=1}^{n} g_{c_j}(t_j - t_{c_j}) \right) dt_1 dt_2 \cdots dt_n \ ,
\end{aligned}$$

where $g_0 := |\widetilde{g}|$ and $g_c = |g|$ for $c \neq 0$ and $d_c(j)$ is the number of those $i \leq n$ for which $c_i = j$. The r.h.s. is the sum $\Phi(\widetilde{m}, m, \widetilde{g}, g)$ of Section 4. It will be proved that it converges *if and only if* the graph of M intersects the line $x = \|g\|\, y$. $\square$

Theorem 5. *If one of the following three conditions holds:*

(a) $T > 0$;

(b) $K \geq \varepsilon \cdot \mathbf{1}$ *for some* $\varepsilon > 0$;

(c) $\|v'\|_\infty \cdot \|K^{-1}q\| \leq \sqrt{2}$,

then $H + V$ *has an eigenvector.*

Proof. For $T > 0$ an eigenvector $\widetilde{\Omega}$ is provided by Araki's time independent perturbation theory of KMS states ([Ara, BrR]) or Connes' cocycle theorem [Con].
If $T = 0$ we may represent $\mathcal{H}$ as the L^2 space of the Gaussian probability space indexed by a real subspace of $\mathcal{K}$ containing q, such as $L^2_{\mathbb{R}}([0,\infty),\nu)$. It is known [Sim] that on such a space the semigroup e^{-tH} is positivity improving and hypercontractive, provided that $K \geq \varepsilon \cdot \mathbf{1}$. Then we can apply Theorem XIII.49 from [ReS4] to the bounded multiplication operator $V = v(Q)$. The conclusion is that $H + V$ has a ground state.
If $T = 0$ and the spectrum of K has no gap above 0, we may approximate K by the operators $K + \varepsilon \cdot \mathbf{1}$, and show that under condition (c) the ground state found above survives the limit $\varepsilon \downarrow 0$. We postpone the proof to Section 5. □

3.3. Interpretation and comments.

Given the structure $(\mathcal{K}, K, q)$ and the measure ν from (9), let us define

$$\frac{1}{m} := \|K^{1/2}q\|^2 = \int_0^\infty x\nu(dx) \ . \tag{12}$$

and

$$\frac{1}{\alpha} := \|K^{-1/2}q\|^2 = \int_0^\infty \frac{1}{x}\nu(dx) \ ; \tag{13}$$

If any of these integrals diverges, we take the corresponding constant m or α to be 0.
If $q \in \mathcal{K}$ is to be interpreted as the position of a harmonic oscillator in some linear environment, then the constants m and α must both be positive, since they play the role of the 'effective' mass and spring constant of the oscillator respectively: suppose that the momentum operator P exists (i.e. $q \in \mathrm{Dom}\,(K)$). Then we should have according to (3),

$$\langle q, Kq\rangle = \mathrm{Im}\,\langle q, iKq\rangle = \mathrm{Im}\,[Q, \frac{1}{m}P] = \frac{1}{m} \ .$$

It is reasonable to persist in the interpretation of m as a mass even if P does not exist. The interpretation of α is justified by the following lemma.

Lemma 6. *Let* $(\mathcal{K}, K, q)$ *be as in (1) and let* $(\mathcal{H}, W, \Omega; H)$ *be the associated linear Bose dynamical system at temperature 0 with* $W(\lambda q) = e^{i\lambda Q}$. *Then the number* $\alpha \geq 0$ *given by (13) is the largest nonnegative number for which*

$$\exists_{c\in\mathbb{R}} : \qquad \tfrac{1}{2}\alpha Q^2 \leq H + c \cdot \mathbf{1} \ . \tag{14}$$

In physical terms, this lemma means that only those degrees of freedom can be forced to take small values by keeping the total energy low, whose phase space vectors are in the domain of $K^{-1/2}$. This explains why temperature states can only be defined on the Weyl algebra of $D_T = \mathrm{Dom}\,(K^{-1/2})$. It also puts a natural restriction on ν.

Proof. For all $f \in \mathrm{Dom}\,(K^{1/2})$ we have by Cauchy-Schwarz,

$$|\langle f, q\rangle|^2 = \left|\int_0^\infty f(x)\nu(dx)\right|^2 \leq \left(\int_0^\infty \frac{1}{x}|f(x)| \cdot x\nu(dx)\right)^2 \leq \frac{1}{\alpha}\|K^{1/2}f\|^2 \ .$$

So, if E_q denotes the orthogonal projection onto $\mathbb{C}q$, we have in the sense of quadratic forms:

$$\alpha E_q \leq K \ .$$

Now, at $T = 0$ we may take $\mathcal{H}$ to be the Fock space over $\mathcal{K}$, and employ the well-known second quantisation operator $d\Gamma$ (not related to the Γ of Prop. 1). Cf. [Sim, Par]. We find:

$$\tfrac{1}{2}\alpha Q^2 \leq \tfrac{1}{2}\alpha\big(\Phi(q)^2 + \Phi(iq)^2\big) = \alpha\big(d\Gamma(E_q) + \tfrac{1}{2}\cdot\mathbf{1}\big) \leq d\Gamma(K) + \tfrac{1}{2}\alpha\cdot\mathbf{1} = H + \tfrac{1}{2}\alpha\cdot\mathbf{1} \ .$$

Conversely, suppose that for some $\alpha' \geq 0$ we have $\frac{1}{2}\alpha' Q^2 \leq H + c\cdot\mathbf{1}$. For $\varepsilon > 0$, let f_ε be given by

$$f_\varepsilon(x) := 1_{[\varepsilon,\infty)}(x)\cdot\frac{1}{x} \ .$$

Then we have for all $\lambda \in \mathbb{R}$, putting $\Psi := W(i\lambda f_\varepsilon)\Omega$,

$$\begin{aligned}\langle \Psi, Q^2\Psi\rangle &= -\frac{d^2}{ds^2}\omega_0\big(W(-i\lambda f_\varepsilon)W(sq)W(i\lambda f_\varepsilon)\big)\,\big|_{s=0} \\ &= -\frac{d^2}{ds^2}\exp\big(\tfrac{i}{2}\lambda s\langle q, f_\varepsilon\rangle - \tfrac{1}{4}s^2\|q\|^2\big)\,\big|_{s=0} = \tfrac{1}{2}\|q\|^2 + \lambda^2\langle q, f_\varepsilon\rangle^2 \ .\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle \Psi, H\Psi\rangle &= \frac{d}{dt}\omega_0\big(W(-i\lambda f_\varepsilon)W(i\lambda S_t f_\varepsilon)\big)\,\big|_{t=0} \\ &= -i\frac{d}{dt}\exp\big(-\tfrac{1}{2}\lambda^2\|f_\varepsilon\|^2 + \tfrac{1}{2}\lambda^2\langle f_\varepsilon, S_t f_\varepsilon\rangle\big)\,\big|_{t=0} = \tfrac{1}{2}\lambda^2\langle f_\varepsilon, K f_\varepsilon\rangle \ .\end{aligned}$$

So our assumption on α' implies that $\alpha'\langle q, f_\varepsilon\rangle^2 \leq \langle f_\varepsilon, K f_\varepsilon\rangle = \langle q, f_\varepsilon\rangle$, i.e. for all $\varepsilon > 0$:

$$\alpha'\int_\varepsilon^\infty \frac{1}{x}\nu(dx) \leq 1 \ .$$

So α is the largest number with property (14). $\square$

Lemma 7. *Let $(\mathcal{K}, K, q)$ be of standard form (1), and let g be given by (2). Then*

$$\lim_{\varepsilon\downarrow 0}\int_0^\infty e^{-\varepsilon t}g(t)dt = \frac{1}{\alpha} \ .$$

In particular, $\frac{1}{\alpha} \leq \|g\|$.

Proof. By Fubini's theorem we have for all $\varepsilon > 0$:

$$\begin{aligned}\int_0^\infty e^{-\varepsilon t} g(t)dt &= \int_0^\infty e^{-\varepsilon t}\left(\operatorname{Im}\int_0^\infty e^{ixt}\nu(dx)\right)dt\\ &= \frac{1}{2i}\int_0^\infty e^{-\varepsilon t}\int_0^\infty \left(e^{ixt}-e^{-ixt}\right)\nu(dx)dt\\ &= \int_0^\infty\left(\frac{1}{2i}\int_0^\infty\left(e^{(-\varepsilon+ix)t}-e^{(-\varepsilon-ix)t}\right)dt\right)\nu(dx)\\ &= \int_0^\infty \frac{x}{\varepsilon^2+x^2}\nu(dx)\ ,\end{aligned}$$

from which the statement follows. □

Lemma 8. *If the condition (11) of Theorem 4 is satisfied, then the effective potential*

$$\tfrac{1}{2}\alpha Q^2 + v(Q)$$

is a strictly convex function of Q.

Proof. If v is constant, then the statement is trivially valid. If it is not, then $\int |\lambda|\mu_+(d\lambda) > 0$, so that $M(0) > 0$, $y > 0$ and on $[0, \|g\|\,y)$ the function M is strictly convex and analytic. It follows that

$$M'(0) < \frac{M(\|g\|\,y) - M(0)}{\|g\|\,y} = \frac{y - M(0)}{\|g\|\,y} < \frac{1}{\|g\|}\ .$$

Therefore by Lemma 7 we have for all $x \in \mathbb{R}$:

$$|v''(x)| \le \int_0^\infty |\lambda|^2\mu_+(d\lambda) = M'(0) < \frac{1}{\|g\|} \le \alpha\ ,$$

so for all x:

$$\frac{d^2}{dx^2}\left(\tfrac{1}{2}\alpha x^2 + v(x)\right) = \alpha + v''(x) > 0\ .$$

□

3.4. An example.

As a typical example let us choose $a, \lambda > 0$ and consider the perturbation

$$V = a\cos\lambda Q = \int_{-\infty}^{\infty} W(\lambda' q)\mu(d\lambda')\ ,$$

where $\mu = \mu_+ = \frac{1}{2}a(\delta_\lambda + \delta_{-\lambda})$. Then $M(x) = a\lambda e^{\lambda x}$, so condition (11) of Theorem 4 reads

$$\exists_{y\ge 0}:\qquad a\lambda e^{\lambda\|g\|y} = y\ .$$

This is equivalent with

$$\exists_{u\ge 0}:\qquad a\lambda^2\,\|g\| = ue^{-u}\ ,$$

i.e., since $\max\{\, ue^{-u} \mid u \ge 0\,\} = 1/e$,

$$a\lambda^2 \ge \frac{1}{e\,\|g\|}\ ,$$

as announced in (4).

4. The main estimate.

In this section we develop the estimate used in the proof of Theorem 4.

4.1. The result.

Let $m_0, m_1, m_2, \cdots$ and $\widetilde{m}_0, \widetilde{m}_1, \widetilde{m}_2, \cdots$ be two sequences of nonnegative numbers, and $g, \widetilde{g}$ two integrable functions $[0,\infty) \to [0,\infty)$. We consider the sum of integrals

$$\begin{aligned} \Phi(\widetilde{m}, m, \widetilde{g}, g) := \sum_{n=0}^{\infty} \quad & \sum_{c_1=0}^{0} \sum_{c_2=0}^{1} \sum_{c_3=0}^{2} \cdots \sum_{c_n=0}^{n-1} \widetilde{m}_{d_c(0)} \left(\prod_{j=1}^{n} m_{d_c(j)} \right) \\ & \times \int \cdots \int_{0 \le t_1 \le \cdots \le t_n} \left(\prod_{j=1}^{n} g_{c_j}(t_j - t_{c_j}) \right) dt_1 dt_2 \cdots dt_n, \end{aligned} \tag{15}$$

where

$$d_c(j) := \#\{\, i \in \{1, 2, \cdots, n\} \,\big|\, c_i = j \,\}$$

and

$$g_c := \begin{cases} \widetilde{g} & \text{if } c = 0, \\ g & \text{if } c \neq 0. \end{cases}$$

Let the generating functions $M, \widetilde{M} : [0,\infty) \to [0,\infty]$ be defined by

$$M(x) := \sum_{k=0}^{\infty} \frac{m_k}{k!} x^k \quad \text{and} \quad \widetilde{M}(x) := \sum_{k=0}^{\infty} \frac{\widetilde{m}_k}{k!} x^k \ , \tag{16}$$

and let $\|g\|, \|\widetilde{g}\|$ denote the integrals of g and $\widetilde{g}$ respectively.

Theorem 9. *The sum $\Phi(\widetilde{m}, m, \widetilde{g}, g)$ in (15) converges if and only if the equation*

$$M(\|g\| y) = y \tag{11}$$

allows a solution y for which $\widetilde{M}(\|\widetilde{g}\|\, y) < \infty$. If y is the least solution, then

$$\Phi(\widetilde{m}, m, \widetilde{g}, g) = \widetilde{M}(\|\widetilde{g}\|\, y) \ . \tag{17}$$

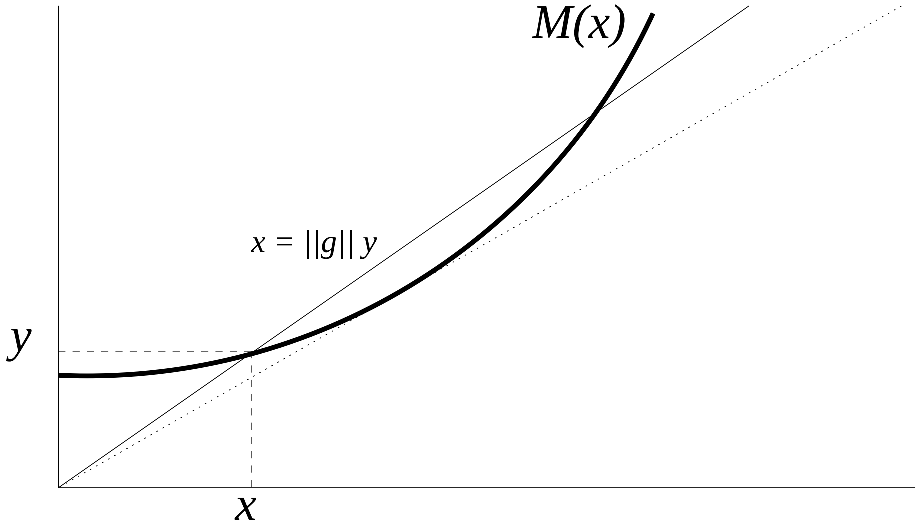


Fig. 1: the main estimate.

4.2. Some consequences.

1. In the special case that

$$\widetilde{m} = m, \qquad \widetilde{g} = g, \qquad \sum_{k=0}^{\infty} \frac{m_k}{k!} = 1, \quad \text{and} \quad \|g\| := \int_0^\infty g(t)dt = 1, \tag{18}$$

this theorem allows the following interpretation. Consider the branching process where at time 0 a single individual of some species is present. At a positive random time (having probability density g) it splits into k new individuals with probability $m_k/k!$. These in their turn live for independent random times — again distributed according to g — and produce independent offspring according to the same law $m_k/k!$, et cetera. Then the sum $\Phi(m, m, g, g)$ in (17) is the total probability measure carried by all possible finite family trees, and is therefore equal to the probability that the progeny of the original individual will eventually die out.
The content of Theorem 9 is then the central statement of Galton-Watson theory, namely that the extinction probability of a branching process equals the lowest fixed point of the generating function of the number of offspring per individual (E.g. [Har]).

In this Section we shall give an independent analytic proof of Theorem 9.
One reason for this effort is, that the interpretation of the right hand side of (17) as the extinction probability of an age-dependent branching process is not immediate.
Another reason is that still a scaling argument would be needed to remove the normalisation restrictions in (18). (To remove the restrictions $\widetilde{m} = m$ and $\widetilde{g} = g$ we only have to allow the first generation to have a different behaviour from its later descendents.)
A third reason is the possibility to generalise the result to distributions in more than one dimension (in the language of Section 3: to a general Weyl integral V).

2. In the case $\widetilde{m}_k = m_k = 1$ $(k \in \mathbb{N})$, $\|g\| = \|\widetilde{g}\| =: t$, we have $M(x) = e^x$, and Theorem 9 implies that the sum $\Phi(t) := \Phi(\widetilde{m}, m, \widetilde{g}, g)$ satisfies

$$\Phi(t) = e^{t\Phi(t)} \ .$$

So, putting $A(t) := t\Phi(t)$ we find

$$A(t) = te^{A(t)}. \tag{19}$$

A and Φ are known as the generating functions of the combinatorial species 'rooted tree' and 'forest' in the sense of Joyal [Joy, BLL]. A is the inverse of the function $z \mapsto ze^{-z}$, which can be found using Lagrange's inversion formula. The result for Φ is then

$$\Phi(t) = \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{n!} \, t^n \ . \tag{20}$$

In particular, the n-th term of the sum in (17) is $(n+1)^{n-1}\|g\|^n/n!$, and the sum converges for $\|g\| \leq 1/e$.
This case was studied by Botvich, Fayolle and Malyshev [BFM] in the context of network theory. Theorem 9 is an improvement compared to the result in [BFM], where the n-th

term was estimated by $(8\|g\|)^n$. In this special case it is only slightly better that the result of [FiL] which gives for the n-th term the estimate

$$2^n t^n \sqrt{e} c^{\frac{n \log\log n}{\log n}} \ , \quad (c > e) \ .$$

Note, however, that (17) and (20) are equalities, not estimates.

3. In [Maa] and [Spo] an estimate was used which required exponential decay of $g(t)$ as $t \to \infty$. Theorem 9 is an important improvement compared to this since exponential decay is the exception rather than the rule.

4.3. Rooted trees.

According to the usual definition a rooted tree is a finite connected graph without cycles and with one distinguished vertex. Here we prefer to use the following, equivalent definition.

A *rooted tree* or *arborescence* is a pair (V, a), where V is a finite set, the set of *vertices*, and a is a map $V \to V$ with the property that a^k becomes eventually constant. The constant is called the *root*, and we shall denote it by $\odot(a)$ or just $\odot$. The least value of k for which a^k is constant is the *height* of the tree.

Drawing an arrow from v to $a(v)$ for each $v \in V \setminus \{\odot\}$, we obtain an oriented graph. The number n of arrows will be called the *order* of the tree. Note that $\#(V) = n + 1$. The vertex set V is partially ordered in a natural way: we say that $v \prec w$ if $v = a^k(w)$ for some $k \in \mathbb{N}$. We think of $a(v)$ as the *parent* of v. By $d_a(v)$ we denote the number $\#a^{-1}(v)$ of offspring of v. By V^* we shall mean $V \setminus \{\odot\}$.

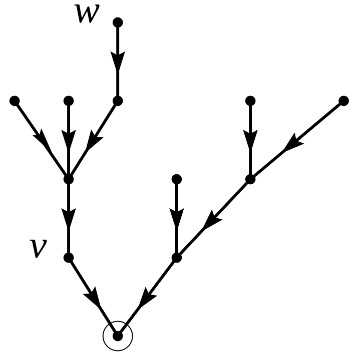


Fig.2: A rooted tree $(v \prec w)$.

Arborescences (V, a) and (W, b) are considered *isomorphic* if there is a bijection $f : V \to W$ such that $b \circ f = f \circ a$. We denote the collection of all isomorphism classes of arborescences of order n by $\mathbb{A}_n$. We write $\mathbb{A}$ for $\bigcup_{n \in \mathbb{N}} \mathbb{A}_n$.

Abusing notation we shall denote elements of $\mathbb{A}$ again by (V, a) or even just by a.

For a rooted tree (V, a) of order n we have the identity

$$\sum_{v \in V} d_a(v) = n. \tag{21}$$

An *automorphism* of $a \in \mathbb{A}$ is an isomorphism $a \to a$. We denote the group of all automorphisms of a by aut (a). By an *indexation* of a rooted tree (V, a) of order n we mean an order-preserving bijection $\iota : V \to \{0, 1, 2, \ldots, n\}$. The set of all indexations of (V, a) will be denoted by $I(a)$. Note that for $\iota \in I(a)$ we always have $i(\odot) = 0$. Most rooted trees have several indexations.

4.4. Proof of the main estimate.

The sum (17) contains a summation over functions $c : \{1, \cdots, n\} \to \{1, \cdots, n\}$ which are *decreasing* in the sense that $c(j) < j$ for all j. We shall call such maps *climbers* of *order* n. Note that a climber is an arborescence with vertex set $\{0, 1, \cdots, n\}$ satisfying

$$i \prec j \quad \Longrightarrow \quad i \leq j \ .$$

We denote the set of all climbers of order n by $\mathcal{C}_n$.
The sum over $\mathcal{C}_n$ occurring in (17) can be replaced by a sum over (indexed) rooted trees.

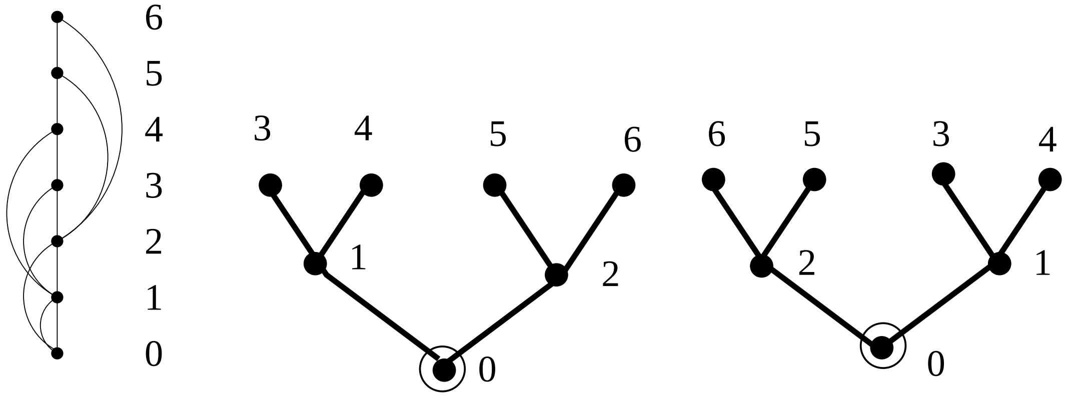


Fig. 3. A climber and two indexed rooted trees.

Lemma 10. *Let* $F : \mathcal{C}_n \to \mathbb{R}$. *Then*

$$\sum_{c \in \mathcal{C}_n} F(c) = \sum_{a \in \mathbb{A}_n} \frac{1}{|\mathrm{aut}\,(a)|} \sum_{\iota \in I(a)} F(\iota \circ a \circ \iota^{-1}) \ .$$

Proof. Choose a climber c of order n, and let (V, a) be its isomorphism class. We must count the number of indexations ι of (V, a) which lead to the climber c. Now, a bijection $\iota : V \to \{0, 1, \cdots, n\}$ enjoys this property iff the following conditions hold:
(i) $v \prec w \Longrightarrow \iota(v) \leq \iota(w)$;
(ii) $c = \iota \circ a \circ \iota^{-1}$.
But (i) is implied by (ii) since the order is determined by the graph. So it suffices to count the maps ι which satisfy (ii). Choosing $V = \{0, 1, \cdots, n\}$ and $a = c$ we see that these are just the automorphisms of a.

□

Now let

$$\Delta_n := \left\{ t = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n \;\middle|\; 0 \leq t_1 \leq \cdots \leq t_n \right\} .$$

The following lemma allows us to raplace the summation over $I(a)$ together with the integration over Δ_n by an easy integral over $[0, \infty)^n$.

Lemma 11. (Packing Lemma). *For any rooted tree* (V, a) *the map*

$$\vartheta_a : \quad I(a) \times \Delta_n \longrightarrow [0, \infty)^{V^*} : \quad (\iota, t) \mapsto r, \quad \textit{where} \quad r_v := t_{\iota(v)} - t_{\iota(a(v))},$$

is bijective up to a subset of $[0,\infty)^{V^*}$ *of measure zero, and has Jacobian 1 on each component* $\{\iota\} \times \Delta_n$.

Proof. Let $n := \#(V^*)$, and choose a point $r \in [0,\infty)^{V^*}$. Allocate a 'branching time' s_v to each vertex $v \in V$ by putting

$$s_v := \sum_{w \prec v} r_w \quad (\text{with } s_\odot := 0).$$

If some of these values s_v coincide, which happens only for a set of points r of measure 0, then r is not in the range of ϑ_a. If they are all different, they determine by their order a unique indexation ι of V:

$$s_v \le s_w \quad \Longleftrightarrow \quad \iota(v) \le \iota(w).$$

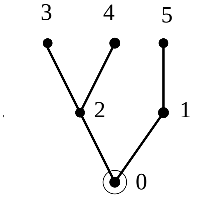
Putting $t_{\iota(v)} := s_v$ we obtain $t \in \Delta_n$ with the property that

$$\vartheta_a(\iota,t)_v := t_{\iota(v)} - t_{\iota(a(v))} = s_v - s_{a(v)} = \sum_{w \prec v} r_w - \sum_{w \prec a(v)} r_w = r_v.$$

So r lies in the range of ϑ_a. Conversely, if $r = \vartheta_a(\iota,t)$, we must have

$$t_{\iota(v)} = r_v + t_{\iota(a(v))} = \cdots = \sum_{w \prec v} r_w = s_v.$$

And since $t_1 < t_2 < t_3 < \cdots < t_n$, the indexation ι is uniquely determined by the order of the 'branching times' s_v, hence by r. So ϑ_a is injective.
Finally, the map $t \mapsto \vartheta_a(\iota,t)$, for $\iota \in I(a)$ fixed can be written as a $V^* \times n$-matrix $(M_{v,k})$, which has 1's at the positions (v,k), where $\iota(v) = k$, (-1)'s at the positions (v,k) with $\iota(a(v)) = k$ and 0's everywhere else. We may put M in standard form by ordering the points in V^* according to the indexation ι, thus putting all the 1's on the diagonal.



$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 4. A indexed rooted tree (a,ι) with its matrix $(M_{v,k})$.

Then since $\iota(a(v)) < \iota(v)$, all the (-1)'s end up below the diagonal. So $\det(M)$ equals 1 in this standard form, and ± 1 in any other ordering of V^*. The Jacobian $|\det(M)|$ of the piecewise linear map ϑ_a equals 1 everywhere.

$\square$

Lemma 12. *The sum of integrals* $\Phi(\widetilde{m}, m, \widetilde{g}, g)$ *in (17) can be written as*

$$\Phi(\widetilde{m}, m, \widetilde{g}, g) = \sum_{(V,a) \in \mathbb{A}} \frac{\|\widetilde{g}\|^{d_a(\odot)}\, \|g\|^{\#(V^*) - d_a(\odot)}}{|\mathrm{aut}\,(a)|} \widetilde{m}_{d_a(\odot)} \left(\prod_{v \in V^*} m_{d_a(v)} \right).$$

Proof. In (17) $\Phi(\widetilde{m}, m, \widetilde{g}, g)$ is written as a sum over n of integrals over $\mathcal{C}_n \times \Delta_n$. First we apply Lemma 10 to replace the sum over $\mathcal{C}_n$ by a sum over indexed rooted trees (a, ι). We obtain

$$\Phi(\widetilde{m}, m, \widetilde{g}, g) = \sum_{n=0}^{\infty} \sum_{(V,a)\in\mathbb{A}_n} \frac{1}{|\mathrm{aut}\,(a)|} \left(\prod_{v\in V} m_{d_a(v)} \right) \times \sum_{\iota\in I(a)} \int_{\Delta_n} \left(\prod_{v\in V^*} g_{\iota(v)}(t_{\iota(v)} - t_{\iota(a(v))}) \right) dt.$$

Then we apply the Packing Lemma (Lemma 11) to replace the sum over ι and the integration over t by an integration over r:

$$\Phi(\widetilde{m}, m, \widetilde{g}, g) = \sum_{(V,a)\in\mathbb{A}} \frac{1}{|\mathrm{aut}\,(a)|} \widetilde{m}_{d_a(\odot)} \left(\prod_{v\in V^*} m_{d_a(v)} \right) \int_{[0,\infty)^{V^*}} \left(\prod_{v\in V^*} g_v(r_v) \right) dr,$$

where $g_v := \widetilde{g}$ if v is the root, otherwise $g_v := g$. The integral over r is now obvious, and the Lemma is proved.

□

Now for $\widetilde{m}, m, \widetilde{g}$ and g fixed, let the *weight* $\widetilde{w}(a)$ of a rooted tree a, as in Lemma 12, be given by

$$\widetilde{w}(a) := \|\widetilde{g}\|^{d_a(\odot)} \, \|g\|^{\#(V^*)-d_a(\odot)} \, \widetilde{m}_{d_a(\odot)} \prod_{v\in V^*} m_{d_a(v)}.$$

By $w(a)$ we shall denote the same weight, but with $\widetilde{g} = g$ and $\widetilde{m} = m$.
Let $\mathbb{A}(h)$ with $h \in \mathbb{N}$ denote the set of all rooted trees of height $\leq h$, and define

$$\Phi_h(\widetilde{m}, m, \widetilde{g}, g) := \sum_{(V,a)\in\mathbb{A}(h)} \frac{\widetilde{w}(a)}{|\mathrm{aut}\,(a)|} \; .$$

Lemma 13. *For all pairs of sequences $\widetilde{m}, m$ of nonnegative numbers, all pairs of functions $g, \widetilde{g} : [0,\infty) \to [0,\infty)$ and all $h \in \mathbb{N}$, $t \geq 0$,*

$$\Phi_{h+1}(\widetilde{m}, m, \widetilde{g}, g) = \widetilde{M}(\|\widetilde{g}\| \, \Phi_h(m, m, g, g)),$$

where M and $\widetilde{M}$ are the generating functions given in (16).

Proof. If a is a rooted tree of height at most $h{+}1$ and root degree $d_a(\odot) = n$, and we cut off its root, then we are left with n rooted trees of height at most h. Therefore, the summation over all $(V, a) \in \mathbb{A}(h{+}1)$ can be replaced by a sum over sequences $\alpha = (\alpha(1), \alpha(2), \cdots, \alpha(n))$ of rooted trees in $\mathbb{A}(h)$, where such sequences are to be identified if they differ by a permutation. So the summation goes over the set $\mathbb{A}^n/S_n$ of orbits

$$\widetilde{\alpha} := \left\{ \, \alpha \circ \pi \; \middle| \; \pi \in S_n \, \right\}$$

of sequences $\alpha \in \mathbb{A}(h)^n$ under S_n. With this correspondence we have

$$\widetilde{w}(a) = \widetilde{m}_n \, \|\widetilde{g}\|^n \cdot \prod_{j=1}^{n} w(\alpha(j)),$$

and

$$|\mathrm{aut}\,(a)| = \#(\widetilde{\alpha}) \cdot \prod_{j=1}^{n} |\mathrm{aut}\,(\alpha(j))|.$$

Now let $N(\alpha) := \#\{\, \pi \in S_n \mid \alpha \circ \pi = \alpha \,\}$ denote the size of the stabilizer of α, (i.e. $N(\alpha) = \prod_{k=1}^{l} n_k!$ if α takes l different tree values $n_1, n_2, \cdots, n_l$ times respectively.) Then

$$\#(\widetilde{\alpha}) = \frac{n!}{N(\alpha)}.$$

We calculate, starting from Lemma 12,

$$\begin{aligned}
\Phi_{h+1}(\widetilde{m}, m, \widetilde{g}, g) &= \sum_{(V,a)\in\mathbb{A}(h+1)} \frac{\widetilde{w}(a)}{|\mathrm{aut}\,(a)|} \\
&= \sum_{n=0}^{\infty} \widetilde{m}_n \,\|\widetilde{g}\|^n \sum_{\widetilde{\alpha}\in\mathbb{A}(h)^n/S_n} \frac{1}{\#(\widetilde{\alpha})} \prod_{j=1}^{n} \frac{w(\alpha(j))}{|\mathrm{aut}\,(\alpha(j))|} \\
&= \sum_{n=0}^{\infty} \frac{\widetilde{m}_n}{n!} \,\|\widetilde{g}\|^n \sum_{\widetilde{\alpha}\in\mathbb{A}(h)^n/S_n} N(\alpha) \prod_{j=1}^{n} \frac{w(\alpha(j))}{|\mathrm{aut}\,(\alpha(j))|} \\
&= \sum_{n=0}^{\infty} \frac{\widetilde{m}_n}{n!} \,\|\widetilde{g}\|^n \sum_{\alpha\in\mathbb{A}(h)^n} \prod_{j=1}^{n} \frac{w(\alpha(j))}{|\mathrm{aut}\,(\alpha(j))|} \\
&= \sum_{n=0}^{\infty} \frac{\widetilde{m}_n}{n!} \,\|\widetilde{g}\|^n \left(\sum_{(V,a)\in\mathbb{A}(h)} \frac{w(a)}{|\mathrm{aut}\,(a)|} \right)^n \\
&= \widetilde{M}\left(\|\widetilde{g}\| \, \Phi_h(m, m, g) \right).
\end{aligned}$$

□

Proof of Theorem 9. First consider the case $\widetilde{m} = m$, $\widetilde{g} = g$. Let M_g denote the map

$$y \mapsto M(\|g\| \, y).$$

By Lemma 13, and since $\Phi_0(m, g) = m_0 = M_g(0)$, we have for all $n \geq 1$:

$$\Phi_{n-1}(m, g) = M_g^{\circ n}(0).$$

Now let us call this number: y_n. Then $y_1, y_2, y_3, \cdots$ is a non-decreasing sequence in $[0, \infty]$, tending to $\Phi(m, g)$ if the sum (17) converges, and to infinity otherwise.
Now suppose that (11) has a solution, i.e. M_g has a fixed point $u \geq 0$. Then, since M_g is non-decreasing, and since $0 \leq u$, we have for all $n \geq 1$:

$$y_n = M_g^{\circ n}(0) \leq M_g^{\circ n}(u) = u.$$

Being bounded above, the sequence converges to a limit $y_\infty \leq u$. As y_∞ must be a fixed point itself, it is the least such point.

On the other hand, if (11) has no solution, the sequence $y_1, y_2, y_3, \cdots$ can have no finite limit, so it must tend to infinity. This proves the theorem in the case $\widetilde{m} = m$, $\widetilde{g} = g$.
In the general case, we define $\widetilde{M}_{\tilde{g}}(y) := \widetilde{M}(\|\widetilde{g}\|\, y)$, and consider the sequence $z_1, z_2, \cdots$, where

$$z_n := \widetilde{M}_{\tilde{g}} \circ M_g^{\circ n}(0) \ .$$

Then by Lemma 13

$$\begin{aligned} \Phi(\widetilde{m}, m, \widetilde{g}, g) &= \lim_{h\to\infty} \Phi_h(\widetilde{m}, m, \widetilde{g}, g) = \lim_{h\to\infty} \widetilde{M}_{\tilde{g}} \circ M_g^{\circ h}(0) \\ &= \lim_{h\to\infty} \widetilde{M}_{\tilde{g}}(y_h) \ , \end{aligned}$$

and the theorem is proved.

□

5. Existence of the Ground State.

In this Section we shall prove part of Theorem 5: we shall show that condition (c) ensures the existence of a ground state vector for $H + V$. Since the case $T > 0$ is already covered by condition (a), we may assume here that $T = 0$, i.e. $\mathcal{H}$ is the Fock space over $\mathcal{K}$, H is the operator $d\Gamma(K)$ and for all $f \in \mathcal{K}$

$$\Phi(f) = \frac{a(f) + a(f)^*}{\sqrt{2}} \ ,$$

in the usual notation regarding Fock space (cf. for example [Sim]). Our main tool is the 'photon number' operator $N := d\Gamma(\mathbf{1})$. Below we shall use the inequality [BFS1]:

$$\|a(f)\psi\|^2 \leq \|f\| \cdot \|N^{1/2}\psi\| \ , \tag{22}$$

and the Lemma

Lemma 14. *Let $(f_l)_{l=1}^{\infty}$ be a complete orthonormal basis for $\mathcal{K}$ with $f_l \in \mathrm{Dom}(K)$. Suppose that $K \geq \varepsilon \cdot \mathbf{1}$ for some $\varepsilon > 0$. Then we have for all $\psi \in \mathrm{Dom}(H^{1/2})$,*

$$\sum_{l=1}^{\infty} \langle a(K^{-1/2} f_l)\psi, a(K^{1/2} f_l)\psi \rangle = \|N^{1/2}\psi\|^2 \ .$$

Proof. Cf. [ArH]. □

Lemma 15. *Let H_n $(n \in \mathbb{N})$ and H be selfadjoint operators on a Hilbert space $\mathcal{H}$ having a common core D such that for all $\psi \in D$, $H_n\psi \to H\psi$ as $n \to \infty$. Let ψ_n be a normalized eigenvector of H_n with eigenvalue E_n, such that $E = \lim_{n\to\infty} E_n$ and the weak limit $\lim_{n\to\infty} \psi_n = \psi$ exists and is nonzero. Then ψ is an eigenvector of H with eigenvalue E. In particular if ψ_n is the ground state of H_n, then ψ is the ground state of H.*

Proof. Cf. [ArH] □

Theorem 16. *Consider $(\mathcal{K}, K, q)$ as defined in Section 3, and let $v : \mathbb{R} \to \mathbb{R}$ be a bounded differentiable function with derivative v'. Suppose that $q \in \operatorname{Dom}(K^{-1})$ and*

$$\|v'\|_\infty \cdot \left\|K^{-1}q\right\| < \sqrt{2}\,.$$

Then $d\Gamma(K) + v(\Phi(q))$ has a ground state.

Proof. For all $\varepsilon > 0$ Theorem 5, condition (b) yields the existence of a ground state vector for $d\Gamma(K+\varepsilon) + v(\Phi(q))$. By the weak compactness of the unit ball in $\mathcal{H}$ there exists a sequence $\varepsilon_n \longrightarrow 0$ and a weakly convergent sequence $\psi_n \longrightarrow \psi$ such that ψ_n is a ground state vector for $\widetilde{H}_n := (d\Gamma(K+\varepsilon_n) + v(\Phi(q))$. Lemma 15 will then ensure the existence of the ground state for $\widetilde{H} = d\Gamma(K) + v(\Phi(q))$, provided that $\psi \neq 0$.
Let $E_n := \inf\operatorname{spec}\widetilde{H}_n$. As $\widetilde{H}_n - E_n \geq 0$ we have for all $f \in \mathcal{K}_{\mathbb{R}} \cap \operatorname{Dom}(\mathcal{K})$:

$$0 \leq \langle a(f)\psi_n, (\widetilde{H}_n - E_n)a(f)\psi_n\rangle,$$

Then using $(\widetilde{H}_n - E_n)\psi_n = 0$, we get $\langle a(f)\psi_n, [\widetilde{H}_n, a(f)]\psi_n\rangle \geq 0$ and the commutator is

$$\begin{aligned}[d\Gamma(K+\varepsilon) + v(\Phi(q)), a(f)] &= -a((K+\varepsilon)f) + v'(\Phi(q))[\Phi(q), a(f)] \\ &= -a((K+e)f) - \frac{1}{\sqrt{2}}v'(\Phi(q))\langle f, q\rangle.\end{aligned}$$

We make the substitition $f = (K+\varepsilon_n)^{-1/2}f_l$, where $(f_l)_{l=1}^\infty$ is the orthonormal basis from Lemma 14, and then we take the sum over all l:

$$\begin{aligned}&\sum_{l=1}^\infty \langle a((K+\varepsilon_n)^{-1/2}f_l)\psi_n, a((K+\varepsilon_n)^{1/2}f_l)\psi_n\rangle \\ &\qquad\leq -\sum_{l=1}^\infty \frac{1}{\sqrt{2}}\langle q, (K+\varepsilon_n)^{-1/2}f_l\rangle\langle a((K+\varepsilon_n)^{-1/2}f_l)\psi_n, v'(\Phi(q))\psi_n\rangle\,.\end{aligned}$$

Using Lemma 14 for the l.h.s. of the inequality and Cauchy-Schwarz and (22) for the r.h.s., we obtain

$$\begin{aligned}\|N^{1/2}\psi_n\|^2 &\leq \frac{1}{\sqrt{2}}\left|\langle a((K+\varepsilon_n)^{-1/2}q)\psi_n, v'(\Phi(q))\psi_n\rangle\right| \\ &\leq \frac{1}{\sqrt{2}}\left\|(K+\varepsilon_n)^{-1}q\right\| \cdot \left\|N^{1/2}\psi_n\right\| \cdot \|v'\|_\infty \\ &\leq \frac{1}{\sqrt{2}}\|K^{-1}q\| \cdot \|N^{1/2}\psi_n\| \cdot \|v'\|_\infty\,.\end{aligned}$$

Now we have assumed that $\left\|N^{1/2}\psi_n\right\| \leq \frac{1}{\sqrt{2}}\left\|K^{-1}q\right\| \cdot \|v'\|_\infty < 1$, say $\|N^{1/2}\psi_n\|^2 \leq 1-\eta$ with $\eta > 0$. This implies in particular that $|\langle\psi_n, \Omega\rangle|^2 > \eta$, where Ω is the ground state in the absence of the perturbation. Finally by taking the limit $n \to \infty$ we get the desired result

$$|\langle\psi, \Omega\rangle| = \lim_{n\to\infty} |\langle\psi_n, \Omega\rangle| > \eta$$

i.e. $\psi \neq 0$.

□

6. Examples.

In this section we list the measures ν associated to some simple linear quantum dynamical systems. More are found in [LeM, Spo, FiL, FKM] and many other sources.

An oscillator in a simple harmonic chain with nearest neighbour interaction [vHe, FiL, GrM] is described by

$$\nu_{\mathrm{chain}}(dx) = \frac{2}{\pi m_0} \cdot 1_{[a,b]}(x) \frac{dx}{\sqrt{(x^2 - a^2)(b^2 - x^2)}} \; .$$

$$a := \sqrt{\frac{\alpha_0}{m_0}}, \quad b := \sqrt{\frac{\alpha_0 + 4\delta}{m_0}} \; .$$

Here m_0 is the mass of each of the oscillators, α_0 the constant of the spring which ties each oscillator to its rest position, and δ the constant of the springs connecting nearest neighbours. Clearly, this model satiesfies condition (b) of Theorem (5), provided that $\alpha_0 > 0$.

The model for Rayleigh scattering treated in [Spo] is described by the measure

$$\nu_{\mathrm{Rayleigh}}(dx) = \frac{1}{\pi} \frac{x^2 \mathrm{Im}\, r(x) dx}{\big((m_0 + \mathrm{Re}\, r(x))x^2 - \alpha_0\big)^2 + x^2 \big(\mathrm{Im}\, r(x)\big)^2} \; ,$$

where again m_0 and α_0 are the mass and spring constant of a harmonic oscillator coupled to the electromagnetic field. The coupling is taken in the dipole approximation, where the oscillator is given a well-behaved charge density function ρ. The function r above is given by

$$r(x) := \lim_{\varepsilon \downarrow 0} \frac{2}{3} \int_{\mathbb{R}^3} |\hat{\rho}(k)|^2 \frac{d_3 k}{\|k\|^2 - x^2 - i\varepsilon} \; .$$

(Cf. [Spo], (B3) or [LeM].) This model satisfies condition (c) of Theorem (5). Typically g drops off exponentially, so that the old condition on v can be used. This leads to extremely small perturbations v.

The ancient model due to Horace Lamb [Lam], which obeys a Langevin equation, is obtained by putting [Maa]

$$\nu_{\mathrm{Lamb}}(dx) = \frac{2\eta x dx}{(m_0 x^2 - \alpha_0)^2 + \eta^2 x^2} \; ,$$

where m_0 and α_0 are the mass and the spring constant of the oscillator, and η is a friction coefficient. Also here exponential decay of the commutator function g occurs, but the conditions of Theorem 5are not satisfied for $T = 0$.

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References

[Ara] Araki, H.: "Positive cone, Radon-Nikodym theorems, relative Hamiltonian and the Gibbs condition in statistical mechanics. An application of the Tomita-Takesaki Theory. In: *C*-Algebras and their applications to statistical mechanics and quantum field theory*, D. Kastler (ed.), North Holland 1976.

[ArH] Arai, A., Hirokawa, M.: "On the Existence and Uniqueness of Ground States of a Generalised Spin-Boson Model", *J. Funct. Anal.* **151**, (1997), 455–503.

[BFM] Botvich, D., Fayolle, G., Malyshev, V.: "Loss Networks in Thermodinamic Limit", In: *11th Conf. Analysis and Optimisation of Systems: discrete event systems*, Sopia-Antipolis France 1994, Springer Verlag Berlin Lect. Notes Control Inf. Sci. **199**, (1994), 465–489.

[BFS1] Bach, V., Fröhlich, J., Sigal, I.M.: "Mathematical Theory of Nonrelativistic Matter and Radiation", Lett. Math. Phys. **34**, (1995), 183–201.

[BFS2] Bach, V., Fröhlich, J., Sigal, I.M.: "Quantum Electrodynamics of Confined Nonrelativistic Particles", Adv. in Math. **137**, (1998), 299–395.

[BLL] Bergeron, F., Labelle, G., Leroux, P.: "Combinatorial specise and tree-like structures", Cambridge University Press 1998.

[BrR] Bratteli, O., Robinson, D.W.: "Operator Algebras and Quantum Statitical Mechanics" vol. 2, Berlin Springer 1997.

[Con] A. Connes, "Une classification des facteurs de type III", Ann. scient. Éc. Norm. Sup. **6** (1973), 133–252.

[FiL] Fidaleo, F., Liverani, C.: "Ergodic Properties for a Quantum Non Linear Dynamics", preprint University of Rome II, 1999.

[FKM] Ford, G., Kac, M., Mazur, P.: "Statistical Mechanics of Assemblies of Coupled Oscillators", J. Math. Phys. **6** (1965), 504–515.

[GrM] S. Graffi, A. Martinez: "Ergodic properties of infinite harmonic crystals: an analytic approach", J. Math. Phys. **37** (1996), 5111–5135.

[HaP] F. Harary, E.M. Palmer, "Graphical enumeration", Academic Press, New York 1973.

[Har] T.E. Harris: "The theory of branching processes", Springer, Berlin 1963.

[JaP1] Jakšić, V., Pillet, C.A.: "On a Model For Quantum Friction.I. Fermi's Golden Rule and Dynamics at Zero Temperatre", Ann. Inst. H. Poincaré **62**, (1995), 289–323.

[JaP2] Jakšić, V., Pillet, C.A.: "On a Model For Quantum Friction.II. Fermi's Golden Rule and Dynamics at Positive Tempereture", Commun. Math. Phys. **176**, (1996), 619–644.

[Joy] Joyal, A: "Une Théorie Combinatoire des Series Formelles", Adv. Math. **42**, (1981), 1–82.

[Lam] H. Lamb, Proc. Lond. math. Soc. **2** (1900) 88.

[LeM] Lewis, J.T., Maassen, H.: "Hamiltonian Models of Classical and Quantum Stochastic Processes", In: *Quantum Probability and Applications to the Quantum Theory of Irreversible Processes*, Lect. Notes Math. **1055**, Springer 1984.

[Maa] Maassen, H.: "Return to Equilibrium by a Solution of a Quantum Langevin Equation", J. Stat. Phys. **34**, (1984), 239–262.

[Par] Parthasarathy, K.R.: "An Introduction to Quantum Stochastic Calculus", Birkhäuser, Basel-Boston-Berlin, 1992.

[ReS1] Reed, M., Simon, B.: "Methods of Modern Mathematical Physics", vol I, Academic Press, New York 1974.

[ReS4] Reed, M., Simon, B.: "Methods of Modern Mathematical Physics", vol IV, Academic Press, New York 1974.

[Sim] Simon, B.: "The $P(\Phi)_2$ Quantum Euclidian Field Theory", Princeton Univ. Press 1974.

[Spo] Spohn, H.: "Asymptotic Completeness for Rayleigh Scattering", J. Math. Phys. **38**, (1997), 2281–2296.

[vHe] van Hemmen, J.L.: "Dynamics and ergodicity of the infinite harmonic crystal", Thesis Groningen, 1976.

[vNe] von Neumann, J.: "Die Eindeutigkeit der Schrödingerschen Operatoren", Math. Annalen **104**, (1931), 570–578.