Hardy Type Estimations with Exponential Weights for Operators of Convolution with Analytic Functions

Marius MĂNTOIU *

Radu PURICE

Institute of Mathematics
of the Romanian Academy
P.O. Box 1-764, Bucharest
RO-70700, ROMANIA
e-mail: purice@stoilow.imar.ro

Abstract In this paper we develop a general procedure to prove Hardy type estimations for an operator that admits a conjugate operator, starting from the Mourre estimation. We use this method for operators of convolution with analytic functions, obtaining Hardy type estimations with exponential weights, for sufficiently small exponents.

* Present address:
Departement de Physique Théorique; Université de Genève;
32, bd. d'Yvoy; CH-1211 Genève 4; SUISSE
e-mail: mantoiu@kalymnos.unige.ch

1 INTRODUCTION

The main aim of this paper is to prove weighted estimations of the type:

$$\|w_1 u\| \leq C\,\|w_2(\lambda(-i\nabla) - E)u\| \tag{1.1}$$

where $\lambda(-i\nabla)$ is the convolution operator with the Fourier transform of the function λ, the norm is the L^2-norm on the space $\mathbb{R}^n$ for some $n \geq 1$, E is a real number that may also belong to the spectrum of the operator $\lambda(-i\nabla)$ in $L^2(\mathbb{R}^n)$, w_1 and w_2 are weight functions that grow at infinity and u is a function in $L^2(\mathbb{R}^n)$ with support far from the origin. In principle one would like the two weight functions w_1 and w_2 to have similar growth at infinity but usually the function w_2 has to grow faster. The technique on which we want to emphasize is that once one can prove a Mourre estimation (see [ABG3] [BP] [GN] [Ar] [M] [BG2] and our Section 2 for the definition and the discussion of these type of estimations) for the commutator of the operator $H = \lambda(-i\nabla)$, or even of a perturbation of such an operator, with a well suited conjugate operator, one can elaborate an abstract procedure leading to weighted estimations of the type (1.1). Concerning the form of the conjugate operator we make some comments in Sections 2 and 4.

The history of this type of inequalities is a long one, starting probably with the well-known Hardy inequality [HLP] and continued by various other authors (see [KT] [ABG2]). In [A1] such an inequality is proven for the Laplace operator ($\lambda(t) = t^2$) and its importance is put into evidence in connection with the problem of existence of eigenvalues in a given real interval for some perturbations of the Laplacian. This type of inequalities is strongly related with the study of the decay at infinity of the eigenfunctions of some classes of perturbations of the operator $H := \lambda(-i\nabla)$ and thus are of great interest also for the analysis of quantum Hamiltonians. In this last context mainly the perturbations of the Laplacian (by two-body [A2] [FHHO] or N-body type potentials [FH]) and the case of the Dirac operator [BG1], [N] have been studied but the case of dissipative Hamiltonians [ABG3], of analytic decomposable Hamiltonians [GN] and of relativistic Schrödinger Hamiltonians [CMS] may motivate the extension of this type of inequalities to larger classes of functions λ. Let us also remind that the case of second order differential operators with variable coefficients is thoroughly treated in [ABG1] and [ABG2]. The case of Hardy inequalities with polinomial weights appears in [AH] and we intend to develop this situation in a forthcomming paper. The case of exponential weights for values of E in the gaps of a Hamiltonian having spectral gaps is discussed in [N] .

Let us shortly comment upon the eigenfunction decay problem and its connection with inequalities of type (1.1). Suppose H is a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^n)$ for which we can prove an estimation of type (1.1) (with $\lambda(-i\nabla)$ replaced by H) for a value E that is an eigenvalue of H with eigenfunction *f*. Denoting by χ the smoothed characteristic function of a ball of sufficiently large radious R in $\mathbb{R}^n$, by $\chi^\perp = 1 - \chi$ and by $u = \chi^\perp f$ we see that:

$$\begin{gathered} \|w_1 f\| \leq \|w_1 u\| + \|w_1 \chi f\| \leq C\,\|w_2(H-E)u\| + \|w_1\chi f\| \\ (H-E)u = (H-E)f - (H-E)\chi f = (H-E)\chi f \end{gathered} \tag{1.2}$$

Thus if we can prove that H applied on functions with compact support takes values in the domain of the weight function w_2 (for example if H is a local operator or a convolution operator with the Fourier transform of a function with some strong regularity properties), then:

$$\|w_1 f\| \leq C\,\|w_2(H-E)\chi f\| + \|w_1\chi f\| \leq C. \tag{1.3}$$

In this way we obtain information on the decay of the eigenfunction *f* of the operator H. This kind of analysis may be of much interest for a large class of Hamiltonians of quantum systems.

Let us still point out that up to now two classes of weight functions have been of interest in connection with the above type of problems: polynomial weights and exponential weights. In order to treat them in a unified setting we shall consider our weight function w as being strictly positive and we shall write it in the form $w := e^{\varphi}$. Then the main difference between the polynomial case and the exponential case comes from the fact that for polynomial weights the derivatives of the phase function φ decay at infinity allowing for some estimation procedures that take advantage of the fact that the function u has support away from the origin, procedure that does not work for exponential weights.

The general method used in the literature [A2] [ABG2] in proving estimations of the type (1.1) consists in making two types of cut-offs: on the function u that is approximated with functions of compact support an on the phase function that is approximated with phase functions that converge to infinity to some finite constant. Then one proves a weighted estimation for this "regularized" situation and finally one removes the cut-offs (first in u and then in the phase function). We shall also adopt this general scheme in our work. In order to be able to treat the "regularized" situation, when the function u has compact support, it is clear that some decay conditions are necessary for the Fourier transform of λ and thus some regularity for the function λ.

In this paper we consider the case when λ belongs to a class of real analytic functions on $\mathbb{R}^n$ with at most some specific polynomial growth at infinity (see the Hypothesis 2.2), case for which we can prove an estimation with exponential weights of the type: $w(x) = e^{\gamma|x|}$, with the constant γ sufficiently small. In a forthcoming paper we shall consider less regular functions λ for which we can prove an estimation with polynomial weights. The body of our paper is organized as follows. In Section 2 we present the general framework needed for the type of calculus we develop and for the statement of our first main result, which is contained in Theorem 2.6. The following two sections are concerned with the proof of Theorem 2.6. In Section 3 we prove a weighted inequality for functions with compact support (away from the origin) and weights belonging to a class containing exponentials with linear phases (with a small exponent γ) together with their bounded approximants that we shall use in the next section. This section contains the main technical points of our procedure of making use of a Mourre type estimation. In Section 4 we give the details of the cut-off procedure and we finish the proof of theorem 2.6.

2 STATEMENT OF THE MAIN RESULT

We shall work in the n-dimensional real space $\mathbb{R}^n$, with the Lebesgue measure denoted by $d^n x$ and we shall use the notations:

$$\begin{gathered} x \cdot y := \sum_{j=1}^n x_j y_j \\ |x| := \sqrt{\sum_{j=1}^n |x_j|^2} \\ B(x;R) := \{y \in \mathbb{R}^n \mid |y-x| < R\}. \end{gathered} \tag{2.1}$$

We shall consider the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^n; d^n x) \equiv L^2(\mathbb{R}^n)$ and we shall set:

$$\begin{gathered} < f, g > := \int_{\mathbb{R}^n} \overline{f(x)} g(x) d^n x \\ \|f\| := \sqrt{< f, f >}. \end{gathered} \tag{2.2}$$

On $L^1(\mathbb{R}^n)$ we consider the Fourier transform:

$$\mathcal{F}(f)(k) \equiv \hat{f}(k) := \int_{\mathbb{R}^n} e^{-ix\cdot k} f(x) \frac{d^n x}{(2\pi)^{n/2}} \equiv \int_{\mathbb{R}^n} e^{-ix\cdot k} f(x) dx \tag{2.3}$$

and we denote in the same way its extension to the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions on $\mathbb{R}^n$. We shall work with two subspaces of $C^\infty(\mathbb{R}^n)$, namely $BC^\infty(\mathbb{R}^n)$ the space of indefinitely differentiable functions on $\mathbb{R}^n$ that are bounded together with all their derivatives and $C^\infty_{pol}(\mathbb{R}^n)$ the space of indefinitely differentiable functions on $\mathbb{R}^n$ that have polynomial growth at infinity as well as their derivatives of all orders. We shall constantly use the standard multiindex notations for monomials in n commuting variables $X := (X_1, ..., X_n)$, for $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$:

$$\begin{gathered} X^\alpha := X_1^{\alpha_1} ... X_n^{\alpha_n} \\ |\alpha| := \alpha_1 + ... + \alpha_n \\ \alpha! := \alpha_1! ... \alpha_n! \\ \alpha + \beta := (\alpha_1 + \beta_1, ..., \alpha_n + \beta_n) \\ \alpha \leq \beta \Longleftrightarrow \alpha_j \leq \beta_j, \quad \forall j \in \{1, ...n\}. \end{gathered} \tag{2.4}$$

We denote by δ_j the multiindex with 1 on position $j \in \{1, ...n\}$ and 0 in rest. We also use the notation:

$$< X > := \sqrt{1 + |X|^2} \equiv \sqrt{1 + \sum_{j=1}^n X_j^2} \tag{2.5}$$

and the following inequalities for x and y in $\mathbb{R}^n$:

$$\begin{gathered} < x + y >^r \leq 2^{r/2} < x >^r < y >^r, \text{ for } r > 0 \\ < x + y >^r \leq C(r) \left(< x >^r + < y >^r \right) \\ \text{with: } C(r) := \begin{cases} 1 \text{ for } r \leq 1 \\ 2^{r-1} \text{ for } r \geq 1 \end{cases} \end{gathered} . \tag{2.6}$$

In $\mathcal{H}$ we shall work with two sets of commuting self-adjoint operators:

$$\begin{gathered} Q := (Q_1, ...Q_n) \\ D := (D_1, ...D_n) \end{gathered} \tag{2.7}$$

where Q_j is the unique self-adjoint extension of the operator:

$$(Q_j f)(x) := x_j f(x), \forall f \in C_0^\infty(\mathbb{R}^n) \tag{2.8}$$

and D_j the unique self-adjoint extension of the operator:

$$D_j f := -i \frac{\partial f}{\partial x_j}, \forall f \in C_0^\infty(\mathbb{R}^n). \tag{2.9}$$

For a fixed $y \in \mathbb{R}^n$ we shall also use the notation:

$$y \cdot D := \sum_{j=1}^n y_j D_j \tag{2.10}$$

for the self-adjoint extension of the operator defined on $C_0^\infty(\mathbb{R}^n)$. For any Borel function $\Phi: \mathbb{R}^n \to \mathbb{C}$ we denote by $\Phi(Q)$, respectively by $\Phi(D)$ the operator defined by the usual functional calculus for commuting families of self-adjoint operators and by $\mathcal{D}(\Phi(Q))$, respectively by $\mathcal{D}(\Phi(D))$ their domains in $\mathcal{H}$. Let us also remind the following well-known intertwining property of the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$:

$$\mathcal{F}^{-1}Q_j\mathcal{F} = D_j \qquad \mathcal{F}^{-1}D_j\mathcal{F} = -Q_j. \tag{2.11}$$

For a function $F \in C^1(\mathbb{R}^n)$ we shall denote by ∇F its gradient.

In the sequel we constantly make use of the functional calculus procedure [ABG3] based on the unitary group generated by a family of n commuting self-adjoint operators $X = (X_1, ...X_n)$ in $\mathcal{H}$, that we denote by:

$$U_X(x) := e^{ix\cdot X} \tag{2.12}$$

More precisely, if F is the Fourier transform of an integrable function, then:

$$F(X) = \int_{\mathbb{R}^n} \hat{F}(x)U_X(x)dx \tag{2.13}$$

defines a bounded normal operator on $\mathcal{H}$ with the integral defined in the weak-operator topology on $\mathcal{H}$. If $F \in C^\infty_{pol}(\mathbb{R}^n)$ then its Fourier transform is a rapidly decaying distribution $\hat{F}$ of a finite order m and in [ABG3] one proves the formula:

$$< f, F(X)g >= \hat{F}[< f, U_X g >] \tag{2.14}$$

for $g \in \mathcal{D}(X_1^m) \cap ...\mathcal{D}(X_n^m)$, with the right hand side interpreted as the value of the rapidly decaying distribution $\hat{F}$ of order m applied on the function:

$$\mathbb{R}^n \ni x \longmapsto < f, U_X(x)g > \in \mathbb{C}. \tag{2.15}$$

Indeed the condition on the vector g implies that the function in (2.15) is of class $C^m(\mathbb{R}^n)$ and bounded together with all its derivatives up to order m. We shall usually use the formula (2.13) having in mind the above interpretation and verifying the domain condition for g.

Let us remark that if in (2.12) we take for X the usual family of derivation operators D, the unitary group they generate is the group of translations in $\mathbb{R}^n$:

$$(U_D(x)f)(y) = f(y + x). \tag{2.16}$$

We shall need this version of the functional calculus in order to make explicit computations of commutators between functions of Q and functions of D. The starting point is the observation that for $f \in \mathcal{D}(F(Q))$ one has:

$$(U_D(x)F(Q)f)(y) = (F(Q)f)(y + x) = F(y + x)f(y + x) = (F(Q + x)U_D(x)f)(y) \tag{2.17}$$

so that formally we can write:

$$U_D(x)F(Q)f = F(Q + x)U_D(x)f. \tag{2.18}$$

This formula is obviously true if F is continuous and f belongs to the domain of the normal operator $F(Q)$. Thus if $F \in C^1(\mathbb{R}^n)$ one gets for $f \in \mathcal{D}(F(Q)) \cap \mathcal{D}(\nabla F(Q))$:

$$[U_D(x), F(Q)]\, f = \{F(Q + x) - F(Q)\}\, U_D(x)f = \int_0^1 ds\, (x \cdot \nabla F(Q + sx))\, U_D(x)f. \tag{2.19}$$

Notations: For a finite complex measure ν on $\mathbb{R}^n$ let us denote by $|\nu|$ its total variation and by $\mathcal{M}(\mathbb{R}^n)$ the space of finite complex measures on $\mathbb{R}^n$ with the norm:

$$\|\nu\|_M := |\nu|(\mathbb{R}^n). \tag{2.20}$$

Observing that $\mathcal{M}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ let us denote by $\mathcal{FM}(\mathbb{R}^n)$ the space of Borel functions on $\mathbb{R}^n$ that are Fourier transforms of measures in $\mathcal{M}(\mathbb{R}^n)$ and by $\mathcal{FM}_1(\mathbb{R}^n)$ the space of functions that belong to $\mathcal{FM}(\mathbb{R}^n)$ together with their first order derivatives. For a function $\mu \in \mathcal{FM}(\mathbb{R}^n)$ we denote by $\hat{\mu}(dk)$ its Fourier transform with the convenient normalization in order to have:

$$\mu(x) = \int_{\mathbb{R}^n} e^{ix\cdot k}\hat{\mu}(dk). \tag{2.21}$$

Evidently, any function in $\mathcal{FM}_1(\mathbb{R}^n)$ belongs to $C^1\mathbb{R}^n)$ and is bounded together with its first order derivatives.

Let us consider now $\mu \in \mathcal{FM}_1(\mathbb{R}^n)$ and $F \in BC^\infty(\mathbb{R}^n)$. Using a formula similar to (2.13) valid for Fourier transforms of measures and (2.19), we see that the commutator $[\mu(D), F(Q)]$ defines a bounded operator on $\mathcal{H}$ and one has the formula:

$$\begin{aligned} [\mu(D), F(Q)] &= \int_{\mathbb{R}^n} \hat{\mu}(dx) \int_0^1 ds\, (x \cdot \nabla F(Q+sx))\, U_D(x) = \\ &= i \int_{\mathbb{R}^n} \int_0^1 ds (\widehat{\nabla\mu}(dx) \cdot \nabla F(Q+sx)) U_D(x) \end{aligned} \tag{2.22}$$

interpreted as equalities of bounded operators on $\mathcal{H}$ with integrals with respect to the weak operator topology. One gets thus the estimation:

$$\|[\mu(D), F(Q)]\| \leq \left\|\widehat{\nabla\mu}\right\|_M \|\nabla F(Q)\|. \tag{2.23}$$

We shall extend now the formulae (2.22) and (2.23) to a more general situation that we shall need in our computations. Let p be a polynomial of degree m on $\mathbb{R}^n$ verifying the relation:

$$\sum_{|\alpha|\leq m} |\partial^\alpha p| \leq C(1+|p|),$$

let $\mu \in \mathcal{FM}_1(\mathbb{R}^n)$ and $\lambda := p\mu$. Let $F \in BC^\infty(\mathbb{R}^n)$, $f \in \mathcal{H}$ and $g \in \mathcal{D}(p(-D))$. For $\beta \in \mathbb{N}^n$ let $\partial^\beta p$ be the derivative of order β of the polynomial p so that due to the hypoellipticity condition on p we have that $g \in \mathcal{D}(\partial^\beta p(-D))$ for any β with $|\beta| \leq m$. For functions H and G_j in $BC^\infty(\mathbb{R}^n)$ with $j \in \{1, ..., n\}$ and $|\alpha| \geq 1$ we have the formulae:

$$\partial_x^\alpha(x \cdot G(x)) = \sum_{j=1}^n \left\{ x_j \partial_x^\alpha G_j(x) + \partial_x^{\alpha-\delta_j} G_j(x) \right\} \tag{2.24}$$

$$\partial_x^\alpha(H(x)U_D(x)g) = \sum_{\beta\leq\alpha} i^{|\beta|} \left(\frac{\alpha!}{\beta!(\alpha-\beta)!}\right) \left(\partial_x^{\alpha-\beta} H(x)\right) U_D(x) D^\beta g \tag{2.25}$$

$$p(i\partial_x)(H(x)U_D(x)g) = \sum_{|\beta|\leq m} \frac{i^{|\beta|}}{\beta!} \left(\partial_x^\beta H(x)\right) U_D(x) \left(\partial^\beta p(-D) g\right). \tag{2.26}$$

Thus the function:

$$\mathbb{R}^n \ni x \longmapsto < f, [U(x), F(Q)]\, g > \in \mathbb{C} \tag{2.27}$$

is of class $C^m(\mathbb{R}^n)$ and we have the equality:

$$
\begin{aligned}
&< f, [\lambda(D), F(Q)]\, g >= \textstyle\int_{\mathbb{R}^n} \hat{\mu}(dx)\, (p(i\partial_x) < f, [U(x), F(Q)]\, g >) = \\
&= \textstyle\int_0^1 ds\ \int_{\mathbb{R}^n} \hat{\mu}(dx)\, (p(i\partial_x) < f, (x \cdot \nabla F(Q+sx)) U_D(x) g >) = \\
&= \textstyle\sum_{|\beta|\leq m} \frac{i^{|\beta|}}{\beta!} i \int_0^1 ds\ \int_{\mathbb{R}^n} \widehat{\nabla\mu}(dx) \left(< f, \partial_x^\beta \nabla F(Q+sx) U_D(x) \left(\partial^\beta p(-D)\right) g >\right) + \\
&+ \textstyle\sum_{|\beta|\leq m} \frac{i^{|\beta|}}{\beta!} \int_0^1 ds\ \int_{\mathbb{R}^n} \hat{\mu}(dx) \left(< f, \partial_x^\beta F(Q+sx) U_D(x) \left(\partial^\beta p(-D)\right) g >\right).
\end{aligned}
\tag{2.28}
$$

From this we derive the estimation:

$$
|< f, [\lambda(D), F(Q)]\, g >| \leq C_F \left(\left\| \widehat{\nabla\mu} \right\|_M + \|\hat{\mu}\|_M \right) \|f\| \, \|p(-D) g\|. \tag{2.29}
$$

We want to analyze now the case when the function F grows polynomially at infinity. To be able to deal with this case we have to impose stronger regularity conditions on λ.

Suppose F is a real function on $\mathbb{R}^n$ such that F and all its derivatives up to order $m+1$ grow at infinity not faster than $< x >^r$. Suppose farther that p is a hypoelliptic polynomial on $\mathbb{R}^n$ and $\mu \in \mathcal{FM}(\mathbb{R}^n)$ is such that $< x >^{r+1} \hat{\mu} \in \mathcal{M}(\mathbb{R}^n)$. Let $\lambda = p\mu$.

Definition 2.1. We define the dense linear subspace:

$$
\mathcal{L} := \mathcal{D}(p(-D)) \cap L^2_{comp}(\mathbb{R}^n). \tag{2.30}
$$

We observe that for $g \in \mathcal{L}$ and $x \in \text{supp } g$ we have that $|x|$ is bounded so that $F(Q)g \in L^2(\mathbb{R}^n)$ and:

$$
\begin{aligned}
|< x >^{-r} \, (F(Q) U_D(x) g)(y)| &\leq \left(\tfrac{<y>}{<x>}\right)^r \left|\tfrac{F(y)}{<y>^r}\right| |g(x+y)| \leq \\
&\leq C_F < x+y >^r |g(x+y)| \leq C_F(g, r).
\end{aligned}
\tag{2.31}
$$

Thus for $f \in \mathcal{H}$ and $g \in \mathcal{L}$ the distribution $\hat{\lambda}$ may be evaluated on the function:

$$
\mathbb{R}^n \ni x \longmapsto < f, [U_D(x), F(Q)]\, g > \in \mathbb{C} \tag{2.32}
$$

(this function being differentiable up to order m and growing not faster than $< x >^{r+1}$ for $x \to \infty$ together with all its derivatives up to order m). In conclusion we can still give sense to the formulae (2.28) and we obtain the estimation:

$$
|< f, [\lambda(D), F(Q)]\, g >| \leq C_F \left\| < x >^{r+1} \hat{\mu} \right\|_M \|f\| \, \|< Q >^r \, p(-D) g\|. \tag{2.33}
$$

In order to be able to deal with exponential weights we shall need to use a variant of our commutator formula for the case when F grows exponentially at infinity. For this situation we have to impose even stronger regularity conditions on the function λ; in fact we shall need analyticity in a strip around $\mathbb{R}^n$ and some specific growth condition at infinity.

For $\delta > 0$ let:

$$
\mathbb{C}^n_\delta := \left\{ z \in \mathbb{C}^n \mid \sum_{j=1}^n |Im(z_j)|^2 < \delta^2 \right\}.
$$

Hypothesis 2.2. For $\delta > 0$ and $m \in \mathbb{N}$ let $\mathcal{O}(\mathbb{C}^n_\delta)$ be the space of analytic functions on $\mathbb{C}^n_\delta$ that are real on $\mathbb{R}^n$ and let $\mathcal{O}^m(\mathbb{C}^n_\delta)$ be the space of functions $\lambda \in \mathcal{O}(\mathbb{C}^n_\delta)$ that are of the form $\lambda = p\mu$ where:

1. p is a polynomial of degree m, verifying :$\sum_{|\alpha|\leq m} |\partial^\alpha p| \leq C(1+|p|)$,
2. μ is analitic and $\mu(.+iy) \in \mathcal{FM}(\mathbb{R}^n_x)$ for any fixed y with $|y| < \delta$,
3. there exists a strictly positive constant κ such that: $\kappa |p| \leq (1+|\lambda|)$.

Let us remark here that a more general class of analytic functions λ can be treated by replacing the polinomial p with a function w belonging to a class of symbols of type m. In fact all one has to do is to use the Taylor polinomial of order $[m]+1$ and control the rests; we propose to develop this procedure in a forthcomming paper.

Lemma 2.3. *If $\mu \in \mathcal{O}(\mathbb{C}^n_\delta)$ and verifies condition (2) in Hypothesis 2.2, then $e^{\gamma|x|}\hat{\mu}_0 \in \mathcal{M}(\mathbb{R}^n)$ for any $\gamma < \delta$.*

Proof. Let us consider $y \in \mathbb{R}^n$ with $|y| < \delta$, let us denote by μ_y the restriction of the function μ to the plane $H_y := \{x+iy \mid x \in \mathbb{R}^n\}$ and let us consider a smooth function f with compact support and define:

$$\mathbb{R}^n \ni x \longmapsto \phi(x) := \int_{\mathbb{R}^n} dk e^{-ixk} f(k) \in \mathbb{C}. \tag{2.34}$$

Then ϕ is analytic and decays rapidly at infinity so that the product function:

$$\mathbb{R}^n \ni x \longmapsto \mu(x+iy)\phi(x) \in \mathbb{C} \tag{2.35}$$

is analytic and goes to zero at infinity. Using Cauchy formula we obtain for f smooth and with compact support:

$$\begin{aligned} \hat{\mu}_y(f) &= \int_{\mathbb{R}^n} d^n k \int_{\mathbb{R}^n} dx \mu(x+iy) e^{-ixk} f(k) = \\ &= \int_{\mathbb{R}^n} d^n x \mu(x+iy) \int_{\mathbb{R}^n} dk e^{-ixk} f(k) = \\ &= \int_{\mathbb{R}^n} d^n x \mu(x+iy)\phi(x) = \\ &= \int_{\mathbb{R}^n} d^n x \mu(x)\phi(x-iy) = \\ &= \int_{\mathbb{R}^n} d^n x \mu(x) \int_{\mathbb{R}^n} dk e^{-ixk} f(k) e^{-yk} = \\ &= (e^{-yk}\hat{\mu})(f). \end{aligned} \tag{2.36}$$

Approaching any $f \in C_\infty(\mathbb{R}^n)$ with functions with compact support we can extend the equality in (2.36) to any $f \in C_\infty(\mathbb{R}^n)$. Now for any $y \in \mathbb{R}^n$ and any $\epsilon \in (0,1)$ we can find a conical set $V_y := \{k \in \mathbb{R}^n \mid y \cdot k \geq (1-\epsilon)|y||k|\}$. Let us choose now a finite family $\{y_1, ..., y_N\} \subset \mathbb{R}^n$ with $|y_j| = \gamma(1-\epsilon)^{-1} < \delta$ and a partition of unity $\{\chi_1, ..., \chi_N\}$ on the unit sphere, such that $k/|k| \in \mathrm{supp}\chi_j$ implies that $k \in V_{y_j}$. We have: $\chi_j e^{-y_j k}\hat{\mu} = \chi_j \hat{\mu}_{y_j}$, so that we also get the equality of the total variations: $\chi_j e^{y_j k}|\hat{\mu}| = \chi_j \left|\hat{\mu}_{y_j}\right|$. Then:

$$\begin{aligned} \left|(e^{\gamma|k|}\hat{\mu})(f)\right| &\leq \left|\left(\sum_{j=1}^N \chi_j e^{-y_j k}\hat{\mu}\right)(f)\right| \leq \left(\sum_{j=1}^N \chi_j \left|\hat{\mu}_{y_j}\right|\right)(|f|) \leq \\ &\leq C_\gamma \left\{\sup_{|y|<\gamma'} \|\hat{\mu}_y\|_M\right\} \|f\|_{L^\infty}. \end{aligned} \tag{2.37}$$

for $\gamma < \gamma' < \delta$. □

Let us suppose now that $\lambda = p\mu \in \mathcal{O}^m_\delta(\mathbb{C}^n)$ and that $F \in C^\infty(\mathbb{R}^n)$ satisfies the estimations:

$$|\partial^\alpha F(x)| \leq C_\alpha e^{a|x|} \tag{2.38}$$

for any $x \in \mathbb{R}^n$ and any α with $|\alpha| \leq m+1$ and some $a \in (0, \delta)$. Repeating now the arguments that led us to (2.33) we conclude that for any $f \in \mathcal{H}$ and any $g \in \mathcal{L}$ the formula (2.28) remains true also for this case and we get the estimation:

$$|< f, [\lambda(D), F(Q)]\, g >| \leq C_F \left\| e^{\gamma|Q|}\hat{\mu} \right\|_M \|f\| \left\| e^{a|Q|} p(-D) g \right\| \tag{2.39}$$

with $\gamma \in (a, \delta)$.

After this elaboration of the main calculus procedure that we shall use let us come back to our problem that we announced in the introduction and formulate our main result and the method we use for proving it. We shall consider convolution operators on $\mathbb{R}^n$ with functions $\lambda = p\mu \in \mathcal{O}^m(\mathbb{C}^n_\delta)$ and we shall prove an estimation of type (1.1) with exponential weights of the form $w(x) = e^{\gamma <x>}$for some sufficiently small positive γ. In order to formulate our theorem we still need some definitions.

Definition 2.4. For a function $\lambda \in \mathcal{O}^m(\mathbb{C}^n_\delta)$ we define its set of *regular values*:

$$\mathcal{E}(\lambda) := \left\{ t \in \mathbb{R} \mid \exists \epsilon > 0, \exists \kappa > 0 \ s.t. \ |\nabla\lambda(k)| \geq \kappa \ \forall k \in \lambda^{-1}((t-\epsilon, t+\epsilon)) \right\}.$$

We call *generalized critical value* any point in the complementary set of $\mathcal{E}(\lambda)$ in $\mathbb{R}$.

Remark 2.5. It is obvious that $\mathcal{E}(\lambda)$ is open in $\mathbb{R}$ and that the image by λ of any zero of $\nabla\lambda$ is a generalized critical value; anyhow it may happen that due to its behaviour at infinity λ may also have some other generalized critical values.

Notations:

1. For any linear subspace $\mathcal{V} \subset \mathcal{H}$ and any $R > 0$ we denote:

$$\mathcal{V}_R := \{ f \in \mathcal{V} \mid \sup p f \cap B(0; R) = \emptyset \}.$$

2. Let us denote by $\mathcal{G}$ the domain of the self-adjoint operator $\lambda(D)$ with the norm:

$$\|f\|^2_{\mathcal{G}} := \|f\|^2 + \|\lambda(D) f\|^2 . \tag{2.40}$$

We are now ready to state our main result concerning the case of exponential weights.

Theorem 2.6. *Let $\delta > 0$, $R \in \mathbb{R}$, $\lambda = p\mu \in \mathcal{O}^m_\delta(\mathbb{C}^n)$ and $E \in \mathcal{E}(\lambda)$. Then there exist a strictly positive constant γ and two positive constants C and R such that for any $f \in \mathcal{D}(p(-D))_R$ we have the estimation:*

$$\left\| e^{\gamma <Q>} f \right\|_{\mathcal{G}} \leq C \left\| \sqrt{< Q >} e^{\gamma <Q>} (\lambda(D) - E) f \right\| .$$

(Let us remark that the constants C and R depend on γ and on E but not on the function f).

The inequality in the above statement is understood in the sense that if the function:

$$x \longmapsto \sqrt{< x >} e^{\gamma <x>} ((\lambda(D) - E) f)(x)$$

is in $L^2(\mathbb{R}^n)$ then the function $e^{\gamma <x>} f(x)$ is also in $L^2(\mathbb{R}^n)$ and we have the stated estimation. Let us remark that in the above statement E may belong to the spectrum of the operator $\lambda(D)$ as well as to its resolvent set as long as it remains a regular value.

Our method to prove the above theorem consists in defining a "conjugate operator" associated to $\lambda(D)$ and generalize the usual ideas used in proving estimations of type (1.1) outside the spectrum [A2], [N] in order to take advantage of the Mourre type estimation that we prove. We dedicate the final part of this section to the definition of the conjugate operator associated to $\lambda(D)$. In choosing it we have been guided by some results in [Ar], [ABG3] and [GN].

The conjugate operator one would like to choose for $\lambda(D)$ would be (see also [Ar] and [GN]):

$$A_0 := \frac{1}{2}\sum_{j=1}^{n}\{Q_j(\partial_j\lambda)(D) + (\partial_j\lambda)(D)Q_j\} \tag{2.41}$$

defined on $C_0^\infty(\mathbb{R}^n)$. Then the commutator of A_0 with $H := \lambda(D)$ is:

$$B_0 := i\,[\lambda(D), A_0] = \sum_{j=1}^{n}(\partial_j\lambda)(D)^2 \tag{2.42}$$

also acting on $C_0^\infty(\mathbb{R}^n)$ but defining a positive sesquilinear form on $\mathcal{D}(p(-D))$ with $\lambda = p\mu$. The form of the operator B_0 given in (2.42) makes clear the reason for the definition of the "regular values". One observes that for $E \in \mathcal{E}(\lambda)$ and for any small neighbourhood J of E contained in $\mathcal{E}(\lambda)$, if one denotes by φ the characteristic function of J, one gets the estimation:

$$\varphi(\lambda(D))B_0\varphi(\lambda(D)) \geq \left\{\inf_{t\in J}\left|(\nabla\lambda)(\lambda^{-1}(t)\right|^2\right\}\varphi(\lambda(D)) \tag{2.43}$$

the constant multiplying $\varphi(\lambda(D))$ in the right hand side being strictly positive.

An essential ingredient in our proof of the weighted estimations, as we shall show in the next section, is the observation (see also [FH] and [ABG1]) that by making use of the explicit form of the conjugate operator, the most singular term appearing in the expression of the sesquilinear form (3.6) has a definite sign. While for polynomial weights this procedure works by considering the conjugate operator given in (2.41), the terms appearing in the remainder being small at infinity due to the behaviour of the phase function, in the case of exponential weights one has to absorb some of these remainders in the expression of the main part and thus one has to consider a more complicate conjugate operator, more intimately connected with the form of the phase function.

Due to our cut-off procedure, we have to work with a class of phase functions containing the linear phase we are interested in, together with its approximants.

Definition 2.7. For any $\gamma \in (0, \delta)$ we define the class of functions:

$$\Phi_{\gamma,m} := \left\{\varphi \in C^\infty\left([1,\infty);\mathbb{R}\right) \mid 0 \leq \varphi' \leq \gamma;\ |\varphi"(t)| \leq \frac{\gamma}{t};\ \left|\varphi^{(l)}(t)\right| \leq \gamma,\ \forall l \leq m+1\right\},$$

where m is the degree of the polynomial p associated to the function λ and we consider weight functions of the form $w(x) := e^{\varphi(<x>)}$ with φ a function belonging to the class $\Phi_{\gamma,m}$.

For any weight function $w(x) := e^{\varphi(<x>)}$ with $\varphi \in \Phi_{\gamma,m}$ we denote:

$$X(x) := \nabla(\varphi(x)) = \frac{x}{< x >}\varphi'(< x >) \tag{2.44}$$

and we define the associated *conjugate operator*:

$$A := \frac{1}{2}\int_0^1 ds \int_{\mathbb{R}^n} dx \sum_{j=1}^{n}\left(\widehat{\partial_j\lambda}\right)(x)\left\{U_D(x)e^{sx\cdot X(Q)}Q_j + Q_je^{-sx\cdot X(Q)}U_D(x)\right\} \tag{2.45}$$

acting on $C_0^\infty(\mathbb{R}^n)$ (with the slight abuse of notation explained after formula (2.13)). Using the same arguments as in the commutator calculus explained above, we verify that the operator in (2.45) is in fact well defined on $C_0^\infty(\mathbb{R}^n)$ with values in $L^2(\mathbb{R}^n)$. All we have to verify is that for any $g \in \mathcal{H}$ and any $f \in C_0^\infty(\mathbb{R}^n)$ the function:

$$\mathbb{R}^n \ni x \longmapsto a_j(x;s) := < g, \left\{ U_D(x) e^{sx\cdot X(Q)} Q_j + Q_j e^{-sx\cdot X(Q)} U_D(x) \right\} f > \in \mathbb{C} \tag{2.46}$$

(for j=1,...n) is of class $C^m(\mathbb{R}^n)$, with $m = \deg(p)$ and that $|(p(i\partial_x)a_j)(x;s)| \leq Ce^{(\gamma+\epsilon)<x>}$ for any $\epsilon > 0$. But:

$$(p(i\partial_x)a_j)(x;s) =$$
$$= \sum_{|\beta|\leq m} i^{|\beta|} \tfrac{1}{\beta!} < g, \partial_x^\beta \left\{ e^{sx\cdot X(Q+x)}(Q_j + x_j) + Q_j e^{-sx\cdot X(Q)} \right\} \left(U_D(x)(\partial^\beta p(-D)) \right) f > \tag{2.47}$$

$$\partial_x^\beta e^{sx\cdot X(Q+x)}(Q_j + x_j) = \partial_x^{\beta-\delta_j} e^{sx\cdot X(Q+x)} + \left(\partial_x^\beta e^{sx\cdot X(Q+x)} \right) (Q_j + x_j) \tag{2.48}$$

$$\partial_x^\beta e^{sx\cdot X(Q+x)} = s^{|\beta|} \sum_{l=1}^{|\beta|} \sum_{\alpha_1+\ldots+\alpha_l=\beta} \partial_x^{\alpha_1}(x\cdot X(Q+x)) \cdot \ldots \cdot \partial_x^{\alpha_l}(x\cdot X(Q+x))\, e^{sx\cdot X(Q+x)} \tag{2.49}$$

$$\partial_x^\alpha x\cdot X(Q+x) = \sum_{j=1}^n \left\{ x_j (\partial_x^\alpha X_j)(Q+x) + \left(\partial_x^{\alpha-\delta_j} X_j \right)(Q+x) \right\} \tag{2.50}$$

$$\partial_x^\beta e^{-sx\cdot X(Q)} = (-s)^{|\beta|} X^\beta(Q) \cdot e^{-sx\cdot X(Q)} \tag{2.51}$$

$$(\partial_x^\alpha X_j)(x) = \partial_x^{\alpha+\delta_j} \varphi(< x >) = \sum_{l\leq|\alpha|+1} b_l(x) \varphi^{(l)}(< x >) \tag{2.52}$$

where the coefficients $b_l(x)$ are symbols of class $S^{l-|\alpha|}(\mathbb{R}^n)$ (see [ABG3]). Putting all these formulae together one gets the estimation:

$$|(p(i\partial_x)a_j)(x;s)| \leq Ce^{(\gamma+\epsilon)<x>} \|g\| \|p(-D)f\| . \tag{2.53}$$

Remark 2.8. Let us notice that this last estimation allows us to extend the operator A to the domain $\mathcal{L}$ by approaching each $f \in \mathcal{L}$ with elements in $C_0^\infty(\mathbb{R}^n)$ and observing that $C_0^\infty(\mathbb{R}^n)$ is an essential domain for $p(-D)$.

We end this section with the remark that although we can prove a "sharp Mourre estimation", similar to (2.43) for the commutator:

$$B := i\,[\lambda(D), A] \tag{2.54}$$

defined on $C_0^\infty(\mathbb{R}^n)$, as we show in the Appendix and although we work with the conjugate operator A given by (2.45), for the estimations we have to prove the inequality (2.43) for B_0 is sufficient. In fact, as it becomes clear by the arguments in Section 3, the inequality (2.43) for B_0 implies a similar inequality for B but with a less precise lower bound.

3 Weighted Estimation for Functions with Compact Support

In this section we begin the proof of our main theorem 2.6 by proving a weighted estimation for compactly supported functions of class $\mathcal{D}(p(-D))_R$ with phase functions of class $\Phi_{\gamma,m}$.

Theorem 3.1. *Let $\lambda = p\mu \in \mathcal{O}^m(\mathbb{C}^n_\delta)$ and $E \in \mathcal{E}(\lambda)$ a regular value for λ; then there exists a constant $\gamma_0 < \delta$ depending only on λ and E such that for any $\gamma \in (0, \gamma_0)$ there are two constants R_γ and C_γ, such that for any phase function $\varphi \in \Phi_{\gamma,m}$ and any $f \in \mathcal{D}(p(-D)) \cap L^2_{comp}(\mathbb{R}^n)_{R_\gamma}$ we have the weighted estimation:*

$$\left\| e^{\varphi(<Q>)} f \right\|^2_{\mathcal{G}} + n^{-1}(b - 1/2) \left\| < Q >^{-1} Ae^{\varphi(<Q>)} f \right\|^2 \leq C_\gamma \left\| \frac{< Q >}{\psi(Q)} e^{\varphi(<Q>)}(\lambda(D) - E) f \right\|^2$$

for any $b \in (1/2, 1)$ and for $\psi(Q) := \sqrt{(1/2n)(1 + 4n < Q > \varphi'(< Q >))}$.

The constants C_γ and R_γ are also depending on the function λ and on E. The rest of this section is devoted to the proof of this theorem.

Proof. Let us first sketch a formal argument concerning this proof. We consider thus a function $f \in \mathcal{D}(p(-D)) \cap L^2_{comp}(\mathbb{R}^n)_{R_\gamma}$ and we compute the following sesquilinear form:

$$2Im < Ae^\varphi f, (\lambda(D) - E)e^\varphi f >= (-i) < e^\varphi f, [A, \lambda(D)]\, e^\varphi f > . \tag{3.1}$$

Taking an interval J containing the value E and contained in the set $\mathcal{E}(\lambda)$ and denoting by ϕ_J the operator $\phi_J(\lambda(D))$ with ϕ_J the characteristic function of the interval J and $\phi_J^\perp := 1 - \phi_J$ we have:

$$\begin{gathered} 2Im < Ae^\varphi f, (\lambda(D) - E)e^\varphi f >= \\ = b < e^\varphi f, B_0 e^\varphi f > +(1 - b) < e^\varphi f, \phi_J B_0 \phi_J e^\varphi f > + \\ +(1 - b) \left\{ < e^\varphi f, B_0 \phi_J^\perp e^\varphi f > + < e^\varphi f, \phi_J^\perp B_0 \phi_J e^\varphi f > \right\} + < e^\varphi f, Re^\varphi f > \end{gathered} \tag{3.2}$$

where $b \in (0, 1)$ and R is a remainder, measuring the difference between $(-i)\,[A, \lambda(D)]$ and B_0 and that has to be estimated. The "localization" with the operator ϕ_J is necessary because we intend to use the Mourre type estimation (2.43) for the second term on the right. For the remainder we shall prove an estimation of the form:

$$|< e^\varphi f, Re^\varphi f >| \leq \gamma C \, \|p(-D)e^\varphi f\|^2 \tag{3.3}$$

for any $f \in \mathcal{D}(p(-D)) \cap L^2_{comp}(\mathbb{R}^n)_{R_\gamma}$.

For the left hand side in (3.1) we consider the equality:

$$\begin{gathered} 2Im < Ae^\varphi f, (\lambda(D) - E)e^\varphi f >= \\ 2Im < Ae^\varphi f, e^\varphi(\lambda(D) - E) f > +2Im < Ae^\varphi f, (\lambda(D) - e^\varphi \lambda(D) e^{-\varphi}) e^\varphi f > . \end{gathered} \tag{3.4}$$

Now:

$$\begin{aligned} 2\,|Im < Ae^\varphi f, e^\varphi(\lambda(D) - E) f >| &\leq 2 \left\| \tfrac{\psi(Q)}{<Q>} Ae^\varphi f \right\| \left\| \tfrac{<Q>}{\psi(Q)} e^\varphi (\lambda(D) - E) f \right\| \leq \\ &\leq \left\| \tfrac{\psi(Q)}{<Q>} Ae^\varphi f \right\|^2 + \left\| \tfrac{<Q>}{\psi(Q)} e^\varphi(\lambda(D) - E) f \right\|^2 = \\ &= \left\langle Ae^\varphi f, \left(\tfrac{\psi(Q)}{<Q>} \right)^2 Ae^\varphi f \right\rangle + \left\| \tfrac{<Q>}{\psi(Q)} e^\varphi(\lambda(D) - E) f \right\|^2 , \end{aligned} \tag{3.5}$$

valid for any function ψ for which the right hand side is bounded. Concerning the second term on the right hand side of (3.4) we shall prove the following formula:

$$\begin{gathered} Im < Ae^\varphi f, (\lambda(D) - e^\varphi \lambda(D) e^{-\varphi}) e^\varphi f >= \\ = - \left\langle Ae^\varphi f, \tfrac{\varphi'(<Q>)}{<Q>} Ae^\varphi f \right\rangle + Im < Ae^\varphi f, \tilde{R} e^\varphi f > \end{gathered} \tag{3.6}$$

with the estimation on the remainder $\tilde{R}$:

$$\left| Im < Ae^\varphi f, \tilde{R} e^\varphi f > \right| \leq \gamma C \, \|p(-D)e^\varphi f\|^2 . \tag{3.7}$$

An important observation is now that the integrand of the term $-\left\langle Ae^{\varphi}f, \frac{\varphi'(<Q>)}{<Q>}Ae^{\varphi}f\right\rangle$, that grows at infinity in the spectral representation of Q (when removing the cut-off on f), has a definite negative sign for $\varphi \in \Phi_{\gamma,m}$. This fact is a reminiscence of the "General Virial Formula" in [FHHO]. Using the Mourre type inequality (2.43), denoting:

$$a := \inf_{t\in J}\left|(\nabla\lambda)(\lambda^{-1}(t)\right|^2 \tag{3.8}$$

and putting everything together we get:

$$\begin{array}{c}\left\|\frac{<Q>}{\psi(Q)}e^{\varphi}(\lambda(D)-E)f\right\|^2 \geq (1-b)a\left\|\phi_J e^{\varphi}f\right\|^2 + \\ +b < e^{\varphi}f, B_0 e^{\varphi}f > + \left\langle Ae^{\varphi}f, \left(2\frac{\varphi'(<Q>)}{<Q>} - \frac{\psi(Q)^2}{<Q>^2}\right)Ae^{\varphi}f\right\rangle + \\ +(1-b)\left\{< e^{\varphi}f, B_0\phi_J^{\perp}e^{\varphi}f > + < e^{\varphi}f, \phi_J^{\perp}B_0\phi_J e^{\varphi}f >\right\} + \\ + < e^{\varphi}f, Re^{\varphi}f > -C\gamma\left\|p(-D)e^{\varphi}f\right\|^2.\end{array} \tag{3.9}$$

With our choice for the function ψ we observe that:

$$\left\langle Ae^{\varphi}f, \left(2\frac{\varphi'(< Q >)}{< Q >} - \frac{\psi(Q)^2}{< Q >^2}\right)Ae^{\varphi}f\right\rangle = -\frac{1}{2n}\left\|< Q >^{-1} Ae^{\varphi}f\right\|^2. \tag{3.10}$$

In order to obtain a positive term we follow an idea of [ABG1] and prove an estimation of the form:

$$< g, B_0 g > \geq \frac{1}{n}\left\|< Q >^{-1} Ag\right\|^2 - \gamma C\left\|p(-D)g\right\|^2 \tag{3.11}$$

for any $g \in C_0^{\infty}(\mathbb{R}^n)$. This is in fact the reason for putting into evidence the first term on the right hand side of (3.2). In order to deal with the fourth term in (3.9) let us consider the explicit formula (2.42) for B_0 and observe that:

$$\begin{array}{c}\nabla\lambda = (\nabla p)\mu + p(\nabla\mu) \\ (\widehat{\nabla\mu})(x) = x\hat{\mu}(x),\end{array} \tag{3.12}$$

so that due to the hypothesis on p and μ the operator B_0 is relatively bounded with respect to $\lambda(D)^2$. Then B_0 defines a bounded sesquilinear form on $\mathcal{G}$ with norm denoted by $|||B_0|||$ so that we have the estimation:

$$\begin{array}{c}\left|< e^{\varphi}f, B_0\phi_J^{\perp}e^{\varphi}f > + < e^{\varphi}f, \phi_J^{\perp}B_0\phi_J e^{\varphi}f >\right| \leq \\ \leq |||B_0|||\left\|\phi_J^{\perp}e^{\varphi}f\right\|_{\mathcal{G}}\left\{\left\|e^{\varphi}f\right\|_{\mathcal{G}} + \left\|\phi_J e^{\varphi}f\right\|_{\mathcal{G}}\right\} \leq 2\,|||B_0|||\left\|\phi_J^{\perp}e^{\varphi}f\right\|_{\mathcal{G}}\left\|e^{\varphi}f\right\|_{\mathcal{G}} \leq \\ \leq |||B_0|||\left\{\theta\left\|\phi_J^{\perp}e^{\varphi}f\right\|_{\mathcal{G}}^2 + \frac{1}{\theta}\left\|e^{\varphi}f\right\|_{\mathcal{G}}^2\right\}.\end{array} \tag{3.13}$$

Let us further observe that:

$$\begin{array}{c}\left\|\phi_J^{\perp}e^{\varphi}f\right\|_{\mathcal{G}}^2 \leq \left\|\phi_J^{\perp}e^{\varphi}f\right\|^2 + \left\|\phi_J^{\perp}\lambda(D)e^{\varphi}f\right\|^2 \leq \\ \leq (1+E^2)\left\|\phi_J^{\perp}e^{\varphi}f\right\|^2 + \left\|(\lambda(D)-E)e^{\varphi}f\right\|^2 \leq \\ \leq (1+E^2)\left\|(\lambda(D)-E)(\lambda(D)-E)^{-1}\phi_J^{\perp}e^{\varphi}f\right\|^2 + \left\|(\lambda(D)-E)e^{\varphi}f\right\|^2 \leq \\ \leq c(E,J)\left\|(\lambda(D)-E)e^{\varphi}f\right\|^2 \leq c(E,J)\left\{\left\|e^{\varphi}(\lambda(D)-E)f\right\|^2 + \left\|R_2 e^{\varphi}f\right\|^2\right\}.\end{array} \tag{3.14}$$

Where:

$$C(E,J) := 1 + \frac{1+E^2}{d(E,J^c)^2} \tag{3.15}$$

and the last term contains a remainder $R_2 := e^{\varphi}\lambda(D)e^{-\varphi} - \lambda(D)$ that has to be estimated. Now we use condition (3) in the Hypothesis 2.2 in order to deal with the last term on the right hand side of (3.9):

$$\begin{aligned} \|p(-D)e^{\varphi}f\|^2 \leq \kappa^{-2}\|e^{\varphi}f\|^2_{\mathcal{G}} = \kappa^{-2}\|\phi_J e^{\varphi}f\|^2_{\mathcal{G}} + \kappa^{-2}\left\|\phi_J^{\perp}e^{\varphi}f\right\|^2_{\mathcal{G}} \leq \\ \leq \kappa^{-2}\left\{\sup_{t\in J} < t >^2\right\}\|e^{\varphi}f\|^2 + \kappa^{-2}\left\|\phi_J^{\perp}e^{\varphi}f\right\|^2_{\mathcal{G}}. \end{aligned} \tag{3.16}$$

We shall denote $c(J) := \sup_{t\in J} < t >^2$. Now let us plug all these estimations in (3.9):

$$\begin{aligned} \left\|\tfrac{<Q>}{\psi(Q)}e^{\varphi}(\lambda(D)-E)f\right\|^2 \geq (1-b)a\|\phi_J e^{\varphi}f\|^2 + \\ + \left(b-\tfrac{1}{2}\right)\tfrac{1}{n}\|< Q >^{-1} Ae^{\varphi}f\|^2 - \gamma C\|p(-D)e^{\varphi}f\|^2 - \\ -(1-b)\,|||B_0|||\left\{\theta\left\|\phi_J^{\perp}e^{\varphi}f\right\|^2_{\mathcal{G}} + \tfrac{1}{\theta}\|e^{\varphi}f\|^2_{\mathcal{G}}\right\} + \\ + < e^{\varphi}f, Re^{\varphi}f > -C\gamma\kappa^{-2}\left(c(J)\|e^{\varphi}f\|^2 + \left\|\phi_J^{\perp}e^{\varphi}f\right\|^2_{\mathcal{G}}\right). \end{aligned} \tag{3.17}$$

Using once again the second inequality in (3.16) for the first term on the right hand side of (3.17) we get:

$$\begin{aligned} \left\|\tfrac{<Q>}{\psi(Q)}e^{\varphi}(\lambda(D)-E)f\right\|^2 \geq \tfrac{(1-b)a}{c(J)}\left(\|e^{\varphi}f\|^2_{\mathcal{G}} - \left\|\phi_J^{\perp}e^{\varphi}f\right\|^2_{\mathcal{G}}\right) \\ + \left(b-\tfrac{1}{2}\right)\tfrac{1}{n}\|< Q >^{-1} Ae^{\varphi}f\|^2 - \gamma C\|e^{\varphi}f\|^2_{\mathcal{G}} - \\ -(1-b)\|B_0\|_{\mathcal{G}}\left\{\theta\left\|\phi_J^{\perp}e^{\varphi}f\right\|^2_{\mathcal{G}} + \tfrac{1}{\theta}\|e^{\varphi}f\|^2_{\mathcal{G}}\right\} + \\ + < e^{\varphi}f, Re^{\varphi}f > -C\gamma\kappa^{-2}\left(c(J)\|e^{\varphi}f\|^2 + \left\|\phi_J^{\perp}e^{\varphi}f\right\|^2_{\mathcal{G}}\right). \end{aligned} \tag{3.18}$$

or equivalently (taking into account (3.3)):

$$\begin{aligned} \left\{\tfrac{(1-b)a}{c(J)} - \left(C\gamma + \tfrac{1-b}{\theta}\|B_0\|_{\mathcal{G}} + 2C\gamma\kappa^{-2}c(J)\right)\right\}\|e^{\varphi}f\|^2_{\mathcal{G}} + \\ + \left(b-\tfrac{1}{2}\right)\tfrac{1}{n}\|< Q >^{-1} Ae^{\varphi}f\|^2 \leq \\ \leq \left\|\tfrac{<Q>}{\psi(Q)}e^{\varphi}(\lambda(D)-E)f\right\|^2 + \left(\tfrac{(1-b)a}{c(J)} + \theta(1-b)\|B_0\|_{\mathcal{G}} + C\gamma\kappa^{-2}\right)\left\|\phi_J^{\perp}e^{\varphi}f\right\|^2_{\mathcal{G}}. \end{aligned} \tag{3.19}$$

From (3.14) we obtain:

$$\begin{aligned} \left\{\tfrac{(1-b)a}{c(J)} - \left(C\gamma + \tfrac{1-b}{\theta}\|B_0\|_{\mathcal{G}} + C\gamma\kappa^{-2}c(J)\right)\right\}\|e^{\varphi}f\|^2_{\mathcal{G}} + \\ + \left(b-\tfrac{1}{2}\right)\tfrac{1}{n}\|< Q >^{-1} Ae^{\varphi}f\|^2 - |< e^{\varphi}f, Re^{\varphi}f >| - \|R_2e^{\varphi}f\|^2 \leq \\ \leq \left\|\tfrac{<Q>}{\psi(Q)}e^{\varphi}(\lambda(D)-E)f\right\|^2 + \\ + \left\{\tfrac{(1-b)a}{c(J)} + \theta(1-b)\|B_0\|_{\mathcal{G}} + C\gamma\kappa^{-2}\right\}c(E,J)\|e^{\varphi}(\lambda(D)-E)f\|^2. \end{aligned} \tag{3.20}$$

Finally using (3.3), (3.16) and similar estimations for the term containing R_2 we get:

$$|< e^{\varphi}f, Re^{\varphi}f >| + \|R_2e^{\varphi}f\|^2 \leq c\gamma\|e^{\varphi}f\|^2_{\mathcal{G}}. \tag{3.21}$$

Choosing $b \in \left[\frac{1}{2}, 1\right)$, γ small enough and θ large enough, we can assure the positivity of the coefficient of the first term on the left hand side of (3.20) and thus we get the desired conclusion. Let us strengthen the fact that the choice of γ and θ only depends on the value of the constant a in the Mourre type estimation and on the function λ and on E.

In conclusion, all that remains to be done in order to finish the proof of the Theorem 2.6 are the following four steps:

1. to give sense to the calculus done in (3.2) and to estimate the remainder term (3.3);
2. to prove (3.6) and to estimate the remainder term in (3.7);
3. to prove the inequality (3.11);
4. to estimate the remainder term R_2 appearing in (3.14).

Step 1: Let us begin by observing that for $f \in \mathcal{L}_{R_\gamma}$ we have $e^{\varphi} f \in \mathcal{L}_{R_\gamma}$. In fact the support condition is obvious and choosing $\eta \in C_0^\infty(\mathbb{R}^n)$ with the property $\eta f = f$ we see that due to the conditions imposed on the derivatives of the phase functions in $\Phi_{\gamma,m}$ and the hypoellipticity of p, the multiplication with the function ηe^{φ} leaves $\mathcal{D}(p(-D))$ invariant. Taking into account the Remark 2.8 and extending A on $\mathcal{L}_{R_\gamma}$, we can thus define the sesquilinear form:

$$\mathcal{L}_{R_\gamma} \times \mathcal{L}_{R_\gamma} \ni (g,f) \longmapsto \mathbf{B}(g,f) := (-i)\left[< Ag, (\lambda(D)-E)f > - < (\lambda(D)-E)g, Af >\right] \in \mathbb{C}. \quad (3.22)$$

As remarked previously B_0 can also be considered as defining a bounded sesquilinear form on $\mathcal{L}_{R_\gamma}$. By abuse of notation we shall also denote this form by $< g, B_0 f >$. Thus:

$$< e^{\varphi} f, Re^{\varphi} f >= \mathbf{B}(e^{\varphi} f, e^{\varphi} f) - < e^{\varphi} f, B_0 e^{\varphi} f > \quad (3.23)$$

and all we have to do is to estimate this difference for $f \in \mathcal{L}_{R_\gamma}$. We shall approach $f \in \mathcal{L}_{R_\gamma}$ with functions in $C_0^\infty(\mathbb{R}^n \setminus B(0,R_\gamma))$ with respect to the norm of $\mathcal{D}(p(-D))$. On $C_0^\infty(\mathbb{R}^n \setminus B(0,R_\gamma))$ we can compute the difference in (3.23) as the sesquilinear form associated to the operator $i[\lambda(D),A] - B_0 = i[\lambda(D),A] - (\nabla\lambda)^2(D)$. But:

$$i[\lambda(D),A] - (\nabla\lambda)^2(D) = \quad (3.24)$$

$$= \frac{1}{2}\int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x) \sum_{j=1}^n x_j \left\{ U_D(x) e^{sx\cdot X(Q)} [\lambda(D), Q_j] + [\lambda(D), Q_j] e^{-sx\cdot X(Q)} U_D(x) \right\} +$$

$$+ \frac{1}{2}\int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x) \sum_{j=1}^n x_j \left\{ U_D(x) \left[\lambda(D), e^{sx\cdot X(Q)}\right] Q_j + Q_j \left[\lambda(D), e^{-sx\cdot X(Q)}\right] U_D(x) \right\} -$$

$$- (\nabla\lambda)^2(D).$$

We begin by calculating the first and the third terms together:

$$\tfrac{1}{2}\int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x) \sum_{j=1}^n x_j \left\{ U_D(x) e^{sx\cdot X(Q)} [\lambda(D), Q_j] + [\lambda(D), Q_j] e^{-sx\cdot X(Q)} U_D(x) \right\} -$$
$$- (\nabla\lambda)(D)^2 =$$

$$= \tfrac{-i}{2}\int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x) \sum_{j=1}^n x_j \{ U_D(x) e^{sx\cdot X(Q)} \partial_j\lambda(D) + \partial_j\lambda(D) e^{-sx\cdot X(Q)} U_D(x) -$$
$$- 2\partial_j\lambda(D) U_D(x) \} =$$

$$= \tfrac{-i}{2}\int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x) \sum_{j=1}^n x_j \{ U_D(x) (e^{sx\cdot X(Q)} - 1) \partial_j\lambda(D) +$$
$$+ \partial_j\lambda(D) (e^{-sx\cdot X(Q)} - 1) U_D(x) \} =$$

$$= \tfrac{-i}{2}\int_0^1 sds \int_0^1 dt \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x) \sum_{j,k=1}^n x_k x_j \{ U_D(x) X_k(Q) e^{tsx\cdot X(Q)} \partial_j\lambda(D) -$$
$$- \partial_j\lambda(D) X_k(Q) e^{-tsx\cdot X(Q)} U_D(x) \} =$$

$$= \frac{-i}{2}\int_0^1 sds\int_0^1 dt\int_{\mathbb{R}^n}\hat{\mu}(dx)p(i\partial_x)\sum_{j,k=1}^n x_k x_j\{X_k(Q+x)e^{tsx\cdot X(Q+x)}\partial_j\lambda(D)-$$
$$-\partial_j\lambda(D)X_k(Q)e^{-tsx\cdot X(Q)}\}U_D(x) =$$
$$= \frac{-i}{2}\int_0^1 sds\int_0^1 dt\int_{\mathbb{R}^n}\hat{\mu}(dx)\int_{\mathbb{R}^n}\hat{\mu}(dy)p(i\partial_x)p(i\partial_y)\sum_{j,k=1}^n x_k x_j y_j\cdot$$
$$\cdot\{X_k(Q+x)e^{tsx\cdot X(Q+x)}U_D(y)-U_D(y)X_k(Q)e^{-tsx\cdot X(Q)}\}U_D(x) =$$
$$= \frac{-i}{2}\int_0^1 sds\int_0^1 dt\int_{\mathbb{R}^n}\hat{\mu}(dx)\int_{\mathbb{R}^n}\hat{\mu}(dy)p(i\partial_x)p(i\partial_y)\sum_{j,k=1}^n x_k x_j y_j\cdot$$
$$\cdot U_D(y)\{X_k(Q+x-y)e^{tsx\cdot X(Q+x-y)}-X_k(Q)e^{-tsx\cdot X(Q)}\}U_D(x). \tag{3.25}$$

Let us treat in a similar way the second term in (3.24). First let us observe that:

$$\left[\lambda(D), e^{\pm sx\cdot X(Q)}\right] = \int_{\mathbb{R}^n}\hat{\mu}(dy)p(i\partial_y)\left[U_D(y), e^{\pm sx\cdot X(Q)}\right] =$$
$$= \int_{\mathbb{R}^n}\hat{\mu}(dy)p(i\partial_y)U_D(y)(e^{\pm sx\cdot X(Q)}-e^{\pm sx\cdot X(Q-y)}) =$$
$$= (\pm s)\int_0^1 dt\int_{\mathbb{R}^n}\hat{\mu}(dy)p(i\partial_y)U_D(y)\sum_{l,k=1}^n x_l y_k(\partial_k X_l)(Q-ty)e^{\pm sx\cdot X(Q-ty)}.$$

so that we obtain for the second term on the right hand side of (3.24):

$$\frac{1}{2}\int_0^1 ds\int_{\mathbb{R}^n}\hat{\mu}(dx)p(i\partial_x)\sum_{j=1}^n x_j\{U_D(x)\left[\lambda(D), e^{sx\cdot X(Q)}\right]Q_j + Q_j\left[\lambda(D), e^{-sx\cdot X(Q)}\right]U_D(x)\} =$$
$$= \frac{1}{2}\int_0^1 sds\int_0^1 dt\int_{\mathbb{R}^n}\hat{\mu}(dx)\int_{\mathbb{R}^n}\hat{\mu}(dy)p(i\partial_x)p(i\partial_y)\sum_{j,l,k=1}^n x_j x_l y_k\cdot$$
$$\cdot U_D(y)\{(\partial_k X_l)(Q-ty+x)e^{sx\cdot X(Q-ty+x)}(Q_j+x_j)-$$
$$-(Q_j-y_j)(\partial_k X_l)(Q-ty)e^{-sx\cdot X(Q-ty)}\}U_D(x). \tag{3.26}$$

Finally, putting (3.25) and (3.26) together, we obtain:

$$i\left[\lambda(D), A\right] - \left(\nabla\lambda\right)^2(D) = \tag{3.27}$$
$$= \frac{-i}{2}\int_0^1 sds\int_0^1 dt\int_{\mathbb{R}^n}\hat{\mu}(dx)\int_{\mathbb{R}^n}\hat{\mu}(dy)p(i\partial_x)p(i\partial_y)U_D(y)\cdot$$
$$\cdot\{\sum_{j,k=1}^n x_k x_j y_j(X_k(Q+x-y)e^{tsx\cdot X(Q+x-y)}-X_k(Q)e^{-tsx\cdot X(Q)}) + \sum_{j,l,k=1}^n x_j x_l y_k\cdot$$
$$\cdot((\partial_k X_l)(Q-ty+x)e^{sx\cdot X(Q-ty+x)}(Q_j+x_j)-$$
$$-(Q_j-y_j)(\partial_k X_l)(Q-ty)e^{-sx\cdot X(Q-ty)})\}U_D(x).$$

We use now formula (2.26) for $p(i\partial_x)$ and for $p(i\partial_y)$ in order to get:

$$i\left[\lambda(D), A\right] - \left(\nabla\lambda\right)^2(D) = \tag{3.28}$$
$$= \left(\frac{-i}{2}\right)\sum_{|\alpha|\leq m}\frac{i^{|\alpha|}}{\alpha!}\sum_{|\beta|\leq m}\frac{i^{|\beta|}}{\beta!}\int_0^1 sds\int_0^1 dt\int_{\mathbb{R}^n}\hat{\mu}(dx)e^{a|x|}\int_{\mathbb{R}^n}\hat{\mu}(dy)e^{a|y|}\cdot$$
$$\cdot\partial^\alpha p(-D)U_D(y)H(Q;x,y)U_D(x)\partial^\beta p(-D)$$

where:

$$H_{\alpha\beta}(Q;x,y) :=$$
$$e^{-a(|x|+|y|)}\left(\partial_x^\beta\partial_y^\alpha\right)\{\sum_{j,k=1}^n x_k x_j y_j(X_k(Q+x-y)e^{tsx\cdot X(Q+x-y)}-$$
$$-X_k(Q)e^{-tsx\cdot X(Q)}) + \sum_{j,l,k=1}^n x_j x_l y_k\cdot \tag{3.29}$$
$$\cdot((\partial_k X_l)(Q-ty+x)e^{sx\cdot X(Q-ty+x)}(Q_j+x_j)-$$
$$-(Q_j-y_j)(\partial_k X_l)(Q-ty)e^{-sx\cdot X(Q-ty)})\}.$$

Using now the formulae (2.48-2.52) and taking into account the definition of X and the conditions on the phase functions $\varphi \in \Phi_{\gamma,m}$ we see that we obtain:

$$\|H_{\alpha\beta}(Q;x,y)\| \leq \gamma C. \tag{3.30}$$

Finally we obtain for any $f \in \mathcal{L}_{R_\gamma}$ the estimation (3.3).

Step 2: For $f \in \mathcal{L}_{R_\gamma}$ and $g \in \mathcal{H}$ let us compute the difference:

$$\begin{array}{c} < g, \lambda(D)f > - < g, e^{\varphi}\lambda(D)e^{-\varphi}f >= \\ = \frac{1}{2} < g, (e^{-\varphi}\lambda(D)e^{\varphi} - e^{\varphi}\lambda(D)e^{-\varphi})f > + \\ + \frac{1}{2} < g, (2\lambda(D) - e^{\varphi}\lambda(D)e^{-\varphi} - e^{-\varphi}\lambda(D)e^{\varphi})f > \end{array} \tag{3.31}$$

We begin with the first term in (3.31):

$$\begin{array}{c} S_1(g,f) :=< g, (e^{-\varphi}\lambda(D)e^{\varphi} - e^{\varphi}\lambda(D)e^{-\varphi})f >= \\ = \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) < g, (e^{-\varphi}U_D(x)e^{\varphi} - e^{\varphi}U_D(x)e^{-\varphi})f >= \\ = \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) < g, \{U_D(x)e^{\varphi(<Q>)}e^{-\varphi(<Q-x>)} - e^{\varphi(<Q>)}e^{-\varphi(<Q+x>)}U_D(x)\}f >= \\ = \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) < g, U_D(x)\{e^{\varphi(<Q>)}e^{-\varphi(<Q-x>)} - e^{x\cdot X(Q)})\}f > + \\ + \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) < g, \{e^{-x\cdot X(Q)} - e^{\varphi(<Q>)}e^{-\varphi(<Q+x>)}\}U_D(x))f > + \\ + \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) < g, \{U_D(x)(e^{x\cdot X(Q)} - 1) - (e^{-x\cdot X(Q)} - 1)U_D(x)\}f > . \end{array} \tag{3.32}$$

Let us denote:

$$\begin{array}{c} Y(Q;s,x) := s\{\varphi(< Q >) - \varphi(< Q + x >)\} - (1-s)x\cdot X(Q) = \\ = s\{\varphi(< Q >) - \varphi(< Q + x >) + x\cdot X(Q)\} - x\cdot X(Q) \end{array} \tag{3.33}$$

so that:

$$S_1(g,f) = \tag{3.34}$$

$$\begin{array}{c} = \int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) < g, U_D(x)\{\varphi(< Q >) - \varphi(< Q - x >) - x\cdot X(Q)\}e^{Y(Q;s,-x)}f > + \\ + \int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) < g, \{\varphi(< Q + x >) - \varphi(< Q >) - x\cdot X(Q)\}e^{Y(Q;s,x)}U_D(x)f > + \\ + \int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) < g, \{U_D(x)e^{sx\cdot X(Q)}x\cdot X(Q) + x\cdot X(Q)e^{-sx\cdot X(Q)}U_D(x)\}f >= \\ = \int_0^1 ds \int_0^1 (1-t)dt \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) \sum_{j,k=1}^n c_{jk}x_jx_k \cdot \\ \cdot < g, \{U_D(x)(\partial_jX_k)(Q - tx)e^{Y(Q;s,-x)} - (\partial_jX_k)(Q + tx)e^{Y(Q;s,x)}U_D(x)\}f > + \\ + \int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) \sum_{j=1}^n x_j < g, \{U_D(x)e^{sx\cdot X(Q)}Q_j\xi(Q) + \xi(Q)Q_je^{-sx\cdot X(Q)}U_D(x)\}f > \end{array}$$

where $Q_j\xi(Q) := X_j(Q) = (\partial_j\varphi)(< Q >) = Q_j < Q >^{-1} \varphi'(< Q >)$. In conclusion, commuting $U_D(x)$ and $\xi(Q)$ in the last term, taking into account the definition of A (2.45) and observing that

$$\widehat{\nabla\mu}(dx) = ix\hat{\mu}(dx),$$

we get:

$$\begin{array}{c} S_1(g,f) = 2i < g, \xi(Q)Af > + \\ + \int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) \sum_{j=1}^n x_j < g, [\xi(Q + x) - \xi(Q)]U_D(x)e^{sx\cdot X(Q)}Q_jf > + \\ + \int_0^1 ds \int_0^1 (1-t)dt \int_{\mathbb{R}^n} \hat{\mu}(dx)p(i\partial_x) \sum_{j,k=1}^n c_{jk}x_jx_k \cdot \\ \cdot < g, \{U_D(x)(\partial_jX_k)(Q - tx)e^{Y(Q;s,-x)} - (\partial_jX_k)(Q + tx)e^{Y(Q;s,x)}U_D(x)\}f > . \end{array} \tag{3.35}$$

But:

$$\xi(Q+x)-\xi(Q)=\int_0^1 dt\sum_{k=1}^n x_k(\partial_k\xi)(Q+tx). \tag{3.36}$$

Using once again the formulae (2.48-2.52) and taking into account the definition of $\tilde{R}$ and the condition on the phase functions $\varphi\in\Phi_{\gamma,m}$ we obtain:

$$ImS_1(Af,f)=2<Af,<Q>^{-1}\varphi'(<Q>)Af>+Im<Af,\tilde{R}_1f> \tag{3.37}$$

with the estimation:

$$\left|<Af,\tilde{R}_1f>\right|\leq\gamma C\left\|p(-D)f\right\|^2 \tag{3.38}$$

where the constant C only depends on the function λ, so that by choosing γ small enough we can make this remainder as small as we like.

Let us concentrate now on the second term in (3.31):

$$S_2(g,f):= \tag{3.39}$$

$$\begin{gathered}
<g,\{2\lambda(D)-e^{\varphi(<Q>)}\lambda(D)e^{-\varphi(<Q>)}-e^{-\varphi(<Q>)}\lambda(D)e^{\varphi(<Q>)}\}f>= \\
=\int_{\mathbb{R}^n}\hat{\mu}(dx)p(i\partial_x)<g,\{2U_D(x)-e^{\varphi(<Q>)}U_D(x)e^{-\varphi(<Q>)}-e^{-\varphi(<Q>)}U_D(x)e^{\varphi(<Q>)}\}f>= \\
=\int_{\mathbb{R}^n}\hat{\mu}(dx)p(i\partial_x)<g,\{2-e^{\varphi(<Q>)-\varphi(<Q+x>)}-e^{-\varphi(<Q>)+\varphi(<Q+x>)}\}U_D(x)f>= \\
=\int_{\mathbb{R}^n}\hat{\mu}(dx)p(i\partial_x)<g,\{(1-e^{\varphi(<Q>)-\varphi(<Q+x>)})+(1-e^{-\varphi(<Q>)+\varphi(<Q+x>)})\}U_D(x)f>= \\
=\int_0^1 ds\int_{\mathbb{R}^n}\hat{\mu}(dx)p(i\partial_x)<g,\{x\cdot X(Q+sx)e^{\varphi(<Q>)-\varphi(<Q+sx>)}- \\
-x\cdot X(Q+sx)e^{-\varphi(<Q>)+\varphi(<Q+sx>)}\}U_D(x)f>= \\
=\int_0^1 ds\int_{\mathbb{R}^n}\hat{\mu}(dx)p(i\partial_x)<g,x\cdot X(Q+sx)(\exp\left\{-sx\cdot\int_0^1 dtX(Q+tsx)\right\}- \\
-\exp\left\{sx\cdot\int_0^1 dtX(Q+tsx)\right\})U_D(x)f>.
\end{gathered}$$

Let us denote:

$$\rho(Q;s,x):=sx\cdot\int_0^1 dtX(Q+tsx). \tag{3.40}$$

For $f\in C_0^\infty(\mathbb{R}^n)$ we have:

$$2ImS_2(Af,f)= \tag{3.41}$$

$$\begin{gathered}
=(-i)\int_0^1 ds\int_{\mathbb{R}^n}\hat{\mu}(dx)p(i\partial_x)\sum_{j=1}^n x_j<f,(\left[A,X_j(Q+tsx)(e^{-\rho(Q;s,x)}-e^{\rho(Q;s,x)})\right]U_D(x)+ \\
+X_j(Q+tsx)(e^{-\rho(Q;s,x)}-e^{\rho(Q;s,x)})\left[A,U_D(x)\right])f>.
\end{gathered}$$

Let us estimate the two commutators appearing in (3.41). If we denote:

$$G(Q;t,s,x):=X(Q+tsx)(e^{-\rho(Q;s,x)}-e^{\rho(Q;s,x)}) \tag{3.42}$$

we see that:

$$(-i)\left[A,G_j(Q;t,s,x)\right]= \tag{3.43}$$

$$= \int_0^1 dt \int_{\mathbb{R}^n} \hat\mu(dy)p(i\partial_y) \sum_{l=1}^n y_l \{U_D(y)(G_j(Q;t,s,x) - G_j(Q-y;t,s,x))e^{tx\cdot X(Q)}Q_l$$
$$+Q_l e^{-tx\cdot X(Q)}U_D(y)(G_j(Q;t,s,x) - G_j(Q-y;t,s,x))\} =$$
$$= \int_0^1 dt \int_0^1 d\tau \int_{\mathbb{R}^n} \hat\mu(dy)p(i\partial_y) \sum_{l,k=1}^n y_l y_k U_D(y)(\partial_k G_j)(Q-(1-\tau)y;t,s,x)\cdot$$
$$\cdot(e^{tx\cdot X(Q)}Q_l + (Q_l - y_l)e^{-tx\cdot X(Q-y)})$$

where the derivative of G is computed with respect to the first variable and is given by:

$$\begin{aligned}(\partial_k G_j)(z;t,s,x) &= (\partial_k X_j)(z+tsx)(e^{-\rho(z;s,x)} - e^{\rho(z;s,x)}) - \\ &-X_j(z+tsx)(\partial_k\rho)(z;t,s,x)(e^{-\rho(z;s,x)} + e^{\rho(z;s,x)}) = \\ &= (\partial_k X_j)(z+tsx)(e^{-\rho(z;s,x)} - e^{\rho(z;s,x)}) - \\ &-sX_j(z+tsx)(e^{-\rho(z;s,x)} + e^{\rho(z;s,x)}) \sum_{l=1}^n x_l \int_0^1 d\tau (\partial_k X_l)(Q+\tau sx).\end{aligned} \tag{3.44}$$

Let us consider now the second commutator:

$$(-i)\,[A, U_D(x)] = \tag{3.45}$$

$$= (-\tfrac{i}{2}) \int_0^1 dt \int_{\mathbb{R}^n} \hat\mu(dy)p(i\partial_y) \sum_{l=1}^n y_l \{U_D(y) \left[e^{ty\cdot X(Q)}Q_l, U_D(x)\right] +$$
$$+ \left[e^{-ty\cdot X(Q)}Q_l, U_D(x)\right] U_D(y)\} =$$
$$= (-\tfrac{i}{2}) \int_0^1 dt \int_{\mathbb{R}^n} \hat\mu(dy)p(i\partial_y) \sum_{l=1}^n y_l \{U_D(y) \int_0^1 d\tau \sum_{r=1}^n x_r [\sum_{k=1}^n t y_k (\partial_r X_k)(Q)Q_l + \delta_{rl}] e^{ty\cdot X(Q)}U_D(x) -$$
$$- \int_0^1 d\tau \sum_{r=1}^n x_r [\sum_{k=1}^n t y_k (\partial_r X_k)(Q)Q_l - \delta_{rl}] e^{-ty\cdot X(Q)}U_D(x)U_D(y)\} =$$
$$= (-\tfrac{i}{2}) \int_0^1 dt \int_{\mathbb{R}^n} \hat\mu(dy)p(i\partial_y) \sum_{l=1}^n y_l \int_0^1 d\tau \sum_{r=1}^n x_r U_D(y)\cdot$$
$$\cdot\{(\sum_{k=1}^n t y_k (\partial_r X_k)(Q)Q_l + \delta_{rl})e^{ty\cdot X(Q)} - (\sum_{k=1}^n t y_k (\partial_r X_k)(Q-y)(Q_l - y_l) - \delta_{rl})e^{-ty\cdot X(Q-y)}\}U_D(x).$$

Putting all these results together we obtain finally:

$$2ImS_2(Af,f) = \tag{3.46}$$

$$= \int_0^1 ds \int_{\mathbb{R}^n} \hat\mu(dx)p(i\partial_x) \sum_{j=1}^n x_j \int_0^1 dt \int_0^1 d\tau \int_{\mathbb{R}^n} \hat\mu(dy)p(i\partial_y) \sum_{l,k=1}^n y_l y_k \cdot$$
$$\cdot < f, U_D(y)(\partial_k G_j)(Q-(1-\tau)y;t,s,x)\{e^{tx\cdot X(Q)}Q_l + (Q_l - y_l)e^{-tx\cdot X(Q-y)}\}U_D(x)f > +$$
$$+ \int_0^1 ds \int_{\mathbb{R}^n} \hat\mu(dx)p(i\partial_x) \sum_{j,r=1}^n x_j x_r \int_0^1 dt \int_0^1 d\tau \int_{\mathbb{R}^n} \hat\mu(dy)p(i\partial_y) \sum_{l=1}^n y_l \cdot$$
$$\cdot < f, U_D(y)X_j(Q+sx-y)(e^{-\rho(Q-y;s,x)} - e^{\rho(Q-y;s,x)})\{(\sum_{k=1}^n t y_k (\partial_r X_k)(Q)Q_l + \delta_{rl})e^{ty\cdot X(Q)} -$$
$$-(\sum_{k=1}^n t y_k (\partial_r X_k)(Q-y)(Q_l - y_l) - \delta_{rl})e^{-ty\cdot X(Q-y)}\}U_D(x)f >=$$
$$= \int_0^1 ds \int_{\mathbb{R}^n} \hat\mu(dx)p(i\partial_x) \sum_{j=1}^n x_j \int_0^1 dt \int_0^1 d\tau \int_{\mathbb{R}^n} \hat\mu(dy)p(i\partial_y) \sum_{l=1}^n y_l \cdot$$
$$\cdot \sum_{k=1}^n \{y_k < f, U_D(y)(\partial_k G_j)(Q-(1-\tau)y;t,s,x)(e^{tx\cdot X(Q)}Q_l + (Q_l - y_l)e^{-tx\cdot X(Q-y)})U_D(x)f > +$$
$$+x_k < f, U_D(y)X_j(Q+sx-y)(e^{-\rho(Q-y;s,x)} - e^{\rho(Q-y;s,x)})[(\sum_{r=1}^n t y_r (\partial_k X_r)(Q)Q_l + \delta_{kl})e^{ty\cdot X(Q)} -$$
$$-(\sum_{r=1}^n t y_r (\partial_k X_r)(Q-y)(Q_l - y_l) - \delta_{kl})e^{-ty\cdot X(Q-y)}]U_D(x)f >\} =$$

$$= \int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x) \sum_{j=1}^n x_j \int_0^1 dt \int_0^1 d\tau \int_{\mathbb{R}^n} \hat{\mu}(dy) p(i\partial_y) \sum_{l=1}^n y_l \cdot$$
$$\cdot \sum_{k=1}^n < f, U_D(y) R_{jlk}(Q; s, t, \tau, x, y) U_D(x) f > .$$

Using the formulae (3.44) and the properties of the phase functions of class $\Phi_{\gamma,m}$ one can easily see that:

$$\sup_{x,y\in\mathbb{R}^n} e^{-a(|x|+|y|)} |p(i\partial_x)p(i\partial_y)x_j y_l < f, U_D(y) R_{jlk}(Q; s, t, \tau, x, y) U_D(x) f >| \leq \gamma C \, \|p(-D)f\|^2 \quad (3.47)$$

for any s,t,τ in the interval $[0,1]$, with C depending only on the function λ. This finishes the proof of the estimation (3.7).

Step 3: Let us look at the operator $A < Q >^{-2} A$ that defines a sesquilinear form on $\mathcal{D}(p(-D))_{R_\gamma}$ that is bounded with respect to the graph-norm of the operator $p(-D)$. We have:

$$< f, A < Q >^{-2} Af >= \quad (3.48)$$

$$= -\frac{1}{4} \sum_{j=1}^n \sum_{l=1}^n \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x) x_j \hat{\mu}(dy) p(i\partial_y) y_l \cdot$$
$$\cdot < f, U_D(x) \left\{ e^{sx\cdot X(Q)} Q_j + e^{-sx\cdot X(Q-x)} (Q_j - x_j) \right\} < Q >^{-2} \cdot$$
$$\cdot \left\{ (Q_l + y_l) e^{ty\cdot X(Q+y)} + Q_l e^{-ty\cdot X(Q)} \right\} U_D(y) f >=$$
$$= -\frac{1}{4} \sum_{j,l=1}^n \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\mu}(dx) \hat{\mu}(dy) p(i\partial_x) p(i\partial_y) x_j y_l (B_{jl} + R_{jl} + T_{jl} + \tilde{T}_{jl} + S_{jl})$$

where:

$$B_{jl} := 4 < f, U_D(y) Q_j < Q >^{-2} Q_l U_D(x) f > \quad (3.49)$$

$$\begin{aligned} R_{jl} :=& < f, U_D(x)(e^{sx\cdot X(Q)} - 1) Q_j < Q >^{-2} Q_l U_D(y) f > + \\ &+ < f, U_D(x) e^{sx\cdot X(Q)} Q_j < Q >^{-2} Q_l (e^{-ty\cdot X(Q)} - 1) U_D(y) f > + \\ &+ < f, U_D(x)(e^{-sx\cdot X(Q+x)} - 1) Q_j < Q >^{-2} Q_l U_D(y) f > + \\ &+ < f, U_D(x) e^{-sx\cdot X(Q+x)} Q_j < Q >^{-2} Q_l (e^{ty\cdot X(Q)} - 1) U_D(y) f > + \\ &+ < f, U_D(x)(e^{-sx\cdot X(Q)} - 1) Q_j < Q >^{-2} Q_l U_D(y) f > + \\ &+ < f, U_D(x) e^{-sx\cdot X(Q)} Q_j < Q >^{-2} Q_l (e^{-ty\cdot X(Q)} - 1) U_D(y) f > + \\ &+ < f, U_D(x)(e^{sx\cdot X(Q+x)} - 1) Q_j < Q >^{-2} Q_l U_D(y) f > + \\ &+ < f, U_D(x) e^{sx\cdot X(Q)} Q_j < Q >^{-2} Q_l (e^{ty\cdot X(Q)} - 1) U_D(y) f > \end{aligned} \quad (3.50)$$

$$\begin{aligned} T_{jl} :=& - < f, U_D(x) e^{sx\cdot X(Q)} Q_j < Q >^{-2} y_l e^{ty\cdot X(Q-y)} U_D(y) f > - \\ &- < f, U_D(x) e^{-sx\cdot X(Q)} Q_j < Q >^{-2} y_l e^{ty\cdot X(Q-y)} U_D(y) f > \end{aligned} \quad (3.51)$$

$$\begin{aligned} \tilde{T}_{jl} :=& < f, U_D(x) e^{-sx\cdot X(Q)} x_j < Q >^{-2} e^{ty\cdot X(Q-y)} U_D(y) f > + \\ &+ < f, U_D(x) e^{-sx\cdot X(Q)} x_j < Q >^{-2} e^{-ty\cdot X(Q-y)} U_D(y) f > \end{aligned} \quad (3.52)$$

$$S_{jl} := - < f, U_D(x) e^{-sx\cdot X(Q+x)} x_j < Q >^{-2} y_l e^{ty\cdot X(Q-y)} U_D(y) f > . \quad (3.53)$$

Now let us discuss each type of term separately. We begin with B_{jl}:

$$\begin{aligned} &\frac{1}{4} \sum_{j,l=1}^n \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\mu}(dx) \hat{\mu}(dy) p(i\partial_x) p(i\partial_y) x_j y_l B_{jl} = \\ &= \sum_{j,l=1}^n \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\mu}(dx) \hat{\mu}(dy) p(i\partial_x) p(i\partial_y) x_j y_l \cdot \\ &\cdot < f, U_D(x) Q_j < Q >^{-2} Q_l U_D(y) f > . \end{aligned} \quad (3.54)$$

But we observe that:

$$\left\| Q_j < Q >^{-2} Q_l \right\| \leq 1 \tag{3.55}$$

so that

$$\begin{aligned} \left| \sum_{j,l=1}^{n} < f_j, Q_j < Q >^{-2} Q_l f_l > \right| \leq \sum_{j,l=1}^{n} \left| < f_j, Q_j < Q >^{-2} Q_l f_l > \right| \leq \\ \leq \sum_{j,l=1}^{n} \|f_j\| \, \|f_l\| \leq n \sum_{j=1}^{n} \|f_j\|^2 . \end{aligned} \tag{3.56}$$

In conclusion we obtain that:

$$\begin{aligned} \frac{1}{4} \sum_{j,l=1}^{n} \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\mu}(dx)\hat{\mu}(dy) p(i\partial_x) p(i\partial_y) x_j y_l B_{jl} \leq \\ \leq n \sum_{j=1}^{n} \left\| \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x) x_j U_D(x) f \right\|^2 = n \sum_{j=1}^{n} \|(\partial_j \lambda)(D) f\|^2 = n < f, B_0 f > . \end{aligned} \tag{3.57}$$

In order to estimate R_{jl} let us consider one of the eight scalar products appearing in its definition:

$$\begin{aligned} | \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\mu}(dx)\hat{\mu}(dy) p(i\partial_x) p(i\partial_y) x_j y_l \cdot \\ \cdot < f, U_D(x) e^{sx\cdot X(Q)} Q_j < Q >^{-2} Q_l (e^{-ty\cdot X(Q)} - 1) U_D(y) f > | \leq \\ \leq \int_0^1 ds \int_0^1 t dt \sum_{j=1}^{n} \int_0^1 d\tau \, \|X_j(Q)\| \int_{\mathbb{R}^n} |\hat{\mu}| (dx) \left\| p(i\partial_x) e^{sx\cdot X(Q)} U_D(-x) f \right\| \cdot \\ \cdot \int_{\mathbb{R}^n} |\hat{\mu}| (dy) \left\| p(i\partial_y) y_j e^{-t\tau y\cdot X(Q)} U_D(y) f \right\| \leq C\gamma \, \|p(-D) f\|^2 ; \end{aligned} \tag{3.58}$$

so that one can prove that:

$$\begin{aligned} \left| \frac{1}{4} \sum_{j,l=1}^{n} \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\mu}(dx)\hat{\mu}(dy) p(i\partial_x) p(i\partial_y) x_j y_l R_{jl} \right| \leq \\ \leq C\gamma \, \|p(-D) f\|^2 . \end{aligned} \tag{3.59}$$

Let us consider now the type of terms appearing in T_{jl} and in $\tilde{T}_{jl}$:

$$\begin{aligned} | \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\mu}(dx)\hat{\mu}(dy) p(i\partial_x) p(i\partial_y) x_j y_k \cdot \\ \cdot < f, U_D(x) e^{sx\cdot X(Q)} Q_j < Q >^{-2} y_l e^{ty\cdot X(Q-y)} U_D(y) f > | \leq \\ \leq \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{\mu}| (dx) \, |\hat{\mu}| (dy) \cdot | \, p(i\partial_x) p(i\partial_y) x_j y_k \\ \cdot < Q_j < Q >^{-1} e^{sx\cdot X(Q)} U_D(-x) f, y_l < Q >^{-1} e^{ty\cdot X(Q-y)} U_D(y) f > | \leq \\ \leq \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{\mu}| (dx) \, |\hat{\mu}| (dy) \cdot \\ \cdot \left\| Q_j < Q >^{-1} p(i\partial_x) x_j e^{sx\cdot X(Q)} U_D(-x) f \right\| \left\| y_l < Q >^{-1} p(i\partial_y) y_k e^{ty\cdot X(Q-y)} U_D(y) f \right\| \end{aligned} \tag{3.60}$$

Let us consider the last norm:

$$\begin{aligned} \left\| y_l < Q >^{-1} p(i\partial_y) y_k e^{ty\cdot X(Q-y)} U_D(y) f \right\| = \\ = \left\| < Q >^{-1} y_l p(i\partial_y) y_k U_D(y) e^{ty\cdot X(Q)} f \right\| \leq \\ \leq \sum_{|\beta| \leq m} \left\| i^{|\beta|} \frac{1}{\beta!} y_l (\partial_y^\beta y_k e^{ty\cdot X(Q)}) U_D(y) < Q - y >^{-1} (\partial^\beta p)(-D) f \right\| \leq \\ \leq C \, |y|^2 \, e^{\gamma |y|} \left\| < Q - y >^{-1} p(-D) f \right\| \end{aligned} \tag{3.61}$$

and take into account the support condition on $f \in \mathcal{L}_{R_\gamma}$ that implies that $p(-D)f = \eta_\gamma p(-D)f$ where η_γ is the characteristic function of the exterior of the ball $B(0; R_\gamma)$; thus:

$$\begin{gathered} \| < Q+y >^{-1} p(-D)f \| \leq \sqrt{2} < y > \| < Q >^{-1} \eta_\gamma p(-D)f \| \leq \\ \leq \sqrt{2} < y > R_\gamma^{-1} \| p(-D)f \| \end{gathered} \tag{3.62}$$

and:

$$\begin{gathered} \left| \frac{1}{4} \sum_{j,l=1}^{n} \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\mu}(dx) \hat{\mu}(dy) p(i\partial_x) p(i\partial_y) x_j y_l (T_{jl} + \tilde{T}_{jl}) \right| \leq \\ \leq C R_\gamma^{-1} \| p(-D)f \|^2 . \end{gathered} \tag{3.63}$$

Now let us finish this step by considering the term S_{jl} and observing that:

$$\begin{gathered} | \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\mu}(dx) \hat{\mu}(dy) p(i\partial_x) p(i\partial_y) x_j y_k \cdot \\ \cdot < f, U_D(x) e^{sx \cdot X(Q)} x_j < Q >^{-2} y_l e^{ty \cdot X(Q-y)} U_D(y) f > | \leq \\ \leq \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{\mu}| (dx) |\hat{\mu}| (dy) \cdot \\ \left\| x_j < Q >^{-1} p(i\partial_x) x_j e^{sx \cdot X(Q)} U_D(-x) f \right\| \left\| y_l < Q >^{-1} p(i\partial_y) y_k e^{ty \cdot X(Q-y)} U_D(y) f \right\| \end{gathered} \tag{3.64}$$

so that finally:

$$\begin{gathered} \left| \frac{1}{4} \sum_{j,l=1}^{n} \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\mu}(dx) \hat{\mu}(dy) p(i\partial_x) p(i\partial_y) x_j y_l S_{jl} \right| \leq \\ \leq C R_\gamma^{-2} \| p(-D)f \|^2 . \end{gathered} \tag{3.65}$$

Step 4: Let us estimate now the remainder term containing R_2 in (3.14):

$$\begin{gathered} \| R_2 f \| := \| (e^{\varphi} \lambda(D) e^{-\varphi} - \lambda(D)) f \| = \\ = \left\| \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x) \left\{ e^{\varphi(<Q>) - \varphi(<Q+x>)} - 1 \right\} U_D(x) f \right\| \leq \\ \leq \int_0^1 ds \int_{\mathbb{R}^n} |\hat{\mu}| (dx) \left\| p(i\partial_x) x \cdot X(Q+sx) \exp \left\{ -sx \cdot \int_0^1 X(Q+tsx) dt \right\} U_D(x) f \right\| \leq \\ \leq C\gamma \| p(-D)f \| . \end{gathered} \tag{3.66}$$

□

4 Proof of the Main Theorem

Let $\gamma > 0$ and $\varphi_0(t) := \gamma t$ for $t \in [1, \infty)$. In order to finish the proof of the Theorem 2.6 we have to extend the estimation in Theorem 3.1 with $\varphi = \varphi_0$ and for γ sufficiently small (as in the statement of Theorem 3.1) to the case when $f \in \mathcal{D}(p(-D)_{R_\gamma}$ such that the function:

$$x \longmapsto \sqrt{< x >} e^{\varphi_0(<x>)} ((\lambda(D) - E)f)(x) \tag{4.1}$$

is of class $L^2(\mathbb{R}^n)$. In order to do this we shall approach the function f with functions with compact support, but in order to control this limit we shall need to work with bounded phase functions $\varphi \in \Phi_{\gamma,m}$ that converge to φ_0. We shall denote (using also the notations of Section 2):

$$\mathcal{M}_R := \left\{ f \in \mathcal{D}(p(-D)_R \mid \sqrt{< Q >} e^{\varphi_0(<Q>)} ((\lambda(D) - E)f) \in L^2(\mathbb{R}^n) \right\} . \tag{4.2}$$

Let us fix $\chi \in C_0^\infty(\mathbb{R})$ such that:

$$0 \leq \chi(t) \leq 1,\ \chi(t) = 0\ for\ |t| \geq 1,\ \chi(t) = 1\ for\ |t| \leq 1/2. \tag{4.3}$$

For $x \in \mathbb{R}^n$ and for $\theta \in (0, 1]$ we denote:

$$\chi_\theta(x) := \chi(\theta < x >). \tag{4.4}$$

For $f \in \mathcal{M}_R$ and $\theta \in (0, 1]$ we denote:

$$f_\theta := \chi_\theta f. \tag{4.5}$$

Let:

$$j(t) := \begin{cases} \left(\int_{\mathbb{R}} e^{-\frac{1}{1-t^2}} dt\right)^{-1} e^{-\frac{1}{1-t^2}},\ for\ |t| < 1 \\ 0,\ for\ |t| \geq 1 \end{cases} \tag{4.6}$$

For $N \in \mathbb{N}$ let:

$$\tilde{\eta}_N(t) := \gamma\ for\ t \leq 2N\ and\ \tilde{\eta}_N(t) := 0\ for\ t > 2N \tag{4.7}$$

$$j_N(t) := \frac{1}{N} j(t/N); \quad \eta_N := j_N * \tilde{\eta}_N; \quad \varphi_N(t) := \int_0^t \eta_N(s) ds,\ \forall t \geq 0. \tag{4.8}$$

Let us remark that we have the following relations:

$$\begin{gathered} j \in C_0^\infty(\mathbb{R}^n),\ 0 \leq j(t) \leq \left(\int_{\mathbb{R}} e^{-\frac{1}{1-t^2}} dt\right)^{-1}, \\ \int_{\mathbb{R}} j(t) dt = 1, \quad \int_{\mathbb{R}} j_N(t) dt = 1, \quad |t| \geq N \Longrightarrow j_N(t) = 0 \end{gathered} \tag{4.9}$$

$$\begin{gathered} \eta_N \in C^\infty(\mathbb{R}),\ \eta_N(t) \leq \gamma, \\ |t(\partial \eta_N)(t)| \leq C_1 \gamma,\ \left|(\partial^k \eta_N)(t)\right| \leq C_k \gamma \ \ \forall t \in \mathbb{R} \end{gathered} \tag{4.10}$$

for k $\in \mathbb{N}$ and with C_k independent of γ;

$$\varphi_N(t) \leq \varphi_0(t), \quad \lim_{N \to \infty} \varphi_N(t) = \varphi_0(t), \quad \forall t \in \mathbb{R}. \tag{4.11}$$

In fact we shall prove only those estimations that are not completely obvious. First:

$$\eta_N(t) = \int_{\mathbb{R}} j_N(t-s) \tilde{\eta}_N(s) ds = \gamma \int_{-\infty}^{2N} j_N(t-s) ds = \gamma \int_{t-2N}^{\infty} j_N(\tau) d\tau \tag{4.12}$$

so that for $t \leq N$ we get $\eta_N(t) = \gamma$ and for $t \geq 3N$ we get $\eta_N(t) = 0$; but in general we have:

$$0 = \inf \tilde{\eta}_N \leq \eta_N(t) \leq \sup \tilde{\eta}_N = \gamma. \tag{4.13}$$

For the first derivative of $\eta_N(t)$ we see that:

$$\begin{gathered} t(\partial \eta_N)(t) = t \int_{\mathbb{R}} (\partial j_N)(t-s) \tilde{\eta}_N(s) ds = t\gamma \int_{-\infty}^{2N} (\partial j_N)(t-s) ds = \\ = t\gamma \int_{t-N}^{2N} (\partial j_N)(t-s) ds = t\gamma \int_{t-2N}^{N} (\partial j_N)(\tau) d\tau = \gamma t \{j_N(N) - j_N(t-2N)\} = \\ = -\gamma t j_N(t-2N) = -\gamma \tfrac{t}{N} j(t/N - 2) = -\gamma \tau j(\tau - 2); \end{gathered} \tag{4.14}$$

but $j(\tau-2)\neq 0$ implies that $1\leq\tau\leq 3$ so that $|t(\partial\eta_N)(t)|\leq 3\gamma\left(\int_{\mathbb{R}}e^{-\frac{1}{1-t^2}}dt\right)^{-1}$. For the higher derivatives we observe that:

$$(\partial^k\eta_N)(t)=-\gamma(\partial^{k-1}j_N)(t-2N)=-\gamma\frac{1}{N^k}(\partial^{k-1}j)(t/N-2) \tag{4.15}$$

so that $\left|(\partial^k\eta_N)(t)\right|\leq C_k\gamma$ for any $k>1$, with the constants C_k independent of γ.

The conclusion of the above analysis is that for any $N\in\mathbb{N}$ the phase function φ_N defined by (4.8) belongs to the class $\Phi_{\gamma',m}$ for some $\gamma'>\gamma$.

We fix now the value of γ small enough (as in the statement of Theorem 3.1), $f\in\mathcal{M}_{R_\gamma}$, $\theta\in(0,1]$ and $N\in\mathbb{N}$ large enough so that we can apply Theorem 3.1 with the phase function φ_N for the function f_θ with compact support. Thus we get the estimation:

$$\|e^{\varphi_N}f_\theta\|^2+\frac{(b-1/2)}{n}\left\|<Q>^{-1}Ae^{\varphi_N}f_\theta\right\|^2\leq C_\gamma\left\|\frac{<Q>}{\psi_N(Q)}e^{\varphi_N}(\lambda(D)-E)f_\theta\right\|^2 \tag{4.16}$$

where ψ_N is given by the same formula as in Section 3 with φ replaced by φ_N. In the sequel we remove the cut-off in f by letting $\theta\to 0$ and using Fatou Lemma on the left hand side of the inequality (4.16) and the Dominated Convergence Theorem on the right hand side. Let us remark that the boundedness of e^{φ_N} is crucial at this step. This leads us to an estimation for any $f\in\mathcal{M}_{R_\gamma}$ with the phase function φ_N. A similar procedure allows us to control the limit $N\to\infty$ and finish the proof of Theorem 2.6.

Lemma 4.1. *There exists a constant C such that for any $N\in\mathbb{N}$ we have:*

$$\frac{<x>e^{\varphi_N(<x>)}}{\psi_N(x)}\leq C\sqrt{<x>}e^{\gamma<x>}.$$

Proof. For $N\in\mathbb{N}$ we define the function:

$$g_N(t):=\frac{te^{\varphi_N(t)}}{\sqrt{1+4nt\varphi_N'(t)}}. \tag{4.17}$$

If $t\leq 2N$ then we see that:

$$\begin{gathered}\varphi_N'(t)=\eta_N(t)=\tfrac{\gamma}{N}\int_{-\infty}^{2N}j((t-s)/N)ds=\gamma\int_{-\infty}^{2}j(t/N-s)ds=\gamma\int_{t/N-2}^{\infty}j(\tau)d\tau\geq\\ \geq\gamma\int_0^{\infty}j(\tau)d\tau=\tfrac{\gamma}{2}\end{gathered} \tag{4.18}$$

so that we have the inequality:

$$\sqrt{1+4nt\varphi_N'(t)}\geq\sqrt{2n\gamma}\sqrt{t} \tag{4.19}$$

and thus for $t\leq 2N$:

$$g_N(t)\leq(\sqrt{2n\gamma})^{-1}\sqrt{t}e^{\varphi_N(t)}\leq(\sqrt{2n\gamma})^{-1}\sqrt{t}e^{\gamma t}. \tag{4.20}$$

If $t\geq 2N$ then we see that:

$$\begin{gathered}\varphi_N(t):=\int_0^t\eta_N(s)ds=\int_0^t ds\int_{\mathbb{R}}j_N(s-\tau)\tilde{\eta}_N(\tau)d\tau=\\ =\gamma\int_0^t ds\int_{-\infty}^{2N}j_N(s-\tau)d\tau=\gamma\int_0^t ds\int_{s-2N}^{N}j_N(\sigma)d\sigma=\gamma N\int_0^{t/N}ds\int_{s-2}^{1}j(\sigma)d\sigma=\\ =\gamma N\int_0^{t/N}\{J(1)-J(s-2)\}\,ds\end{gathered} \tag{4.21}$$

where:

$$J(t) := \int_{-\infty}^{t} j(s)ds = \int_{-1}^{t} j(s)ds. \tag{4.22}$$

Thus, for $t \geq 2N$ we obtain:

$$\begin{aligned} \varphi_N(t) &= \gamma N \left[s(J(1) - J(s-2))\right]_0^{t/N} + \gamma N \int_0^{t/N} sj(s-2)ds = \\ &= \gamma t - \gamma N \int_{-1}^{t/N-2} (t/N - 2 - s)j(s)ds = \xi(\tau)\gamma t \end{aligned} \tag{4.23}$$

where:

$$\begin{aligned} \tau &:= t/N \geq 2 \\ \xi(\tau) &:= 1 - (1/\tau) \int_{-1}^{\tau-2} (\tau - 2 - s)j(s)ds. \end{aligned} \tag{4.24}$$

We observe that for $\tau \geq 3$:

$$\xi(\tau) := 1 - (1/\tau) \int_{-1}^{1} (\tau - 2 - s)j(s)ds = 2/\tau \leq 2/3 \tag{4.25}$$

and for $2 \leq \tau \leq 3$ there exists a strictly positive constant ξ_0 such that:

$$(1/\tau) \int_{-1}^{\tau-2} (\tau - 2 - s)j(s)ds \geq \xi_0 > 0 \tag{4.26}$$

so that: $\xi(\tau) \leq 1 - \xi_0 < 1$. We conclude that there exists a constant $\alpha_0 < 1$ such that for any $\tau \geq 2$ one has $\xi(\tau) \leq \alpha_0 < 1$ and thus:

$$g_N(t) \leq te^{\varphi_N(t)} \leq te^{(1-\xi_0)\gamma t} \leq \sqrt{t}e^{\gamma t}(\sqrt{t}e^{-\xi_0\gamma t}) \leq \kappa\sqrt{t}e^{\gamma t} \tag{4.27}$$

with $\kappa := \sup_{t\geq 1} (\sqrt{t}e^{-\xi_0\gamma t})$. □

Evidently $\lim_{\theta\to 0} f_\theta(y) = f(y)$, for a.e. y in $\mathbb{R}^n$. Moreover:

$$\partial^\alpha(\chi_\theta f) = \sum_{\beta\leq\alpha} C_\beta^\alpha (\partial^{\alpha-\beta}\chi_\theta)(\partial^\beta f). \tag{4.28}$$

But:

$$(\partial^\beta\chi_\theta)(x) = \theta^{|\beta|} \sum_{p\leq|\beta|} b_p(x)(\partial^p\chi)(\theta < x >) \tag{4.29}$$

with $b_p(x)$ the symbols defined in (2.52). Thus we see that $\lim_{\theta\to 0} (\partial^\beta\chi_\theta)(x) = 1$ for $\beta = 0$ and this limit vanishes if $|\beta| \geq 1$, so that $\lim_{\theta\to 0} (\partial^\alpha f_\theta)(y) = (\partial^\alpha f)(y)$ for a.e. y in $\mathbb{R}^n$.

Let us consider the limit for $\theta \to 0$ of the right hand side of (4.16):

$$\begin{aligned} &\frac{<Q>}{\psi_N(Q)} e^{\varphi_N}(\lambda(D) - E)f_\theta = \\ &= \chi_\theta(Q)\frac{<Q>}{\psi_N(Q)} e^{\varphi_N}(\lambda(D) - E)f + \frac{<Q>}{\psi_N(Q)} e^{\varphi_N} \left[\lambda(D), \chi_\theta(Q)\right] f. \end{aligned} \tag{4.30}$$

We observe that the first term converges in L^2-norm to $\frac{<Q>}{\psi_N(Q)}e^{\varphi_N}(\lambda(D)-E)f$ and we shall prove that the second one converges in L^2-norm to zero. In fact for any integer N we can find a finite constant (diverging with N) such that:

$$\left\| \frac{e^{\varphi_N}}{\psi_N(Q)} \right\| \leq C_N \tag{4.31}$$

so that it is enough to analyze the family of L^2-functions: $\{< Q > [\lambda(D), \chi_\theta(Q)]\, f\}_{\theta>0}$. We have:

$$\begin{gathered} < Q > [\lambda(D), \chi_\theta(Q)]\, f = \\ = \int_{\mathbb{R}^n} \hat{\mu}(dx) < Q > p(i\partial_x)\,[U_D(x), \chi_\theta(Q)]\, f = \\ = \sum_{j=1}^{n} \int_0^1 dt \int_{\mathbb{R}^n} \hat{\mu}(dx) < Q > p(i\partial_x)\, \{x_j\, (\partial_j \chi_\theta(Q+tx))\, U_D(x) f\} = \\ = \sum_{j=1}^{n} \int_0^1 dt \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x) \left\{ x_j \frac{Q_j+tx_j}{<Q_j+tx_j>} \theta < Q > (\partial\chi)(\theta < Q+tx >) U_D(x) f \right\}. \end{gathered} \tag{4.32}$$

Thus:

$$\begin{gathered} \|< Q > [\lambda(D), \chi_\theta(Q)]\, f\|^2 = \\ \int_{\mathbb{R}^n} dy \left| \int_{\mathbb{R}^n} \hat{\mu}(dx) \sum_{j=1}^{n} \int_0^1 dt p(i\partial_x) \left\{ x_j \frac{Q_j+tx_j}{<Q_j+tx_j>} \theta < Q > (\partial\chi)(\theta < Q+tx >) U_D(x) f \right\}(y) \right|^2 \leq \\ \leq \{|\hat{\mu}|\,(\mathbb{R})\} \int_{\mathbb{R}^n} |\hat{\mu}|\,(dx) \int_{\mathbb{R}^n} dy \sum_{j=1}^{n} \int_0^1 dt \cdot \\ \cdot \left| p(i\partial_x) \left\{ x_j \frac{Q_j+tx_j}{<Q_j+tx_j>} \theta < Q > (\partial\chi)(\theta < Q+tx >) U_D(x) f \right\}(y) \right|^2 \end{gathered} \tag{4.33}$$

by using Jensen formula. Making use of (2.25) we obtain:

$$\begin{gathered} p(i\partial_x) \left\{ x_j \frac{Q_j+tx_j}{<Q_j+tx_j>} \theta < Q > (\partial\chi)(\theta < Q+tx >) U_D(x) f \right\} = \\ = \theta < Q > \sum_{|\beta|\leq m} \frac{i^{|\beta|}}{\beta!} \left\{ \partial_x^\beta (x_j \frac{Q_j+tx_j}{<Q_j+tx_j>} (\partial\chi)(\theta < Q+tx >)) \right\} U_D(x) (\partial^\beta p)(-D) f, \end{gathered} \tag{4.34}$$

$$\begin{gathered} \partial_x^\beta (x_j \frac{Q_j+tx_j}{<Q_j+tx_j>} (\partial\chi)(\theta < Q+tx >)) = \\ = \sum_{\gamma\leq\beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} \left(\partial_x^\gamma x_j \frac{Q_j+tx_j}{<Q_j+tx_j>} \right) \left\{ \partial_x^{\beta-\gamma} (\partial\chi)(\theta < Q+tx >) \right\}, \end{gathered} \tag{4.35}$$

$$\begin{gathered} \partial_x^\gamma x_j \frac{Q_j+tx_j}{<Q_j+tx_j>} = x_j (Q_j+tx_j) \partial_x^\gamma \frac{1}{<Q_j+tx_j>} + t x_j \partial_x^{\gamma-\delta_j} \frac{1}{<Q_j+tx_j>} + \\ + (Q_j+tx_j) \partial_x^{\gamma-\delta_j} \frac{1}{<Q_j+tx_j>} + t \partial_x^{\gamma-2\delta_j} \frac{1}{<Q_j+tx_j>} \end{gathered} \tag{4.36}$$

(where the terms with a negative order of derivation have to be considered zero because in fact they do not appear). Further we see that:

$$\left| \partial_x^\alpha \frac{1}{< Q_j + tx_j >} \right| \leq t^{|\alpha|} \frac{1}{< Q_j + tx_j >^{1+|\alpha|}} \tag{4.37}$$

so that:

$$\left| \partial_x^\alpha x_j \frac{Q_j + tx_j}{< Q_j + tx_j >} \right| \leq C < x > . \tag{4.38}$$

For the second factor on the right hand side of (4.35) we have:

$$\partial_x^\gamma (\partial\chi)(\theta < Q+tx >) = t^{|\gamma|} \sum_{p=1}^{1+|\alpha|} \theta^{p-1} s_p(Q+tx) (\partial^p \chi)(\theta < Q+tx >) \tag{4.39}$$

so that:

$$|\partial_x^\gamma(\partial\chi)(\theta < Q + tx >)| \le Ct^{|\gamma|}\zeta_\theta(< Q + tx >) \tag{4.40}$$

where ζ_θ is the characteristic function of the set: $\left\{\tau \in \mathbb{R}_+ \mid \frac{1}{2\theta} \le t \le \frac{1}{\theta}\right\}$. In conclusion, making also use of the first inequality in (2.6) we obtain:

$$\begin{aligned} &\int_{\mathbb{R}^n} dy \sum_{j=1}^n \int_0^1 dt \left|p(i\partial_x)\left\{x_j \tfrac{Q_j+tx_j}{<Q_j+tx_j>}\theta < Q > (\partial\chi)(\theta < Q + tx >)U_D(x)f\right\}(y)\right|^2 \le \\ &\le C < x >^2 \int_{\mathbb{R}^n} \left|\theta < Q + tx > \zeta_\theta(< Q + tx >) \sum_{|\beta|\le m} ((\partial^\beta p)(D)f)(y + x)\right|^2 dy \le \\ &\le C < x >^2 \int_{\frac{1}{2\theta}\le|y|\le\frac{1}{\theta}} \left|(\partial^\beta p)(D)f)(y + x)\right|^2 dy \underset{\theta\to 0}{\to} 0. \end{aligned} \tag{4.41}$$

where we took into account the fact that $\theta < y >$ is bounded by 1 on the support of $\zeta_\theta(y)$. Using the Dominated Convergence Theorem in (4.33), the above relation implies now that

$$\| < Q > [\lambda(D), \chi_\theta(Q)]\, f\| \underset{\theta\to 0}{\to} 0$$

and we can control the right hand side of (4.16).

Let us analyze now the left hand side of the inequality (4.16). The integrand in the first term converges pointwise:

$$\lim_{\theta\to 0} e^{\varphi_N(y)} f_\theta(y) = e^{\varphi_N(y)} f(y). \tag{4.42}$$

For the second term on the left side of (4.16) let us remark that if $f \in \mathcal{D}(p(-D))_{R_\gamma}$ and $N \in \mathbb{N}$, then $e^{\varphi_N} f \in \mathcal{D}(p(-D))_{R_\gamma}$ too and we can extend the operator $< Q >^{-1} A$ to the domain $\mathcal{D}(p(-D))_{R_\gamma}$ (see the estimations following the definition of A in Section 2). Moreover we have:

$$\begin{aligned} &\left\| < Q >^{-1} A e^{\varphi_N}(f_\theta - f)\right\|^2 = \\ &= \int_{\mathbb{R}^n} \Big| \tfrac{1}{2<y>} \sum_{j=1}^n \int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x)\cdot \\ &\cdot\left\{x_j(y_j e^{-sx\cdot X(y)} + (x_j + y_j)e^{sx\cdot X(x+y)})e^{\varphi_N(<x+y>)}(\chi_\theta(x + y) - 1)(U_D(x)f)(y)\right\} \Big|^2 dy \le \\ &\le \int_{\mathbb{R}^n} \Big| \tfrac{1}{2<y>} \sum_{j=1}^n \int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx) x_j\cdot \\ &\cdot(\chi_\theta(x + y) - 1)p(i\partial_x)\left\{y_j e^{-sx\cdot X(y)} + (x_j + y_j)e^{sx\cdot X(x+y)}\right\} e^{\varphi_N(<x+y>)}(U_D(x)f)(y) \Big|^2 dy + \\ &+ \sum_{1\le|\beta\le m|} \tfrac{i^{|\beta|}}{\beta!} \int_{\mathbb{R}^n} \Big| \tfrac{1}{2<y>} \sum_{j=1}^n \int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx) x_j\cdot \\ &\cdot(\partial_x^\beta \chi_\theta)(x + y)(\partial^\beta p)(i\partial_x)(y_j e^{-sx\cdot X(y)} + (x_j + y_j)e^{sx\cdot X(x+y)})e^{\varphi_N(<x+y>)}(U_D(x)f)(y) \Big|^2 dy. \end{aligned} \tag{4.43}$$

and let us observe that we also have the following bound uniform in $\theta \in (0, 1]$:

$$\begin{aligned} &\int_{\mathbb{R}^n} \Big| \tfrac{1}{2<y>} \sum_{j=1}^n \int_0^1 ds \int_{\mathbb{R}^n} \hat{\mu}(dx) p(i\partial_x)\cdot \\ &\cdot\left\{x_j(y_j e^{-sx\cdot X(y)} + (x_j + y_j)e^{sx\cdot X(x+y)})e^{\varphi_N(<x+y>)}(\chi_\theta(x + y) - 1)(U_D(x)f)(y)\right\} \Big|^2 dy \le \\ &\le C(\chi) \sum_{|\beta|\le m} \sum_{j=1}^n \left(\int_{\mathbb{R}^n} |\hat{\mu}|(dx)e^{a|x|}\right)\left(\int_{\mathbb{R}^n} |\hat{\mu}|(dx)e^{a|x|}\right) \int_{\mathbb{R}^n} dy\cdot \\ &\cdot e^{-2a|x|} \Big| (\partial^\beta p)(i\partial_x)\tfrac{1}{2<y>}\left\{y_j e^{-sx\cdot X(y)} + (x_j + y_j)e^{sx\cdot X(x+y)}\right\} e^{\varphi_N(<x+y>)}(U_D(x)f)(y)) \Big|^2 \le \\ &\le C_N'(\chi) \left(\int_{\mathbb{R}^n} |\hat{\mu}|(dx)e^{a|x|}\right)^2 \|p(-D)f\|^2. \end{aligned} \tag{4.44}$$

In order to study the pointwise convergence of the integrand in the last expression in (4.43) let us observe that the expression in the fifth line obviously goes to zero pointwise while for any β with $|\beta| \geq 1$ we have:

$$\left|(\partial_x^\beta \chi_\theta)(x+y)\right| \leq C(\chi)\theta^{|\beta|} \tag{4.45}$$

so that the last line of (4.43) converges to zero also. From the uniform bound (4.44) we immediately obtain the L^2-convergence to zero of the first line of (4.43) and hence:

$$\lim_{\theta\to 0} \left\|< Q >^{-1} A e^{\varphi_N} f_\theta\right\| = \left\|< Q >^{-1} A e^{\varphi_N} f\right\| \tag{4.46}$$

for any $N \in \mathbb{N}$. Using now Fatou lemma for the first term in (4.16) we finally get:

$$\begin{array}{c} \|e^{\varphi_N} f\|^2 + \frac{(b-1/2)}{n} \|< Q >^{-1} A e^{\varphi_N} f\|^2 \leq \\ \leq \liminf_{\theta\to 0} \left(\|e^{\varphi_N} f_\theta\|^2 + \frac{(b-1/2)}{n} \|< Q >^{-1} A e^{\varphi_N} f_\theta\|^2\right) \leq \\ \leq C_\gamma \liminf_{\theta\to 0} \left\|\frac{<Q>}{\psi_N(Q)} e^{\varphi_N}(\lambda(D)-E) f_\theta\right\|^2 = \\ = \left\|\frac{<Q>}{\psi_N(Q)} e^{\varphi_N}(\lambda(D)-E) f\right\|^2 \end{array} \tag{4.47}$$

for any $N \in \mathbb{N}$.

Now we still have to study the behaviour of the inequality (4.47) when $N \to \infty$. Using Lemma 4.1 we see that the right hand side is uniformly bounded by:

$$\left\|\frac{< Q >}{\psi_N(Q)} e^{\varphi_N}(\lambda(D)-E) f\right\|^2 \leq C \left\|\sqrt{< Q >} e^{\gamma< Q >}(\lambda(D)-E) f\right\|^2, \forall N \in \mathbb{N} \tag{4.48}$$

with C independent of N, the right hand side being finite due to the hypothesis $f \in \mathcal{M}_{R_\gamma}$. But evidently:

$$\frac{< x >}{\psi_N(x)} e^{\varphi_N(< x >)} \underset{N\to\infty}{\to} \sqrt{< x >} e^{\gamma < x >} \tag{4.49}$$

so that by the Dominated Convergence Theorem we obtain the convergence of the right hand side in (4.47). For the first term on the left hand side one can immediately use the Fatou lemma in a way similar to the argument we gave for the $\theta \to 0$ limit. Thus we obtain the desired inequality:

$$\left\|e^{\gamma< Q >} f\right\| \leq C \left\|\sqrt{< Q >} e^{\gamma< Q >}(\lambda(D)-E) f\right\|. \tag{4.50}$$

References

[A1] S.Agmon: "Spectral Properties of Schrödinger Operators and Scattering Theory", *Ann. Scuola Norm. Sup. di Pisa, Cl. Sci.* (4) **2**, 151-218, (1975).

[A2] S.Agmon: "Lectures on Exponential Decay of Solutions of Second Order Elliptic Equations", Princeton Univ. Press, (1982).

[AH] S.Agmon, L.Hörmander: "Asymptotic Properties of Solutions of Differential Equations with Simple Characteristics", *J. Analyse Math.*, **30**, 1-38, (1976).

[ABG1] W.Amrein, Anne Boutet de Monvel, V.Georgescu: "Hardy and Carleman type Estimates for Second Order Differential Operators with Operator-Valued Coefficients", preprint UGVA-DPT 1983, 12-414.

[ABG2] W.Amrein, Anne Boutet de Monvel, V.Georgescu: "Hardy type Inequalities for Abstract Differential Operators", *Memoirs of the American Mathematical Society*, (375) **70**, 1-119, (1987).

[ABG3] W.Amrein, Anne Boutet de Monvel, V.Georgescu: "*C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*", Birkhäuser, (1996).

[BG1] Anne Berthier, V. Georgescu: "On the Point Spectrum of Dirac Operators", *J. Func. Analysis*, **71**, 309-338, 1987.

[BG2] Anne Boutet de Monvel, V.Georgescu: "Spectral Theory and Scattering Theory by the Conjugate Operator Method", *Algebra i Analyz*, **4**, 73-116, (1992); (=*St. Petersburg Math. J.*, (3)**4**, 469-501, (1993)).

[BP] Anne Boutet de Monvel, R.Purice: "The Conjugate Operator Method: application to Dirac Operators and to Stratified Media", IMA Preprint no. 1341, 1995, Univ. of Minnesota.

[CMS] R.Carmona, W.C.Masters, B.Simon: "Relativistic Schrödinger Operators: Asymptotic Behaviour of the Eigenfunctions", *J. Func. Anal.*, **91**, 117-142, (1990).

[FH] R.Froese, I.Herbst: "Exponential Bounds and Absence of Positive Eigenvalues for N-Body Schrödinger Operators, *Comm. Math. Phys.*, **87**, 429-447, (1982).

[FHHO] R.Froese, I.Herbst, Maria Hoffmann-Ostenhof, T. Hoffmann-Ostenhof: "L^2-Exponential Lower Bounds to Solutions of the Schrödinger Equation", *Comm. Math. Phys.*, **87**, 265-286, (1982).

[GN] Ch. Gérard, F. Nier: "Théorie de la Diffusion pour les Opérateurs Analytiquement Décomposables", preprint Ecole Polytechnique Palaiseau, no. 1134, 1996.

[Ar] G. Arsu: "Spectral Analysis for Simply Characteristic Operators by Mourre's Method; II", *J. Operator Theory*, **29**, 287-305, 1993.

[HLP] G.H.Hardy, J.E.Littlewood, G.Polya: "*Inequalities*", Cambridge Univ. Press, (1952).

[KT] A.Kufner, H.Triebel: "Generalizations of Hardy's Inequality", *Confer. Sem. Mat. Univ. Bari*, **156**, 1-21, (1978).

[M] E.Mourre: "Absence of Singular Continuous Spectrum for Certain Self-Adjoint Operators", *Comm. Math. Phys*, **78**, 391-408, (1981).

[N] Gh. Nenciu: "On Exponential Decay of Solutions of Schrödinger and Dirac Equations: Bounds of Eigenfunctions Corresponding to Energies in the Gaps of Essential Spectrum", lecture at St. Jean de Monts, 1994.