A MODEL OF PERTURBATION THEORY

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Abstract

We consider a model of the perturbation theory where the Hamiltonian $\mathbf{H}$ is a sum of a multiplication operator $\mathbf{H}_0$ by $|x|^{2l}$ in the space $L_2(\mathbb{R}^d)$ and of an integral operator $\mathbf{V}$ with oscillating kernel. We find conditions for the coincidence of the essential spectra of the operators $\mathbf{H}_0$ and $\mathbf{H}$ and for the existence and completeness of the wave operators relating $\mathbf{H}_0$ and $\mathbf{H}$.

1. Our aim here is to introduce a new model of the perturbation theory. Let $\mathbf{H}_0$ be multiplication by $|x|^{2l}$, $l > 0$, in the space $L_2(\mathbb{R}^d)$ and let $\mathbf{V}$ be an integral operator defined by

$$(\mathbf{V}u)(x) = \int_{\mathbb{R}^d} v(\langle x, y\rangle)u(y)dy, \tag{1}$$

where v is a real periodic function

$$v(t) = \sum_{n=-\infty}^{\infty} v_n e^{int}, \quad v_{-n} = \bar{v}_n, \quad v_0 = 0. \tag{2}$$

The operator $\mathbf{V}$ can also be viewed as a Fourier integral operator. For example, if

$$v(t) = \gamma \cos t \quad \text{or} \quad v(t) = \gamma \sin t, \tag{3}$$

then (1) is a linear combination of the classical Fourier operator and of its adjoint. In this paper we study spectral properties of the Hamiltonian $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$ in the space $L_2(\mathbb{R}^d)$. The case of a constant function $v(t)$ corresponds [1] to the point interaction and is excluded by the condition $v_0 = 0$. Passing to the spherical coordinates, considering the space $L_2(\mathbb{R}^d)$ as $L_2(\mathbb{R}_+; L_2(\mathbb{S}^{d-1}))$ and making the change of variables $\lambda = |x|^{2l}$, we can fit the operator $\mathbf{H}$ into the framework of the Friedrichs model. However, the results of the paper [5] are not applicable to perturbations (1) (even in the case $d = 1$) since the kernels obtained do not satisfy its assumptions either for $\lambda = 0$ or for $\lambda \to \infty$ (or for both these points).

For the case (3), the model we consider first appeared in [4]. It was checked in [4] that $\sigma_{ess}(\mathbf{H}) = \sigma_{ess}(\mathbf{H}_0)$ for arbitrary $l > 0$ and that the wave operators $W_\pm(\mathbf{H}, \mathbf{H}_0)$ exist and are complete if $2l > d$. The discrete spectrum of $\mathbf{H}$ for such functions $v(t)$ was studied in [8].

We show that under the assumption

$$\sum_{n=1}^{\infty} |v_n| n^{-d/2} < \infty \tag{4}$$

the operator $\mathbf{V}$ is bounded in $L_2(\mathbb{R}^d)$ and $\sigma_{ess}(\mathbf{H}) = \sigma_{ess}(\mathbf{H}_0)$ for arbitrary $l > 0$. Moreover, if

$$\sum_{n=1}^{\infty} |v_n| n^{-d/2+1+\varepsilon} < \infty \tag{5}$$

for some $\varepsilon > 0$, then the wave operators $W_{\pm}(\mathbf{H}, \mathbf{H}_0)$ exist and are complete provided $2l > 1$. Note that both conditions (4) and (5) are getting less restrictive as d increases.

We found also necessary and sufficient conditions for the negative spectrum of the operator $\mathbf{H}$ to be finite and calculated explicitly the total number of its negative eigenvalues. These results will be published elsewhere.

2. Let us first of all rewrite (1) in operator notation. Let Φ,

$$(\Phi u)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} u(y)dy,$$

be the Fourier operator and let $\mathbf{T}_n$,

$$(\mathbf{T}_n u)(x) = n^{d/2} u(nx),$$

be the unitary operator of dilations in $L_2(\mathbb{R}^d)$. Then equality (1) can be rewritten as

$$\mathbf{V} = (2\pi)^{d/2} \sum_{n=1}^{\infty} n^{-d/2} \mathbf{T}_n \left(v_n \Phi^* + \bar{v}_n \Phi \right). \tag{6}$$

To be precise, we take (6) for the definition of $\mathbf{V}$. Under assumption (4), series (6) is convergent in norm so that the operator $\mathbf{V}$ is bounded. This implies of course that the operator $\mathbf{H}$ is self-adjoint on the domain $\mathcal{D}(\mathbf{H}_0)$ of the operator $\mathbf{H}_0$.

As a side remark, we note that the operator

$$\tilde{\mathbf{V}} = \Phi \mathbf{V} \Phi^* = (2\pi)^{d/2} \sum_{n=1}^{\infty} n^{-d/2} \mathbf{T}_{1/n} \left(v_n \Phi^* + \bar{v}_n \Phi \right)$$

has the structure similar to that of $\mathbf{V}$. In particular, if (2) contains only a finite number of terms, then $\tilde{\mathbf{V}}$ can be written as an integral operator (1) with kernel $\tilde{v}(\langle x, y\rangle)$ where $\tilde{v}(t)$ is again a periodic function of t (but of course its period is not necessarily 2π). Thus, all our results can be reformulated in terms of the Hamiltonian $\tilde{\mathbf{H}} = \tilde{\mathbf{H}}_0 + \tilde{\mathbf{V}}$ where $\tilde{\mathbf{H}}_0 = (-\Delta)^l$.

Let Q be multiplication in the space $L_2(\mathbb{R}^d)$ by some bounded function $q(x)$ which tends to 0 as $|x| \to \infty$. Then the operator $Q\Phi Q$ is compact. It follows that operators $Q\mathbf{T}_n\Phi Q$ and $Q\mathbf{T}_n\Phi^* Q$ are compact for all n. According to (6), under assumption (4), this implies compactness of operators $Q\mathbf{V}Q$ and, in particular, of $(\mathbf{H}_0 + cI)^{-1}\mathbf{V}(\mathbf{H}_0 + cI)^{-1}$. Let us now use the resolvent identity

$$(\mathbf{H} + cI)^{-1} - (\mathbf{H}_0 + cI)^{-1} = -\Big((\mathbf{H} + cI)^{-1}(\mathbf{H}_0 + cI)\Big) \Big((\mathbf{H}_0 + cI)^{-1}\mathbf{V}(\mathbf{H}_0 + cI)^{-1}\Big), \tag{7}$$

where $c > 0$ is sufficiently large. The first factor in the right-hand side is bounded because $\mathbf{V}$ is bounded and the second is compact. Hence operator (7) is also compact which, by the Weyl theorem, ensures that the essential spectra of the operators $\mathbf{H}_0$ and $\mathbf{H}$ coincide. This yields the first result.

Theorem 1 *Let condition* (4) *hold. Then the operator* $\mathbf{V}$ *is bounded and the operator* $\mathbf{H}$ *is self-adjoint on the domain of the operator* $\mathbf{H}_0$. *Moreover,* $\sigma_{ess}(\mathbf{H}) = [0, \infty)$.

3. The operators $\mathbf{H}_0$ and $\mathbf{V}$ are rotationally invariant. This allows us to separate variables in the spherical coordinates which is necessary for a more detailed study of the operator $\mathbf{H}$. Let $\mathfrak{h}_m$ be the subspace of spherical functions $Y_m(\omega)$, $\omega \in \mathbb{S}^{d-1}$, of order m. Let $\mathcal{K}$ be the L_2-space with weight r^{d-1} of functions defined on $\mathbb{R}_+$ and let $\mathfrak{H}_m = \mathcal{K} \otimes \mathfrak{h}_m$. To put it differently, $\mathfrak{H}_m \subset \mathbb{R}^d$ is the subspace spanned by functions

$$u_m(x) = |x|^{-\delta} f(|x|) Y_m(\hat{x}), \quad \hat{x} = x|x|^{-1}, \quad \delta = (d-1)/2, \tag{8}$$

where $f \in L_2(\mathbb{R}_+)$ and $Y_m \in \mathfrak{h}_m$. Then

$$L_2(\mathbb{R}^d) = \bigoplus_{m=0}^{\infty} \mathfrak{H}_m, \quad \mathfrak{H}_m = \mathcal{K} \otimes \mathfrak{h}_m, \tag{9}$$

and every subspace $\mathfrak{H}_m$ is invariant with respect to the Fourier operator Φ, which reduces to the Fourier-Bessel transform on $\mathfrak{H}_m$. More precisely, set

$$(\Phi_m f)(r) = \int_0^\infty (rs)^{1/2} \mathcal{I}_{m+(d-2)/2}(rs) f(s) ds, \tag{10}$$

where $\mathcal{I}_{m+(d-2)/2}$ is the Bessel function. Then, for function (8),

$$(\Phi u_m)(x) = i^{-m} |x|^{-\delta} (\Phi_m f)(|x|) Y_m(\hat{x}). \tag{11}$$

The operator Φ_m is of course unitary on $L_2(\mathbb{R}_+)$ and $\Phi_m = \Phi_m^*$. Clearly,

$$(\mathbf{T}_n u_m)(x) = |x|^{-\delta} (T_n f)(|x|) Y_m(\hat{x}), \tag{12}$$

where T_n,

$$(T_n f)(r) = n^{1/2} f(r),$$

is the unitary operator of dilations in $L_2(\mathbb{R}_+)$.

Set

$$v_m(t) = (2\pi)^{d/2} \sum_{n=1}^{\infty} n^{-\delta} (v_n i^m + \bar{v}_n i^{-m}) (nt)^{1/2} \mathcal{I}_{m+(d-2)/2}(nt)$$

and

$$(V_m f)(r) = \int_0^\infty v_m(rs) f(s) ds.$$

According to (10)

$$V_m = (2\pi)^{d/2} \sum_{n=1}^{\infty} n^{-d/2} (v_n i^m + \bar{v}_n i^{-m}) T_n \Phi_m. \tag{13}$$

This equality should be considered as the precise definition of the operator V_m.

Obviously, $\mathfrak{H}_m$ is invariant with respect to the operator $\mathbf{H}_0$. It follows from (8), (11) and (12) that every subspace $\mathfrak{H}_m$ is also invariant with respect to operator (6). It reduces on $\mathfrak{H}_m$ to the operator $\Omega V_m \Omega^* \otimes I_m$ where $\Omega : L_2(\mathbb{R}_+) \to \mathcal{K}$ is a unitary operator defined by

$(\Omega g)(r) = r^{-\delta} g(r)$ and I_m is the identity operator in the space $\mathfrak{h}_m$. Let H_0 be multiplication by x^{2l} and $H^{(m)} = H_0 + V_m$ in the space $L_2(\mathbb{R}_+)$. Then

$$\mathbf{H} = \bigoplus_{m=0}^{\infty} \Omega H^{(m)} \Omega^* \otimes I_m, \tag{14}$$

and hence the spectral analysis of the operator $\mathbf{H}$ reduces to that for the family of operators $H^{(m)}$.

4. Now we can check that the wave operators

$$W_\pm(\mathbf{H}, \mathbf{H}_0) = s - \lim_{t \to \pm\infty} \exp(i\mathbf{H}t) \exp(-i\mathbf{H}_0 t)$$

exist and are complete, that is their ranges coincide with the absolutely continuous subspace of the operator $\mathbf{H}$. According to (9), (14), to that end it suffices to verify the same facts for all pairs $H_0, H^{(m)}$, $m = 0, 1, \ldots$. By the Birman-Krein theorem [2] (see also [7]), we need only to show that operator

$$(H^{(m)} + cI)^{-1} - (H_0 + cI)^{-1} = -\Big((H^{(m)} + cI)^{-1}(H_0 + cI)\Big) \Big((H_0 + cI)^{-1} V_m (H_0 + cI)^{-1}\Big),$$

belongs to the trace class $\mathfrak{S}_1$ or that

$$(H_0 + I)^{-1} V_m (H_0 + I)^{-1} \in \mathfrak{S}_1. \tag{15}$$

According to (13) it suffices to check that the operators $(H_0+I)^{-1} T_n \Phi_m (H_0+I)^{-1}$ belong to $\mathfrak{S}_1$ and the corresponding trace norms satisfy

$$\sum_{n=1}^{\infty} |v_n| n^{-d/2} ||(H_0 + I)^{-1} T_n \Phi_m (H_0 + I)^{-1}||_1 < \infty. \tag{16}$$

Since

$$(H_0 + I)^{-1} T_n = n^{2l} T_n (H_0 + n^{2l} I)^{-1},$$

condition (16) is equivalent to

$$\sum_{n=1}^{\infty} |v_n| n^{-d/2+2l} ||(H_0 + n^{2l} I)^{-1} \Phi_m (H_0 + I)^{-1}||_1 < \infty. \tag{17}$$

Next we use the following

Lemma 2 *Let the operator Φ_m be defined by formula (10) and let P and Q be multiplications by functions p and q. Then the operator $P\Phi_m Q$ acting in the space $L_2(\mathbb{R}_+)$ belongs to the trace class and*

$$||P\Phi_m Q||_1^2 \le C \int_0^\infty |p(r)|^2 (1 + r^2)^\sigma dr \int_0^\infty |q(s)|^2 (1 + s^2)^\sigma ds, \quad C = C(m, \sigma), \tag{18}$$

provided the right-hand side is finite for some $\sigma > 1/2$.

A proof can be obtained similarly to the case when the role of Φ_m is played by the usual Fourier transform (see [3] or [6]), so it will be given in a short form. Actually, we follow closely the proof of T. Kato published in [6]. Let G be mulptiplication by $(1+r^2)^{1/2}$. We have that $P\Phi_m Q = K_1 L K_0$ where

$$K_1 = PG^\sigma \Phi_m G^{-\sigma}, \quad L = G^\sigma \Phi_m G^{-\sigma} \Phi_m G^{-\sigma} \Phi_m G^\sigma, \quad K_0 = G^{-\sigma} \Phi_m G^\sigma Q.$$

Clearly, the Hilbert-Schmidt norm

$$||K_0||_2^2 = \int_0^\infty \int_0^\infty (1+r^2)^{-\sigma} rs \mathcal{I}^2_{m+(d-2)/2}(rs)(1+s^2)^\sigma |q(s)|^2 dr ds \leq C \int_0^\infty |q(s)|^2 (1+s^2)^\sigma ds,$$

since the function $t\mathcal{I}^2_{m+(d-2)/2}(t)$ is bounded. Quite similarly,

$$||K_1||_2^2 \leq C \int_0^\infty |p(r)|^2 (1+r^2)^\sigma dr.$$

To conclude the proof of (18), it remains to check that the operator $L = L_\sigma$ is bounded. Let $\mathbf{G}$ be multiplication by $(1+|x|^2)^{1/2}$ in the space $L_2(\mathbb{R}^d)$ and

$$\mathbf{L}_\sigma = \mathbf{G}^\sigma \Phi \mathbf{G}^{-\sigma} \Phi^* \mathbf{G}^{-\sigma} \Phi \mathbf{G}^\sigma.$$

By a direct calculation, one easily verifies that the operator $\mathbf{L}_2$ is bounded in $L_2(\mathbb{R}^d)$. Then, by the complex interpolation, this results extends to all $\sigma \in (0,2)$. The restriction of the operator $\mathbf{L}_\sigma$ on the subspace $\mathfrak{H}_m$ equals $i^{-m}\Omega L_\sigma \Omega^* \otimes I_m$. Hence the operator L_σ is bounded in $L_2(\mathbb{R}_+)$.

It follows from Lemma 2 that, in the case $2l > 1$, all operators $(H_0 + n^{2l} I)^{-1} \Phi_m (H_0 + I)^{-1}$ belong to the trace class and

$$||(H_0 + n^{2l} I)^{-1} \Phi_m (H_0 + I)^{-1}||_1 \leq C n^{-2l+1+\varepsilon}, \quad C = C(m, \varepsilon),$$

for any $\varepsilon > 0$. Therefore, under assumption (5), the series (17) converges which implies (15). Let us formulate the result obtained.

Theorem 3 *Let $l > 1/2$ and let condition* (5) *hold. Then the wave operators $W_\pm(\mathbf{H}, \mathbf{H}_0)$ exist and are complete. In particular, the absolutely continuous part of $\mathbf{H}$ is unitarily equivalent to the operator $\mathbf{H}_0$.*

Compared to [4] we allow sufficiently more general functions $v(t)$ and relax the condition $2l > d$ imposed in [4] because separation of variables was not taken into account there.

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