INTERFACE FLUCTUATIONS FOR THE D=1 STOCHASTIC GINZBURG-LANDAU EQUATION WITH NON-SYMMETRIC REACTION TERM

STELLA BRASSESCO[1], PAOLO BUTTÀ[2]

[1] Instituto Venezolano de Investigaciones Científicas
[2] Rutgers University, New Jersey, USA

Abstract. We consider a Ginzburg-Landau equation in the interval $[-\varepsilon^{-1}, \varepsilon^{-1}]$, $\varepsilon > 0$, with Neumann boundary conditions, perturbed by an additive white noise of strength $\sqrt{\varepsilon}$, and reaction term being the derivative of a function which has two equal depth wells at ± 1, but is not symmetric. When $\varepsilon = 0$, the equation has equilibrium solutions that are increasing, and connect -1 with $+1$. We call them instantons, and we study the evolution of the solutions of the perturbed equation in the limit $\varepsilon \to 0^+$, when the initial datum is close to an instanton. We prove that, for times that may be of the order of ε^{-1}, the solution stays close to some instanton, whose center, suitably normalized, converges to a Brownian motion plus a drift. This drift is known to be zero in the symmetric case, and, using a perturbative analysis, we show that, if the non-symmetric part of the reaction term is sufficiently small, it determines the sign of the drift.

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Introduction.

We consider a stochastic perturbation of the one dimensional Ginzburg-Landau (G-L) equation with Neumann boundary conditions (N.b.c.) in the interval $[-\varepsilon^{-1}, \varepsilon^{-1}]$, where ε is a parameter that will go to zero. The potential term is a double well function V with equal minima at ± 1, and the stochastic perturbation is given by a space time white noise, with intensity $\sqrt{\varepsilon}$. The two minima of the potential give rise to two equilibrium homogeneous solutions of the G-L equation, the constant functions identically equal to ± 1. They are stable, and there are several other equilibrium configurations, all of them unstable. The more stable ones are $\pm\phi^{(\varepsilon)}$, where $\phi^{(\varepsilon)}$ is an increasing function of the spatial variable x, which is close to 1 if x is close to ε^{-1} and close to -1 if x is close to $-\varepsilon^{-1}$, for ε small. The associated unstable manifolds have dimension 1, and the flow on it consists on a pair of orbits connecting $\phi^{(\varepsilon)}$ with the functions ± 1, as $t \to \infty$. The corresponding eigenvalue goes to zero with ε, and the motion along this manifold is in consequence very slow (see [4] and [12]).

Reaction diffusion equations like the G-L equation appear when modelling phenomena such as phase transitions and evolution of interfaces (see for instance [2], [3] and [11] and the references therein for physical motivations), and a natural question is then how the motion along the unstable manifold of $\pm\phi^{(\varepsilon)}$ (that corresponds to the interface) is affected by the noise. In [1], [2], [3] and [11], the behaviour as $\varepsilon \to 0^+$ of the solution of the perturbed G-L equation was investigated, when the initial condition is close to $\phi^{(\varepsilon)}$. Roughly, it was shown that this solution is, at time T_ε, very close to $\phi^{(\varepsilon)}(x + \sqrt{\varepsilon}B_{T_\varepsilon})$, where B_t is a Brownian motion. In [2], T_ε can be of the order of ε^{-1}. This is the typical time needed to see a finite shift, and it results much shorter than that needed for the equation without noise. In [3] this result was extended for times that may grow as ε^{-k}, when the lenght of the interval also is of the order of an inverse power of ε. In both cases, the potential term V was supposed to be symmetric, and the identification of the asymptotic behaviour of the displacement relies on that symmetry. The techniques of Funaki in [11] are different (also the result and the setup are slightly different), but he also requires this symmetry, and he posed the problem of investigating what is, if any, the effect of a deformation on the form of the wells of V, still if they have the same depth.

We prove here that, in the time scale ε^{-1}, if the potential V is non-symmetric, the solution of the stochastic G-L equation has a drift along the unstable manifold of $\phi^{(\varepsilon)}$. That is, the configuration with initial condition close to $\phi^{(\varepsilon)}$ is, at time $\varepsilon^{-1}t$, close to $\phi^{(\varepsilon)}\big(x + B(t) - \beta t\big)$, for some β, whose sign, at least for a sufficiently small deformation, depends on an explicit integral of the non-symmetric part (see Theorem 1.1 for a precise statement).

In Section 1 we introduce precisely the model, recall the results quoted here, and state our result (Theorem 1.1). Using the techniques in [2], it is not difficult to guess what the drift β should be, and to prove that the shift is given by a Brownian motion plus this drift. This is done in Section 2. But it is not clear how to conclude from the expression of β we obtain, that it is not zero. In Section 3 we compute the derivative of β with respect to λ at

$\lambda = 0$, where λ is the parameter that controlls the deformation. This derivative is given in terms of the trace of the operator of the linearized symmetric G-L equation (see Proposition 3.1). After computing this trace, we are left with an expression for the derivative that yields, for small deformations, the sign of the drift.

The problem of studying the motion of the travelling front if we add a small linear factor to the potential can also be treated with the same techniques. In this case, the wells are not of equal depth, and there is a travelling wave instead of the instanton. It can be seen that there is an extra velocity term, but up to now we are not able to deduce, from the expression for this velocity, whether it is against or in the same sense as the velocity of the front for the equation without noise.

1. Preliminaries and statement of the main results.

We consider the family of processes given as solutions of the initial value problem for the stochastic partial differential equation

$$\frac{\partial m_t}{\partial t} = \frac{1}{2}\frac{\partial^2 m_t}{\partial x^2} - V'(m_t) + \sqrt{\varepsilon}\alpha, \quad t \geq 0 \quad x \in [-\varepsilon^{-1}, \varepsilon^{-1}] \tag{1.1}$$

with Neumann boundary conditions (N.b.c.) in the interval $[-\varepsilon^{-1}, \varepsilon^{-1}]$, where ε is a small parameter that will go to zero. The term $\alpha = \alpha(t,x)$ is a standard space-time white noise, and V' is the derivative of the real valued function V,

$$V(m) = \frac{m^4}{4} - \frac{m^2}{2} + \lambda\Gamma(m), \tag{1.2}$$

where Γ is a smooth positive function with support contained in $(0,1)$, and $\lambda \in [0,1]$ is a parameter so small that V has two equal absolute minima at ± 1. We think of V as a non-symmetric perturbation of the polynomial $V_0(m) = m^4/4 - m^2/2$.

We say that m_t is a solution of (1.1) with initial condition m_0 if it satisfies the integral equation

$$\begin{aligned} m_t(x) = &\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} dy\, H_t^{(\varepsilon)}(x,y)\, m_0(y) - \int_0^t ds \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} dy\, H_{t-s}^{(\varepsilon)}(x,y)\, V'(m_s(y)) \\ &+ \sqrt{\varepsilon}\int_0^t ds \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} dy\, \alpha(s,y)\, H_{t-s}^{(\varepsilon)}(x,y) \quad t \geq 0, \quad x \in [-\varepsilon^{-1}, \varepsilon^{-1}], \end{aligned} \tag{1.3}$$

where $H_t^{(\varepsilon)}$ is the heat operator with N.b.c. in $[-\varepsilon^{-1}, \varepsilon^{-1}]$ and $H_t^{(\varepsilon)}(x,y)$ is the corresponding kernel. The stochastic integral has to be understood in the sense of [13] and

$$Z_t(x) \doteq \int_0^t ds \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} dy\, \alpha(s,y)\, H_{t-s}^{(\varepsilon)}(x,y) \tag{1.4}$$

is a Gaussian process with Hölder continuous paths (see [13] for details). If m_0 is continuous and satisfies N.b.c., there exists a unique continuous solution of (1.3), as follows from [8]. In the sequel we will denote it by m_t.

In the case $\varepsilon = 0$, equation (1.1) is a deterministic evolution equation in $\mathbb{R}$, known as Ginzburg-Landau equation, which admits a stationary solution $\phi(x)$ satisfying

$$\frac{\phi''}{2} = V'(\phi), \qquad \phi(\pm\infty) = \pm 1, \tag{1.5}$$

and the centering condition

$$\phi(0) = 0. \tag{1.6}$$

(We are using f' to denote the derivative of a function f.)

We call ϕ the *instanton*. Note that the translates

$$\phi_{x_0}(x) \doteq \phi(x - x_0), \quad x_0 \in \mathbb{R} \tag{1.7}$$

satisfy (1.5) so they are also stationary solutions of the G-L equation in $\mathbb{R}$.

Set

$$\rho = \left(\int dx\, (\phi')^2 \right)^{-1} \tag{1.8}$$

and, for any function f, let

$$\|f\| \doteq \sup_{x \in \mathbb{R}} |f(x)| \quad \text{and} \quad \|f\|_\varepsilon \doteq \sup_{x \in [-\varepsilon^{-1}, \varepsilon^{-1}]} |f(x)|. \tag{1.9}$$

Before stating our result, we need to introduce the kernel $g(x,y,t)$, which is the fundamental solution of the linearization of the G-L equation about the instanton,

$$\frac{\partial u}{\partial t} = Lu \quad \text{with} \quad L = \frac{1}{2}\frac{\partial^2}{\partial x^2} - V''(\phi_{x_0}).$$

1.1 Theorem.

Let m_t be the solution of equation (1.1) with initial condition m_0. Let $\zeta \in (0,1)$ and suppose that m_0 is continuous, satisfies N.b.c. and

$$\|m_0 - \phi_{x_0}\|_\varepsilon \le \sqrt{\varepsilon} \text{ for some } x_0 \text{ with } |x_0| \le (1-\zeta)\varepsilon^{-1}$$

Then, there exists a process ξ_t in $D([0,T])$, adapted to m_t, such that

$$\lim_{\varepsilon \to 0+} P\left(\sup_{x \in [-\varepsilon^{-1}, \varepsilon^{-1}]} \left| m_t(x) - \phi_{x_0}(x - \xi_t) \right| \le \varepsilon^{1/4} \quad \forall\, t \le \varepsilon^{-1} T \right) = 1 \tag{1.10}$$

for any given $T > 0$, and

$$X_t^\varepsilon \doteq \xi_{\varepsilon^{-1}t} - x_0 \tag{1.11}$$

converges weakly in $D([0,T])$ to

$$X_t = W_t + \beta t \tag{1.12}$$

where W_t is a Brownian motion starting from 0 and

$$E[(W_1)^2] = \rho$$

with ρ as defined in (1.8)*, while the drift β is finite and given by*

$$\beta = \frac{\rho}{2} \int_0^\infty dt \int dx \int dy \left(g(x,y,t) - \rho\phi'(x)\phi'(y)\right)^2 \phi'(x) V'''(\phi(x)) \tag{1.13}$$

Finally (recall that $\beta = \beta(\lambda)$), we have that

$$\frac{d\beta}{d\lambda}(0) = \frac{27}{5} \int_0^1 du \, \frac{\Gamma(u)}{(1-u^2)^3} u^2 (1-u)^2 \tag{1.14}$$

To explain how the process ξ_t is defined, we recall the definition of center of a function, introduced in [2] for $\lambda = 0$.

1.2 Definition.

Given $m \in C(\mathbb{R})$, we say that x_0 is the center of m if

$$\int dx \, (m(x) - \phi_{x_0}(x))\phi'_{x_0}(x) = 0.$$

The connection of this definition with the G-L equation will be more clear later, by the moment let us just recall that the center of ϕ_{x_0} is x_0, which is the point where $\phi_{x_0} = 0$. For functions that are close to some instanton, the center is well defined. The precise statement is given in the next lemma.

1.3 Lemma.

Let $m \in C(\mathbb{R})$, with $\|m\|_\infty \le 2$, and $\zeta \in (0,1)$ be given. Then, there exists $\delta > 0$ such that, if

$$\|m - \phi_{x_0}\|_\varepsilon < \delta$$

for some x_0 satisfying $|x_0| \le (1-\zeta')\varepsilon^{-1}$ for some $\zeta' \in (\zeta, 1)$, then m has a center ξ, which is unique in the set $\{x : |x| \le (1-\zeta)\varepsilon^{-1}\}$. Moreover, there exists $c > 0$ such that

$$|x_0 - \xi| \le c\left(\|m - \phi_{x_0}\|_\varepsilon + e^{-\varepsilon^{-1}\zeta}\right).$$

Finally, if $m \in C_N(\mathbb{R})$ (see (2.1) below), then m has a unique center ξ in the whole interval $[-\varepsilon^{-1}, \varepsilon^{-1}]$.

Now we set, as in [2],

1.4 Definition.

Given $m \in C(\mathbb{R})$, we define the function $\xi(m)$ as the center of m in case m satisfies the conditions of Lemma 1.3 for some $\zeta \in (0,1)$ and we say that $\xi(m)$ is "proper". Otherwise we set $\xi(m) = 0$ and we say that it is not proper. We define

$$\xi_t = \xi(m_t), \tag{1.15}$$

As we will explain later, the process m_t has a proper center with probability going to 1 as $\varepsilon \to 0^+$, up to times t of order ε^{-1}.

Remark. *A result analogous to Theorem 1.1 was obtained in [2] in the symmetric case, that is, when $\lambda = 0$. In that case, the drift $\beta = 0$, and the limiting process X_t is a Brownian motion. In fact, the proof of (1.10) given in [2] for the case $\lambda = 0$, with the process ξ_t defined as above can be adapted to our case. But the symmetry of V_0 is crucial in [2] to identify the limiting law of X_t^ε, and it was not clear, at least for us, if β was different from zero for some choice of Γ, and the way was to compute the first term of an expansion of β in λ. Observe in fact that, from (1.14), one concludes that if Γ is positive, then β is positive (for small λ), which is not obvious only from (1.13). Also Funaki, [11], requires that V_0' is odd, and the drift that appears in his result comes out from the fact that he considers a non-homogeneous noise.*

2. Proof of Theorem 1.1: First part.

For proving Theorem 1.1, we linearize the equation (1.1) around the instanton, and use its stability with respect to the same equation (1.1) in $\mathbb{R}$, as in [1], [2] and [3]. It is then convenient to extend the process m_t to $\mathbb{R}$, and to consider several integral equations equivalent to (1.1). We will mention here these equations for completeness, the proofs and details can be found in Section 2 of [2].

For any continuous function f defined on the interval $[-\varepsilon^{-1}, \varepsilon^{-1}]$, we denote by $\check{f}$ the extension of f to a function on the whole $\mathbb{R}$ obtained by successive reflections around the points $(2n+1)\varepsilon^{-1}$, $n \in \mathbb{Z}$, and we call $C_N(\mathbb{R})$ the space of functions so obtained, that is

$$\begin{aligned} C_N(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \,:\, & f \text{ is continuous and invariant} \\ & \text{by reflections around the points } (2n+1)\varepsilon^{-1},\ n \in \mathbb{Z} \} \end{aligned} \tag{2.1}$$

Consider then the integral equation

$$m_t = H_t m_0 - \int_0^t ds\, H_{t-s} V'(m_s) + \sqrt{\varepsilon} Z_t, \quad t \ge 0, \quad x \in \mathbb{R}, \tag{2.2}$$

where H_t is the heat operator in $\mathbb{R}$, and the last term Z_t is really $\check{Z}_t$, with the process Z_t defined as in (1.4). Equation (2.2) has a unique continuous solution m_t if the initial datum

m_0 is bounded and continuous. Given m_0 defined in $[-\varepsilon^{-1},\varepsilon^{-1}]$ and satisfying N.b.c., m_t is a solution of (1.3) with initial datum m_0 if and only if its extension to $\mathbb{R}$, $\check{m}_t$, solves (2.2), with initial condition and noise being the extensions of m_0 and Z_t respectively. In case $m_0 \in C_N(\mathbb{R})$, we have thus two ways of representing the process m_t, one as the restriction to $[-\varepsilon^{-1},\varepsilon^{-1}]$ of the solution of (2.2), and the other as the solution of (1.3). In general we will not use ˇ and denote by the same a function in $C_N(\mathbb{R})$ and its restriction to $[-\varepsilon^{-1},\varepsilon^{-1}]$.

Let us now introduce another integral equation, that results from linearizing the G-L equation around the instanton. If we expand $V'(m_s)$ around ϕ_{x_0}, and use the fact that ϕ_{x_0} is stationary for the G-L equation, we obtain from (2.2) that $m_t - \phi_{x_0}$ satisfies the equation

$$(m_t - \phi_{x_0})(x) = \int dy\, g_{x_0}(x,y,t) m_0(y) - \int_0^t ds \int dy\, g_{x_0}(x,y,t-s) \\ \times \big(V'(m_s) - V'(\phi_{x_0}) - V''(\phi_{x_0})(m_s - \phi_{x_0})\big)(x) + \sqrt{\varepsilon}\hat{Z}_{x_0}, \qquad (2.3)$$

where $g_{x_0}(x,y,t)$ is the fundamental solution of

$$\frac{\partial u}{\partial t} = L_{x_0} u \quad \text{with} \quad L_{x_0} = \frac{1}{2}\frac{\partial^2}{\partial x^2} - V''(\phi_{x_0}).$$

The process $\hat{Z}_{x_0}$ is the extension (as a function of $C_N(\mathbb{R})$) to $\mathbb{R}$ of the Gaussian process given by

$$\hat{Z}_{x_0}(x,t) = \int_0^t ds \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} dy\, \alpha(s,y) \\ \times \sum_{k\in\mathbb{Z}} \big(g_{x_0}(x, y + 4k\varepsilon^{-1}, t-s) + g_{x_0}(x, 4k\varepsilon^{-1} + 2\varepsilon^{-1} - y, t-s)\big) \qquad (2.4)$$

Finally, the following result is a consequence of the operator L_{x_0} having a spectral gap. We refer to [2] and [9] for details.

2.1 Lemma.

There are $\gamma > 0$ and $c > 0$ such that, for any continuous bounded function f and any $x_0 \in \mathbb{R}$,

$$\left\| \int dy \left(g_{x_0}(\cdot,y,t) - \rho\phi'_{x_0}(\cdot)\phi'_{x_0}(y)\right) f(y) \right\|_\infty \le c e^{-\gamma t} \left\| f - \rho\phi'_{x_0} \int dx\, \phi'_{x_0}(x) f(x) \right\|_\infty$$

Let us again remark that (1.10) follows from the proof of the corresponding result for $\lambda = 0$ given in Theorem 3.6 of [2], which does not depend on the symmetry. Consequently, to prove the first part of Theorem 1.1, we only need to show that

$$X_t^\varepsilon \to X_t \qquad (2.5)$$

for X_t^ε and X_t as defined in (1.11) and (1.12), with β as in (1.13). We need a preliminary lemma. In (2.42) below we will fix T_ε, by the moment just recall that it is of the form $\varepsilon^{-\tilde{a}}$ with $\tilde{a}$ small and positive.

2.2 Lemma.

Given $\zeta > 0$, let m_t be the solution of (1.3) *with initial condition $m_0 = \phi_{x_0}$, for some $|x_0| < (1-\zeta/2)\varepsilon^{-1}$. Then, for any positive p, there exists a positive constant c_p such that*

$$E|\xi_{T_\varepsilon} - x_0|^p \le c_p(\varepsilon T_\varepsilon)^{p/2} \tag{2.6}$$

and, given a small, there exists a positive constant $k > 0$ such that

$$\left|E\xi_{T_\varepsilon} - x_0 - \beta\varepsilon T_\varepsilon\right| \le kT_\varepsilon^{1/2}\varepsilon^{1-a}. \tag{2.7}$$

The estimate (2.6) follows from Lemma 4.2 of [2], since the proof given there works also for non-symmetric potentials. In that lemma it is also given an estimate for $|E\xi_{T_\varepsilon} - x_0|$, which is stronger than that given in (2.7) for $|E\xi_{T_\varepsilon} - x_0 - \beta\varepsilon T_\varepsilon|$. The symmetry of V_0 is crucial for that, and the corresponding result is no longer valid in our case. However (2.7) is enough for our purposes.

Proof of Lemma 2.2.

According to the above discussion, we only need to prove (2.7). In the sequel we denote by k_i, $i = 1, 2, \dots$, suitable positive constants. Define, for each positive δ and x_0 as in the statement, the set

$$S_{x_0}(\delta) = \{m \in C_N(\mathbb{R}) : \|m\| \le 2,\ \|m - \phi_{x_0}\|_\varepsilon \le \delta\}. \tag{2.8}$$

We know (see Lemma 1.3) that, if δ is sufficiently small, any function $m \in S_{x_0}(\delta)$ has a unique center ξ in the interval $[-\varepsilon^{-1}, \varepsilon^{-1}]$. In that case, $\xi - x_0$ is the unique zero in $[-\varepsilon^{-1}, \varepsilon^{-1}]$ of the function

$$C(y) = \int dx\, [m(x) - \phi_{x_0}(x-y)]\phi'_{x_0}(x-y)$$

In other words, given $m \in S_{x_0}(\delta)$, if δ is sufficiently small, its center is defined through

$$\xi - x_0 = C^{-1}(0) \tag{2.9}$$

The function C^{-1} is differentiable at the point 0. Indeed, we have (recall the definition (1.8) of ρ),

$$\begin{aligned}(C^{-1})'(0) = (C'(\xi - x_0))^{-1} &= \left[\int dx\, \phi'(x)^2 - \int dx\, [m(x) - \phi_\xi(x)]\phi''_\xi(x)\right]^{-1} \\ &= \left[\frac{1}{\rho} - \int dx\, [m(x) - \phi_\xi(x)]\phi''_\xi(x)\right]^{-1}\end{aligned} \tag{2.10}$$

Since

$$\begin{aligned}|m(x) - \phi_\xi(x)| &\le |m(x) - \phi_{x_0}(x)| + |\phi_{x_0}(x) - \phi_\xi(x)| \\ &\le \delta + k_1|x_0 - \xi| \le \delta + k_1\delta,\end{aligned}$$

and $\phi''_{x_0}(x)$ converges to zero exponentially as $x \to \pm\infty$, see [9], it is clear from (2.10) that for δ small,

$$C'(\xi - x_0) > \frac{1}{2\rho} \tag{2.11}$$

Then, we have a third order Taylor expansion, valid for $|z|$ small,

$$C^{-1}(z) - C^{-1}(0) = z\,(C^{-1})'(0) + \frac{z^2}{2}(C^{-1})''(0) + \frac{z^3}{6}(C^{-1})'''(\theta), \qquad |\theta| \le |z| \tag{2.12}$$

Recall that

$$|C(0)| = \left|\int dx\,[m(x) - \phi_{x_0}(x)]\phi'_{x_0}(x)\right| \le k_2\delta \tag{2.13}$$

Then, for δ small, we can take $z = C(0)$ in (2.12), and from (2.9) and (2.10) we obtain

$$\begin{aligned}\xi - x_0 = &- \frac{\rho\int dx\,[m(x) - \phi_{x_0}(x)]\phi'_{x_0}(x)}{1 - \rho\int dx\,[m(x) - \phi_\xi(x)]\phi''_\xi(x)} \\ &- \frac{\left(\int dx\,[m(x) - \phi_{x_0}(x)]\phi'_{x_0}(x)\right)^2}{2}(C^{-1})''(0) \\ &- \frac{\left(\int dx\,[m(x) - \phi_{x_0}(x)]\phi'_{x_0}(x)\right)^3}{6}(C^{-1})'''(\theta)\end{aligned} \tag{2.14}$$

By Lemma 4.1 of [2], the process m_t starting from ϕ_{x_0} satisfies, for suitable positive constants c_n and any integer n,

$$P\big(F_\varepsilon(x_0)\big) \ge 1 - c_n\varepsilon^n \tag{2.15}$$

where

$$F_\varepsilon(x_0) \doteq \{m_t \in S_{x_0}(\varepsilon^{1/4}) \quad \forall\, t \le T_\varepsilon\} \tag{2.16}$$

Then, for any ε small enough, we can take $m = m_{T_\varepsilon}$ in formula (2.14) in the set $F_\varepsilon(x_0)$. Taking expectations, and observing that $|\xi_{T_\varepsilon} - x_0 - \beta\varepsilon T_\varepsilon| \le k_1\varepsilon^{-1}$, from (1.8), (2.10) and after computing $(C^{-1})''(0)$ we get that, for any given n,

$$\begin{aligned}&|E(\xi_{T_\varepsilon} - x_0 - \beta\varepsilon T_\varepsilon)| \\ &\le E\big[|\xi_{T_\varepsilon} - x_0 - \beta\varepsilon T_\varepsilon|\mathbf{1}\big((F_\varepsilon(x_0))^c\big)\big] + \big|E\big[(\xi_{T_\varepsilon} - x_0 - \beta\varepsilon T_\varepsilon)\mathbf{1}(F_\varepsilon(x_0))\big]\big| \\ &\le o(\varepsilon^n) + \left|E\left[\rho\mathbf{1}(F_\varepsilon(x_0))\int dx\,(m_{T_\varepsilon} - \phi_{x_0})\phi'_{x_0} + \beta\varepsilon T_\varepsilon\right]\right| \\ &+ \rho E\left|\frac{\mathbf{1}(F_\varepsilon(x_0))\int dx\,(m_{T_\varepsilon} - \phi_{x_0})\phi'_{x_0}\int dx\,(m_{T_\varepsilon} - \phi_{\xi_{T_\varepsilon}})\phi''_{\xi_{T_\varepsilon}}}{C'(\xi_{T_\varepsilon} - x_0)}\right| \\ &+ E\left|\frac{\mathbf{1}(F_\varepsilon(x_0))\left(\int dx\,(m_{T_\varepsilon} - \phi_{x_0})\phi'_{x_0}\right)^2 C''(\xi_{T_\varepsilon} - x_0)}{2(C'(\xi_{T_\varepsilon} - x_0))^3}\right| \\ &+ E\left|\mathbf{1}(F_\varepsilon(x_0))\left(\int dx\,(m_{T_\varepsilon} - \phi_{x_0})\phi'_{x_0}\right)^3\frac{(C^{-1})'''(\theta)}{6}\right|,\end{aligned} \tag{2.17}$$

where we use $o(\varepsilon^n)$ to indicate a term that goes to zero faster than ε^n. Let us see now how to estimate the last three expectations in the r.h.s. of (2.17). From (2.11), if ε is sufficiently small,

$$\left| \frac{\mathbf{1}(F_\varepsilon(x_0))}{C'(\xi_{T_\varepsilon} - x_0)} \right| \le 2\rho. \tag{2.18}$$

Also, recall that $m_t - \phi_{x_0}$ can be approximated by the Gaussian process $\sqrt{\varepsilon}\hat{Z}_{x_0}$. More precisely, for any $a \in (0,1)$, we have (see formula (4.83) in [2])

$$\mathbf{1}(G_\varepsilon(a, x_0))\|m_t - \phi_{x_0} - \sqrt{\varepsilon}\hat{Z}_{x_0}(\cdot, t)\|_\varepsilon \le k_3 \varepsilon^{1-2a} T_\varepsilon^2 \quad \forall t \le T_\varepsilon, \tag{2.19}$$

where the set $G_\varepsilon(a, x_0)$ satisfies

$$G_\varepsilon(a, x_0) \subset \{\|\hat{Z}_{x_0}(\cdot, t)\| \le \varepsilon^{-a}(t \vee 1)^{1/2}, \quad \forall t \le \varepsilon^{-2}\} \tag{2.20}$$

and there are positive constants c_n such that, for any integer n,

$$P\Big(G_\varepsilon(a, x_0)\Big) \ge 1 - c_n \varepsilon^n. \tag{2.21}$$

We refer to Proposition 5.4 of [2] for proofs and details. Also, from (3.10) of [2], we know that, given a as above, the set

$$A = \{\|m_{T_\varepsilon} - \phi_{\xi_{T_\varepsilon}}\|_\varepsilon \le \varepsilon^{1/2-a}\} \tag{2.22}$$

satisfies

$$P(A) \ge 1 - c_n \varepsilon^n. \tag{2.23}$$

Finally, one computes

$$C''(\xi_{T_\varepsilon} - x_0) = \int dx\, (m_{T_\varepsilon} - \phi_{\xi_{T_\varepsilon}})\phi'''_{\xi_{T_\varepsilon}} \tag{2.24}$$

Hereafter we will assume $a \in (0, 1/4)$, it will be taken smaller later. From (2.19) and (2.21), we get that, for any $p \ge 1$, there exist constants c_p such that

$$\begin{aligned} E&\left| \mathbf{1}(F_\varepsilon(x_0)) \left(\int dx\, (m_{T_\varepsilon} - \phi_{x_0})\phi'_{x_0} \right)^p \right| \\ &\le c_p E\left| \mathbf{1}(F_\varepsilon(x_0) \cap G_\varepsilon(a, x_0)) \left(\int dx\, (m_{T_\varepsilon} - \phi_{x_0} - \sqrt{\varepsilon}\hat{Z}_{x_0}(\cdot, T_\varepsilon))\phi'_{x_0} \right)^p \right| \\ &+ c_p \varepsilon^{p/2} E\left| \int dx\, \hat{Z}_{x_0}(\cdot, T_\varepsilon)\phi'_{x_0} \right|^p + o(\varepsilon^n) \le c_p \left(k_3[\varepsilon^{1-2a}T_\varepsilon^2]^p + [\varepsilon T_\varepsilon]^{p/2} \right) \end{aligned} \tag{2.25}$$

where in the last line we have used that $\int dx\, \hat{Z}_{x_0}(x,t)\phi'_{x_0}(x)$ is a Brownian motion with diffusion coefficient D_ε, satisfying

$$E\left[\int dx\, \phi'_{x_0}(x)\hat{Z}_{x_0}(x,t) \right] = 0, \quad E\left[\int dx\, \phi'_{x_0}(x)\hat{Z}_{x_0}(x,1) \right]^2 = D_\varepsilon, \text{ and } \lim_{\varepsilon \to 0^+} \left| D_\varepsilon - \frac{1}{\rho} \right| = 0 \tag{2.26}$$

(as follows from (2.4), or see Proposition 5.4 of [2]). Inequality (2.25) for $p=3$ yields, after observing that $(C^{-1})'''$ is bounded in a neighbourhood of zero, that the last expectation in the r.h.s. of (2.17) is less than $k_4[(\varepsilon^{1-2a}T_\varepsilon^2)^3+(\varepsilon T_\varepsilon)^{3/2}]$. From the Cauchy-Schwartz inequality, (2.18), (2.22), (2.23), (2.24) and (2.25), we get that the second and third expectations in the r.h.s. of (2.17) are less than $k_5[\varepsilon^{1-2a}T_\varepsilon^2+(\varepsilon T_\varepsilon)^{1/2}]\varepsilon^{1/2-a}$. We have then proved that, for a small,

$$|E\xi_{T_\varepsilon}-x_0-\beta\varepsilon T_\varepsilon|\le O(\varepsilon T_\varepsilon^{1/2}\varepsilon^{-a})+\left|E\left[\rho\mathbf{1}(F_\varepsilon(x_0))\int dx\, m_{T_\varepsilon}(x)\phi'_{x_0}(x)+\beta\varepsilon T_\varepsilon\right]\right| \tag{2.27}$$

We are left with the estimate of the last term in (2.27); we multiply both sides of the integral equation (2.3) by ϕ'_{x_0}, we expand the potential and recall that

$$\int dx\, g_{x_0}(x,y,s)\phi'_{x_0}(y)=\phi'_{x_0}(x)\quad \forall\, s\ge 0,\ \forall\, x_0\in\mathbb{R}. \tag{2.28}$$

We obtain then the equation

$$\begin{aligned}\int dx\,(m_{T_\varepsilon}-\phi_{x_0})\phi'_{x_0}&=-\int_0^{T_\varepsilon}dt\int dx\,(m_t-\phi_{x_0})^2\frac{V'''(\phi_{x_0})}{2}\phi'_{x_0}\\&-\int_0^{T_\varepsilon}dt\int dx\,(m_t-\phi_{x_0})^3\frac{V''''(\theta_t)}{6}\phi'_{x_0}+\sqrt{\varepsilon}\int dx\,\phi'_{x_0}\hat{Z}_{x_0}(\cdot,T_\varepsilon)\end{aligned} \tag{2.29}$$

with θ_t a suitable continuous function on $\mathbb{R}$ such that $\phi_{x_0}\wedge m_t\le\theta_t\le\phi_{x_0}\vee m_t$.

Then (2.15), (2.26) and the Cauchy-Schwartz inequality yield

$$\left|E\left[\mathbf{1}(F_\varepsilon(x_0))\int dx\,\phi'_{x_0}(x)\hat{Z}_{x_0}(x,T_\varepsilon)\right]\right|\le(T_\varepsilon c_n\varepsilon^n)^{1/2} \tag{2.30}$$

Taking expectations, from (2.29) and (2.30) we get

$$\begin{aligned}&\left|E\left[\rho\mathbf{1}(F_\varepsilon(x_0))\int dx\,(m_{T_\varepsilon}-\phi_{x_0})\phi'_{x_0}+\beta\varepsilon T_\varepsilon\right]\right|\\&\le\left|E\left[\rho\mathbf{1}(F_\varepsilon(x_0))\int_0^{T_\varepsilon}dt\int dx\,(m_t-\phi_{x_0})^2\frac{V'''(\phi_{x_0})}{2}\phi'_{x_0}-\beta\varepsilon T_\varepsilon\right]\right|\\&+\rho\left|E\left[\mathbf{1}(F_\varepsilon(x_0))\int_0^{T_\varepsilon}dt\int dx\,(m_t-\phi_{x_0})^3\frac{V''''(\theta_t)}{6}\phi'_{x_0}\right]\right|+o(\varepsilon^n)\end{aligned} \tag{2.31}$$

Let us estimate now the first expectation on the r.h.s. of (2.31). Since in $F_\varepsilon(x_0)$ the integral inside the expectation is bounded by k_6T_ε, from (2.21), with an error $o(\varepsilon^n)$ we can restrict the expectation to the set $F_\varepsilon(x_0)\cap G_\varepsilon(a,x_0)$. Then, by adding and subtracting $\sqrt{\varepsilon}\hat{Z}_{x_0}(\cdot,t)$

to $(m_t - \phi_{x_0})$ and by estimating as in (2.25), from (2.19) and (2.21) we get

$$
\begin{aligned}
&\left| E\left[\rho \mathbf{1}(F_\varepsilon(x_0)) \int_0^{T_\varepsilon} dt \int dx\, (m_t - \phi_{x_0})^2 \frac{V'''(\phi_{x_0})}{2} \phi'_{x_0} - \beta\varepsilon T_\varepsilon \right] \right| \\
&\le \rho \left| E\left[\mathbf{1}(F_\varepsilon(x_0) \cap G_\varepsilon(a, x_0)) \int_0^{T_\varepsilon} dt \int dx\, (m_t - \phi_{x_0} - \sqrt{\varepsilon}\hat{Z}_{x_0}(\cdot, t))^2 \frac{V'''(\phi_{x_0})}{2} \phi'_{x_0} \right] \right| \\
&+ \rho\sqrt{\varepsilon} \left| E\left[\mathbf{1}(F_\varepsilon(x_0) \cap G_\varepsilon(a, x_0)) \int_0^{T_\varepsilon} dt \int dx\, (m_t - \phi_{x_0} - \sqrt{\varepsilon}\hat{Z}_{x_0}(\cdot, t))\hat{Z}_{x_0}(\cdot, t) V'''(\phi_{x_0})\phi'_{x_0} \right] \right| \\
&+ \varepsilon \left| E\left[\rho \mathbf{1}(F_\varepsilon(x_0) \cap G_\varepsilon(a, x_0)) \int_0^{T_\varepsilon} dt \int dx\, \hat{Z}_{x_0}(\cdot, t)^2 \frac{V'''(\phi_{x_0})}{2} \phi'_{x_0} - \beta T_\varepsilon \right] \right| + o(\varepsilon^n)
\end{aligned}
\tag{2.32}
$$

By (2.19) and (2.20) we conclude that the first two expectations on the r.h.s. of (2.32) are $o(\varepsilon)$ if we take a small. By the same reasoning, it can be seen that the same holds for the last expectation in (2.31). According to that, from (2.27) and (2.31), the proof of (2.7) is complete if we show that

$$
\left| E\left[\rho \mathbf{1}(F_\varepsilon(x_0) \cap G_\varepsilon(a, x_0)) \int_0^{T_\varepsilon} dt \int dx\, \hat{Z}_{x_0}(\cdot, t)^2 \frac{V'''(\phi_{x_0})}{2} \phi'_{x_0} - \beta T_\varepsilon \right] \right| = O(1) \tag{2.33}
$$

as $\varepsilon \to 0^+$. From (2.15), (2.21) and (2.26), (2.33) is equivalent to

$$
\left| \int_0^{T_\varepsilon} dt \left(\int dx \frac{V'''(\phi_{x_0})}{2} \phi'_{x_0} \rho E[\hat{Z}_{x_0}(\cdot, t)^2] - \beta \right) \right| = O(1) \quad \text{as } \varepsilon \to 0^+ \tag{2.34}
$$

Next, observe that

$$
\begin{aligned}
E\left[\hat{Z}_{x_0}(x, t)^2\right] &= \int_0^t ds \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} dy \left(\sum_{j \in \mathbb{Z}} g_{x_0}(x, y + 4j\varepsilon^{-1}, s) + g_{x_0}(x, 4j\varepsilon^{-1} + 2\varepsilon^{-1} - y, s) \right)^2 \\
&= \sum_{i=0}^{6} I_i(x, t)
\end{aligned}
\tag{2.35}
$$

where

$$
\begin{aligned}
I_0(x,t) &= \int_0^t ds \int dy\, g_{x_0}(x,y,s)^2 \\
I_1(x,t) &= -\int_0^t ds \int_{|y|>\varepsilon^{-1}} dy\, g_{x_0}(x,y,s)^2 \\
I_2(x,t) &= \int_0^t ds \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} dy\, g_{x_0}(x,2\varepsilon^{-1}-y,s)^2 \\
I_3(x,t) &= 2\int_0^t ds \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} dy\, g_{x_0}(x,2\varepsilon^{-1}-y,s) g_{x_0}(x,y,s) \\
I_4(x,t) &= 2\int_0^t ds \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} dy\, g_{x_0}(x,y,s) \\
&\quad \times \Big(\sum_{j\neq 0} g_{x_0}(x,y+4j\varepsilon^{-1},s) + g_{x_0}(x,4j\varepsilon^{-1}+2\varepsilon^{-1}-y,s) \Big) \\
I_5(x,t) &= 2\int_0^t ds \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} dy\, g_{x_0}(x,2\varepsilon^{-1}-y,s) \\
&\quad \times \Big(\sum_{j\neq 0} g_{x_0}(x,y+4j\varepsilon^{-1},s) + g_{x_0}(x,4j\varepsilon^{-1}+2\varepsilon^{-1}-y,s) \Big) \\
I_6(x,t) &= \int_0^t ds \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} dy\, \Big(\sum_{j\neq 0} g_{x_0}(x,y+4j\varepsilon^{-1},s) + g_{x_0}(x,4j\varepsilon^{-1}+2\varepsilon^{-1}-y,s) \Big)^2
\end{aligned}
$$

It is not difficult to see that

$$
\lim_{\varepsilon\to 0+} \int_0^{T_\varepsilon} dt \int dx\, I_i(x,t) V'''(\phi_{x_0}(x))\phi_{x_0}(x) = 0 \qquad \forall\, i=1,2,3,4,5,6 \tag{2.36}
$$

so (2.34) follows once we prove

$$
\left| \rho \int_0^{T_\varepsilon} dt \left(\int_0^t ds \int dx \int dy\, g_{x_0}(x,y,s)^2 \frac{V'''(\phi_{x_0}(x))}{2} \phi'_{x_0}(x) - \beta \right) \right| = O(1) \quad \text{ as } \varepsilon\to 0^+ \tag{2.37}
$$

Recalling (1.8) and (2.28), we get that

$$
\begin{aligned}
&\int dx \int dy\, g_{x_0}(x,y,s)^2 \frac{V'''(\phi_{x_0}(x))}{2} \phi'_{x_0}(x) \\
&= \int dx \int dy\, \big(g_{x_0}(x,y,s) - \rho\phi'_{x_0}(x)\phi'_{x_0}(y) \big)^2 \frac{V'''(\phi_{x_0}(x))}{2} \phi'_{x_0}(x) \\
&+ \rho \int dx\, \frac{V'''(\phi_{x_0}(x))}{2} \phi'_{x_0}(x)^3
\end{aligned} \tag{2.38}
$$

Since the derivatives of ϕ_{x_0} vanish at $\pm\infty$ (see [9] for general properties of travelling fronts), integrating by parts twice and using (1.5) we have

$$
\int dx\, V'''(\phi_{x_0})(\phi'_{x_0})^3 = \int dx\, \phi'''_{x_0}\phi''_{x_0} = 0 \tag{2.39}
$$

so that, recalling also that $g_{x_0}(x+x_0, y+x_0, s) = g(x,y,s)$ for any $s \geq 0$, from the definition (1.13), (2.38) and (2.39), (2.37) becomes

$$\frac{\rho}{2}\left|\int_0^{T_\varepsilon} dt \int_t^\infty ds \int dx \int dy\, V'''(\phi(x))\phi'(x)\big(g(x,y,s) - \rho\phi'(x)\phi'(y)\big)^2\right| = O(1) \tag{2.40}$$

as $\varepsilon \to 0^+$. We have now

$$\begin{aligned}
&\left|\int_0^{T_\varepsilon} dt \int_t^\infty ds \int dx \int dy\, V'''(\phi(x))\phi'(x)\big(g(x,y,s) - \rho\phi'(x)\phi'(y)\big)^2\right| \\
&\leq \left|\int_0^{1} dt \int_t^1 ds \int dx \int dy\, V'''(\phi(x))\phi'(x)\big(g(x,y,s) - \rho\phi'(x)\phi'(y)\big)^2\right| \\
&+ \left|\int_0^{1} dt \int_1^\infty ds \int dx \int dy\, V'''(\phi(x))\phi'(x)\big(g(x,y,s) - \rho\phi'(x)\phi'(y)\big)^2\right| \\
&+ \left|\int_1^{T_\varepsilon} dt \int_t^\infty ds \int dx \int dy\, V'''(\phi(x))\phi'(x)\big(g(x,y,s) - \rho\phi'(x)\phi'(y)\big)^2\right|
\end{aligned} \tag{2.41}$$

From the Feynman-Kac representation for g, see (3.15) below, it is easy to see that the first integral in the r.h.s. of (2.41) is bounded. For the other two, since $\sup_{t\geq 1}\sup_{x,y\in\mathbb{R}} g(x,y,t)$ is finite, we can apply Lemma 2.1 to get

$$\left|\int dx \int dy\, V'''(\phi(x))\phi'(x)\big(g(x,y,s) - \rho\phi'(x)\phi'(y)\big)^2\right| \leq k e^{-\gamma s}$$

Then, the l.h.s of (2.41) is bounded and so (2.7) is proved.

Proof of Theorem 1.1.

Let us consider, as in [2], a discretization of the process X_t^ε. Fix

$$b < \frac{1}{10}, \quad n_\varepsilon = [\varepsilon^{b-\frac{1}{10}}], \quad T_\varepsilon = n_\varepsilon \varepsilon^{-b}, \quad t_n = nT_\varepsilon, \tag{2.42}$$

define

$$Y_t = \xi_{t_n} - x_0, \qquad \text{for } t_n \leq t < t_{n+1} \tag{2.43}$$

and set

$$Y_\tau^\varepsilon = Y_{\varepsilon^{-1}\tau}. \tag{2.44}$$

Recall that Y_τ^ε is precisely X_τ^ε when $\tau = \varepsilon t_n$. We shall see that

$$Y_t^\varepsilon \to X_t \qquad \text{as } \varepsilon \to 0^+ \tag{2.45}$$

weakly in $D[0,T]$, with X as in (1.12). Following [2], let

$$M_\tau^\varepsilon = Y_\tau^\varepsilon - \int_0^\tau dt\, \gamma_{1,\tau}(t) \tag{2.46}$$

and

$$N_\tau^\varepsilon = \left(M_\tau^\varepsilon\right)^2 - \int_0^\tau dt\, \gamma_{2,\tau}(t), \tag{2.47}$$

where

$$\gamma_{i,\tau}(t) = \begin{cases} \gamma_i(t_n) & \text{if } \varepsilon t_n \le t < \varepsilon t_{n+1} \le \tau \\ 0 & \text{otherwise} \end{cases}$$

with

$$\gamma_1(t_n) = (\varepsilon T_\varepsilon)^{-1} E\big[Y_{t_{n+1}} - Y_{t_n} \big| \mathcal{F}_{t_n}\big]$$

and

$$\gamma_2(t_n) = (\varepsilon T_\varepsilon)^{-1} \Big(E\big[Y_{t_{n+1}}^2 - Y_{t_n}^2 \big| \mathcal{F}_{t_n}\big] - E\big[Y_{t_{n+1}} - Y_{t_n} \big| \mathcal{F}_{t_n}\big] E\big[Y_{t_{n+1}} + Y_{t_n} \big| \mathcal{F}_{t_n}\big]\Big) \tag{2.48}$$

It is not difficult to see that M_τ^ε and N_τ^ε are martingales with respect to the filtration $\mathcal{F}_t^\varepsilon$ generated by the noise up to time $\varepsilon^{-1}t$ (we refer to [2] for details). Moreover, by Theorem 2.6.2 of [6], the condition

$$\sup_{t_n \le \varepsilon^{-1} T} E\big(\gamma_i^2(t_n)\big) \le c, \quad i = 1, 2 \tag{2.49}$$

for some positive constant c, implies the tightness of the family P^ε of the laws of Y_τ^ε in the space $D[0,T]$, and the condition

$$\lim_{\varepsilon \to 0^+} \sup_{t_n \le \varepsilon^{-1} T} (\varepsilon T_\varepsilon)^{-1} E\big[(Y_{t_{n+1}} - Y_{t_n})^4\big] = 0 \tag{2.50}$$

implies that any limit point P of P^ε has support in the space of continuous functions (see Theorem 2.7.8 in [6]). We will see that (2.49) and (2.50) are satisfied in our case. To identify the limit, we will show that

$$\lim_{\varepsilon \to 0^+} \sup_{t_n \le \varepsilon^{-1} T} E\big[(\varepsilon T_\varepsilon)^{-1} E\big[Y_{t_{n+1}} - Y_{t_n} - \beta \varepsilon T_\varepsilon \big| \mathcal{F}_{t_n}\big]\big] = 0 \tag{2.51}$$

and

$$\lim_{\varepsilon \to 0^+} \sup_{t_n \le \varepsilon^{-1} T} E\big|\rho - (\varepsilon T_\varepsilon)^{-1} E[Y_{t_{n+1}}^2 - Y_{t_n}^2 \big| \mathcal{F}_{t_n}]\big| = 0 \tag{2.52}$$

From (2.51) and (2.52) it is easy to conclude that $Y_t - \beta t$ and $(Y_t - \beta t)^2 - \rho t$ are $\mathcal{F}_t$ martingales and so, from Levy's characterization of Brownian motion, the limiting law is unique, and given by (1.12). Let us prove now (2.49), (2.50), (2.51) and (2.52).

Proof of (2.49).

From (4.46) of [2] (that holds also for non-symmetric potentials), the strong Markov property, the fact that $|\xi_t| \le \varepsilon^{-1}$, and Lemma 4.1 of [2], it is enough to prove

$$E[\gamma_i^2(t_0)] \le c, \quad i = 1, 2 \tag{2.53}$$

in the case the process starts from some instanton ϕ_{x_0}, with $|x_0| \le (1-\zeta/2)\varepsilon^{-1}$. But

$$E[\gamma_1^2(t_0)] \le 2(\varepsilon T_\varepsilon)^{-2}\left(|E\xi_{T_\varepsilon} - x_0 - \beta_\varepsilon T_\varepsilon|^2 + \beta^2(\varepsilon T_\varepsilon)^2\right),$$

which is bounded from (2.7) if we choose $a < 1/20$. By (2.48)

$$E[\gamma_2^2(t_0)] = (\varepsilon T_\varepsilon)^{-2}\left(E[(\xi_{T_\varepsilon} - x_0)^2] - (E[\xi_{T_\varepsilon} - x_0])^2\right)^2$$

so (2.53) for $i = 2$ follows from (2.6).

Proof of (2.50).

By the same reasoning used in the beginning of the proof of (2.49) it is enough to show that

$$\lim_{\varepsilon\to 0^+} (\varepsilon T_\varepsilon)^{-1} E|\xi_{T_\varepsilon} - x_0|^4 = 0 \tag{2.54}$$

with initial condition an instanton. But (2.54) follows from (2.6).

Proof of (2.51) *and* (2.52).

Taking $a < 1/20$ in (2.7), we obtain from (2.42) that

$$\lim_{\varepsilon\to 0^+} (\varepsilon T_\varepsilon)^{-1} \left|E\left[Y_{t_1} - \beta\varepsilon T_\varepsilon\right]\right| = 0 \tag{2.55}$$

As in the beginning of the proof of (2.49), it can be seen that (2.55) implies (2.51). Analogously (2.52) follows from

$$\lim_{\varepsilon\to 0^+} E\left|\rho - (\varepsilon T_\varepsilon)^{-1}[\xi_{T_\varepsilon} - x_0]^2\right| = 0. \tag{2.56}$$

To prove (2.56), square equation (2.14), with m_{T_ε}, restricting then to the set $F_\varepsilon(x_0)$ as in (2.17). It can be seen, after estimates similar to that of the terms in (2.17), that

$$E[\xi_{T_\varepsilon} - x_0]^2 = \rho^2 E\left[\left(\int dx\, (m_{T_\varepsilon} - \phi_{x_0})\phi'_{x_0}\right)^2\right] + \mathcal{R}$$

where

$$\lim_{\varepsilon\to 0^+} (\varepsilon T_\varepsilon)^{-1} |\mathcal{R}| = 0 \tag{2.57}$$

To get (2.56) from (2.57) it is then enough to prove that

$$\lim_{\varepsilon\to 0^+} (\varepsilon T_\varepsilon)^{-1} E\left[\left(\int dx\, (m_{T_\varepsilon} - \phi_{x_0})\phi'_{x_0}\right)^2\right] = \frac{1}{\rho} \tag{2.58}$$

But, from (2.29) we have that

$$\begin{aligned} &E\left[\left(\int dx\, (m_{T_\varepsilon} - \phi_{x_0})\phi'_{x_0}\right)^2\right] \\ &= E\left[S^2 + \varepsilon\left(\int dx\, \phi'_{x_0}\hat{Z}_{x_0}(\cdot, T_\varepsilon)\right)^2 - 2\sqrt{\varepsilon} S \int dx\, \phi'_{x_0}\hat{Z}_{x_0}(\cdot, T_\varepsilon)\right] \end{aligned}$$

where S stands for the sum of the first two integrals on the r.h.s. of (2.29). From (2.26),

$$\lim_{\varepsilon \to 0+} (\varepsilon T_\varepsilon)^{-1} E \left[\varepsilon \left(\int dx\, \phi'_{x_0} \hat{Z}_{x_0}(\cdot, T_\varepsilon) \right)^2 \right] = \frac{1}{\rho}$$

Also, proceeding as in the proof of (2.57) it is not difficult to see that

$$\lim_{\varepsilon \to 0+} (\varepsilon T_\varepsilon)^{-1} E S^2 = 0 \tag{2.59}$$

which implies (2.58), and so (1.12) is proved.

It is an easy consequence of Lemma 2.1 that the drift β is finite. But it could be zero in principle. We will show in the next section that this is not the case in general, and that (1.14) holds, thus concluding the proof of Theorem 1.1.

3. Proof of Theorem 1.1: Last part.

We will prove (1.14) in this section. The expression (1.13) for β involves a rather complicated dependence on λ and Γ and, as we have already explained, we could not say anything about the sign of β directly, even for particular simple Γ. In what follows, we suppose Γ is given, and we write explicitely the dependence on λ of β, ρ, g and ϕ:

$$\beta = \beta(\lambda); \quad \rho = \rho(\lambda); \quad g(x,y,t) = g_{(\lambda)}(x,y,t); \quad \phi(x) = \phi_{(\lambda)}(x).$$

With some abuse of notation with respect to the previous sections, we simply put

$$\beta(0) = \beta_0; \quad \rho(0) = \rho_0; \quad g_{(0)} = g_0; \quad \phi_{(0)} = \phi_0$$

The expression (1.13) for the drift $\beta = \beta(\lambda)$ looks then

$$\beta(\lambda) = \frac{\rho(\lambda)}{2} \int_0^\infty dt \int dx \int dy \left(g_{(\lambda)}(x,y,t) - \rho(\lambda)\phi'_{(\lambda)}(x)\phi'_{(\lambda)}(y) \right)^2 \phi'_{(\lambda)}(x) V'''(\phi_{(\lambda)}(x)) \tag{3.1}$$

Then, we proceed to compute the right derivative of β with respect to λ. Since $\beta_0 = 0$, the sign of this derivative gives the sign of β in some small interval $(0, \lambda_0)$.

3.1 Lemma.

$$\begin{aligned}
&\frac{d\beta}{d\lambda}(0) = \\
&\frac{9}{16} \int_0^\infty dt \int dx \int dy \left(\psi(x) V_0'''(\phi_0(x)) + \Gamma''(\phi_0(x)) \right) \phi_0'(x) \phi_0''(y) \left(g_0(x,y,t) - \frac{3}{4}\phi_0'(x)\phi_0'(y) \right) \\
&-\frac{9}{16} \int_0^\infty dt \int dx \int dy\, \phi_0'(x)^2 V_0'''(\phi_0(x)) \psi'(y) \left(g_0(x,y,t) - \frac{3}{4}\phi_0'(x)\phi_0'(y) \right)
\end{aligned} \tag{3.2}$$

where $V_0(m) = m^4/4 - m^2/2$ *and*

$$\psi(x) = 2\phi_0'(x) \int_0^x dy\, \frac{\Gamma(\phi_0(y))}{\phi_0'(y)^2} \tag{3.3}$$

Proof.

Recall that we are using primes ($'$) to denote spatial derivatives. Differentiating (3.1) we obtain

$$\begin{aligned}\frac{d\beta}{d\lambda}(0) = \frac{3}{8}\int_0^\infty dt \int dx \int dy\, 2\left(g_0(x,y,t) - \frac{3}{4}\phi_0'(x)\phi_0'(y)\right)\phi_0'(x)V_0'''(\phi_0(x))\\ \times\left(\tilde{g}(x,y,t) - \tilde{\rho}\phi_0'(x)\phi_0'(y) - \frac{3}{4}\psi'(x)\phi_0'(y) - \frac{3}{4}\phi_0'(x)\psi'(y)\right)\\ + \frac{3}{8}\int_0^\infty dt \int dx \int dy\, \left(g_0(x,y,t) - \frac{3}{4}\phi_0'(x)\phi_0'(y)\right)^2 \frac{d}{d\lambda}\left(\phi_{(\lambda)}'V'''(\phi_{(\lambda)})\right)\Big|_{\lambda=0}(x)\end{aligned} \tag{3.4}$$

where we used that

$$\rho_0 = 3/4 \tag{3.5}$$

and set

$$\tilde{g}(x,y,t) = \frac{d}{d\lambda}g_{(\lambda)}(x,y,t)\Big|_{\lambda=0}, \tag{3.6}$$

$$\psi(x) = \frac{d}{d\lambda}\phi_{(\lambda)}(x)\Big|_{\lambda=0} \quad \text{and} \quad \tilde{\rho} = \frac{d}{d\lambda}\rho(\lambda)\Big|_{\lambda=0}. \tag{3.7}$$

Recalling (2.28), and observing that

$$\frac{d}{d\lambda}\left(\phi_{(\lambda)}'V'''(\phi_{(\lambda)})\right)\Big|_{\lambda=0}(x) = \left[\psi V_0'''(\phi_0) + \Gamma''(\phi_0)\right]'(x),$$

(3.4) becomes

$$\frac{d\beta}{d\lambda}(0) = \frac{3}{8}(I_1 + I_2) \tag{3.8}$$

where

$$\begin{aligned}I_1 \doteq 2\int_0^\infty dt \int dx \int dy\, \left(g_0(x,y,t) - \frac{3}{4}\phi_0'(x)\phi_0'(y)\right)\phi_0'(x)V_0'''(\phi_0(x))\\ \times\left(\tilde{g}(x,y,t) - \frac{3}{4}\phi_0'(x)\psi'(y)\right),\end{aligned} \tag{3.9}$$

$$I_2 \doteq \int_0^\infty dt \int dx \int dy\, \left(g_0(x,y,t) - \frac{3}{4}\phi_0'(x)\phi_0'(y)\right)^2 \left[\psi V_0'''(\phi_0) + \Gamma''(\phi_0)\right]'(x) \tag{3.10}$$

Let us now compute ψ and $\tilde{g}$.

Computation of ψ.

Recall that $\phi_{(\lambda)}$ is the solution of (1.5) that satisfies $\phi_{(\lambda)}(0) = 0$, and that $\phi_{(\lambda)}' > 0$ (see [9] for details). Multiplying both sides of the equation in (1.5) by $\phi_{(\lambda)}'$ we obtain

$$\left((\phi_{(\lambda)}')^2\right)' = 4\big(V(\phi_{(\lambda)})\big)'. \tag{3.11}$$

Since from our hypothesis $V(\phi_{(\lambda)}(-\infty)) = V(-1) = -1/4$, by integrating (3.11) we get

$$(\phi'_{(\lambda)})^2 = 4\big(V(\phi_{(\lambda)}) + 1/4\big),$$

so that, recalling that $V(u) > -1/4$,

$$\frac{\phi'_{(\lambda)}}{2\big(V(\phi_{(\lambda)}) + 1/4\big)^{1/2}} = 1.$$

Integrating this last expression from 0 to x, and replacing V by its definition, we obtain an implicit formula for $\phi_{(\lambda)}$:

$$\int_0^{\phi_{(\lambda)}(x)} dz\, H(\lambda, z) = x \tag{3.12}$$

where

$$H(\lambda, z) \doteq \frac{1}{2}\left(\frac{z^4}{4} - \frac{z^2}{2} + \frac{1}{4} + \lambda\Gamma(z)\right)^{-1/2}$$

Differentiating (3.12) with respect to λ and evaluating at 0, by using the definition (3.7), we get

$$H(0, \phi_0(x))\psi(x) + \int_0^{\phi_0(x)} dz\, \frac{\partial H}{\partial \lambda}(0, z) = 0$$

Recalling the definition of H and substituting in the above formula, we have

$$\psi(x) = \frac{(1 - \phi_0^2(x))}{4} \int_0^{\phi_0(x)} dz\, \frac{\Gamma(z)}{(1/2 - z^2/2)^3} \tag{3.13}$$

Since $\phi_0 = \tanh$,

$$1 - \phi_0^2(x) = \phi_0'(x) \tag{3.14}$$

which substituting in (3.13) yields (3.3).

Computation of $\tilde{g}$.

By the Feynman-Kac formula,

$$g_{(\lambda)}(x, y, t)dy = E\left[e^{-\int_0^t dl\, V''(\phi_{(\lambda)}(B_l^x))}\mathbf{1}(B_t^x \in dy)\right], \tag{3.15}$$

for B_t^x a Brownian motion starting at x, and $\mathbf{1}(\cdot)$ the indicator function of a set. Differentiating (3.15) with respect to λ and evaluating at $\lambda = 0$ we obtain

$$\tilde{g}(x, y, t)dy = -E\left[e^{-\int_0^t dl\, V_0''(\phi_0(B_l^x))}\int_0^t ds\, \Big(V_0'''(\phi_0(B_s^x))\psi(B_s^x) + \Gamma''(\phi_0(B_s^x))\Big)\mathbf{1}(B_t^x \in dy)\right] =$$
$$-\int_0^t ds \int dz\, \frac{e^{-\frac{(x-z)^2}{2s}}}{\sqrt{2\pi s}}$$
$$\times\, E\left[e^{-\int_0^t dl\, V_0''(\phi_0(B_l^x))}\Big(V_0'''(\phi_0(B_s^x))\psi(B_s^x) + \Gamma''(\phi_0(B_s^x))\Big)\mathbf{1}(B_t^x \in dy)\Big| B_s^x = z\right],$$

Now, decomposing the integral and using standard properties of Brownian motion, we obtain

$$E\Big[e^{-\int_0^t dl\, V_0''(\phi_0(B_l^x))}\Big(V_0'''(\phi_0(B_s^x))\psi(B_s^x)+\Gamma''(\phi_0(B_s^x))\Big)\mathbf{1}(B_t^x\in dy)\Big|B_s^x=z\Big]$$

$$=\big(V_0'''(\phi_0(z))\psi(z)+\Gamma''(\phi_0(z))\big)E\left[e^{-\int_0^s dl\, V_0''(\phi_0(B_l^x))}\Big|B_s^x=z\right]$$

$$\times E\left[e^{-\int_0^{t-s} dl\, V_0''(\phi_0(B_l^z))}\mathbf{1}(B_{t-s}^z\in dy)\right],$$

so we have

$$\tilde g(x,y,t)=-\int_0^t ds\int dz\,\big(V_0'''(\phi_0(z))\psi(z)+\Gamma''(\phi_0(z))\big)g_0(z,y,t-s)g_0(x,z,s) \tag{3.16}$$

Now, we are going to obtain a simpler expression for I_1 (see (3.9)). From the equation for g_0 it is not difficult to obtain that

$$\int_0^t ds\int dx\,V_0'''(\phi_0(x))\phi_0'(x)g_0(z,x,t-s)g_0(y,x,s)=-\left(\frac{\partial}{\partial z}g_0(z,y,t)+\frac{\partial}{\partial y}g_0(z,y,t)\right). \tag{3.17}$$

From the expression (3.16) for $\tilde g$, after interchanging the order of the integrals and using (2.28), from (3.17) we get

$$2\int dy\int dx\left(g_0(x,y,t)-\frac{3}{4}\phi_0'(x)\phi_0'(y)\right)V_0'''(\phi_0(x))\phi_0'(x)\tilde g(x,y,t)=$$

$$-2\int dy\int dx\left(g_0(x,y,t)-\frac{3}{4}\phi_0'(x)\phi_0'(y)\right)V_0'''(\phi_0(x))\phi_0'(x)$$

$$\times\int_0^t ds\int dz\,\big(V_0'''(\phi_0(z))\psi(z)+\Gamma''(\phi_0(z))\big)g_0(z,y,t-s)g_0(x,z,s)=$$

$$-2\int dz\,\big(V_0'''(\phi_0(z))\psi(z)+\Gamma''(\phi_0(z))\big)\int dy\left(g_0(z,y,t)-\frac{3}{4}\phi_0'(z)\phi_0'(y)\right)$$

$$\times\int_0^t ds\int dx\,V_0'''(\phi_0(x))\phi_0'(x)g_0(x,y,t-s)g_0(x,z,s)=$$

$$2\int dz\,\big(V_0'''(\phi_0(z))\psi(z)+\Gamma''(\phi_0(z))\big)\int dy\left(g_0(z,y,t)-\frac{3}{4}\phi_0'(z)\phi_0'(y)\right)$$

$$\times\left(\frac{\partial}{\partial z}g_0(z,y,t)+\frac{\partial}{\partial y}g_0(z,y,t)\right) \tag{3.18}$$

Next, recall that

$$\int dy\left(g_0(z,y,t)-\frac{3}{4}\phi_0'(z)\phi_0'(y)\right)\frac{\partial}{\partial z}g_0(z,y,t)$$

$$=\frac{\partial}{\partial z}\int dy\frac{g_0(z,y,t)}{2}\left(g_0(z,y,t)-\frac{3}{4}\phi_0'(z)\phi_0'(y)\right)=\frac{1}{2}\frac{\partial}{\partial z}\left(g_0(z,z,2t)-\frac{3}{4}\phi_0'(z)^2\right) \tag{3.19}$$

and

$$\int dy\left(g_0(z,y,t)-\frac{3}{4}\phi_0'(z)\phi_0'(y)\right)\frac{\partial}{\partial y}g_0(z,y,t)=\frac{3}{4}\phi_0'(z)\int dy\,\phi_0''(y)g_0(z,y,t). \tag{3.20}$$

From (3.18), (3.19) and (3.20), we obtain for I_1 (see (3.9))

$$
\begin{aligned}
I_1 = 2\int_0^\infty dt \int dz\, \left(\psi(z)V_0'''(\phi_0(z)) + \Gamma''(\phi_0(z))\right) \\
\times \left[\frac{1}{2}\frac{\partial}{\partial z}\left(g_0(z,z,2t) - \frac{3}{4}\phi_0'(z)^2\right) + \frac{3}{4}\phi_0'(z)\int dy\, \phi_0''(y) g_0(z,y,t)\right] \\
- \frac{3}{2}\int_0^\infty dt \int dx \int dy\, \phi_0'(x)^2 V'''(\phi_0(x))\psi'(y)\left(g_0(x,y,t) - \frac{3}{4}\phi_0'(x)\phi_0'(y)\right).
\end{aligned}
\tag{3.21}
$$

Turning now to I_2 , we integrate (3.10) with respect to y, using the semigroup property, and then integrate by parts with respect to x. This yields

$$
I_2 = -\int_0^\infty dt \int dz\, \left(\psi(z)V_0'''(\phi_0(z)) + \Gamma''(\phi_0(z))\right)\frac{\partial}{\partial z}\left(g_0(z,z,2t) - \frac{3}{4}\phi_0'(z)^2\right) \tag{3.22}
$$

Recalling that $\int dy\, \phi_0''(y)\phi_0'(y) = 0$, inserting (3.21) and (3.22) into (3.8), one gets (3.2). □

We need to simplify (3.2). To do so, we integrate first with respect to t.

3.2 Proposition.

The function

$$
G(x,y) = \int_0^\infty dt\, \left(g_0(x,y,t) - \frac{3}{4}\phi_0'(x)\phi_0'(y)\right)
$$

is the generalized Green function for the operator

$$
L_0 = \frac{1}{2}\frac{d^2}{dx^2} - V_0(\phi_0(x))
$$

acting on the functions bounded at $\pm\infty$.

Remark. *Since* 0 *is an eigenvalue for* L_0*, the Green function for* L_0 *does not exist. The generalized Green function permits to invert* L_0 *in the subspace orthogonal to that generated by the eigenfunctions corresponding to* 0*, in our case,* ϕ_0'*. (See pp 353–357 of [5] for details.)*

Proof of Proposition 3.2.

According to [5], it is enough to check that it is continuous as a function of (x,y), of class C^2 if $x \neq y$, bounded at $\pm\infty$ as a function of x for each fixed y, and:

$$
L_0 G = \frac{3}{4}\phi_0'(x)\phi_0'(y) \quad \text{if } x \neq y \tag{3.23}
$$

$$
\left.\frac{dG}{dx}\right|_{x=y-0}^{x=y+0} = -2 \tag{3.24}
$$

$$
\int dx\, G(x,y)\phi_0'(x) = 0, \tag{3.25}
$$

and this is easy.

3.3 Proposition.

The generalized Green function for the operator L_0 (with the boundary conditions stipulated above) is:

$$G(x,y) = \begin{cases} \frac{3}{4}\left(\phi_0'(y)u_0(x) + \phi_0'(x)u_1(y) + \frac{5}{2}\phi_0(x)\phi_0'(y)\right) & \text{if } x \leq y \\ \frac{3}{4}\left(\phi_0'(y)u_1(x) + \phi_0'(x)u_0(y) + \frac{5}{2}\phi_0(x)\phi_0'(y)\right) & \text{if } x \geq y, \end{cases} \tag{3.26}$$

where u_0 and u_1 are defined below in (3.29) *and* (3.28) *respectively.*

Proof.

To obtain the Green function we need to solve

$$L_0 u = \phi_0' \tag{3.27}$$

If we suppose that $u = \phi_0' F$ for some function F, using that $\phi_0'' = 2V_0'(\phi_0)$, we get

$$(F' {\phi_0'}^2)' = 2{\phi_0'}^2,$$

which can be integrated twice, yielding

$$u(x) = \phi_0'(x) \int_0^x dy \left(\frac{c}{{\phi_0'}^2(y)} + \frac{2}{{\phi_0'}^2(y)} \int_0^y dz\, {\phi_0'}^2(z) \right),$$

where c is a constant that we will choose conveniently. Using (3.14), we have that

$$u(x) = \phi_0'(x) \int_0^x dy \left(\frac{c + 2\phi_0(y) - 2/3\phi_0^3(y)}{{\phi_0'}^2(y)} \right)$$

Taking $c = -4/3$, we get a solution of (3.27), which is bounded at $+\infty$. Let us call it u_1.

$$u_1(x) = \frac{2}{3}\phi_0'(x) \int_0^x dy \left(\frac{\phi_0'\phi_0(y) - 2(1-\phi_0(y))}{{\phi_0'}^2(y)} \right)$$

Moreover, after a change of variables, and performing the integrals,

$$\begin{aligned} u_1(x) &= \frac{2}{3}\phi_0'(x) \int_0^{\phi_0(x)} \frac{w\,dw}{(1-w^2)^2} - \frac{4}{3}\phi_0'(x) \int_0^{\phi_0(x)} \frac{dw}{(1-w)^2(1+w)^3} \\ &= \frac{e^{-2x}}{6} + \frac{1}{2} - \frac{2}{3}\phi_0'(x) - \frac{\phi_0(x)}{2} - \frac{x\phi_0'(x)}{2} \end{aligned} \tag{3.28}$$

Clearly, the function $u_0(x) = u_1(-x)$ is a solution of (3.27) bounded at $-\infty$:

$$u_0(x) = \frac{e^{2x}}{6} + \frac{1}{2} - \frac{2}{3}\phi_0'(x) + \frac{\phi_0(x)}{2} + \frac{x\phi_0'(x)}{2}. \tag{3.29}$$

Now, consider the function $\tilde{G}(x,y)$

$$\tilde{G}(x,y) = \begin{cases} \frac{3}{4}\big(\phi_0'(y)u_0(x) + \phi_0'(x)u_1(y)\big) \text{ if } x \leq y \\ \frac{3}{4}\big(\phi_0'(y)u_1(x) + \phi_0'(x)u_0(y)\big) \text{ if } x \geq y \end{cases}$$

It is easy to see that $\tilde{G}$ is continuous in $(x,y), C^\infty$ if $x \neq y$, and satisfies (3.23). For (3.24), we have

$$\frac{d\tilde{G}(x,y)}{dx}\Big|_{x=y-0}^{x=y+0} = \frac{3}{4}\big[\phi_0'(y)(u_1 - u_0)'(y) - \phi_0''(y)\big(u_1(y) - u_0(y)\big)\big].$$

Since $L_0\phi_0' = L_0(u_1 - u_0) = 0$, the previous expression is constant. Evaluating it for instance at 0, we get easily (3.24) from (3.28) and (3.29).

Then, observe that if we take

$$G(x,y) = \tilde{G}(x,y) + \phi_0'(x)A(y), \tag{3.30}$$

this G satisfies (3.23), (3.24) and the regularity conditions prescribed, for any regular A. To obtain the generalized Green function we must choose A such that (3.30) satisfies (3.25). This gives

$$A(y) = -\frac{\int dx\, \tilde{G}(x,y)\phi_0'(x)}{\int dx\, \phi_0'(x)^2} = -\frac{3}{4}\int dx\, \tilde{G}(x,y)\phi_0'(x). \tag{3.31}$$

It is easy to see from the definitions that $A(y) \to 0$ as $y \to \pm\infty$, and also that $L_0 A = 0$, so it has to be $A(y) = a\phi_0'(y)$ for some constant a. To evaluate a, let us compute $-3/4\int dx\, \tilde{G}(x,0)\phi'(x)$. Since $u_0(0) = u_1(0) = 0$ and $u_0(x) = u_1(-x)$,

$$\begin{aligned}\int dx\, \tilde{G}(x,0)\phi_0'(x)dx = \frac{3}{2}\Bigg[&\frac{1}{6}\int_0^\infty dx\, e^{-2x}\phi_0'(x) + \frac{1}{2}\int_0^\infty dx\, \phi_0'(x) \\ &- \frac{2}{3}\int_0^\infty dx\, \phi_0'(x)^2 - \frac{1}{2}\int_0^\infty dx\, \phi_0'(x)\phi_0(x) - \frac{1}{2}\int_0^\infty x\phi_0'(x)^2\Bigg]\end{aligned} \tag{3.32}$$

Let us compute the previous integrals. Recalling that $\phi_0 = \tanh$, and $\phi_0' = 1 - {\phi_0}^2$,

$$\begin{aligned}
&\frac{1}{6}\int_0^\infty dx\, e^{-2x}\phi_0'(x) = \frac{1}{3}\int_0^1 du\left(\frac{1}{1+u} - \frac{1}{(1+u)^2}\right) = \frac{1}{3}\log 2 - \frac{1}{6} \\
&\frac{1}{2}\int_0^\infty dx\, \phi_0'(x) = \frac{1}{2} \\
&\frac{2}{3}\int_0^\infty dx\, \phi_0'(x)^2 = \frac{2}{3}\int_0^\infty dx\, \phi_0'(x)(1 - \phi_0(x)^2) = \frac{4}{9} \\
&\frac{1}{2}\int_0^\infty dx\, \phi_0'(x)\phi_0(x) = \frac{1}{4} \\
&\frac{1}{2}\int_0^\infty dx\, x\phi_0'(x)^2 = \frac{x\sinh(x)}{6\cosh^3(x)} + \frac{1}{12\cosh(x)} \\
&+ \frac{1}{3}\left(\frac{-2xe^{-x}}{\cosh(x)} + \log 2 - \log(1 + e^{-2x})\right)\Big|_0^\infty = \frac{1}{3}\log 2 - \frac{1}{12}.
\end{aligned}$$

Putting all this together, we have

$$\int dx\, \tilde{G}(x,0)\phi_0'(x) = -\frac{5}{2},$$

which, from (3.30), (3.31) and the discussion above yields (3.26).

Now, we are able to simplify (3.2).

Proof of (1.14).

The expression (3.2) in terms of G is

$$\begin{aligned}\frac{d\beta}{d\lambda}(0) =& \frac{9}{16}\int dx \int dy\, \big(\psi(x)V_0'''(\phi_0(x)) + \Gamma''(\phi_0(x))\big)\phi_0'(x)\phi_0''(y)G(x,y)\\ &- \frac{9}{16}\int dx \int dy\, \phi_0'(x)^2 V_0'''(\phi_0(x))\psi'(y)G(x,y)\end{aligned} \tag{3.33}$$

From (3.26), after the computation of the corresponding integrals, (that we do not report here), one obtains

$$\int dy\, \phi_0''(y)G(x,y) = -x\phi_0'(x) \tag{3.34}$$

and

$$\int dx\, \phi_0'(x)^2 V_0'''((\phi_0(x))G(x,y) = -\phi_0''(y) \tag{3.35}$$

Recall that $x\phi_0'(x)$ is orthogonal to $\phi_0'(x)$, and that $L_0(x\phi_0'(x)) = \phi_0''(x)$. Then, from Proposition 3.2 and the subsequent Remark, (3.34) has to be true. The same reasoning applies to (3.35). From (3.34) and (3.35) we have

$$\frac{d\beta}{d\lambda}(0) = -\frac{9}{16}\Big(\int dx\, \big(\psi(x)V_0'''(\phi_0(x)) + \Gamma''(\phi_0(x))\big)x\phi_0'(x)^2 - \int dx\, \psi'(x)\phi_0''(x)\Big) \tag{3.36}$$

Then, from (3.3), it is not difficult to compute

$$\int dx\, \psi'(x)\phi_0''(x) = \frac{4}{15}\int_0^\infty dx \frac{\Gamma(\phi_0(x))}{\phi_0'(x)^2}(1-\phi_0(x))^2\left(4-7\phi_0(x)+18\phi_0(x)^2-9\phi_0(x)^3\right), \tag{3.37}$$

$$\int \Gamma''(\phi_0(x))x\phi_0'(x)^2 = -2\int dx\, \frac{\Gamma(\phi_0(x))}{\phi_0'(x)^2}\big(\phi_0(x) + x\phi_0'(x)\big)\left(1-\phi_0(x)^2\right)^2 \tag{3.38}$$

and finally, using (3.3) and $V_0'''(u) = 6u$,

$$\begin{aligned}&\int dx\, V_0'''(\phi_0(x))\psi(x)x\phi_0'(x)^2 = 12\int dx\, \phi_0(x)x\phi_0'(x)^3 \int_0^x dy\, \frac{\Gamma(\phi_0(y))}{\phi_0'(y)^2}\\ &= 12\int_0^\infty dy\, \frac{\Gamma(\phi_0(y))}{\phi_0'(y)^2}\\ &\times\left[y\left(\frac{1}{6} - \frac{\phi_0(y)^2}{2} + \frac{\phi_0(y)^4}{2} - \frac{\phi_0(y)^6}{6}\right) + \frac{4}{45} - \frac{\phi_0(y)^5}{30} - \frac{\phi_0(y)}{6} + \frac{\phi_0(y)^3}{9}\right]\end{aligned} \tag{3.39}$$

Substituting (3.37), (3.38) and (3.39) in (3.36), and changing variables, one gets (1.14).

3.4 Remark.

The Green function $G(x, y)$ is only used to compute (3.34) and (3.35), so one could have only written these two formulas, that can be checked (once one knows them!) by observing that they satisfy the corresponding equations plus the orthogonality conditions. We think however that it is worth exhibiting the explicit form, as well as the derivation of G, since it seems to be not known, and may be useful in another contexts.

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References

[1] S. Brassesco, *Stability of the instanton under small random perturbations*, Stoch. Proc. Appl. **54** (1994), 309–330.

[2] S. Brassesco, A. De Masi, E. Presutti, *Brownian fluctuations of the interface in the d=1 Ginzburg-Landau equation with noise*, Ann. Inst. H. Poincaré B **31** (1995).

[3] S. Brassesco, P. Buttá, A. De Masi, E. Presutti, *Interface fluctuations and couplings in the d=1 Ginzburg-Landau equation with noise*, to appear in Journal of Theoretical Probability (1997).

[4] J. Carr, B. Pego, *Metastable patterns in solutions of $u_t = \varepsilon^2 u_{xx} + u(1-u^2)$*, Commun. Pure Applied Math. **42** (1989), 523–576.

[5] R. Courant, D. Hilbert, *Methods for mathematical phisiscs*, Intercience Publishers Inc, New York (1953).

[6] A. De Masi, E. Presutti, *Mathematical methods for hydrodynamical limits,*, Lecture Notes in Mathematics **1501** (1991), Springer-Verlag.

[7] C.R. Doering, *Nonlinear parabolic stochastic differential equations with additive colored noise on $\mathbb{R}^d \times \mathbb{R}_+$: a regulated stochastic quantization.*, Comm. Math. Phys. **109** (1987), 537–561.

[8] W. Faris, G. Jona Lasinio, *Large fluctuations for a nonlinear heat equation diffusion equation with noise*, J. Phys. **A 15** (1982), 3025–3055.

[9] P. Fife, J.B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Rat. Mech. Anal. **65** (1977), 335–361.

[10] ———, *A phase plane discussion of convergence to travelling fronts for nonlinear diffusion*, Arch. Rat. Mech. Anal. **75** (1981), 281–314.

[11] T. Funaki, *The scaling limit for a stochastic PDE and the separation of phases*, Prob. Theory Relat. Fields **102** (1995), 221–288.

[12] G. Fusco, J. Hale, *Slow-motion manifolds, dormant instability and singular perturbations*, J. Dynamics Differential equations **1** (1989), 75–94.

[13] J.B. Walsh, *An introduction to stochastic partial differential equations*, Lecture Notes in Mathematics **1180** (1984), Springer-Verlag, 265–437.

S. Brassesco, Instituto Venezolano de Investigaciones Científicas, Apartado Postal 21827, Caracas 1020-A, Venezuela
E-mail address: `sbrasses@ivic.ivic.ve`

P. Buttà, Center for Mathematical Sciences Research, Rutgers, the State University of New Jersey, 110 Frelinghuysen rd, Piscataway NJ 08854–8019, USA
E-mail address: `pbutta@math.rutgers.edu`