Upper bounds for quantum dynamics governed by Jacobi matrices with self-similar spectra

I. Guarneri, H. Schulz-Baldes

International Centre for the Study of Dynamical Systems,

Università di Milano, sede di Como, via Lucini 3, 22100 Como, Italy

Instituto Nazionale di Fisica della Materia, Unitã di Milano, Via Celoria 16, 20133 Milano, Italy

Abstract

We study a class of one-sided Hamiltonian operators with spectral measures given by invariant and ergodic measures of dynamical systems of the interval. We analyse dimensional properties of spectral measures, and prove upper bounds for the asymptotic spread in time of wavepackets. These bounds involve the Hausdorff dimension of the spectral measure, multiplied by a correction calculated from the dynamical entropy, the density of states, and the capacity of the support. For Julia matrices, the correction disappears and the growth is ruled by the fractal dimension.

1 Introduction.

One-particle Schrödinger operators with almost-periodic potentials display a rich variety of spectral types, including singular continuous spectra. In the latter situation, there is compelling numerical evidence [14, 8, 23, 12, 20] that transport is typically sub-ballistic and anomalous. More precisely, this means that the second moment of the position operator asymptotically grows in time as $t^{2\beta}$ where β can take any value in $[0, 1]$ depending on the specific model and on parameter values. Determining the transport exponent β from the Hamiltonian is an interesting and important task both from the mathematical and the physical viewpoint. Concerning the latter, let us point out that the metal-insulator transition in quasicrystals occurs by anomalous transport [13] and furthermore that the transport exponent enters in the anomalous Drude formula [25].

On a rigorous level, some connections have been established between dimensional properties of the local density of states (LDOS) and asymptotic transport properties. The asymptotic decay of the time-averaged staying probability at the initial site is ruled by the correlation dimensions of the LDOS [8, 15, 2, 19, 25]. As to the growth of the αth moment of the position operator, a general argument bounds it below by $\alpha d/D$ where d is the Hausdorff dimension of the LDOS and D the dimension of physical space [9, 10, 19, 2, 25]. Obtaining upper bounds and sharp estimates appears to require more detailed information than just the dimensional properties of the LDOS. Improved lower bounds involving the structure of eigenfunctions have been heuristically derived and numerically verified in [17].

In general, there cannot be a tight connection between dimensional properties and the behavior of moments of the position operator; for example, there are models with spectra of arbitrarily small Hausdorff dimension and ballistic ($\beta = 2$) propagation [18], and also models with zero-dimensional spectra which display ballistic motion on arbitrarily large time scales [19, 18, 11]. On the other hand, information about eigenfunctions may be redundant at least in the case of Jacobi matrices, whose generalized eigenfunctions are given by the orthogonal polynomials of the LDOS.

When investigating such issues on a given one-particle Hamilton operator, one is actually confronted with two different problems. First, one has to analyse the spectral measures; then, as a second step, comes the derivation of transport properties. These two problems are in principle different and have to be tackled separately. It is convenient to circumvent the first problem by constructing the Hamilton operator as the Jacobi matrix associated with a LDOS prescribed in advance as self-similar fractal measure. This strategy has already lead to numerical results [12] and to further numerical-theoretical analysis [20]. By the very same strategy, we obtain in this paper rigorous upper bounds for transport in purely spectral terms. As a prototype class of self-similar measures, we consider measures constructed by non-linear, disjoint iterated function systems (IFS). The spectral analysis of such measures can be performed relatively easily: the Hausdorff dimension of the measure is given by the quotient of the dynamical entropy and the Lyapunov exponent of the associated dynamical system. The corresponding Jacobi matrices are believed to be almost-periodic [20], but a rigorous proof only exists for the special case of Julia matrices, which are known to be limit-periodic [4].

The basic quantity considered here for the study of quantum transport is the minimal carrier of a wave packet originally localized at the origin. It is defined, at all times, as the radius of the smallest ball centered at the origin and carrying a fixed percentage of the time-averaged presence probability at the given time. Upper bounds on its algebraic growth in time translate into upper bounds for the growth of other quantities measuring the spatial extent of wave packets, such as inverse participation ratios [10], entropic widths [10] and (inverse) negative moments of the position operator; however, positive moments are out of reach.

In our approach the dynamical spreading of wavepackets over increasing time scales is controlled by an appropriate *renormalization dynamics* which resolves the spectrum on accordingly decreasing energy scales. It is exactly the existence of a dynamical system generating the spectrum which makes the renormalization dynamics accessible. Our upper bound for the growth exponent of the minimal carrier is then given by the Hausdorff dimension of the spectral measure multiplied by a correction factor calculated from the dynamical entropy, the density of states (DOS) and the (logarithmic) capacity of its fractal support. The DOS enters our estimates because it controls the exponential growth of orthogonal polynomials via a formula of the Herbert-Jones-Thouless type. This is the only information about eigenfunctions we use.

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2 Statement of the results.

In this chapter, we introduce the operators studied in this work, quickly review some general facts from the theory of orthogonal polynomials, and finally state our main results.

2.1 Spectral measures.

The self-similar fractal measures considered in this work are constructed by non-linear, disjoint iterated function systems (IFS) [16]. The construction is as follows. Let $I_1^1 < I_2^1 < \ldots < I_L^1$ be a finite sequence of pairwise disjoint closed intervals all contained in a closed interval I^0. Let S be a smooth real function such that, for all $l = 1 \ldots L$, the restriction S_l of S to I_l^1 is bijective from I_l^1 to I^0 with smooth inverse S_l^{-1} (in particular, we assume the derivative of S to be bounded away from 0 and ∞ on the intervals I_l^1). We call *codes* the one-sided sequences of symbols taken from $\{1, \ldots, L\}$ and denote the set of codes of length N by Σ_L^N and the set of codes of infinite length by Σ_L. For all $N \in \mathbf{N}$, $S^{-N}(I^0)$ consists of L^N closed, disjoint intervals $I_\sigma^N = S_{\sigma_1}^{-1} \circ \ldots \circ S_{\sigma_N}^{-1}(I^0)$, $\sigma = (\sigma_1 \ldots \sigma_N) \in \Sigma_L^N$, which we call the intervals of the Nth generation.

We further assume that there exist positive constants $a < 1$ and b so that, for any $N \in \mathbf{N}$, all intervals of Nth generation satisfy $|I_\sigma^N| \leq b\, a^N$ where $|I|$ denotes the length of the interval I. A non-negative (rescaled) Schwarzian derivative of S is a sufficient condition for this basic contraction property to hold (see Misiurewicz' theorem in [7]). We note that the contraction property can hold even if the S_l's are not everywhere contractive, but are locally expansive.

Now $J = \bigcap_{N \geq 0} S^{-N}(I^0)$ is a fractal set which is invariant under S, *i.e.* $S(J) = J$. The dynamical system (J, S) is conjugated to the shift on Σ_L by the coding map $E \in J \mapsto \sigma(E) \in \Sigma_L$. The associated pullback allows to construct invariant, ergodic measures for the dynamical system (J, S) as pre-images of shift-invariant, ergodic measures on Σ_L. For measures μ on J obtained in this way, the pointwise dimensions

$$d_\mu(E) \;=\; \lim_{\epsilon \to 0} \frac{\log(\mu([E - \epsilon, E + \epsilon]))}{\log(\epsilon)} \tag{1}$$

exist μ-almost surely and are μ-almost surely equal to the information or Hausdorff dimension $\dim_{\mathrm{H}}(\mu)$ of μ (see Appendix 1 for a proof and references to the literature). Moreover,

$$\dim_{\mathrm{H}}(\mu) \;=\; \frac{\mathcal{E}(\mu)}{\Lambda(\mu)} \tag{2}$$

where $\mathcal{E}(\mu)$ and $\Lambda(\mu)$ are the dynamical (Kolmogorov-Sinai) entropy and the Lyapunov exponent of the dynamical system (J, S, μ).

We shall always suppose that the sequences $(\log(\mu(I_\sigma^N))/N)_{N \in \mathbf{N}}$ are bounded from below, uniformly in $\sigma \in \Sigma_L$. This assumption holds true in all cases explicitely considered below.

2.2 Jacobi matrices.

Once the measure μ on J is fixed, we construct the Hamiltonian as the Jacobi matrix of μ. Let P_n, $n \geq 0$, denote the orthogonal and normalized polynomials associated to μ. They are a Hilbert basis $\mathcal{B} = (P_n)_{n \in \mathbf{N}}$ in $L^2(\mathbf{R}, \mu)$ and satisfy a three term recurrence relation $EP_n(E) = t_{n+1}P_{n+1}(E) + v_nP_n(E) + t_nP_{n-1}(E)$, $n \geq 1$, and $EP_0(E) = t_1P_1(E) + v_0P_0(E)$

where $v_n \in \mathbf{R}$ and $t_n \geq 0$ are bounded sequences. Therefore the isomorphism of $L^2(\mathbf{R}, \mu)$ onto $\ell^2(\mathbf{N})$ associated with the basis $\mathcal{B}$ carries the operator of multiplication by E in $L^2(\mathbf{R}, \mu)$ into the self-adjoint finite difference operator H defined on $\ell^2(\mathbf{N})$ by:

$$H|n\rangle \;=\; t_{n+1}|n+1\rangle + v_n|n\rangle + t_n|n-1\rangle \;, \qquad n \geq 1 \;, \tag{3}$$

and $H|0\rangle = t_1|1\rangle + v_0|0\rangle$. Then μ is the spectral measure of H associated to $|0\rangle$, also called its local density of states (LDOS). That means,

$$\langle 0|f(H)|0\rangle \;=\; \int_J f(E)\, d\mu(E)$$

for all continuous functions f. The vector $|0\rangle$ is a cyclic vector for H, therefore the measure μ dominates all spectral measures of H associated to other states in Hilbert space.

The second important measure associated with the operator H is the density of states (DOS). It is defined by

$$\mathcal{N} \;=\; \text{w-}\lim_{n\to\infty} \frac{1}{n} \sum_{E,\, P_n(E)=0} \delta_E \;, \tag{4}$$

whenever the weak limit of point measures appearing on the right-hand side exists. Conditions for this and interesting consequences thereof follow from the work of Stahl and Totik [26] which we shall review next.

2.3 Orthogonal polynomials.

We first need to review some basic notions from (logarithmic) potential theory (see, *e.g.*, [1] or the appendix of [26]). The (logarithmic) capacity $\text{cap}(K)$ of a compact set $K \subset \mathbf{R}$ is defined by

$$\log(\text{cap}(K)) \;=\; \sup \left\{ \int d\nu(E) \int d\nu(E') \, \log(|E-E'|) \;\middle|\; \text{supp}(\nu) \subset K \right\} \;,$$

where the supremum is taken over all Borel probability measures ν carried by K. The capacity of a Borel set Y is given by the supremum over the capacities of compact sets contained in Y. If $\text{cap}(Y) > 0$, then there exists a unique Borel measure ω_Y for which the supremum is attained. It is called the Forstmann equilibrium measure. Next, the Green's function of Y with pole at infinity is the (unique) non-negative subharmonic function in $\mathbf{C}$, harmonic in $\mathbf{C}\backslash(Y \cup \{\infty\})$, and satisfying $g_Y(E) = 0$ for all $E \in Y\backslash Y'$ where $\text{cap}(Y') = 0$. If $\text{cap}(Y) > 0$, then

$$g_Y(z) \;=\; \int d\omega_Y(E) \log(|E-z|) \;-\; \log(\text{cap}(Y)) \;. \tag{5}$$

We shall verify in Chapter 4 that the capacity of the sets J in Section 2.1 is positive and that any of the self-similar measures μ on J is asymptotically regular in the sense of Stahl and Totik [26] (see Appendix 2 for a definition). By the results of [26] stated in Appendix 2, these facts imply the following.
(i) The DOS is well defined and coincides with the Forstmann equilibrium measure of J:

$$\mathcal{N} \;=\; \omega_J \;. \tag{6}$$

(ii) The capacity can be calculated by the formula

$$\log(\mathrm{cap}(J)) \;=\; \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log(t_j) \;. \tag{7}$$

(iii) If we introduce the transfer matrices (by convention, $t_0 = 0$)

$$\Phi_n(z) \;=\; A_n(z)\cdots A_0(z) \;, \qquad A_j(z) \;=\; \begin{pmatrix} (z-v_j)/t_{j+1} & -t_j/t_{j+1} \\ 1 & 0 \end{pmatrix} \;,$$

then the following Herbert-Jones-Thouless type formula holds:

$$\lim_{n\to\infty} \frac{1}{n} \log(\|\Phi_n(z)\|) \;=\; g_J(z) \;, \tag{8}$$

where the convergence is locally uniform in $\mathbf{C}\backslash I^0$ (a sequence of functions f_n converges locally uniformly in an open set $G \subset \mathbf{C}$ to a function f if for all $z \in G$ and sequences $z_n \to z$, one has $f_n(z_n) \to f(z)$). Moreover, (8) with the equality replaced by $\leq$ holds locally uniformly in $\mathbf{C}$. Thus g_J can be seen as the Lyapunov exponent of the one-dimensional lattice Hamiltonian H. Moreover, the spectrum is characterized as the set of points where this Lyapunov exponent vanishes (except for a set of vanishing capacity). Note that the main modification with respect to the usual Herbert-Jones-Thouless formula with discrete Laplacian ($t_n = 1$ for all $n \geq 1$) is the capacity term in (5).

2.4 Growth exponents.

Under the dynamics $e^{-\imath tH}$, a wave packet initially localized on the state $|0\rangle$ spreads out over the basis $\mathcal{B}$. To study this spreading, we introduce the *minimal carrier* as

$$\overline{n}(\epsilon, T) \;=\; \min\left\{ \overline{n} \in \mathbf{N} \;\middle|\; \sum_{n\geq\overline{n}} \int_0^T \frac{dt}{T} |\langle n|e^{-\imath Ht}|0\rangle|^2 \leq \epsilon \right\} \;,$$

and the corresponding growth exponents:

$$\beta_0^+(\epsilon) \;=\; \limsup_{T\to\infty} \frac{\log(\overline{n}(\epsilon,T))}{\log(T)} \;, \qquad \beta_0^-(\epsilon) \;=\; \liminf_{T\to\infty} \frac{\log(\overline{n}(\epsilon,T))}{\log(T)} \;, \tag{9}$$

as well as $\beta_0^\pm = \lim_{\epsilon\to 0} \beta_0^\pm(\epsilon)$. These transport exponents are linked to the growth exponents of the moments of the position operator in the basis $\mathcal{B}$. For $\alpha \neq 0$, the latter are defined by

$$\beta_\alpha^+ \;=\; \limsup_{T\to\infty} \frac{\log\left(\sum_{n\geq 0} n^\alpha\, p_n(T)\right)}{\log(T^\alpha)} \;, \qquad \beta_\alpha^- \;=\; \liminf_{T\to\infty} \frac{\log\left(\sum_{n\geq 0} n^\alpha\, p_n(T)\right)}{\log(T^\alpha)} \;. \tag{10}$$

where $p_n(T) \;=\; \int_0^T dt\, |\langle n|e^{-\imath Ht}|0\rangle|^2/T$. We have $\beta_\alpha^+ \;\leq\; \beta_0^-$ whenever $\alpha \;<\; 0$ and $\beta_\alpha^- \;\geq\; \beta_0^+$ whenever $\alpha > 0$. Furthermore [10], β_0^- is an upper bound for the entropic growth exponent in the basis $\mathcal{B}$ defined by

$$\beta_{\mathrm{ent}}^+ \;=\; \limsup_{T\to\infty} \frac{-\sum_{n\geq 0} p_n(T)\log(p_n(T))}{\log(T)} \;.$$

Finally, we refer the reader to [10] for a definition of inverse participation ratios and a proof that also their growth exponents are bounded from above by β_0^-.

The lower bound $\beta_0^- \geq \dim_{\rm H}(\mu)$ holds for any self-adjoint H and for any Hilbert basis $\{|n\rangle\}$ [9, 10, 19, 2]. In addition, for operators in the class (3), we have the *a priori* ballistic upper bound $\beta_0^+ \leq 1$. One way of deriving this is observing that (3) is a bounded operator in the Banach space X_α ($\alpha > 0$) of $\ell^2(\mathbf{N})$-vectors ψ such that $\|\psi\|_\alpha = \sup_n |\langle\psi|n\rangle| \exp(\alpha n) < \infty$. Then at any time t, the minimal carrier can be estimated from $|\langle n| \exp(-iHt)|0\rangle|^2 \leq \exp(-2\alpha n + 2t\|H\|_\alpha)$, which directly yields the ballistic bound.

2.5 Main results.

Our main results give upper bounds on the exponent β_0^+.

Theorem 1 *Let S be an analytic map and let $R_c = |I^0|/2$ denote the spectral radius and E_c the center of the spectrum. Then*

$$\beta_0^+ \;\leq\; \frac{1}{\Lambda(\mu)} \log\left(\inf_{R>R_c} \frac{\max_{z\in\Gamma_R}\; g_J(S(z))}{\log(R)-\log(R_c)}\right)\;, \tag{11}$$

where Γ_R is the circle of radius R around E_c.

Theorem 2 *For any $l = 1\dots L$, let the branch S_l have an analytic continuation $\hat{S}_l$ given by a polynomial map of degree D_l. We set $D = \max_{l=1\dots L} D_l$ and $\gamma = \mathrm{arcsinh}(\Delta(4R_c)^{-1})$, where $R_c = |I^0|/2$ is the spectral radius and Δ is the size of the smallest gap at the first generation. Then*

$$\beta_0^+ \;\leq\; \frac{1}{\Lambda(\mu)} \log\left(\frac{1}{\gamma} \max_{l=1\dots L}\; \sup_{E\in J}\; g_J(\hat{S}_l(E)) + D\right)\;. \tag{12}$$

Remark 1 When combined with the rigorous lower bound $\beta_0^- \geq \dim_{\rm H}(\mu)$ from [9, 10, 19, 2], these results prove that quantum transport in presence of singular continuous spectra can actually be anomalous (as it will become more explicit in the applications presented below). To our knowledge, the only other examples of quantum models in which the motion can be rigorously shown to be other than ballistic or localized are the following: for finite rank perturbation of a localized model, the moments of the position operator diverge at most logarithmically [24]; next, less pathological, the quantum motion is diffusive in the Anderson model with free random variables [25] (*i.e.* the growth exponent of the disorder averaged second moment of the position operator equals 1); the latter hamiltonian model can be identified with the coherent potential approximation of the usual Anderson model, and a special case of it is the Wegner n-orbital model in the limit $n \to \infty$ [21].

Remark 2 Neither of the bounds (11) and (12) is optimal in general. In the case of (12) this becomes particularly evident when J is a linear Cantor set, see Remark 4 below. Our present proof may possibly be improved at several places; nevertheless, it yields optimal results in the case of Julia sets, as shown in Section 2.6 below.

Remark 3 Using equation (2), the above bounds can be written as the product of the Hausdorff dimension of the measure times a correction factor involving the DOS, the dynamical entropy, and the capacity.

2.6 Applications and comments.

One class of shift-invariant, ergodic measures on Σ_L are the measures for which the σ_n, $n \in \mathbf{N}$, are independent random variables with same distribution $\mathrm{Prob}\{\sigma_n = l\} = p_l$, $l = 1 \dots L$, $\sum_{l=1}^{L} p_l = 1$. The corresponding measures μ on J will be called Bernoulli measures with weights p_l. For such measures, $\mathcal{E}(\mu) = -\sum_i p_i \log p_i$. Our present applications only consider this class.

First, we treat Julia matrices. A real Julia set is the fractal J associated to a polynomial mapping S of degree L satisfying the hypothesis imposed in the construction of J. Putting this into (11) and (5), we find that $\max_{z\in\Gamma_R} g_J(S(z)) \sim L\log(R)$ at large R. Taking the limit $R \to \infty$, we get the upper bound $\log(L)/\Lambda(\mu) = \dim_{\mathrm{H}}(\mu)\log(L)/\mathcal{E}(\mu)$ because of equation (2). The correction factor with respect to the Hausdorff dimension disappears if μ is the maximal entropy measure, for which $\mathcal{E}(\mu) = \log(L)$. This is the the Bernoulli measure with equal weights; the corresponding Jacobi matrices are called Julia matrices. Recalling the general lower bound $\beta_0^- \geq \dim_{\mathrm{H}}(\mu)$ [10], we obtain:

Corollary 1 *If μ is the Bernoulli measure with equal weights on a real Julia set J, then*

$$\beta_0^- \;=\; \beta_0^+ \;=\; \dim_{\mathrm{H}}(\mu)\;.$$

Thus the dimensional properties of the LDOS completely determine the transport properties in Julia matrices. Formal arguments and numerical verifications go actually farther [20]: the scaling exponents $\beta_\alpha^\pm$ defined in (10) satisfy $\beta_\alpha^\pm = D_{1-\alpha}$ for $\alpha > 0$ where $(D_q)_{q\in\mathbf{R}}$ denotes the family of multifractal dimensions. The main reason is presumably the exact renormalization property of the orthogonal polynomials ($\langle n|m \circ S\rangle = \delta_{n,Lm}$ [3]). Our proof makes no use of this fact; however, using it considerably simplifies the proof, as explained in Remark 5 in Section 3.2. Nor do we use the fact that the LDOS and the DOS of Julia matrices coincide (*i.e.*, the Forstmann equilibrium measure is the Bernoulli measure with equal weights [6]).

As a second example, we consider the usual Cantor set $\mathcal{C}$ in $[0,1]$ where $\hat{S}_1(E) = \Lambda E$ and $\hat{S}_2(E) = \Lambda - \Lambda E$ with $\Lambda > 2$. Let us replace this in (12). We first note the maximum over branches becomes irrelevant because of symmetry, next that the supremum over $E \in J = \mathcal{C}$ is actually taken when E is the outer border of the Cantor set. Therefore we obtain from (12):

$$\beta_0^+ \;\leq\; \frac{1}{\log(\Lambda)}\;\log\left(\left(\operatorname{arcsinh}\left(\frac{\Lambda-2}{4}\right)\right)^{-1}\left(\int d\mathcal{N}(E)\,\log(\Lambda - E) - \log(\mathrm{cap}(\mathcal{C}))\right) + 1\right)\;. \tag{13}$$

The DOS and the capacity can be calculated numerically by standard procedures. Nevertheless a crude estimate of $\mathrm{cap}(\mathcal{C})$ (see Remark 6 in Chapter 4) shows that the argument of the second logarithm in (13) grows logarithmically with Λ. This proves that β_0^+ tends to 0 in the limit $\Lambda \to \infty$.

Remark 4 The bound (13) for the case of linear Cantor sets does not depend on the weights p_l, but there is numerical evidence that growth exponents do (for momenta at least) [12]. In addition, as the gaps shrink on decreasing Λ, the bound can become larger than 1, the *a priori* ballistic bound. Our estimates are therefore far from optimal in this situation.

Apart from the appendices giving complementary results, the rest of the article contains the proof of Theorems 1 and 2. Chapter 3 concludes with Proposition 3 which, when combined with the results of Sections 4.1 and 4.2, directly imply Theorems 1 and 2. All $C_i(\xi,\ldots)$ appearing below denote quantities only depending on constants $\xi,\ldots$.

3 Proof of upper bounds.

In this chapter, we will bound β_0^+ under a hypothesis formulated in Section 3.2. This hypothesis will be verified in the next chapter.

3.1 Resolving the spectrum.

In this section we state and prove a general result making no reference to the specific structure of the Hamiltonian. Then follows a corollary allowing to deduce upper bounds on the dynamics for the Jacobi matrices constructed in Sections 2.1 and 2.2.

Let $|\psi\rangle$ be a vector in some separable Hilbert space $\mathcal{H}$, μ its spectral measure, $J = \mathrm{supp}(\mu)$, and $(|n\rangle)_{n\in\mathbf{N}}$ a Hilbert basis in the cyclic subspace of ψ. Suppose that, for any integer N, we have a finite covering of J by intervals Δ_j^N, $j = 1,\ldots n_N$, with pairwise disjoint interiors and satisfying

$$0 \;<\; \lim_{N\to\infty} \frac{1}{N}\log\left(\frac{1}{|\Delta^N_{j(E)}|}\right) \;=\; \Lambda \;<\; \infty \qquad \mu\text{-a.e. }, \tag{14}$$

where, for given N and $E \in J$, $\Delta^N_{j(E)}$ is the interval containing E. Finally, let $\chi_j^N(E)$ the characteristic function of Δ_j^N. Using the notation $|\chi_j^N\rangle = \chi_j^N(H)|\psi\rangle$, the following holds true:

Proposition 1 *Fix $\lambda \in (0,\Lambda)$ and $\epsilon > 0$. Let $K \subseteq J$ be a compact with $\mu(K) > 1-\epsilon$. To a given time T, we associate $N = N(T)$ by:*

$$e^{(N-1)(\Lambda-\lambda)} \;<\; \frac{T}{\sqrt{\epsilon}} \;\leq\; e^{N(\Lambda-\lambda)}\ , \tag{15}$$

and let $\mathcal{J}_N(K) = \{j \,|\, K\cap\Delta_j^N \neq \emptyset\}$. Then there are $\overline{T}(\lambda,\epsilon)$ and $C_1(\Lambda,\lambda)$ so that, for all $T > \overline{T}(\lambda,\epsilon)$, and for any family of indices $\mathcal{F}\subset\mathbf{N}$:

$$\sum_{n\in\mathcal{F}}\int_0^T \frac{dt}{T}|\langle n|e^{-\imath Ht}|\psi\rangle|^2 \;\leq\; 18\,\epsilon + \frac{C_1(\Lambda,\lambda)}{\sqrt{\epsilon}}e^{2N\lambda}\sum_{n\in\mathcal{F}}\sum_{j\in\mathcal{J}_{N(T)}(K)}|\langle n|\chi_j^N\rangle|^2\ . \tag{16}$$

Proof. Because $\log(1/|\Delta^N_{j(E)}|)/N$ is a measurable function of E, it follows from (14) and Lusin's theorem that there is a $\overline{N}(\lambda,\epsilon)$ and a compact $K_\epsilon \subseteq J$ with $\mu(K_\epsilon) > 1-\epsilon$ so that

$$e^{-N(\Lambda+\lambda)} \;<\; |\Delta_j^N| \;<\; e^{-N(\Lambda-\lambda)} \tag{17}$$

for all $N > \overline{N}(\lambda,\epsilon)$ and for all $j \in \mathcal{J}_N(K_\epsilon)$. Let us define $K_1 = K\cap K_\epsilon$; then $\mu(K_1) > 1-2\epsilon$. From now on, we shall assume N and T to be related to each other via (15). For $0 \leq t \leq T$, we approximate $e^{-\imath Ht}|\psi\rangle$ by:

$$|\psi_T(t)\rangle \;=\; \sum_{j\in\mathcal{J}_N(K_1)} e^{-\imath tE_j^N}|\chi_j^N\rangle$$

where E_j^N is any point in Δ_j^N. Then we have:

$$\|\,|\psi_T(t)\rangle - e^{-\imath tH}|\psi\rangle\|^2 \;\leq\; \sum_{j\in\mathcal{J}_N(K_1)} \int_{\Delta_j^N} d\mu(E)\, t^2|E-E_j^N|^2 + 4 \sum_{j\notin\mathcal{J}^N(K_1)} \mu(\Delta_j^N)\;.$$

As $|E-E_j^N|\leq|\Delta_j^N|$ for $j\in\mathcal{J}_N(K_1)$ and $E\in\Delta_j^N$, using (17) we get that the latter expression is not larger than

$$\sum_{j\in\mathcal{J}_N(K_1)} \mu(\Delta_j^N)\, t^2 e^{-2N(\Lambda-\lambda)} + 8\,\epsilon\;,$$

which, for $0\leq t\leq T$, is not larger than $9\,\epsilon$ because of (15). Hence,

$$\sum_{n\in\mathcal{F}} \int_0^T \frac{dt}{T}|\langle n|e^{-\imath tH}|\psi\rangle|^2 \;\leq\; 18\,\epsilon + 2\sum_{n\in\mathcal{F}} \int_0^T \frac{dt}{T}|\langle n|\psi_T(t)\rangle|^2\;. \tag{18}$$

We now choose the points E_j^N so that the latter term can be easily estimated. Let ℓ be the integer part of $(1+\exp(N(\Lambda+\lambda))$. As all intervals Δ_j^N, $j\in\mathcal{J}_N(K_1)$, have length not less than $\exp(-N(\Lambda+\lambda))$, everyone of them contains at least one integer multiple of $1/\ell$. Let E_j^N be such a point. Then

$$\int_0^T \frac{dt}{T}|\langle n|\psi_T(t)\rangle|^2 \;\leq\; \int_0^{2\pi\ell} \frac{dt}{T}|\langle n|\psi_N(t)\rangle|^2 \;=\; \sum_{j,l\in\mathcal{J}_N(K_1)} \langle n|\chi_{\Delta_j^N}\rangle\langle\chi_{\Delta_l^N}|n\rangle \int_0^{2\pi\ell} \frac{dt}{T} e^{-\imath(E_j^N-E_l^N)t}\;.$$

The latter integral yields $2\pi\ell\delta_{jl}$. Recalling the definition of ℓ, and $K_1\subseteq K$, we directly get inequality (16). $\Box$

We now restrict μ to be a fractal measure as constructed in Section 2.1, $(|n\rangle)_{n\in\mathbf{N}}$ the basis of its orthogonal polynomials and H the associated Jacobi matrix. As intervals Δ_j^N we choose the intervals I_σ^N of the Nth generation. As shown by equation (33) in the proof of Theorem 3, in that case equation (14) holds with Λ given by the Lyapunov exponent $\Lambda(\mu)$. Further, we set $K=J$ and $\mathcal{F}=(n)_{n\geq\overline{n}}$ in Proposition 1. Hence we obtain:

Corollary 2 *Let μ and H be as constructed in* Sections 2.1 *and* 2.2*. Fix $\lambda\in(0,\Lambda(\mu))$ and $\epsilon>0$. To a given time T, we associate a generation index N by*

$$e^{(\Lambda(\mu)-\lambda)(N-1)} \;<\; \frac{T}{\sqrt{\epsilon}} \;\leq\; e^{(\Lambda(\mu)-\lambda)N}\;. \tag{19}$$

If $\overline{n}=\overline{n}(\epsilon,N)$ is chosen such that

$$\sum_{n\geq\overline{n}} \sum_{\sigma\in\Sigma_L^N} |\langle 0|\chi_{I_\sigma^N}(H)|n\rangle|^2 \;\leq\; \epsilon^{3/2}\, e^{-2\lambda N}\;, \tag{20}$$

then there is a constant $C_2(\Lambda,\lambda)$ so that, for all T larger than some $T(\lambda,\epsilon)$, we have

$$\sum_{n>\overline{n}} \int_0^T \frac{dt}{T}|\langle n|e^{-\imath tH}|0\rangle|^2 \;<\; C_2(\Lambda,\lambda)\,\epsilon\;. \tag{21}$$

3.2 A bound of quasi-ballistic type.

The aim of this section is determine $\overline{n}(\epsilon, N)$ such that bound (20) holds under the following hypothesis, which will be verified in Chapter 4. Let $\mathbf{S}$ be the isometric operator defined in $L^2(\mathbf{R}, \mu)$ by $\mathbf{S}\phi = \phi \circ S$.

Hypothesis: *There exist constants* $0 < A \leq B$ *and* C *such that*

$$|\langle n|\mathbf{S}|m\rangle| \;\leq\; \exp\left(-An + Bm + C\right) \; . \tag{22}$$

Proposition 2 *Assume* (22) *to be valid and let* $\xi > 1$. *Then there is a constant* $C_3(\xi, \lambda)$ *so that the bound* (20) *holds if*

$$\overline{n}(\epsilon, N) \;\geq\; C_3(\xi, \lambda)\, \epsilon^{1-\xi}\, e^{N\xi \log(B/A)} \; . \tag{23}$$

We call this a quasi-ballistic bound because it is directly given by the growth rate of the matrix elements in (22). Apart from hypothesis (22), the essential ingredient of the proof is that the spectrum contains gaps at all scales (self-similarity). In the next subsection, we set the stage for the proof of Proposition 2. Subsections 3.2.2 and 3.2.3 give technical details which allow to conclude the proof in 3.2.4.

3.2.1 A renormalization operator.

In order to estimate the sum on the left hand side of (20) we introduce suitable operators $\mathcal{P}_l$, $l = 1, ..., L$, and $\mathcal{S}$ on the Schatten ideal $\mathcal{L}^1$ of trace class operators on $L^2(\mathbf{R}, \mu)$. For $\rho \in \mathcal{L}^1$, these are defined by

$$\mathcal{P}_l(\rho) \;=\; \chi_{I_l^1}(H)\rho\, \chi_{I_l^1}(H) \; , \qquad \mathcal{S}\rho \;=\; \mathbf{S}\, \rho\, \mathbf{S}^\dagger \; .$$

We call $\mathcal{S}$ the scaling operator, $\mathcal{P} = \sum_{l=1}^{L} \mathcal{P}_l$ the reduction operator, and $\mathcal{R} = \mathcal{P} \circ \mathcal{S}$ the renormalization operator. One easily verifies that $\mathcal{R}$ is trace and positivity preserving. Let us further introduce the density matrices (positive and of unit trace)

$$\rho_N \;=\; \sum_{\sigma \in \Sigma_L^N} \chi_{I_\sigma^N}(H)|0\rangle\langle 0|\chi_{I_\sigma^N}(H) \; .$$

Because $\chi_{I_\sigma^N}(S(E)) = \sum_{l=1}^{L} \chi_{S_l^{-1}(I_\sigma^N)}(E)$, we have $\mathcal{R}(\rho_N) = \rho_{N+1}$ and the expression in (20) can be written as

$$\sum_{n \geq \overline{n}} \sum_{\sigma \in \Sigma_L^N} |\langle 0|\chi_{I_\sigma^N}(H)|n\rangle|^2 \;=\; \sum_{n \geq \overline{n}} \langle n|\mathcal{R}^N(\rho_0)|n\rangle \; . \tag{24}$$

We hence have to estimate $\mathcal{R}$ and for this purpose we introduce the norms

$$||\rho||_a = \sup_{n,m \geq 0} e^{a(n+m)}\, |\langle n|\rho|m\rangle| \; , \qquad a > 0 \; .$$

For any $a > 0$, the set $X_a = \{\rho \in \mathcal{L}^1 |\; ||\rho||_a < \infty\}$ is a Banach space. Under hypothesis (22), we shall show that $\mathcal{P}, \mathcal{S}, \mathcal{R}$ are continuous operators in $\bigcup_{a > \overline{a}} X_a$ for some $\overline{a}$, and we shall thereby estimate their norms, finally obtaining the proof of Proposition 5.

The following lemma is based on the well known Combes-Thomas argument. Improving the lower bound on θ defined below would lead to a better bound in Theorem 2, because it enters the estimates of Proposition 5.

Lemma 1 *Let I be any interval of the first generation and Δ be the smallest gap in the first generation. Using the notation $\Gamma_{nm}(I) = \langle n|\chi_I(H)|m\rangle$, we define:*

$$\theta(I) \;=\; \liminf_{n,m\to\infty} \frac{1}{|n-m|}\log\left(\frac{1}{|\Gamma_{nm}|}\right)\,, \qquad \theta_0(\Delta) \;=\; \operatorname{arcsinh}\left(\frac{\Delta}{4R_c}\right)\,.$$

Then

$$\theta_0(\Delta) \;\le\; \theta(I) \;<\; \infty\,.$$

Proof. We have $\chi_I(E)P_m(E) = \sum_{n\ge 0}\Gamma_{nm}(I)P_n(E)$ in $L^2(\mathbf{R},\mu)$. Suppose $\theta(I)=\infty$. Then $\Gamma_{nm}(I)$ decays faster than exponentially as $n\to\infty$. Since the theorem in Appendix 2 sets an uniform bound on the exponential growth of $P_n(E)$, $E\in I^0$, the series is uniformly convergent in I^0 and its sum is analytic there. This cannot be true, because $\chi_I(E)P_m(E)$ vanishes in $I^0\setminus I$.

For $\theta\in\mathbf{R}$, we define the operator U_θ in $\ell^2(\mathbf{N})$ by $\langle n|U_\theta|m\rangle = e^{-\theta n}\delta_{n,m}$ and set $H_\theta = U_\theta^{-1}HU_\theta$ and $Q_\theta = H_\theta - H$. Let further Γ denote the circle of radius $(|I|+\Delta)/2$ around the center of I. By a standard argument, the operator $(z-H_\theta)^{-1}$ is bounded whenever $\|Q_\theta(z-H)^{-1}\|<1$. Since $\|(z-H)^{-1}\| > 2/\Delta$ for all $z\in\Gamma$ and

$$\|Q_\theta\| \;\le\; (\max_{j\ge 1} t_j)\,|e^\theta - e^{-\theta}| \;\le\; R_c\,|e^\theta - e^{-\theta}|\,,$$

this is guaranteed with a uniform bound in $z\in\Gamma$ whenever

$$\sinh(|\theta|) \;<\; \frac{\Delta}{4R_c}\,.$$

Therefore we have

$$e^{\theta(n-m)}\langle n|\chi_I(H)|m\rangle \;=\; \int_\Gamma \frac{dz}{2\pi\imath}\,\langle n|U_\theta^{-1}(z-H)^{-1}U_\theta|m\rangle \;=\; \int_\Gamma \frac{dz}{2\pi\imath}\,\langle n|(z-H_\theta)^{-1}|m\rangle,$$

and the last expression is bounded by some constant. This determines a minimal rate of exponential decay. □

3.2.2 Norms of the reduction operator.

It is easily seen that, if $\rho\in X_a$,

$$\|\mathcal{P}_l(\rho)\|_b^{1/2} \;\le\; \sup_{n\ge 0}\, e^{nb}\sum_{k\ge 0}|\langle n|\chi_{I_l^1}(H)|k\rangle|\,e^{-ak}\,\|\rho\|_a^{1/2}\,.$$

The proof of Lemma 1 shows that $|\langle n|\chi_{I_l^1}(H)|k\rangle| \le c(\Delta)\exp(-\theta_0(\Delta)|n-k|)$. A short computation shows that $\mathcal{P}_l$ is a bounded operator in X_a if $a<\theta_0(\Delta)$, and that

$$\|\mathcal{P}_l\|_{a\to a}^{1/2} \;\le\; \frac{2c(\Delta)}{\theta_0(\Delta)-a}\,. \tag{25}$$

3.2.3 Norms of the scaling operator.

Hypothesis (22) is used at this very point. If $\rho \in X_a$,

$$\|\mathcal{S}(\rho)\|_b^{1/2} \;\leq\; \sup_{n\geq 0} e^{bn} \sum_{k\geq 0} |\langle n|\mathbf{S}|k\rangle|\, e^{-ak}\, \|\rho\|_a^{1/2}\ .$$

Using Hypothesis (22) for $k \leq B^{-1}(An - C)$, and $|\langle n|\mathbf{S}|k\rangle| \leq 1$ for $k > B^{-1}(An - C)$, one easily finds that, if $a < B$, then $\mathcal{S}$ is bounded from X_a to $X_{aA/B}$, and that

$$\|\mathcal{S}\|_{a\to aA/B} \;\leq\; \frac{B}{a(B-a)}\, e^{aC/B}\ . \tag{26}$$

Remark 5 If we restrict ourselves to Julia matrices (S is a polynomial of degree L and μ is defined as the Bernoulli measure with equal weights), a bound on the norm of $\mathcal{S}$ can be obtained more directly by using the renormalization property $\langle n|\mathbf{S}|m\rangle = \delta_{n,Lm}$ of the associated orthogonal polynomials [3]. It is then immediate that $\mathcal{S}$ has unit norm from X_a to $X_{a/L}$. This considerably simplifies the arguments in Subsection 3.2.4. For the case of Julia matrices, it is even not necessary to go through the developments of Chapter 4.

3.2.4 Conclusion of the proof of the quasi-ballistic bound.

According to (24), we have

$$\sum_{n\geq \overline{n}} \sum_{\sigma\in\Sigma_N^L} |\langle 0|\chi_{I_\sigma^N}(H)|n\rangle|^2 \;\leq\; \|\mathcal{R}^N(\rho_0)\|_a \sum_{n\geq\overline{n}} e^{-an} \;\leq\; \frac{1}{a}\, \|\mathcal{R}^N(\rho_0)\|_a\, e^{-a(\overline{n}+1)}\ . \tag{27}$$

We now choose $a = (A/B)^N a_0$ with $a_0 < \min\{B, \theta_0(\Delta)\}$, so that the bounds (25) and (26) are applicable, and estimate $\|\mathcal{R}^N(\rho_0)\|_a$ by:

$$\|\rho_0\|_{a_0} \|\mathcal{P}\|_{a_0\to a_0} \|\mathcal{S}\|_{a_0\to a_0 A/B} \cdots \|\mathcal{P}\|_{a_0(A/B)^{N-1}\to a_0(A/B)^{N-1}} \|\mathcal{S}\|_{a_0(A/B)^{N-1}\to a_0(A/B)^N}\ .$$

Substituting the bounds (25), (26), and $\|\rho_0\|_{a_0} = 1$ leads to

$$\|\mathcal{R}^N(\rho_0)\|_{a_0(A/B)^N} \;\leq\; Q^N \left(\frac{B}{A}\right)^{2N^2} \;\equiv\; \left(\frac{4c^2(\Delta) B^2 L e^{2a_0 C/B}}{(\theta_0(\Delta)-a_0)^2 (B-a_0)^2 a_0^2}\right)^N \left(\frac{B}{A}\right)^{2N^2}\ .$$

Now, in order for (27) with a replaced by $a_0(A/B)^N$ to be less than $\epsilon^{3/2} e^{-2\lambda N}$, it is sufficient that

$$\overline{n} \;\geq\; \frac{1}{a_0} \left(\frac{B}{A}\right)^N \log\left(\frac{Q^N\, e^{2\lambda N}}{\epsilon^{3/2} a_0} \left(\frac{B}{A}\right)^{2N^2+N}\right)\ .$$

For this, it is in turn sufficient that $\overline{n}$ satisfies (23) where the constant $C_3(\xi,\lambda)$ may also depend on A, B, Q, and a_0, but not on ϵ and N.

3.3 Upper bound under hypothesis (22).

Replacing (19) in (23), and using definition (9) of the transport exponent β_0^+, one gets:

Proposition 3 *Suppose that the orthogonal polynomials of* μ *satisfy* (22). *Then*

$$\beta_0^+ \;\leq\; \frac{\log(B/A)}{\Lambda(\mu)} \;.$$

4 Asymptotic properties of orthogonal polynomials.

In this chapter, we prove two propositions which, when combined with Propostion 3, directly lead to Theorems 1 and 2. We shall use the main results of the work of Stahl and Totik [26] reviewed in Appendix 2. For this purpose, let us first verify that the capacity of J is positive and that the LDOS is asymptotically regular in the sense of [26]. The definition of this property is recalled in Appendix 2. Let us point out that the measure μ on J need not be defined via an invariant and ergodic measure on the symbolic dynamics, but we do use the hypothesis imposed at the very end of Section 2.1.

Lemma 2 *Let* J *be a set constructed as in* Section 2.1. *Then* $\mathrm{cap}(J) > 0$.

Lemma 3 *The measures* μ *on* J *constructed in* Section 2.1 *are asymptotically regular.*

Knowing that $\mathrm{cap}(J) > 0$ and that μ is asymptotically regular, the theorem in Appendix 2 directly implies that formulas (6), (7) and (8) in Section 2.3 hold.

Proof of Lemma 2 (adapted from [1]). Let μ be a probability measure carried by J. Its potential is

$$\Phi_\mu(E) \;=\; \int d\mu(E') \, \log\left(\frac{1}{|E-E'|}\right) \;.$$

We show that, if μ is the Bernoulli measure with equal weights, then $|\Phi_\mu(E)| \leq C < \infty$ for $E \in J$. This implies that $\mathrm{cap}(J) \geq \exp(-C)$.

Denoting $I^N_{\sigma(E)}$ the Nth generation interval containing $E \in J$, we have the identity $I^0 = \bigcup_{N\geq 1}(I^{N-1}_{\sigma(E)} \backslash I^N_{\sigma(E)})$. Thus

$$\Phi_\mu(E) \;=\; \sum_{N\geq 1} \int_{I^{N-1}_{\sigma(E)} \backslash I^N_{\sigma(E)}} d\mu(E') \, \log\left(\frac{1}{|E-E'|}\right) \;. \tag{28}$$

As long as E' ranges in $I^{N-1}_{\sigma(E)} \backslash I^N_{\sigma(E)}$, $|E-E'|$ cannot be less than the gaps $|G^N|$ of generation N adjacent to $I^N_{\sigma(E)}$. From Lemma 4 in Appendix 1, we get $|G^N| \geq a/\lambda^N$ where $\lambda = \max_{E\in J} |S'(E)|$ and a is some positive constant. Inserting this in (28), we obtain

$$|\Phi_\mu(E)| \;\leq\; \sum_{N\geq 1} (L-1) \, \frac{1}{L^N} \log\left(\frac{\lambda^N}{a}\right) \;, \tag{29}$$

which directly implies the result. □

Remark 6 The proof of Lemma 2 yields a lower estimate for the capacity of the Cantor set $\mathcal{C}$ in $[0,1]$ generated with $L = 2$ and $\hat{S}_1(E) = \Lambda E$, $\hat{S}_2(E) = \Lambda(1-E)$, $\Lambda > 2$. In that case, $|G_N| = (\Lambda-2)\Lambda^{-N}$, so equation (29) yields $\mathrm{cap}(\mathcal{C}) \geq (\Lambda-2)\Lambda^{-2}$. This crude estimate was used in the discussion in Section 2.6.

Proof of Lemma 3. Here we use the hypothesis that $-\log(\mu(I_\sigma^N))/N$ has a uniform upper bound, say h. We shall use the criterion (37) and verify that the sets on the left and right hand-side of (37) coincide. For that purpose, we use the formula for the upper pointwise dimensions given in Theorem 3 in which the denominator is replaced by equation (33). As $|I_\sigma^N| \leq ba^N$ with $a < 1$, we therefore obtain for any $E \in J$, $\overline{d}_\mu(E) \leq h/\log(1/a) < \infty$ which proves the lemma. □

4.1 The analytic case.

Proposition 4 *Let S be an analytic map, and let E_c and $R_c = |I^0|/2$ be the the center of the spectrum and its radius, respectively. For any $R > R_c$, the bound (22) holds for any $\eta > 0$ with*

$$A \;=\; \log\left(\frac{R}{R_c}\right)\;, \qquad B \;=\; \max_{z\in\Gamma_R}\, g_J(S(z)) + \eta\;, \tag{30}$$

where Γ_R is the circle of radius R around E_c and g_J denotes the Green's function of J.

Proof. Using analyticity of S and orthogonality of the polynomials, we first write

$$\langle n|\mathbf{S}|m\rangle \;=\; \sum_{k\geq n} \frac{1}{2\pi\imath}\int_\Gamma dz\, \frac{P_m(S(z))}{(z-E_c)^{k+1}}\, \int d\mu(E)\, P_n(E)\,(E-E_c)^k\;,$$

where Γ is some integration path around the spectrum with positive orientation. For given $R > R_c$, let us choose Γ to be Γ_R. Furthermore, we apply the Cauchy-Schwarz inequality to the last term and bound $(E - E_c)$ by R_s for E in the spectrum. Thus summing up, we get

$$|\langle n|\mathbf{S}|m\rangle| \;\leq\; \left(\frac{R_c}{R}\right)^n \,\frac{1}{1-R_c/R}\, \max_{z\in\Gamma_R} |P_m(S(z))|\;.$$

Finally we use (34) in order to get an upper bound on the growth of the orthogonal polynomials. For any $\eta > 0$, there exists an $M(\eta)$ such that

$$|P_m(S(z))| \;\leq\; \exp(m\, g_J(S(z)) + \eta)\;, \tag{31}$$

for all $m \geq M(\eta)$. Hence (22) holds for any $m \geq M(\eta)$ and the choices (30). We can choose $C = C(\eta)$ in such a way that it holds for all $m \in \mathbf{N}$. □

4.2 The polynomial IFS case.

Proposition 5 *For any $l = 1\ldots L$, let the branch S_l have analytic continuation $\hat{S}_l$ given by a polynomial of degree D_l. Set $D = \max_l D_l$. Let furthermore $R_c = |I^0|/2$ be the spectral radius and Δ the size of the smallest gap at the first generation. Then, for any $\eta > 0$, the bound (22) holds with*

$$A \;=\; \operatorname{arcsinh}\left(\frac{\Delta}{4R_c}\right) \;, \qquad B \;=\; \max_{l=1\ldots L}\, \sup_{E\in J}\, g_J(\hat{S}_l(E)) + D \operatorname{arcsinh}\left(\frac{\Delta}{4R_c}\right) + \eta \;.$$

Proof. We start from the following identity:

$$\begin{aligned}\langle n|\mathbf{S}|m\rangle \;&=\; \sum_{l=1}^{L} \int_{\mathbf{R}} d\mu(E)\, \chi_{I_l^1}(E)\, P_n(E)\, P_m(\hat{S}_l(E)) \\ &=\; \sum_{l=1}^{L} \sum_{k\geq 0} \int_{\mathbf{R}} d\mu(E)\, \Gamma_{k0}(I_l^1)\, P_k(E)\, P_n(E)\, P_m(\hat{S}_l(E)) \;,\end{aligned}$$

with $\Gamma_{k0}(I_l^1)$ defined as in Lemma 1. By orthogonality, the sum over k is restricted to $k \geq |n - Dm|$. Using the bound (34) (with the same precautions as in the proof of Proposition 4), plus the Schwarz inequality, we obtain by Lemma 1

$$|\langle n|\mathbf{S}|m\rangle| \;\leq\; L\, c(\Delta)\, \exp\left(m \max_{l=1\ldots L}\, \sup_{E\in J}\, g_J(\hat{S}_l(E)) + \eta\right) \sum_{k\geq |n-Dm|} e^{-\theta_0(\Delta)k}$$

and the announced result follows. □

Appendix 1: pointwise dimensions of IFS measures.

The lower and upper pointwise dimensions of a Borel measure μ at a point $E \in \mathbf{R}$ are defined by

$$\underline{d}_\mu(E) \;=\; \liminf_{\epsilon\to 0} \frac{\log(\mu([E-\epsilon, E+\epsilon]))}{\log(\epsilon)} \;, \qquad \overline{d}_\mu(E) \;=\; \limsup_{\epsilon\to 0} \frac{\log(\mu([E-\epsilon, E+\epsilon]))}{\log(\epsilon)} \;. \quad (32)$$

Furthermore, the Hausdorff dimension $\dim_{\mathrm{H}}(\mu)$ of μ is defined by the infimum of the Hausdorff dimensions of all Borel subsets $\Delta \subset \mathbf{R}$ satisfying $\mu(\Delta) = 1$. It is equal to $\mu-\mathrm{esssup}_{E\in\mathbf{R}}\, \underline{d}_\mu(E)$ [10, 2].

In this appendix we give a proof that the pointwise dimensions of the IFS measures described in Section 2.1 coincide and are constant μ-almost surely. Although this can be deduced from more general results in the literature ([22], but there probably exist other works), we give here a short, independent proof for the sake of completeness and also because it leads to a slightly more general result, used in Chapter 4.

Theorem 3 *If μ is any measure supported by J, then its pointwise dimensions are given by*

$$\underline{d}_\mu(E) \;=\; \liminf_{N\to\infty} \frac{-\frac{1}{N}\log(\mu(I^N_{\sigma(E)}))}{\frac{1}{N}\sum_{i=0}^{N-1}\log(|S'(S^{\circ i}(E))|)} \;, \qquad \overline{d}_\mu(E) \;=\; \limsup_{N\to\infty} \frac{-\frac{1}{N}\log(\mu(I^N_{\sigma(E)}))}{\frac{1}{N}\sum_{i=0}^{N-1}\log(|S'(S^{\circ i}(E))|)} \;.$$

It immediately follows from the Breiman-Shannon-McMillan theorem and Birkhoff's ergodic theorem [5], that the upper and lower pointwise dimension coincide, and are constant μ-almost surely whenever the measure μ is ergodic. This gives formula (2).

For the proof of Theorem 3, we shall need several reformulations of the basic contraction properties of S stating that there exist finite positive constants b_1 and $a_1 < 1$ such that $|I^N_\sigma| \leq b_1\, a_1^N$. We denote the gap to the right of I^N_σ by G^N_σ.

Lemma 4 *There exist finite positive constants a_2, a_3, a_4 and $a_5 < 1$ such that for any $N \in \mathbf{N}$ and $\sigma \in \Sigma_L$ the following properties hold.*
i) *For all $E_N, E'_N \in I^N_\sigma$, one has $|(S^{\circ N})'(E'_N)|/a_2 \leq |(S^{\circ N})'(E_N)| \leq a_2\, |(S^{\circ N})'(E'_N)|$.*
ii) *For any $E_N \in I^N_\sigma$, one has $1/a_3 \leq |(S^{\circ N})'(E_N)|\, |I^N_\sigma| \leq a_3$.*
iii) *If I is either of the intervals of generation N adjacent to G^N_σ, then $|I|/a_4 \leq |G^N_\sigma| \leq a_4|I|$.*
iv) $|I^N_\sigma| \leq a_5 |I^{N-1}_\sigma|$.

Proof. i) It is sufficient to show that for any choice of $E_j,\ E'_j \in I^j_\sigma$, $j = 0 \ldots N-1$, one has $\prod_{j=0}^{N-1} \frac{|S'(E_j)|}{|S'(E'_j)|} \leq a_2$. In fact, by the chain rule $(S^{\circ N})'(E_N) = \prod_{j=0}^{N-1} S'(S^{\circ j}(E_N))$ and $S^{\circ j}(E_N) \in I^{N-j}_\sigma$ and this implies i). Next we note that

$$|S'(E_j)| \;\leq\; |S'(E'_j)| + \max_{E\in J} |S''(E)|\, |I^j_\sigma| \;\leq\; |S'(E'_j)| + b_2\, b_1\, a_1^j \ ,$$

for some finite constant b_2. By one of the hypothesis in Section 2, $|S'(E)| \geq b_3$ and it follows that

$$\prod_{j=0}^{N-1} \frac{|S'(E_j)|}{|S'(E'_j)|} \;\leq\; \prod_{j=0}^{N-1} \left(1 + \frac{b_2\, b_1\, a_1^j}{b_3}\right) \;\leq\; \exp\left(\frac{b_2\, b_1}{(1-a_1)\, b_3}\right) \ .$$

ii) follows directly from $\min_{E'\in I^0} |(S^{-1}_{\sigma_1} \circ \ldots \circ S^{-1}_{\sigma_N})'(E')|\, |I^0| \leq |I^N_\sigma| \leq \max_{E'\in I^0} |(S^{-1}_{\sigma_1} \circ \ldots \circ S^{-1}_{\sigma_N})'(E')|\, |I^0|$ and i).

Concerning iii), we denote the gap by G and the intervals to its left and right by I_+ and I_-. Let us suppose that the gap opened at generation N, notably that there exists an σ such that $I^{N-1}_\sigma = S^{-1}_{\sigma_1} \circ \ldots \circ S^{-1}_{\sigma_{N-1}}(I^0)$ is the disjoint union of I_+, G and I_-. The situation where G opened at some earlier generation can be treated in an analogous way. Now I_+, G and I_- are respectively the images under the application $S^{-1}_{\sigma_1} \circ \ldots \circ S^{-1}_{\sigma_{N-1}}$ of the intervals I^1_l, G^1_l and I^1_{l+1} of the first generation for some $l = 1 \ldots L$. Fix $E_{N-1} \in I^{N-1}_\sigma$. By the same argument as in ii), we obtain $|I^1_l|/a_1 \leq |(S^{\circ N-1})'(E_{N-1})|\, |I_+| \leq a_1 |I^1_l|$, a similar inequality for I_- as well as $|G^1_l|/a_1 \leq |(S^{\circ N-1})'(E_{N-1})|\, |G| \leq a_1 |G^1_l|$. From these inequalities we can easily deduce iii).

To prove iv), we use the same notations and suppose that the interval of the Nth generation is I_- (the case I_+ being again similar). Then $|I^{N-1}_\sigma| = |I_+| + |G| + |I_-|$. The above estimates now lead to

$$|I^{N-1}_\sigma| \;\geq\; \left(\frac{|I^1_l|}{a_1^2\, |I^1_{l+1}|} + \frac{|G^1_l|}{a_1^2\, |I^1_{l+1}|} + 1\right)\, |I_-| \ .$$

$\square$

Proof of Theorem 3. For $\epsilon > 0$, let $N(\epsilon)$ be the smallest positive integer satisfying $I^{N(\epsilon)}_{\sigma(E)} \subset [E-\epsilon, E+\epsilon]$. Clearly $\mu(I^{N(\epsilon)}_{\sigma(E)}) \leq \mu([E-\epsilon, E+\epsilon])$ and $|I^{N(\epsilon)}_{\sigma(E)}| \leq 2\epsilon$. Moreover, $I^{N(\epsilon)-1}_{\sigma(E)}$ contains either $[E-\epsilon, E]$ or $[E, E+\epsilon]$. Let us show that there exists an $r \in \mathbf{N}$ (independent of ϵ) such that $\mu([E-\epsilon, E+\epsilon]) \leq \mu(I^{N(\epsilon)-r}_{\sigma(E)})$. For this purpose, we note that $I^{N(\epsilon)-s}_{\sigma(E)}$ is an in $s \geq 1$ increasing sequence of intervals satisfying $|I^{N(\epsilon)-s}_{\sigma(E)}| \geq a_5^{-s+1} |I^{N(\epsilon)-1}_{\sigma(E)}| \geq a_5^{-s+1}\epsilon$ with a_5 as given in Lemma 4.

The latter lemma moreover implies that the gap $G^{N(\epsilon)-s}$ of generation $N(\epsilon)-s$ having possibly non-empty intersection with $[E-\epsilon, E+\epsilon]$ satisfies $|G^{N(\epsilon)-s}| \geq a_4^{-1}|I_{\sigma(E)}^{N(\epsilon)-s}| \geq a_4^{-1}a_5^{-s+1}\epsilon$. Now we choose r such that $a_4^{-1}a_5^{-r+1} \geq 1$. Then $[E-\epsilon, E+\epsilon]$ is covered by $I_{\sigma(E)}^{N(\epsilon)-r}$ and $G^{N(\epsilon)-r}$ and clearly we have $\mu([E-\epsilon, E+\epsilon]) \leq \mu(I_{\sigma(E)}^{N(\epsilon)-r})$ as well as $|I_{\sigma(E)}^{N(\epsilon)-r}| > \epsilon$. Consequently

$$\frac{\log(\mu(I_{\sigma(E)}^{N(\epsilon)-r}))}{\log(|I_{\sigma(E)}^{N(\epsilon)}|/2)} \;\leq\; \frac{\log(\mu([E-\epsilon, E+\epsilon]))}{\log(\epsilon)} \;\leq\; \frac{\log(\mu(I_{\sigma(E)}^{N(\epsilon)}))}{\log(|I_{\sigma(E)}^{N(\epsilon)-r}|)} \,.$$

Now $N(\epsilon) \to \infty$ as $\epsilon \to 0$ and therefore it suffices to show that

$$\liminf_{N\to\infty} \frac{-\frac{1}{N}\log(\mu(I_{\sigma(E)}^{N}))}{-\frac{1}{N}\log(|I_{\sigma(E)}^{N+r}|/2)} \;=\; \liminf_{N\to\infty} \frac{-\frac{1}{N}\log(\mu(I_{\sigma(E)}^{N}))}{-\frac{1}{N}\log(|I_{\sigma(E)}^{N-r}|)} \;,$$

and that their common value is that in the statement, as well as a similar statement for the superior limit. But because $E \in I_{\sigma(E)}^{N}$ for all $N \in \mathbf{N}$, Lemma 4 ii) implies that for $M = N$ and $M = N - r$

$$-\frac{\log(|I_{\sigma(E)}^{M}|)}{N} \;=\; \frac{1}{N}\sum_{i=1}^{N}\log(|S'(S^{\circ i-1}(E))|) + \mathcal{O}\left(\frac{1}{N}\right) \;, \tag{33}$$

and this concludes the proof. □

Appendix 2: the theory of Stahl and Totik.

The following theorem summarizes the main results of [26] used in this work. Let us recall that a sequence of functions f_n converges locally uniformly in an open set $G \subset \mathbf{C}$ to a function f if for all $z \in G$ and sequences $z_n \to z$, one has $f_n(z_n) \to f(z)$. In a similar way, one defines a bound $f_n \leq f$ to hold locally uniformly in G.

Theorem [26, Theorems 3.1.1 and 4.2.1] *Let μ be a Borel probability measure on $\mathbf{R}$ with compact support $J = \mathrm{supp}(\mu)$ and $I \subset \mathbf{R}$ be the smallest interval containing J. Furthermore $(P_n)_{n\geq 0}$ and $(t_n, v_n)_{n\geq 0}$ denote the associated orthogonal polynomials and coefficients of the Jacobi matrix. Then the following three assertions are equivalent:*

1. *the limit*
$$\lim_{n\to\infty} |P_n(z)|^{1/n} \;=\; \exp(g_J(z)) \tag{34}$$
holds true locally uniformly in $\overline{\mathbf{C}}\backslash I$;

2. *the limit*
$$\lim_{n\to\infty} |P_n(E)|^{1/n} \;=\; 1 \tag{35}$$
holds true for all $E \in J\backslash Y$ where $\mathrm{cap}(Y) = 0$;

3. *the formula*
$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}\log(t_j) \;=\; \log(\mathrm{cap}(J)) \tag{36}$$
holds.

If the above assertions are satisfied, the measure μ *is called asymptotically regular. This also implies that* (34) *holds locally uniformly in* $\overline{\mathbf{C}}$ *when the equality is replaced by an inequality* $\leq$*. Furthermore, if* $\mathrm{cap}(J) > 0$*, then*

$$\text{w-}\lim_{n\to\infty} \frac{1}{n} \sum_{E,\, P_n(E)=0} \delta_E \;=\; \omega_J \;.$$

A sufficient condition for a measure μ *to be asymptotically regular is the following equality:*

$$\mathrm{cap}\left(\left\{E \in \mathbf{R} \mid \overline{d}_\mu(E) < \infty \right\}\right) \;=\; \mathrm{cap}(J) \;, \tag{37}$$

where $\overline{d}_\mu(E)$ *is the upper pointwise dimension of* ν *at* E *defined in* (32).

References

[1] H. Aikawa, M. Essén, *Potential Theory: Selected Topics*, Lecture Notes in Mathematics 1633, (Springer, Berlin, 1995).

[2] J.-M. Barbaroux, J.M. Combes, R. Montcho, *Remarks on the relation between quantum dynamics and fractal spectra*, J. Math. Anal and Appl. 213, 698-722 (1997).

[3] M. F. Barnsley, J. S. Geronimo, A. N. Harrington, *Infinite Dimensional Jacobi Matrices Associated with Julia Sets for Polynomials*, Proc. Am. Math. Soc. **88** (4), 625-630 (1983).

[4] J. Bellissard, *Stability and instability in quantum mechanics*, in *Trends and developments in the eighties*, Eds. S Albeverio, Ph. Blanchard, (World Scientific, Singapore, 1985).

[5] P. Billingsley, *Ergodic Theory and Information*, (Wiley, New York, 1965).

[6] H. Brolin, *Invariant sets under iteration of rational functions*, Arkiv för Matematik, Band 6 nr 6, 103-144 (1965).

[7] P. Collet, J.-P. Eckmann, *Iterated maps on the interval as dynamical systems*, (Birkhäuser, Basel, 1980).

[8] R. Ketzmerick, G. Peschel, and T. Geisel, *Slow decay of temporal Correlations in quantum systems with Cantor spectra*, Phy. Rev. Lett., **69**, 695-698, (1992).

[9] I. Guarneri, *Spectral properties of quantum diffusion on discrete lattices*, Europhys. Lett., **10**, 95-100, (1989); *On an estimate concerning quantum diffusion in the presence of a fractal spectrum*, Europhys. Lett., **21**, 729-733, (1993); and I. Guarneri, G. Mantica, *On the asymptotic properties of quantum dynamics in the presence of fractal spectrum*, Ann. Inst. H. Poincaré, **61**, 369-379, (1994).

[10] I. Guarneri, *Singular continuous spectra and discrete wave packet dynamics*, J. Math. Phys. **37**, 5195-5206, (1996).

[11] I. Guarneri, *On the Dynamical Meaning of Spectral Dimensions*, to appear in Ann. Inst. H. Poincaré, 1997.

[12] I. Guarneri, G. Mantica, *Multifractal Energy Spectra and Their Dynamical Implications*, Phys. Rev. Lett. **73**, 3379-3383 (1994).

[13] F. Hippert, D. Gratias, Eds., *Lectures on quasicrystals*, (Les éditions de physique, Les Ulis, 1994).

[14] H. Hiramoto, S. Abe, *Dynamics of an Electron in Quasiperiodic Systems. I. Fibonacci Model*, J. Phys. Soc. Japan **57**, 230-240 (1988); and *Dynamics of an Electron in Quasiperiodic Systems. II. Harper Model*, J. Phys. Soc. Japan **57**, 1365-1372 (1988).

[15] M. Holschneider, *Fractal wavelet dimension and localization*, Comm. Math. Phys. **160**, 457-474 (1994).

[16] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30**, 713-747 (1981).

[17] R. Ketzmerick, K. Kruse, S. Kraut, T. Geisel, *What determines the spreading of a wave packet?*, Phy. Rev. Lett. **79**, 1959-1962 (1997).

[18] Y. Last, private communication.

[19] Y. Last, *Quantum dynamics and decomposition of singular continuous spectra*, J. Funct. Ana. **142**, 402-445 (1996).

[20] G. Mantica, *Quantum intermittency in almost periodic systems derived from their spectral properties*, Physica **D 103**, 576-589, (1997); *Wave Propagation in Almost-Periodic Structures*, Physica **D 109**, 113-127 (1997).

[21] P. Neu, R. Speicher, *Rigorous mean field model for the CPA: Anderson model with free random variables*, J. Stat. Phys. **80**, 1279-1308 (1995).

[22] Y. Pesin, H. Weiss, *The Multifractal Analysis of Gibbs Measures: Motivation, Mathematical Foundation, and Examples*, Chaos **7**, 89-106 (1997).

[23] F. Piéchon, *Anomalous Diffusion Properties of Wave Packets on Quasiperiodic Chains*, Phys. Rev. Lett. **76**, 4372-4375 (1996).

[24] R. del Rio, S. Jitomirskaya, Y. Last, B. Simon, *Operators with singular continuous spectrum: IV. Hausdorff dimension, rank-one perturbations and localization*, J. d'Analyse Math. **69**, 153-200 (1996).

[25] H. Schulz-Baldes, J. Bellissard, *Anomalous transport: a mathematical framework*, Rev. Math. Phy. **10**, 1-46 (1998); H. Schulz-Baldes, J. Bellissard, *A Kinetic Theory for Quantum Transport in Aperiodic Media*, to appear in J. Stat. Phys., June 1998.

[26] H. Stahl, V. Totik, *General Orthogonal Polynomials*, (Cambridge University Press, Cambridge, 1992)