Big islands in dispersing billiard-like potentials.

Vered Rom-Kedar
The Department of Applied Mathematics and Computer Science,
The Weizmann Inst. of Science, P.O.B. 26, Rehovot 76100, Israel.
Tel: 972 8 934 3170, Fax: 972 8 9344122 vered@wisdom.weizmann.ac.il
and
Dmitry Turaev
Weierstrass Institute for Applied Analysis and Stochastics,
Mohrenstr. 39, 10117 Berlin, Germany.
turaev@wias-berlin.de

May 4, 1998

Abstract

We derive a rigorous estimate of the size of islands (in both phase space and parameter space) appearing in smooth Hamiltonian approximations of scattering billiards. The derivation includes the construction of a local return map near singular periodic orbits for an arbitrary scattering billiard and for the general smooth billiard potentials. Thus, *universality* classes for the local behavior are found. Moreover, for all scattering geometries and for many types of natural potentials which limit to the billiard flow as a parameter $\epsilon \to 0$, islands of *polynomial* size in ϵ appear. This suggests that the loss of ergodicity via the introduction of the physically relevant effect of smoothening of the potential in modeling, for example, scattering molecules, may be of physically noticeable effect.

AMS No: 58F15, 82C05, 34C37, 58F05, 58F13, 58F14.

Keywords: singular Hamiltonian systems, elliptic islands, ergodicity, mixed dynamics.

1 Introduction

When an integrable Hamiltonian system is perturbed, non-integrable motion appears [12]. What happens when a highly chaotic Hamiltonian system is perturbed? In particular, consider an ergodic mixing flow - can a perturbation ruin these properties? Clearly, for uniformly hyperbolic systems the answer is negative, e.g. for geodesic flows on the surfaces of negative curvature [1]. The other basic example for hyperbolic behavior in the Hamiltonian setting is the class of scattering billiards [14]; Billiard motion corresponds to a point particle traveling with a constant speed in a region, undergoing elastic collisions at the region's boundary. When the boundary is concave, causing neighboring trajectories to diverge upon reflection, the billiard is called scaterring. Due to the divergence instability the scattering billiards are ergodic and mixing systems [14, 15, 7]. Thus, these have been suggested [14] as a first step models for substantiating the basic assumption of statistical mechanics - the ergodic hypothesis of Boltzmann (see especially the discussion and references in [15, 16]).

The scattering billiard corresponds to a non-smooth dynamical system which may formally be thought of as a Hamiltonian motion in a singular potential. However, classical molecules move in smooth (though very steep) potentials. Therefore, it is natural to consider the effect of smoothening of the potential on scattering billiards, and in particular its effect on the ergodic properties. In [17] we proved that smooth scattering billiard-like potentials may give rise to islands, ruining ergodicity (see also [4] for related results concerning the "finite range potentials" problem). Thus, we demonstrated that proving Boltzmann hypothesis for the hard sphere model is a-priori insufficient for proving the achievement of thermodynamical equilibrium for a gas.

Islands appearing in highly chaotic regimes were observed in numerous numerical simulations [9, 20]. While many of these simulations were carried for the standard map, only recently it has been shown that for large K it has a dense set of parameter values for which elliptic islands exist [5]. It is tempting to assume that the numerically observed chaotic sea is a nonuniformly hyperbolic set of positive measure; however no results in this direction have appeared to the best of our knowledge.

What is clearly known and has thus attracted much attention [9, 10, 11, 6, 19] is that the appearance of islands in large "chaotic sea" is of vast significance; First, it implies that the tail of the correlation function has finite range oscillations for initial conditions falling in the island. Moreover, around the islands there may be a "stickiness" region, which influences the temporal correlation function of initial conditions from the "chaotic sea". It is suspected that this stickiness may cause power-law instead of exponential decay of correlations even when the initial conditions lie in the chaotic sea.

Islands may be useful; When living in a chaotic sea, the islands correspond to stable dynamics in a highly noisy environment. This observation may be used in the context of control of chaos (see focus issue in Chaos 7 (4),1997), where it is desirable to switch quickly from highly ordered to highly chaotic behavior. Our construction enables to locate islands in both phase space and parameter space of the smooth billiard-like flow, allowing for a control of such switching.

In all applications the size of the islands and their periodicity matters; these define the space and time scales on which the transition from chaotic to periodic motion is observable. For example, consider the time correlation function

$$C(T; x(0)) = \lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} f(x(t)) f(x(t+T)) dt \tag{1.1}$$

This function will essentially be N-periodic for $x(0) \in I_N$ where I_N denotes an island of period N. Thus clearly, the difference between the chaotic sea and island behavior will not be seen before time of order N - demonstrating the crucial dependence on the periodicity of the islands. Averaging (1.1) over $x(0)$ gives the correlation function for a general ensemble of initial conditions - clearly the contribution of the time correlation function associated with the island is proportional to its area.

Here we consider the two degrees of freedom Hamiltonian flow associated with

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V(x, y; \gamma, \epsilon). \tag{1.2}$$

where, as $\epsilon \to 0$, the billiard-like potential $V(x, y; \gamma, \epsilon)$ vanishes in the interior of the scattering billiard domain D_γ and is of finite height on its boundary (see the next section for exact conditions on V). The parameter ϵ controls the steepness of the potential whereas γ controls the billiard geometry. We assume that at $\gamma = 0$ the billiard has a *simple singular periodic orbit* - namely, a periodic orbit which is tangent to the billiard boundary at exactly one reflection point. In [17] we proved that in such a setting a linearly stable periodic orbit appears in (1.2) for $\epsilon > 0$. Here, we find that *for typical potentials V there exist elliptic islands around these periodic orbits and that their size depends on ϵ polynomially* - see Theorem 1 and the following discussion.

Moreover, the proof of theorem 1 includes the construction of the local return map near the periodic singular orbit, see (2.10) and discussion section. This map *divides the family of smooth scattering billiard-like potentials to universality classes, supplying for each class a computational tool for estimating the size and location of elliptic islands.*

The paper is ordered as follows; in section 2 we formulate the main results of our paper, in section 3 we construct the local return map near singular periodic orbit (the proofs are in appendix A and B) in section 4 we compute the Birkhoff normal form for the local return map and prove that the first Birkhoff coefficient is not identically zero. In section 5 we discuss the geometrical interpretation of the universality classes which emerge from the local return map and conjecture regarding the visibility and typical period of the islands we have found.

2 Main result.

Consider a plane scattering billiard - a billiard in a domain $D \subset R^2$ which is a complement to a union of a finite number of strictly convex regions. The boundary of D consists of a finite number of smooth (C^{r+1}, $r \geq 2$) curves $S_1, S_2, \ldots$ joined at the corner points. The angles between the boundary arcs S_i at the corner points are non-zero and the curvature of S_i is bounded away from zero along the arcs.

The billiard defines a dynamical system (the billiard flow b_t): inertial motion of a pointwise particle inside D and elastic reflections at the boundary - the angle of reflection equals the angle of incidence. We consider a billiard which possesses *a (simple) singular* periodic orbit.

A basic tool in our analysis is the construction of a regularized billiard to a billiard with a singular periodic orbit. Its construction naturally leads to a new definition of stability or multipliers of the singular orbit. We observe that these multipliers play an essential role in determining the dynamics near the singular periodic orbit. Thus, we define (see figure 3.1):

Definition: *Consider the billiard flow in D which attains a singular periodic orbit L. The domain D_L is a* regularization of D w.r.t. L *if L is a regular periodic orbit of D_L, namely D_L differs from D by local smooth deformation of D near the tangencies of L and these deformations are bounded away from any regular reflection point of L.*

Definition: *Consider the billiard flow in D which attains a singular periodic orbit L. The* singular multipliers *of L are the multipliers of L in a regularization of D w.r.t. L.*

Notice that while a singular billiard has many regularizing billiards, the singular multipliers of a singular periodic orbit are uniquely defined.

2.1 Smooth billiard potentials.

Consider a smooth Hamiltonian approximation of the given billiard. Namely, consider a two parameter family $h_{t,\epsilon,\gamma}$ of C^r Hamiltonian flows associated with

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V(x, y; \gamma, \epsilon). \tag{2.1}$$

We begin with recalling the sufficient conditions [17] satisfied by the potentials V which ensure that the Hamiltonian flow of (2.1) approximates the billiard flow as $\epsilon \rightarrow 0$ (for a fixed γ). We call the potentials satisfying these conditions *smooth billiard potentials.* Furthermore, below we formalize the required generic dependence on γ.

The potential $V(x, y; \gamma, \epsilon)$ tends to zero inside a region D_γ as $\epsilon \rightarrow 0$ (along with all derivatives, uniformly in any compact subregion of D_γ) and it tends to infinity outside D_γ, where $D_{\gamma=0} = D$.

Thus varying γ corresponds to changing the shape of the limiting billiard whereas the parameter ϵ governs the rate at which the Hamiltonian flow approximates the billiard flow in D_γ.

To have the proper reflection law in the limit, it is necessary that the gradient of the potential stays normal to the boundary of the billiard as $\epsilon \rightarrow 0$; namely, we require that *in a neighborhood of* $(\partial D \backslash C)$ *(*C *is the set of corner points) there exists* a pattern function $Q(x, y; \gamma, \varepsilon)$ *whose level lines coincide with the level lines of* V *in a small neighborhood of the boundary arcs without the corners and which has (along with all derivatives) a* finite *limit as* $\varepsilon \to 0$ *such that at* $\gamma = 0$ *the boundary of the billiard is composed of level lines of* $Q(x, y; 0, 0)$:

$$Q(x, y; 0, 0)|_{(x,y) \in S_i} \equiv const. \tag{2.2}$$

Here, we further assume that these level lines depend smoothly on γ. Thus, at $\gamma \neq 0$ they define a deformed billiard region D_γ so that the original billiard is now embedded in the one-parameter family of scattering billiards. This is done to unfold the appearance of the singular (tangent) periodic orbit in the original billiard flow. Indeed, if the dependence of D_γ on γ is in general position (satisfies (3.13)), then the tangent periodic orbit disappears, say, at $\gamma < 0$ and there is no other periodic orbit in its small neighborhood, whereas at the opposite sign of γ two periodic orbits are born, one passing near the former point of tangency without hitting the boundary and the other has a regular reflection close to that point (see [17] for more details).

The existence of the pattern function implies that for each boundary arc S_i there exists *a barrier function* $W_i(Q, \epsilon)$ such that near the boundary (in a small neighborhood N_i of S_i without the corners)

$$V(x, y; \epsilon, \gamma) = W_i(Q(x, y; \gamma, \epsilon), \epsilon). \tag{2.3}$$

We assume that ∇V does not vanish in a finite neighborhood of the boundary arcs, thus:

$$\nabla Q|_{(x,y) \in N_i} \neq 0 \tag{2.4}$$

and

$$\frac{d}{dQ} W_i(Q, \varepsilon) \neq 0. \tag{2.5}$$

Note that the barrier function does not depend explicitly on (x, y) and γ. It describes the growth of the potential across the boundary arcs and we consider it as afore-hand given and unchanged (though we allow for small C^{r+1} perturbations of the function Q which describes the geometry of the billiard). We consider a large class of possible barrier functions; however, some restrictions are imposed.

First, the barrier function W_i must, as $\epsilon \to 0$, tend to zero at each Q that lies to the inner side of S_i and it must tend to infinity on the outer side. Now note that by (2.5) the value of Q may be considered as a function of W (and ε) near the boundary arc. At small ε, a finite change in W corresponds to a small change in Q. Therefore, the following condition makes sense:

as $\varepsilon \to +0$, *for any finite, strictly positive* W_1 *and* W_2, *the function* $Q(W;\varepsilon)$ *tends to zero uniformly on the interval* $[W_1, W_2]$ *along with all* $(r+1)$ *derivatives.*

Potentials of the above structure will be called *smooth billiard potentials.* For example, the following barrier functions give rise to such potentials:

$$\frac{\epsilon}{Q^\alpha}, \quad (1-Q^\alpha)^{\frac{1}{\epsilon}}, \quad \epsilon e^{-\frac{1}{Q^\alpha}}, \quad e^{-\frac{Q}{\epsilon}}, \quad \epsilon|\ln Q|^\alpha, \quad \epsilon \ln \ldots |\ln Q|, \qquad \alpha > 0, \tag{2.6}$$

It is proved in [17] that, as $\epsilon \to 0$, the Hamiltonian flows (2.1) with smooth billiard potentials *r-converge to the billiard flow in D_γ*, for any finite r. Namely, these converge C^r smoothly to the billiard flow outside any neighborhood of singular orbits and C^0 converge near tangent trajectories.

2.2 The scaling assumption, asymptotic normal form and elliptic islands.

In this paper we impose an additional restriction on the class of smooth billiard potentials which is satisfied by all typical examples (e.g. (2.6)), as demonstrated in section 2.3. Assume that the barrier functions $W(Q,\epsilon)$ satisfy the following *scaling assumptions* with respect to rescaling of $Q = \delta\tilde{Q} + \beta$ to $\tilde{Q}$:

[S] *For some* $\delta = \delta(\epsilon, H) > 0$, $\beta = \beta(\epsilon, H)$ *and* $\nu(\epsilon, H)$ *such that* $\delta \to 0$, $\beta \to 0$, $\nu/H \to 0$ *as* $\epsilon \to 0$, *the function*

$$\tilde{W}_\epsilon(\tilde{Q}) = \frac{W(\delta\tilde{Q} + \beta, \epsilon) - \nu}{H\delta^{3/2}} \tag{2.7}$$

converges as $\epsilon \to 0$ *to a* C^{r+1} *function* $\tilde{W}_0(\tilde{Q})$, *either at* $\tilde{Q} > 0$ *or at all real* $\tilde{Q}$. *The convergence is* C^{r+1}*-uniform on any closed finite interval of values of* $\tilde{Q}$ *from the domain of definition. Furthermore, the integral*

$$\int_1^{+\infty} \tilde{W}'_\epsilon(q)\frac{dq}{\sqrt{q}} \tag{2.8}$$

converges uniformly, for all sufficiently small ϵ.

In this definition H is the value of energy, see remarks after theorem 1.

The convergence of the integral (2.8) allows one to consider a function

$$F(v) = -\int_0^{+\infty} \tilde{W}_0'(v+x^2)dx \equiv -\frac{1}{2}\int_v^{+\infty} \tilde{W}_0'(Q)\frac{dQ}{\sqrt{Q-v}}. \qquad (2.9)$$

It is defined either on R$^+$ or on R (i.e., on the domain of definition of $\tilde{W}_0$) and it is, obviously, C^r-smooth. We take a convention that the pattern function Q increases across the billiard boundary when moving inwards, therefore W' is negative so F is a positive function.

The following theorem is the main result of the paper:

Theorem 1: *Consider a scattering billiard which has a simple singular periodic orbit. Consider a two parameter family of C^r, $r \geq 5$, smooth Hamiltonian flows $h_t(\epsilon,\gamma)$ of the form (2.1) with a smooth billiard potential, approximating the billiard flow as $(\epsilon,\gamma) \to 0$. Assume that the barrier function near the point of tangency satisfies the scaling assumption [S] for some $\delta(\epsilon, H)$ and that the associated function F is such that the range of values of $F'(v)$ includes* R$^-$. *Finally, assume that Q dependence on γ is in general position (i.e., (3.13) holds).*

Then, for small ϵ, in the (γ,ϵ) plane there exists a wedge $C^-\delta(\epsilon,H) < \gamma < C^+\delta(\epsilon,H)$ (with some constant $C^{\pm}$) such that for the parameter values in this wedge, on the energy level H there exist elliptic islands of width proportional to $\delta(\epsilon,H)$.

Proof outline: The proof occupies Sections 3,4. Roughly speaking, we first show (Proposition 3.1) that the dynamics of the Hamiltonian flow in an $O(\delta)$-neighborhood of the tangent periodic trajectory is described, as $\epsilon \to 0$, by the two-dimensional map

$$\bar{u} = v, \quad \bar{v} = \xi(v + \frac{a}{\sqrt{\kappa}}F(v)) - (\xi - 2)\Gamma - u. \qquad (2.10)$$

It is the limit of the two-dimensional return map of the Hamiltonian flow to a local cross-section near the tangent trajectory, written in some rescaled coordinates. κ is the billiard's curvature at the tangency and the quantities

ξ and a ($|\xi| > 2$, $a > 0$) are defined by the billiard flow at $\epsilon = 0, \gamma = 0$; In fact, ξ is just the sum of the singular multipliers of L; Γ is the rescaled parameter γ.

We call (2.10) *the asymptotic normal form* of the local Poincaré map near the tangent trajectory. In proposition 4.1, we show that the map (2.10) has an elliptic island of finite size (in the rescaled to δ variables u and v) for a finite range of values of Γ. Returning to the original, non-scaled variables gives the δ-size island which appears in the wedge as stated. $\square$.

Few remarks are now in order; First, the theorem implies that given a billiard with a simple singular periodic orbit there exist one parameter families of Hamiltonian flows $h_t(\epsilon, \gamma(\epsilon))$ which r-converge to the billiard flow and for which elliptic islands of size $\delta(\epsilon, H)$ exist for all $\epsilon < \epsilon_0$.

Second, the size of the island depends, of course, on the choice of the coordinates - we choose the most natural coordinates so the theorem applies directly to the islands size as observed in the phase space coordinates (2.1). To be specific, given the singular periodic orbit L, we choose the origin as the point of tangency, and the x-axis parallel to L at that point. A local Poincaré map in a neighborhood of L is defined by taking $x = const$, and the co-ordinates on the cross-section are the variables y and $y' = p_y/p_x$ (here $p_x \approx \sqrt{H}$, hence, when large H values are considered this additional factor arises).

Thirdly, the billiard flow is clearly independent of H, hence, using assumption [S] it is possible to obtain local return map which is H-independent. It follows that for the Hamiltonian flow islands appear for all H values (and their size depends on H via $\delta(\epsilon, H)$ and $\sqrt{H}$ as explained above). For greater genericity, we also allow the chosen value of H to depend on ϵ.

Finally, the scaling assumption [S] is satisfied by all potentials listed in (2.6). Moreover, given a barrier function W, in many cases, the scaling constant δ and the function F involved in the above estimate of the size of elliptic islands are fairly easy to compute. Before embarking into the details of the proof, we demonstrate the above statement for typical examples of the list (2.6).

2.3 Islands size for specific potentials.

Consider the barrier function

$$W(Q,\epsilon) = \frac{\epsilon}{Q^\alpha}.$$

It satisfies assumption [S] provided

$$\delta(\epsilon, H) = \left(\frac{\epsilon}{H}\right)^{\frac{1}{\alpha+3/2}}, \quad \nu = 0,\ \beta = 0, \tag{2.11}$$

for which

$$\tilde{W}_0(\tilde{Q}) = \frac{1}{\tilde{Q}^\alpha}.$$

Integration of (2.9) gives in this case

$$F(v) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha)}\frac{1}{v^{\alpha+1/2}}.$$

The function F satisfies the condition of the theorem so for such potentials the size of elliptic islands is given by δ of (2.11), namely it is polynomial in ϵ/H. Note that the size of islands is larger (asymptotically) for larger α (though the limit $\alpha \to \infty$ cannot be taken in (2.11)).

Choosing

$$W(Q,\epsilon) = \exp\left(-Q/\epsilon\right)$$

will satisfy the scaling assumption, provided

$$\delta = \epsilon, \quad \nu = 0,\ \beta = -\epsilon \ln(\delta^{\frac{3}{2}} H), \tag{2.12}$$

for which

$$\tilde{W}_0(\tilde{Q}) = \exp\left(-\tilde{Q}\right)$$

and

$$F(v) = \frac{\sqrt{\pi}}{2}e^{-v}.$$

Thus, in this case the elliptic islands are of size ϵ, independent asymptotically of the value of H (in the (y, y') coordinates).

An example which has been used by many authors [8, 3] is:

$$W(Q,\epsilon) = (1-Q^\alpha)^{1/\epsilon}.$$

It is obviously equivalent to

$$W(Q,\epsilon) = \exp\left(-Q^{\alpha}/\epsilon\right)$$

which is analogous to the previous example; if $\epsilon^{\frac{3}{2\alpha}} H \to 0$, one may assume

$$\delta = \frac{\epsilon^{1/\alpha}}{\alpha}(-\ln(\epsilon^{\frac{3}{2\alpha}} H))^{(1-\alpha)/\alpha}, \quad \nu = 0,\ \beta = \left(-\epsilon \ln(\delta^{\frac{3}{2}} H)\right)^{1/\alpha}. \tag{2.13}$$

Here again

$$\tilde{W}_0(\tilde{Q}) = \exp\left(-\tilde{Q}\right)$$

and

$$F(v) = \frac{\sqrt{\pi}}{2} e^{-v}.$$

Thus the islands size is again polynomial in ϵ. The critical exponent of $H \propto \epsilon^{-3/2\alpha}$ arises from the analysis - if H grows faster - different asymptotics applies. Also, notice the different behavior for $\alpha > 1$ vs $\alpha < 1$, again we see that steeper decay of the potential gives rise to larger islands.

Finally, for

$$W(Q,\epsilon) = \epsilon |\ln Q|^{\alpha}$$

take

$$\delta = \left(\frac{\alpha\epsilon}{H}\right)^{2/3} \left(\frac{2}{3}|\log\frac{\alpha\epsilon}{H}|\right)^{\frac{2(\alpha-1)}{3}}, \quad \nu = \epsilon|\ln\delta|^{\alpha}, \quad \beta = 0, \tag{2.14}$$

for which

$$\tilde{W}_0(\tilde{Q}) = -\ln\tilde{Q}$$

and

$$F(v) = \frac{\pi}{2\sqrt{v}}.$$

3 The Poincaré map near a tangent periodic orbit.

Here we formulate and prove Proposition 3.1 describing the local return map near a simple singular periodic orbit. Consider the two parameter family of

Hamiltonian systems (2.1) with smooth billiard potentials obeying the scaling assumption [S] (by which $\delta, \beta, \tilde{W}(\tilde{Q})$ are determined), which r-converge to the embedding family of the billiard flow with the simple singular periodic orbit L as above.

Choose the coordinates (x, y) such that the origin is at the point of tangency and the x-axis is tangent to the boundary - i.e., it is parallel to the tangent piece of the singular trajectory L. On a fixed energy level H take a small two-dimensional cross section $\Sigma = \{x = -c\}$ to this piece of L, where c is some fixed small positive number (see figure 3.1). For greater genericity, we assume that c may depend on ϵ and γ and it tends to some small positive value as $\epsilon, \gamma \to 0$.

The variables y and $y' \equiv p_y/p_x$ serve as the coordinates on Σ: given fixed y', the value of p_x is found from the energy constrain because $\frac{\partial H}{\partial p_x} \equiv p_x(1+y'^2)$ does not vanish on Σ due to the choice of the coordinate frame.

Proposition 3.1 *There is a region on the cross-section Σ on the energy level H where the local return map B_ϵ of the smooth Hamiltonian flow may be reduced, by an affine transformation $(y, y') \mapsto (u, v)$ which has a finite limit (with non-zero Jacobian) as $\epsilon \to 0$, to the form:*

$$\left\{ \begin{array}{l} \bar{u} = v + \dots \\ \bar{v} = \xi(v + \frac{a}{\sqrt{\kappa}}\delta F\left(\frac{v + \dots}{\delta}\right)) - (\xi - 2)v_\gamma - u + \dots \end{array} \right. \tag{3.1}$$

where the function F is defined by (2.9), $\delta(\epsilon, H)$ is the scaling constant from (2.7), κ is the curvature of the billiard at the tangency point at $\gamma = 0$, the quantities a, ξ and v_γ have a finite limit as $\epsilon \to 0$ where $|\xi| > 2$, $a > 0$, and v_γ vanishes at $\gamma = 0, \epsilon = 0$. The map (3.1) is defined at $\|(u, v)\| \leq K\delta$ (v must be strictly positive when the domain of definition of F is R^+) where K may be taken arbitrarily large provided ϵ is small enough. The dots in (3.1) stand for smooth functions of $(u/\delta, v/\delta)$ which, along with all derivatives, are $o(\delta)$ as $\epsilon \to 0$.

Proof: Similar to [17], we construct the return map B_ϵ from two artificial maps $B_\epsilon^{(r)}$ and $B_\epsilon^{(s)}$, both defined as the maps of the cross-section Σ to itself. In sections 3.1 and 3.2 we describe these constructions (the details of the corresponding proofs are deferred to appendices A and B) and in section 3.3 we prove that their composition may be transformed to (3.1).

3.1 The regular map.

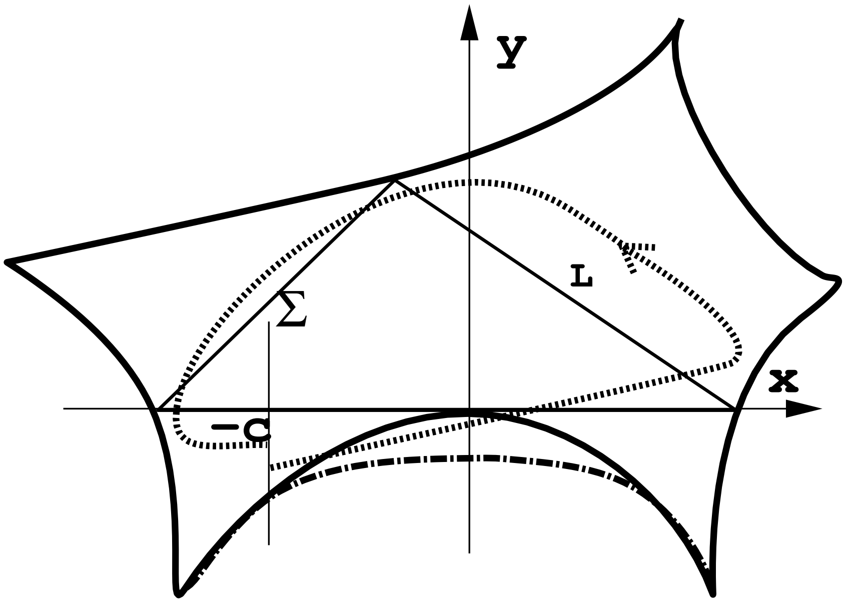

Figure 3.1: Auxilary regular local map near tangency.
$\cdot - \cdot - \cdot - \cdot - \cdot$ is the regularization of D w.r.t. L.
$\ldots$ is a trajectory of the auxiliary Hamiltonian flow.

The regular map $B_\epsilon^{(r)} : \Sigma \rightarrow \Sigma$ is constructed in appendix A by ignoring the passage near the tangent point at the origin. Namely, we locally modify the function Q near the point of tangency so that the billiard's boundary $Q(x, y; \gamma, 0) = 0$ is moved outward, see figure 3.1. Given a c, the modified Q is changed at most for $|x| < c$, near $Q = 0$. For this regularized billiard, the periodic orbit L survives and becomes non-singular. We consider an auxiliary Hamiltonian flow which corresponds to the modified function Q in (2.1),(2.3). The regular map $B_\epsilon^{(r)}$ is the Poincaré map for the auxiliary Hamiltonian flow. Since the auxiliary Hamiltonian flow is still generated by a smooth billiard potential, it r-converges [17] to the corresponding regularized billiard flow. In particular, it C^r-converges to the regularized billiard flow near the now non-singular periodic orbit L. Thus, $B_\epsilon^{(r)}$ is a C^r-diffeomorphism which, as $\epsilon \rightarrow 0$, approaches a finite limit, namely, the Poincaré map $B_0^{(r)}$ of the regularized billiard flow near L. Thus, using the hyperbolic structure of the regularized billiard map $B_0^{(r)}$, we prove in appendix A the following:

Lemma 3.1 *The map $B_\epsilon^{(r)} : \Sigma \to \Sigma$ is given by:*

$$B_\epsilon^{(r)} \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y_{\gamma,\epsilon} \\ p_{\gamma,\epsilon} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y - y_{\gamma,\epsilon} \\ y' - p_{\gamma,\epsilon} \end{pmatrix} + \dots \tag{3.2}$$

where $(y_{\gamma,\epsilon}, p_{\gamma,\epsilon})$ are the coordinates of the hyperbolic fixed point of $B_\epsilon^{(r)}$ corresponding to the periodic orbit L at $(\epsilon = 0, \gamma = 0)$ and the dots stand for non-linear terms. Then, $y_{\gamma,\epsilon}, p_{\gamma,\epsilon}$ have a finite limit as $\epsilon \to 0$ with $y_{0,0} = p_{0,0} = 0$, and the linearization matrix $\mathcal{B} = \{b_{ij}(\gamma, \epsilon)\}$ satisfies the following:

- *The sign of all the coefficients b_{ij} is identical:*

$$sign\, b_{ij} = sign\, \mathcal{B}. \tag{3.3}$$

- *Moreover,*

$$sign(b_{12} - cb_{11}) = sign(b_{22} - cb_{21}) = sign\, \mathcal{B}. \tag{3.4}$$

- *The determinant of $\mathcal{B}$ is 1.*
- *The magnitude of the trace of $\mathcal{B}$ is larger than 2.*

3.2 The singular map.

In appendix B the singular part $B_\epsilon^{(s)} : \Sigma \to \Sigma$ is constructed; Define the cross-section $\Sigma' = \{x = c\}$. Recall that the auxiliary Hamiltonian flow coincides with the original flow outside the size c neighborhood of the point of tangency. Now define $B_\epsilon^{(s)}$ as the composition of the map $\Sigma \to \Sigma'$ by the orbits of the original Hamiltonian flow and, then, the map $\Sigma' \to \Sigma$ defined by the backward orbits of the auxiliary Hamiltonian flow, see figure 3.2.

By construction, the local Poincaré map B_ϵ is the superposition

$$B_\epsilon = B_\epsilon^{(s)} \circ B_\epsilon^{(r)}. \tag{3.5}$$

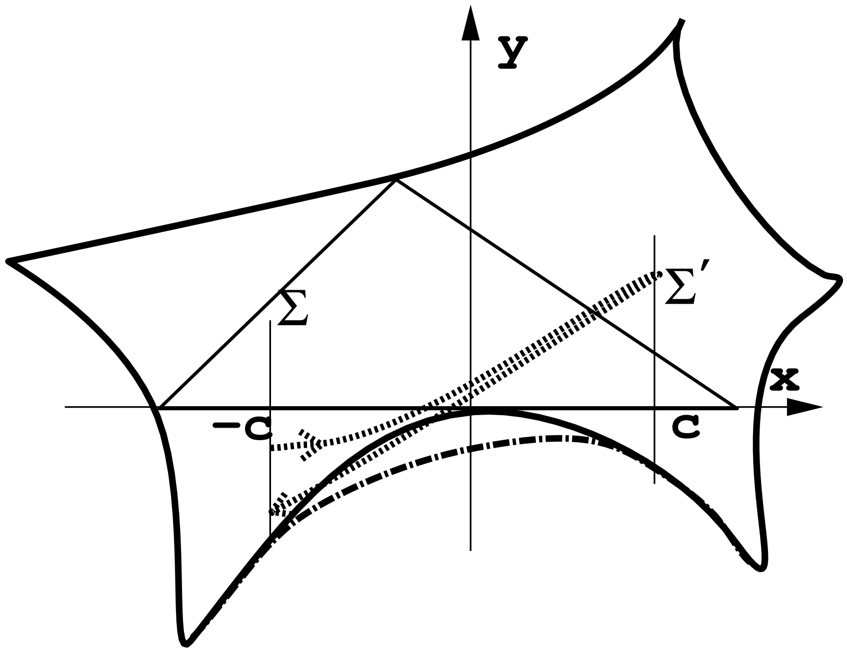

Figure 3.2: Singular local map near tangency.

The singular part is computed locally, using the scaling property. More precisely, in the appendix we prove the following:

Lemma 3.2 *The map* $B_\epsilon^{(s)} : \Sigma \rightarrow \Sigma$ *is given by:*

$$\begin{aligned} \bar{y} &= y_0(\epsilon) + y - c\frac{\delta}{\sqrt{\kappa}} F\left(\frac{y + cy' + o(\delta)}{\delta}\right) + o(\delta) \\ \bar{y}' &= p_0(\epsilon) + y' + \frac{\delta}{\sqrt{\kappa}} F\left(\frac{y + cy' + o(\delta)}{\delta}\right) + o(\delta), \end{aligned} \tag{3.6}$$

where $y' = p_y/p_x$*,* F *is defined by (2.9), and* $y_0, p_0 \rightarrow 0$ *as* $\epsilon \rightarrow 0$*; the quantity* κ *is half the curvature at the point of tangency. The formula (3.6) is valid for* $\|(y/\delta, y'/\delta)\| \leq K$ *where* K *may be taken arbitrarily large, provided* ϵ *is small enough; also, if* F *is defined only on* R^+*, the allowed values of* y *and* y' *are such that* $(y + cy')/\delta$ *is strictly positive. The symbol* $o(\delta)$ *denotes functions of* $(y/\delta, y'/\delta)$ *which, along with all derivatives with respect to* $(y/\delta, y'/\delta)$*, decay to zero faster than* $\delta(\epsilon)$ *as* $\epsilon \rightarrow 0$*.*

3.3 The local return map.

It follows from our construction (lemma 3.1 and 3.2) that the composition of the singular flow and the regularized billiard gives the desired return map (see (3.5)).

Now, superposition of $B_\epsilon^{(r)}$ (see (3.2)) and $B_\epsilon^{(s)}$ (see (3.6)) gives the following formula for the Poincaré map B_ϵ:

$$\left\{\begin{array}{l} \bar{y} = y_{\gamma,\epsilon} + b_{11}(y - y_{\gamma,\epsilon} + y_0) + b_{12}(y' - p_{\gamma,\epsilon} + p_0) + \\ \qquad (b_{12} - cb_{11})\frac{\delta}{\sqrt{\kappa}}F\left(\frac{y + cy' + o(\delta)}{\delta}\right) + o(\delta) \\ \bar{y}' = p_{\gamma,\epsilon} + b_{21}(y - y_{\gamma,\epsilon} + y_0) + b_{22}(y' - p_{\gamma,\epsilon} + p_0) + \\ \qquad (b_{22} - cb_{21})\frac{\delta}{\sqrt{\kappa}}F\left(\frac{y + cy' + o(\delta)}{\delta}\right) + o(\delta) \end{array}\right. \tag{3.7}$$

Making the affine transformation

$$\begin{array}{l} v = y + cy', \\ u = (b_{22} - b_{21}c)(y - y_{\gamma,\epsilon}) - (b_{12} - b_{11}c)(y' - p_{\gamma,\epsilon}) + y_{\gamma,\epsilon} - y_0 + c(p_{\gamma,\epsilon} - p_0), \end{array} \tag{3.8}$$

brings the map to the form (3.1), where

$$\xi = tr\mathcal{B}, \tag{3.9}$$

$$a = \frac{(1, c)\mathcal{B}(-c, 1)^T}{\xi} \tag{3.10}$$

and

$$v_\gamma = y_{\gamma,\epsilon} + cp_{\gamma,\epsilon} - \frac{(b_{11} + b_{21}c - 1)y_0 + (b_{12} + b_{22}c - c)p_0}{\xi - 2}. \tag{3.11}$$

Moreover, $a > 0$ by (3.4) and $|\xi| > 2$ by (3.9) (see Lemma 3.1).

Finally, the determinant of the transformation is given by:

$$J = \frac{\partial(u, v)}{\partial(y, y')} = (1, c)\mathcal{B}(-c, 1)^T = \xi a, \tag{3.12}$$

hence it clearly has a finite, non-vanishing limit as $\epsilon \to 0$, completing the proof of Proposition 3.1. □.

3.4 Non-degenerate dependence on γ.

Here we establish that under the assumptions of Proposition 3.1, and if additionally, the dependence of the smooth billiard potential on γ is in general position, then v_γ of (3.1) can be replaced by γ.

Indeed, consider the billiard flow ($\epsilon = 0$); then from (3.11) $v_\gamma = 2(y_{\gamma,\epsilon=0} + cp_{\gamma,\epsilon=0})$ where the point $M_\gamma = (y_{\gamma 0}, p_{\gamma 0})$ is the hyperbolic fixed point of the regular map which corresponds to the regularized billiard. Notice that on the cross-section $\Sigma : \{x = -c\}$ the straight line $v = y + cy' = 0$ is tangent to the *singularity line* (the curve of initial conditions corresponding to tangent trajectories). Moreover, the point M_0 belongs to the singularity line by assumption, and by definition $M_0 = (0, 0)$. Thus, $v_0 = 0$. We make the following *non-degenericity assumption*:

The fixed point M_γ of the regular map $B_0^{(r)}$ crosses the singularity line with non-zero velocity as γ varies.

Since v measures the distance from the singularity line, it is equivalent to

$$\frac{\partial v_\gamma}{\partial \gamma} \neq 0. \tag{3.13}$$

Thus γ may be rescaled so that

$$v_\gamma \equiv \gamma. \tag{3.14}$$

Notice that the non-degenericity assumption is formulated in terms of the billiard flow solely; thus, it describes how the boundary of the billiard depends on γ. It is a condition of general position: if it is not satisfied, it may be easily achieved by a small smooth perturbation of the function $Q(x, y; \gamma, 0)$: the position of the singularity line is determined only by the local behavior of Q near the point of tangency whereas the position of the auxiliary fixed point M is determined only by the local behavior of Q at the points of regular collision of L with the boundary.

The non-degeneracy condition is simple and natural when the dependence on γ is localized near the tangent point (e.g., when artificially embedding the singular billiard in a family of billiards as in regularization); In this case the periodic orbit of the auxiliary billiard does not move as γ varies. This means that $y'_\gamma = 0$. Since the coordinates on Σ are defined so that the orbit starting at $(y=0, y'=0)$ is tangent to the boundary, it follows that $\partial y_\gamma / \partial \gamma \neq 0$ provided the boundary near the point of tangency moves, as γ varies, with non-zero velocity in the normal direction (e.g., see figure 3.1).

4 Islands.

To complete the proof of Theorem 1, we now establish that the Poincaré map found in Proposition 3.1 attains elliptic islands as stated in the Theorem. Let us consider the following rescaling of (3.1):

$$u' = \frac{u}{\delta}, \quad v' = \frac{v}{\delta}, \quad \Gamma = \frac{v_\gamma}{\delta}. \tag{4.1}$$

In the new variables the map (3.1) has the map (2.10) as a limit as $\epsilon \to 0$. Since $O(1)$ intervals in the rescaled variables correspond to $O(\delta)$ intervals in the original variables, it follows that once we show that for a finite range of values of Γ the map (2.10) has an elliptic island of a finite size, we are done.

Thus, theorem 1 reduces to the following statement.

Proposition 4.1: *Assume the range of values of $F'(v)$ includes R^-. Then, for any $|\xi| > 2$ and $a > 0$ there exists an interval of values of Γ for which the map (2.10) has an elliptic fixed point with non-zero first coefficient in the Birkhoff normal form.*

Proof: Fixed points of (2.10) are:

$$u_f = v_f, \quad \Gamma_f(v_f) = v_f + \frac{a\xi}{\sqrt{\kappa}(\xi - 2)} F(v_f) \tag{4.2}$$

which defines $\Gamma_f(v_f)$ for all v_f in the domain of definition of F (e.g. R^+). The fixed points are linearly stable if and only if:

$$v_f \in I_s = \{v_f | \; -(1 + \frac{2}{|\xi|}) < \frac{a}{\sqrt{\kappa}} F'(v_f) < -(1 - \frac{2}{|\xi|})\}. \tag{4.3}$$

Namely I_s is a union of open intervals of stability in v_f, which is non-empty if, for example, the range of $F'(v)$ includes R^- as required, leading to intervals of the corresponding Γ_f values.

On I_s, the eigenvalues of the fixed point are $e^{\pm i\omega}$ where

$$\cos\omega = \frac{\xi}{2}(1 + \frac{a}{\sqrt{\kappa}}F'(v_f)). \tag{4.4}$$

Since F' takes all negative values, there exists at least one interval of linear stability from I_s on which the value of ω runs from 0 to π.

To prove the lemma we need to show that at least for one v_f from the given interval the first coefficient of the Birkhoff normal form at the elliptic fixed point is non-zero (according to Moser theorem this will prove that at the corresponding value of Γ - and for all close Γ - there exists an elliptic island surrounding the fixed point).

The sought Birkhoff coefficient equals to

$$B(v_f) = -\frac{1}{8\sin^3\omega}\left(\frac{\xi a}{2\sqrt{\kappa}}F'''(v_f) + \left(\frac{\xi a}{2\sqrt{\kappa}}F''(v_f)\right)^2 \frac{(1+4\cos\omega)}{(1-\cos\omega)(1+2\cos\omega)}\right). \tag{4.5}$$

Indeed, the linear transformation $z = e^{i\omega}(v - v_f) - (u - u_f)$ brings the map (2.10) to the following form near the fixed point:

$$\bar{z} = e^{i\omega}z - \frac{\xi a}{\sqrt{\kappa}}F''(v_f)\frac{e^{i\omega}}{8\sin^2\omega}(z - z^*)^2 + i\frac{\xi a}{\sqrt{\kappa}}F'''(v_f)\frac{e^{i\omega}}{48\sin^3\omega}(z - z^*)^3 + \dots \tag{4.6}$$

where the dots stand for higher order terms.

An arbitrary two-dimensional conservative map near an elliptic fixed point is written as

$$\bar{z} = e^{i\omega}z + e^{i\omega}\left(\alpha z^2 - 2\alpha^* z z^* + \eta z^{*2}\right) + e^{i\omega}\beta z^2 z^* + \dots \tag{4.7}$$

where

$$Re\beta = \alpha\alpha^* - \eta\eta^* \tag{4.8}$$

(which is necessary to have the Jacobian equal to one) and the dots stand for the rest of cubic and higher order terms. In the case of map (4.6)

$$\alpha = \eta = -\frac{\xi a F''(v_f)}{8\sqrt{\kappa}\sin^2\omega} \tag{4.9}$$

and

$$\beta = -i\frac{\xi a F'''(v_f)}{16\sqrt{\kappa}\sin^3\omega}. \tag{4.10}$$

If $\omega \neq 2\pi/3$, a symplectic transformation of the form

$$z' = z - \left[\frac{e^{i\omega}\alpha}{e^{2i\omega} - e^{i\omega}} z^2 - 2\frac{e^{i\omega}\alpha^*}{1 - e^{i\omega}} z z^* + \frac{e^{i\omega}\eta}{e^{-2i\omega} - e^{i\omega}} z^{*2}\right] + \left[\frac{\frac{1}{2}\eta\eta^*}{1 - \cos 3\omega} - \frac{\frac{1}{2}\alpha\alpha^*}{1 - \cos\omega}\right] z^2 z^* + \dots \tag{4.11}$$

eliminates all quadratic terms in (4.7) and brings it to the form (dropping $'$)

$$\bar{z} = e^{i\omega} z + i e^{i\omega} B z^2 z^* + \dots \tag{4.12}$$

where B is the first Birkhoff coefficient:

$$B = Im\beta + \frac{\sin\omega}{1 - \cos\omega}\left[\eta\eta^* \frac{1 - 2\cos\omega}{1 + 2\cos\omega} - 3\alpha\alpha^*\right]. \tag{4.13}$$

Plugging (4.9) and (4.10) in (4.13) gives (4.5).

If $\omega \neq 2\pi/3$, $\omega \neq \pi/2$, all the other cubic and quartic terms in (4.12) are non-resonant, hence they can be eliminated by the further normalizing transformation. Then, in the symplectic polar coordinates $z = \sqrt{r}e^{i\theta}$ the map becomes

$$\bar{r} = r + O(r^3), \quad \bar{\theta} = \theta + \omega + Br + O(r^2). \tag{4.14}$$

When $\bar{\theta}'_r|_{r=0} \equiv B \neq 0$, the KAM-theory is applied to this map at small r, which gives the existence of the sought elliptic island.

Now note that the quantity $B(v_f)$ given by (4.5) cannot be identically zero on the interval of values of v_f under consideration. Indeed, if for example $F''(v_f)$ never vanishes, then $B(v_f)$ tends to infinity when approaching the bifurcation point $v_f = v_f^1$ which corresponds to $\cos\omega = 1$. One can check that even if $F''(v_f^1)$ vanishes but the zero is of finite order, then still $B(v_f)$ is not identically zero near $v_f = v_f^1$. Indeed, in this case, for some finite $n \geq 1$ and $d \neq 0$,

$$\begin{aligned} F'(v_f) &= F'(v_f^1) + d(v_f - v_f^1)^n + \dots, \\ F''(v_f) &= nd(v_f - v_f^1)^{n-1} + \dots \\ F'''(v_f) &= n(n-1)d(v_f - v_f^1)^{n-2} + \dots \end{aligned} \tag{4.15}$$

then, from (4.4)

$$\cos\omega = 1 + \frac{da\xi}{2\sqrt{\kappa}}(v_f - v_f^1)^n + \dots \tag{4.16}$$

hence, from (4.5)

$$B(v_f) = \frac{da\xi}{16\sqrt{\kappa}\sin^3\omega} n(1 + \frac{8}{3}n)(v_f - v_f^1)^{n-2} + \ldots \not\equiv 0. \tag{4.17}$$

This proofs the lemma for the case where the function F is analytic (as in all our examples). In this case, it follows that B may vanish only at isolated points in the interval of stability.

Without the assumption of the analyticity, if $B(v_f)$ were identically zero on the interval under consideration, we would have by (4.5)

$$\frac{\xi a}{2\sqrt{\kappa}} F'''(v_f) + \left(\frac{\xi a}{2\sqrt{\kappa}} F''(v_f)\right)^2 \frac{(1 + 4\cos\omega)}{(1 - \cos\omega)(1 + 2\cos\omega)} = 0 \tag{4.18}$$

or, in view of (4.4),

$$\frac{d^2\cos\omega}{dv_f^2} + \left(\frac{d\cos\omega}{dv_f}\right)^2 \frac{(1 + 4\cos\omega)}{(1 - \cos\omega)(1 + 2\cos\omega)} = 0. \tag{4.19}$$

This equation is easily integrated; on the interval $\cos\omega \in [-\frac{1}{2}, 1]$ we get

$$\frac{d\cos\omega}{dv_f} = c\sqrt[3]{(1 - \cos\omega)^5(1 + 2\cos\omega)}, \tag{4.20}$$

where c is some constant which cannot be zero because $\cos\omega$ must run all the interval under consideration as v_f varies. At $\cos\omega$ close to 1 we have

$$\frac{d\cos\omega}{dv_f} \sim (1 - \cos\omega)^{5/3}. \tag{4.21}$$

This means that ω never approaches zero at finite v_f which contradicts the initial assumption (that the range of F' includes all negative values, see (4.4)).

$\square$.

As an example, consider numerical simulations of the map (2.10) for $W(Q) = \epsilon/Q$ which confirm the existence of $O(1)$ islands; the map (2.10) becomes:

$$\begin{cases} \bar{u} = v \\ \bar{v} = \xi v - (\xi - 2)\Gamma - u + \xi\frac{\pi}{4}\frac{a}{\sqrt{\kappa}}\frac{1}{v^{3/2}} \end{cases} \tag{4.22}$$

Taking $\xi = 4$, $\pi over4\frac{a}{\sqrt{\kappa}} = 1$ makes $I_s = (1, 3^{0.4} \approx 1.5518)$ and thus an interval of stability for $\Gamma \in (\frac{5}{3}3^{0.4} \approx 2.586, 3.)$, figure 4.1 confirms the existence of elliptic island for $\Gamma = 2.6, 2.8$. In fact it is seen that even for $\Gamma = 3.$ the stability island still exists.

5 Discussion.

Our main result, Theorem 1, together with assumption [S], supplies a computational tool for estimating the size of elliptic islands for specific potentials. For natural Hamiltonians which appear in physical setting, the estimated size of the islands is polynomial in ϵ, where ϵ is the small parameter which controls the steepness of the potential near the core. Moreover, the dependence of the islands size on the energy has also been explicitly computed (see sections 2.2 and 2.3) and is also typically polynomial.

Furthermore, we introduced the notion of singular multipliers for tangent periodic orbits of billiard. These measure the impact of the singular periodic orbit. In our setting these multipliers play a central role in the dynamics as explained next. We believe billiards with singular periodic orbits are dense among scattering billiards hence that the concept of singular multipliers may be useful in other investigations of billiard problems.

While proving the rigorous estimates for the islands size, we have constructed the following local map near singular periodic orbits (2.10) :

$$\begin{cases} \bar{u} = v \\ \bar{v} = \xi v + \frac{a\xi}{\sqrt{\kappa}} F(v) - u + (\xi - 2)\Gamma. \end{cases} \tag{5.1}$$

This map defines universality classes - large classes of different billiard geometries and different potentials give rise to exactly the same local map. The Jacobian of the transformation to this map is $J = \frac{\xi a}{\delta^2}$ and the unfolding parameter $\Gamma = v_\gamma/\delta$, see (3.9,3.10,3.12). Now δ and $F(\cdot)$ are the rescaling parameter and rescaling function which depend only on the rate at which the smooth billiard potential approaches a billiard - they are independent of the billiard geometry (see section 2.3 for examples of δ, F). Conversely, κ, ξ, a are parameters which depend only on the billiard flow (the geometry): κ is the local curvature near the tangent periodic orbit (local parameter). ξ is the sum of the singular multipliers measuring the instability of the singular periodic orbit (global parameter, independent of the local structure near tangency). The parameter a measures, roughly, how the regular flow rotates the singularity line - the larger the a the closer the image of the singularity line to its normal. The larger the ξ the closer this image is to the unstable eigenvector of the regularized billiard return map, thus for large ξ, the quantity a measures the alignment between this eigenvector and the normal to the singularity line.

Summarizing, the coefficient of the nonlinear term in (3.1) is given by $\frac{1}{\sqrt{\kappa}}\xi a$ which shows how the curvature, the singular multipliers and the global billiard geometry effects (respectively) scale together to create a significant nonlinear term.

5.1 Are the islands observable?

The disappearance of two hyperbolic periodic orbits via a simple singular periodic orbit is a codimension one bifurcation as explained in the proof of lemma 3.3. This implies, in particular, that such a bifurcation persists in any one parameter family of scattering billiard. In [17], we conjectured that billiards with simple singular periodic orbits are dense among the scattering billiards; If true, the above observation implies that any one parameter family of scattering billiards must undergo such a bifurcation along a dense set of values of the parameter γ. Then, by theorem 1, islands exist in a $\delta(\epsilon)$ wedge emanating from each such γ.

Clearly the width of the wedge in parameter space and the size of the island depend on its period N. Indeed, examine the map (2.10). It depends on the two parameters ξ and $\xi a/\sqrt{\kappa}$. For periodic orbits of long period, one expects large singular multipliers hence large ξ: if λ_L is the Lyapunov exponent of the billiard map, we expect that for large N

$$\xi = \Lambda + \frac{1}{\Lambda} \approx \exp(N\lambda_L). \tag{5.2}$$

On the other hand, as discussed above, in this limit the parameter a asymptotes a constant positive value. It follows (see (4.2), (4.3)) that the stability interval in γ values is proportional to $\frac{\delta}{\xi} = \delta(\epsilon)\exp(-N\lambda_L)$. Finally, recall that the transformation to (2.10) involved a Jacobian $\xi\frac{1}{\delta^2}a$ thus it follows that order one areas in (2.10) correspond to order $\delta^2\exp(-N\lambda_L)$ areas in the physical variables (y, y').

We see that the period of the singular periodic orbit plays a crucial role in the islands' size estimates as well. Consider a value γ^* for which no singular periodic orbits exist. Consider an ϵ-ball in phase space around a point lying on the singularity line. Using the exponential stretching rates of the billiard, one can argue that in this ball there exists a periodic point of period $N = -2\frac{\ln\epsilon}{\lambda_T}$, where λ_T denotes the exponential growth rate of lines under the billiard flow[1]. Now, assuming that ϵ size perturbations of the billiard geometry (i.e. in γ) may be made so that the periodic orbit of period N does not hit the corner points (this is another formulation of the density assumption), we obtain that there is a singular periodic orbit of period of order N for a γ value which is ϵ-close to γ^*. Summarizing, provided our density conjecture is correct, in an ϵ ball around $(\gamma,\epsilon) = (\gamma^*, 0)$, for any value of γ^* *there exist a polynomial (in ϵ) size set of parameter values for which polynomial size islands of period $O(\ln\epsilon)$ exist.* So, the islands are observable.

Acknowledgment

We thank H. Primack, Y. Sinai and U. Smilansky for useful discussions. This work has been supported by a research grant from the Henri Gutwirth fund for research. RK also acknowledges the support of the Jakubskind-Cymerman Prize.

A The regular map $B_\epsilon^{(r)}$.

Here we prove Lemma 3.1 regarding the form of the regular auxiliary map. The map $B_\epsilon^{(r)} : \Sigma \to \Sigma$ is constructed by ignoring the passage near the tangent point at the origin, by locally moving the boundary outward as shown in figure 3.1. For the obtained auxiliary billiard the singular periodic orbit L becomes regular. Therefore [17], the map $B_\epsilon^{(r)}$ which is the Poincaré map for for the auxiliary Hamiltonian flow is C^r-close to the Poincaré map $B_0^{(r)} : \Sigma \to \Sigma$ for the auxiliary billiard flow.

Since L is now regular and the auxiliary billiard is still scattering, the fixed point $(y = 0, y' = 0)$ of $B_0^{(r)}$ at $\gamma = 0$ is hyperbolic. Thus, for small γ and ϵ it persists. Denoting the fixed point as $M_{\gamma,\epsilon} = (y_{\gamma,\epsilon}, p_{\gamma,\epsilon})$, let us write the Taylor expansion for $B_\epsilon^{(r)}$ at $M_{\gamma,\epsilon}$:

$$B_\epsilon^{(r)} \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y_{\gamma,\epsilon} \\ p_{\gamma,\epsilon} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y - y_{\gamma,\epsilon} \\ y' - p_{\gamma,\epsilon} \end{pmatrix} + \dots \tag{A.1}$$

---

[1] In fact, line elements may grow only quasi-exponentially in the billiard flow [2]. However, this only slightly changes our estimates, producing islands of period $O((\ln\epsilon)^\beta)$ instead of simple logarithim in ϵ.

where the dots stand for nonlinear terms. Now, we establish some inequalities on the coefficients of the linearization matrix $\mathcal{B} = \{b_{ij}(\gamma, \epsilon)\}$ which will be used in constructing the asymptotic normal form.

The linearization matrix $\mathcal{B} = \{b_{ij}(\gamma, \epsilon)\}$, depends continuously on the parameters. At $\epsilon = 0$, it is the linearization matrix of the Poincaré map for the billiard flow, therefore it must fit the specific hyperbolic structure of the scattering billiard. Thus, it is known [13, 18] that the tangent to the unstable manifold of the fixed point M lies in the unstable cone $dy \cdot dy' > 0$ and the tangent to the stable manifold lies in the stable cone $dy \cdot dy' < 0$. Moreover, the unstable cone is mapped inside itself by the linearized map, and the stable cone is mapped inside itself by the inverse to the linearized map. It follows that all the b_{ij} are of same sign, which proves (3.3).

In fact, the invariance of the stable and unstable cones of this particular form holds for the map from any cross-section to any other (near each of the cross-sections the x-axis must be parallel to the piece of the orbit which is cut by the given cross-section). It is geometrically evident for the scattering billiard that upon each regular reflection with the boundary the orientation is changed on the unstable (as well as on the stable) manifold: after the reflection the positive part of the unstable cone is mapped into the negative one and vice versa. Since the sign of b_{ij} determines whether the linearized map preserves orientation on the unstable manifold of M or not, it follows that

$$sign\ b = (-1)^N \tag{A.2}$$

where N is the number of regular collisions of L with the boundary.

At $\gamma = 0$ the fixed point M on the cross-section $\Sigma : \{x = -c\}$ is $(y = 0, y' = 0)$. Infinitesimal increments dy and dy' which satisfy $dy + cdy' = 0$ correspond to a family of rays which focus at $(x = 0, y = 0)$. It is the point of tangency of L to the boundary of the original billiard. For the auxiliary billiard, the boundary is pushed outward near this point. Thus, in the auxiliary billiard, the family of rays pass the focusing point without reflection and becomes divergent. This means that after the first regular reflection the image of the vector $(dy = -cdy', dy')$ must belong to the unstable cone. Moreover, it is geometrically evident that it belongs to the same (positive or negative) part of the unstable cone as the image of the vector $(dy = 0, dy')$ for the same dy'. This means that (see another proof in [17])

$$\text{sign}(b_{12} - cb_{11}) = \text{sign}(b_{22} - cb_{21}) = \text{sign}\ \mathcal{B}, \tag{A.3}$$

namely (3.4) holds.

The map $B^{(r)}$ is a measure-preserving diffeomorphism, so

$$b_{11}b_{22} - b_{12}b_{21} = 1.$$

Finally, the trace of the matrix $\{b_{ij}\}$ is the sum of the multipliers of the hyperbolic fixed point, therefore,

$$|b_{11} + b_{22}| > 2. \tag{A.4}$$

B The singular map $B_\epsilon^{(s)}$.

Here we prove Lemma 3.2 regarding the form of the singular map.

The equations of motion near the boundary are found from (2.1):

$$\begin{array}{ll} \dot{x} = p_x, & \dot{p}_x = -W'(Q)Q_x \\ \dot{y} = p_y, & \dot{p}_y = -W'(Q)Q_y \end{array} \tag{B.1}$$

It is convenient to move the origin to the level $Q(x,y;\gamma,\epsilon) = \beta(\epsilon, H)$ where β is defined by the scaling assumption [S], to the point where the x-axis is tangent to this level; such point is unique for all small ϵ and γ because the curvature of the boundary never vanishes. Hence:

$$Q_x|_{(0,0;\gamma,\epsilon)} = 0, \quad Q_y|_{(0,0;\gamma,\epsilon)} = 1. \tag{B.2}$$

Since the curvature of the boundary is non-zero, it follows that

$$Q = \beta + y + \kappa(\gamma,\epsilon)x^2 + O(xy, y^2, x^3) \tag{B.3}$$

where the curvature factor κ is strictly positive.

The move of the origin leads to a small shift of the cross-section Σ and of the position of the origin on the cross-section; this, obviously, does not affect Proposition 3.1.

Let us prove that the map from $\Sigma : \{x = -c\}$ to $\Sigma' : \{x = c\}$ by the forward flow is written as

$$\begin{aligned} y_{out} &= y_{in} + c(y'_{in} + y'_{out}) + o(\delta) \\ y'_{out} &= y'_{in} + \frac{\delta}{\sqrt{\kappa}} F\left(\frac{y_{in} + cy'_{in} + o(\delta)}{\delta}\right) + o(\delta), \end{aligned} \tag{B.4}$$

where "in" refers to Σ and "out" refers to Σ'; the function F is defined by (2.9) and $o(\delta)$ has the same sense as in Lemma 3.2.

The map $B^{(s)}$ is the composition of (B.4) and the backward motion from Σ' to Σ with the auxiliary Hamiltonian flow. In the auxiliary Hamiltonian flow the billiard's boundary is pushed outside on a finite distance so the potential asymptotically vanish everywhere between Σ' and Σ. Therefore, the auxiliary Hamiltonian flow is here C^r-close to the motion with a constant speed as $\epsilon \to 0$. Thus, the backward map is written as

$$\begin{aligned} \bar{y}_{in} &= y_0(\epsilon) + y_{out} - 2cy'_{out} + o(\delta) \\ \bar{y}'_{in} &= p_0(\epsilon) + y'_{out} + o(\delta). \end{aligned} \tag{B.5}$$

(This is just the Taylor expansion near $(y_{out}, y'_{out}) = 0$: since $(y_{out}, y'_{out}) = O(\delta)$ in Lemma 3.1, the nonlinearities are $o(\delta)$ in the sense defined there.) The composition of (B.5) and (B.4) gives (3.6). Thus, to prove the lemma, it is sufficient to prove the formula (B.4).

Notice that $p_x > 0$ near the tangent trajectory, thus one may use x as the new time variable. Fixing the energy implies:

$$2(H - W(Q,\epsilon)) = p_x^2(1 + y'^2) \tag{B.6}$$

and equations (B.1) are rewritten as

$$\frac{dy}{dx} = p_y/p_x, \quad \frac{d^2y}{dx^2} = -W'(Q,\epsilon)\frac{Q_y(x,y) - Q_x(x,y)\dfrac{dy}{dx}}{2(H - W(Q,\epsilon))}(1 + \left(\frac{dy}{dx}\right)^2) \tag{B.7}$$

Let us rescale the variables, taking into account the smallness of y and y':

$$y = \delta\tilde{y}, x = \sqrt{\delta}\tilde{x}, \frac{dy}{dx} = \delta p. \tag{B.8}$$

¿From (B.3) it follows that

$$Q = \beta + \delta(\tilde{y} + \kappa\tilde{x}^2 + O(\sqrt{\delta})). \tag{B.9}$$

Let us denote $\tilde{Q} = (Q - \beta)/\delta$:

$$\tilde{Q} = \tilde{y} + \kappa\tilde{x}^2 + O(\sqrt{\delta}). \tag{B.10}$$

Now, the scaling assumption [S] is designed so that (B.7) becomes, in the rescaled variables,

$$\begin{aligned} \tilde{y}' &= \sqrt{\delta} p \\ p' &= -\frac{1}{2}\tilde{W}_\epsilon'(\tilde{Q})G(\tilde{x}, \tilde{y}, p; \epsilon) \end{aligned} \tag{B.11}$$

where $'$ in the left-hand side denotes the differentiation with respect to $\tilde{x}$; the function G is

$$G = \frac{Q_y(\sqrt{\delta}\tilde{x}, \delta\tilde{y}) - \delta Q_x(\sqrt{\delta}\tilde{x}, \delta\tilde{y})p}{1 - \frac{\delta}{1-\nu/H}\sqrt{\delta}\tilde{W}_\epsilon(\tilde{Q})} \; \frac{1+\delta^2 p^2}{1-\nu/H}. \tag{B.12}$$

For each fixed $(\tilde{x}, \tilde{y}, p)$, as $\epsilon \to 0$, this system has the following limit (we use (B.2),(B.10) and that $\nu/H \to 0$ by the scaling assumption):

$$\tilde{y}' = 0, \quad p' = -\frac{1}{2}\tilde{W}_0'(\tilde{y} + \kappa\tilde{x}^2). \tag{B.13}$$

Let us take some $X(\epsilon)$ which tends to infinity slowly (with respect to δ) so that the solution of (B.11) still approach (along with all derivatives with respect to the initial conditions) the solution of (B.13) as $\epsilon \to 0$, on the unbounded interval $\tilde{x} \in [-CX, CX]$ for any fixed C, provided $\tilde{y}$ and p stay bounded (if $\tilde{W}_0$ is defined on R^+ only, $\tilde{y}$ should also stay positive and separated from zero), see figure B.1. Then, take a cross-section $\Sigma_\delta = \{Q = \beta + \delta X^2\}$. By assumption, the system (B.11) inside Σ_δ is well approximated by the system (B.13). The latter is easily integrated so one can see that the flow inside Σ_δ defines the map $(\tilde{y}_{in,\delta}, p_{in,\delta}) \mapsto (\tilde{y}_{out,\delta}, p_{out,\delta})$ from any finite size neighborhood of $\tilde{x} = -X/\sqrt{\kappa}$ to a neighborhood of $\tilde{x} = X/\sqrt{\kappa}$, which is written as

$$\begin{aligned} \tilde{y}_{out,\delta} &= \tilde{y}_{in,\delta} + o(1) \\ p_{out,\delta} &= p_{in,\delta} + \frac{1}{\sqrt{\kappa}}F(\tilde{y}_{in,\delta}) + o(1). \end{aligned} \tag{B.14}$$

Comparison of (B.14) with (B.4) shows that to prove the lemma it is sufficient to verify that the flow outside Σ_δ is essentially a constant speed motion. Namely, we will prove that for negative $\tilde{x}$ the flow from Σ to Σ_δ in the region of finite $\tilde{y}$ and p defines the map

$$\tilde{y}_{in,\delta} \;=\; \frac{1}{\delta}(y_{in} + cy_{in}' + o(\delta)) \tag{B.15}$$

$$p_{in,\delta} \;=\; \frac{1}{\delta}(y'_{in}+o(\delta)), \tag{B.16}$$

and for positive $\tilde{x}$ the flow from Σ_δ to Σ' in the region of finite $\tilde{y}$ and p defines the map

$$\begin{aligned} y_{out} &= \delta(\tilde{y}_{out,\delta}+cp_{out,\delta})+o(\delta) && \text{(B.17)}\\ y'_{out} &= \delta p_{out,\delta}+o(\delta). && \text{(B.18)}\end{aligned}$$

These two formulas are absolutely symmetric, therefore, we prove only (B.16). It is convenient to start on the cross-section $\Sigma_Q : \{Q(x,y) = Q(-c,0)\}$ rather than on $\{x=-c\}$. For the orbits which start with y_{in} and y'_{in} of order $O(\delta)$ the distance in x between these two cross-sections is $O(\delta)$ too. The flow near Σ is close to the constant speed motion, being on a finite distance on the billiard boundary. Thus, the transition from Σ to Σ_Q adds only $o(\delta)$ terms to y', and $y'\Delta x$ is added to the value of y. Correspondingly, the formula (B.16) which is to be proved changes as follows (we also use the rescaled variables on Σ_Q):

$$\begin{aligned} \tilde{y}_{in,\delta} &= \tilde{y}_{in}-\sqrt{\delta}\tilde{x}_{in}p_{in}+o(1) && \text{(B.19)}\\ p_{in,\delta} &= p_{in}+o(1). && \text{(B.20)}\end{aligned}$$

Let us rewrite the system (B.11) as

$$\frac{d}{d\tilde{Q}}\tilde{y}=\sqrt{\delta}\frac{p}{\tilde{Q}'}, \qquad p'=-\frac{1}{2\tilde{Q}'}\tilde{W}'_\epsilon(\tilde{Q})G(\tilde{x},\tilde{y},p;\epsilon) \tag{B.21}$$

where

$$\tilde{Q}'\equiv\frac{d\tilde{Q}}{d\tilde{x}}=\frac{1}{\sqrt{\delta}}Q_x(\sqrt{\delta}\tilde{x},\delta\tilde{y})+\sqrt{\delta}Q_y(\sqrt{\delta}\tilde{x},\delta\tilde{y})p. \tag{B.22}$$

It follows from (B.3) that if c is not large, then for all $x\in[-c,c]$

$$|Q_x|\geq C|x|-O(y)=C\sqrt{\delta}|\tilde{x}|+O(\delta) \tag{B.23}$$

for some positive constant C, whence

$$|\tilde{Q}'|\geq\frac{C}{2}|\tilde{x}|. \tag{B.24}$$

Recall, that we consider the region of large $\tilde{Q}$: $X^2 \equiv Q_1 \leq \tilde{Q} \leq Q_2 \equiv (Q(-c,0)-\beta)/\delta$ for finite $\tilde{y}$ and p. In this region $\tilde{x}$ is large hence separated from zero so $\tilde{Q}' \neq 0$ and $\tilde{Q}$ may be taken in (B.21) as a new independent variable indeed. The variable $\tilde{x}$ is now a function of $\tilde{Q}$ and $\tilde{y}$. For fixed $\tilde{Q}$

$$\frac{\partial \tilde{x}}{\partial \tilde{y}} = -\frac{\sqrt{\delta}}{Q_x} Q_y. \tag{B.25}$$

It follows from (B.23) that the ratio $\sqrt{\delta}/Q_x(\sqrt{\delta}\tilde{x}, \delta\tilde{y})$ is bounded along with all derivatives with respect to $\tilde{x}$ and $\tilde{y}$. Hence, $\dfrac{\partial \tilde{x}}{\partial \tilde{y}}$ is bounded, along with all derivatives with respect to $\tilde{y}$, in the region under consideration.

As it is seen from (B.22), all the derivatives of $\tilde{Q}'$ with respect to $\tilde{x}$ and $\tilde{y}$ are bounded. So, it follows now that all the derivatives of $\tilde{Q}'$ with respect to $\tilde{y}$ at fixed $\tilde{Q}$ are bounded too. Moreover (see (B.24) and (B.10)),

$$\frac{\partial^k}{\partial \tilde{y}^k} \frac{1}{\tilde{Q}'} \leq const \cdot \frac{1}{\sqrt{\tilde{Q}}}, \quad k = 0, \ldots, r. \tag{B.26}$$

When evaluating the solution $(\tilde{y}(\tilde{Q}), p(\tilde{Q}))$ of (B.21) we may a priory assume that it is bounded along with all derivatives with respect to the initial conditions $(\tilde{y}_{in}, p_{in})$ (because this is true at the starting moment and because we prove at the end that the flow map stays within the bounds which are independent of those assumed initially; see (B.28)).

Rewrite the system as

$$\begin{aligned} \tilde{y}(\tilde{Q}) &= \tilde{y}_{in} + \sqrt{\delta} p_{in} \int_{Q_2}^{\tilde{Q}} \frac{dq}{\tilde{Q}'} + \sqrt{\delta} \int_{Q_2}^{\tilde{Q}} (p - p_{in}) R \frac{dq}{\sqrt{q}} \\ p(\tilde{Q}) &= p_{in} - \frac{1}{2} \int_{Q_2}^{\tilde{Q}} R G \tilde{W}_\epsilon'(q) \frac{dq}{\sqrt{q}} \end{aligned} \tag{B.27}$$

where $R \equiv \sqrt{\tilde{Q}}/\tilde{Q}'$. The functions R and G are uniformly bounded along with all derivatives with respect to $(\tilde{y}_{in}, p_{in})$. The boundedness follows from (B.26), from the *a priory* assumed boundedness of $\tilde{y}$ and p and from the boundedness of Q_x, Q_y and $\dfrac{\partial \tilde{x}}{\partial \tilde{y}}$. Note also that as $\tilde{Q}$ grows, the function $\tilde{W}_\epsilon$ which enter the expression (B.12) for the function G may, in principle, grow unbounded. However, since the integral of $\frac{\tilde{W}'_\epsilon}{\sqrt{\tilde{Q}}}$ is bounded by assumption [S], the function $\tilde{W}_\epsilon$ is at most linear with respect to $\sqrt{\tilde{Q}}$. Now, since $\tilde{Q} = O(1/\delta)$ by definition, it follows that the product $\sqrt{\delta}\tilde{W}_\epsilon$ is uniformly bounded, so the term $\left(1 - \frac{\delta}{1-\nu/H}\sqrt{\delta}\tilde{W}_\epsilon(\tilde{Q})\right)^{-1}$ in G is indeed bounded.

Integration of (B.27) gives (we use that $\sqrt{\delta}\int \frac{dq}{\sqrt{q}} = O(\sqrt{\delta\tilde{Q}})$ is bounded since $\tilde{Q}$ is $O(\frac{1}{\delta})$ by definition):

$$\begin{aligned} \tilde{y}(\tilde{Q}) &= \tilde{y}_{in} + \sqrt{\delta}p_{in}(\tilde{x} - \tilde{x}_{in}) + O(1) \cdot \max_q ||p(q) - p_{in}||_{C^r} \\ p(\tilde{Q}) &= p_{in} + O(1) \cdot \int_{Q_1}^{Q_2} \left|\tilde{W}'_\epsilon(q)\right| \frac{dq}{\sqrt{q}} \end{aligned} \tag{B.28}$$

where $O(1)$ denotes functions bounded, along with all derivatives with respect to $(\tilde{y}_{in}, p_{in})$; we denote as $||\cdot||_{C^r}$ the maximum of the absolute values of the quantity itself and of all its derivatives with respect to $(\tilde{y}_{in}, p_{in})$.

By the scaling assumption [S], the integral $\int_1^\infty \tilde{W}'_\epsilon(q)\frac{dq}{\sqrt{q}}$ converges uniformly. Now, since $Q_1, Q_2 \to \infty$ as $\epsilon \to 0$, it follows that the integral in the right hand side of the second equation of (B.28) tends to zero. Thus, though the estimates for the $O(1)$-factors in (B.28) may depend on the *a priory* assumed bounds for y and p, their contribution is canceled by the factor which can be taken arbitrarily small as $\epsilon \to 0$. This justifies the *a priory* assumption.

Putting $\tilde{Q} = Q_1$ in the left-hand side of (B.28) gives the map $\Sigma_Q \to \Sigma_\delta$. Since $|\tilde{x}(Q_1)| \leq CX$ and $X(\epsilon)$ is assumed to grow sufficiently slow, the term $\sqrt{\delta}p_{in}\tilde{x}(Q_1)$ in the first equation of (B.28) is $o(1)$. Thus, we are in complete agreement with (B.20). The lemma is proved. □.

References

[1] D. V. Anosov. Geodesic flows on Riemann manifolds with negative curvature. *Proc. Steklov Inst. Math.*, 90, 1967.

[2] L. Bunimovich and Y. Sinai. Markov partitions for dispersed billiards. *Comm. Math. Phys.*, 78(2):247–280, 1980/81.

[3] P. Collas, D. Klein, and H.-P. Schwebler. Convergence of hamiltonian systems to billiards. *Chaos*, 1998.

[4] V. Donnay. Elliptic islands in generalized sinai billiards. *Ergod. Th. & Dynam. Sys.*, 16:975–1010, 1996.

[5] P. Duarte. Plenty of elliptic islands for the standard family of area preserving maps. *Ann. Inst. H. Poincar Anal. Non Linaire*, 11(4):359–409, 1994.

[6] R. Fleischmann, T. Geisel, and R. Ketzmerick. Quenched and negative hall effect in periodic media: application to antidot superlattices. *Europhys. lett.*, 25(3):219–224, 1994.

[7] G. Gallavotti and D. Ornstein. Billiards and Bernoulli schemes. *Comm. Math. Phys.*, 38:83–101, 1974.

[8] K. Hansen. Bifurcations and complete chaos for the diamagnetic kepler problem. *Phys. Rev. E.*, 51(3):1838–1844, 1995.

[9] C. Karney. Long time correlations in the stochastic regime. *Physica D*, 8:360, 1983.

[10] R. MacKay, J. Meiss, and I. Percival. Resonances in area preserving maps. *Physica D*, 27:1–20, 1987.

[11] J. Meiss and E. Ott. Markov tree model of transport in area-preserving maps. *Physica D*, 20:387–402, 1986.

[12] H. Poincaré. Sur les équations de la dynamique et le problème de trois corps. *Acta Math.*, 13:1–270, 1890.

[13] Y. Sinai. On the foundations of the ergodic hypothesis for dynamical system of statistical mechanics. *Dokl. Akad. Nauk. SSSR*, 153:1261–1264, 1963.

[14] Y. Sinai. Dynamical systems with elastic reflections: Ergodic properties of scattering billiards. *Russian Math. Sur.*, 25(1):137–189, 1970.

[15] Y. Sinai and N. Chernov. Ergodic properties of some systems of two-dimensional disks and three-dimensional balls. *Uspekhi Mat. Nauk*, 42(3(255)):153–174, 256, 1987. In Russian.

[16] D. Szász. Boltzmann's ergodic hypothesis, a conjecture for centuries? *Studia Sci. Math. Hungar.*, 31(1–3):299–322, 1996.

[17] D. Turaev and V. Rom-Kedar. Islands appearing in near-ergodic flows. *Nonlinearity*, 11(3):575–600, 1998. to appear.

[18] M. Wojtkowski. Principles for the design of billiards with nonvanishing lyapunov exponents. *Comm. Math. Phys.*, 105(3):391–414, 1986.

[19] G. Zaslavsky and M. Edelman. Maxwell's demon as a dynamical model. *Phys. Rev. E*, 56(5):5310–5320, 1997.

[20] G. Zaslavsky, M. Edelman, and B. Niyazov. Self-similarity, renormalization and phase space nonuniformity of hamiltonian chaotic dynamics. *Chaos*, 7:159, 1997.

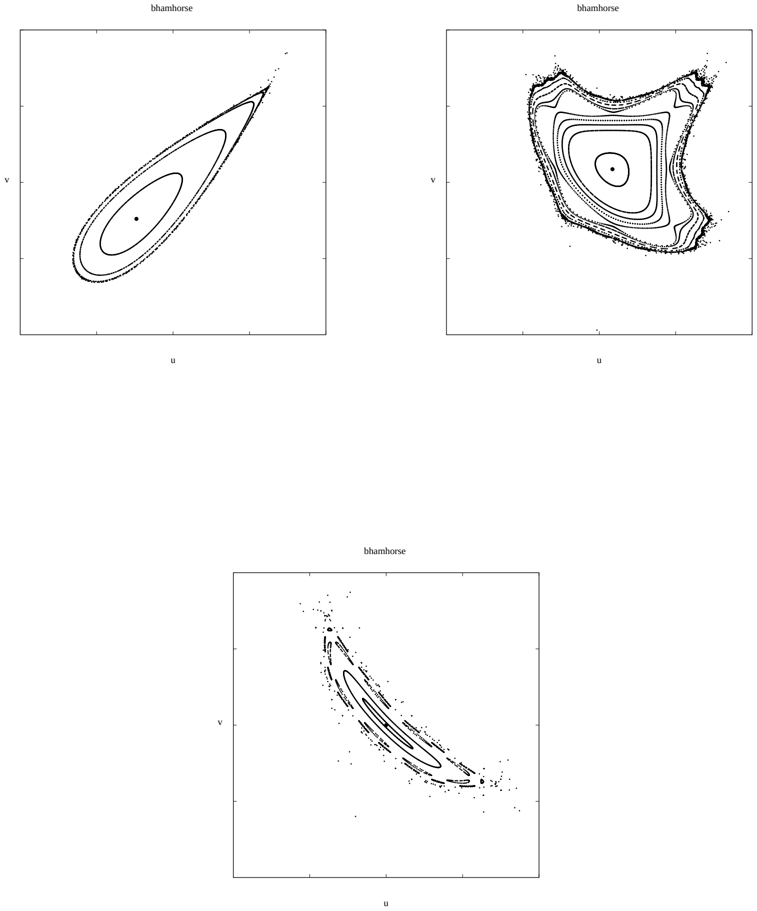

Figure 4.1: Rescaled local map near singular periodic orbit.
a. $\Gamma = 2.6$ b. $\Gamma = 2.8$ c. $\Gamma = 3$.

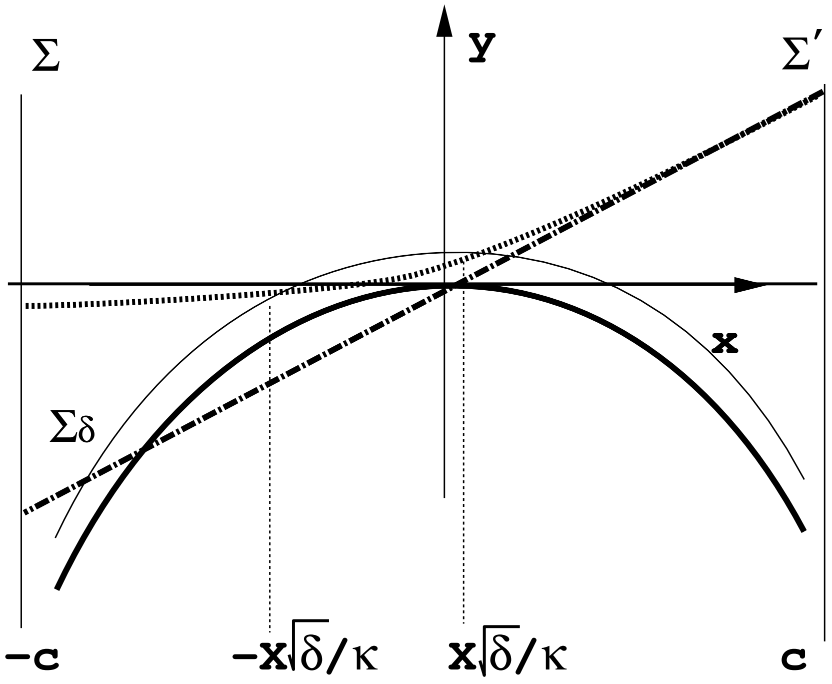

Figure B.1: Details of singular local map construction.