# Almost Gibbsian versus Weakly Gibbsian

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**Abstract**

We consider various extensions of the standard definition of Gibbs states for lattice spin systems. When a random field has conditional distributions which are almost surely continuous (almost Gibbsian field), then there is a potential for that field which is almost surely summable (weakly Gibbsian field). This generalizes the standard Kozlov-Sullivan theorems. The converse is not true in general. We give (counter)examples illustrating the relation between topological and measure-theoretic aspects of generalized Gibbs definitions.

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## 1 Situation.

The standard definition of a Gibbs random field for lattice spin systems starts from an interaction potential. This interaction is sufficiently local. With the interaction and with a given a priori (reference) measure, one constructs the

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so called Gibbs specification. This is the collection of finite volume Gibbs measures with all possible boundary configurations fixed outside the volume. The infinite volume Gibbs measures are the random fields consistent with this specification. This means that their (local) conditional distributions have a version given by the Boltzmann-Gibbs factor built from the interaction. This is the famous Dobrushin-Lanford-Ruelle equation. More details and the development of the ensuing Gibbs theory can be found in the standard [Georgii88], [Simon93], [Ruelle78], [Preston76], [Sinai82].

An important question is to understand what random fields are Gibbsian. In other words, when is there a potential from which this random field can be constructed as the corresponding Gibbs measure. This potential provides then an interesting physical parametrization from which, via the standard Gibbs formalism, other global characterizations can be drawn such as those related to the large deviation behavior, the Gibbs phase rule or global aspects of the phase diagram of the system.

An important characterization of Gibbs fields goes via the theorems of Sullivan and Kozlov. Roughly speaking, these theorems give conditions on the locality of the conditional distributions under which the field is Gibbsian for some nicely behaving potential. The notion of quasi-locality plays a central role here. It expresses the continuity of the conditional distributions and, under some extra conditions, a quasi-local random field allows a translation invariant absolutely and uniformly summable interaction potential. We will come back to this later on.

In the last ten years, there has been an intense study of the limits of the Gibbsian formalism. Physically relevant non-Gibbsian fields were constructed. They have mostly appeared as image measures under renormalization group transformations of Gibbs measures. In other cases (like for stationary measures of interacting particle systems modeling some non-equilibrium situation), it is not at all clear whether bona fide Gibbs measures appear and in some cases this was disproven. The situation as in 1993 was summarized in [van Enter *et al.*93] and we refer to this article for further back-ground. Even before however, Roland Dobrushin stressed that one should allow for more general definitions of a Gibbs state than is usually done. He made the analogy with unbounded spins where the summability of the potential cannot be understood uniformly. One should look for the 'good' configurations on which the usual Gibbsian game can be played. This was illustrated in his last conference talk where he showed how to give a Gibbsian characterization of a non-Gibbsian state, [Dobrushin95], [Dobrushin *et al.*97]. This program was carried further in [Bricmont *et al.*98] and [Maes *et al.*97]. Many questions remain however (see e.g. [van Enter *et al.*97], [Maes *et al.*98], [Lörinczi *et al.*97b], [Lörinczi *et al.*97a]). In short, what are possible generalizations of the standard Gibbsian framework, how do they relate, how typical are they and what remains of the Gibbs formalism ?

In the present paper, we take a closer look at two of the main generalizations that have appeared. We call them almost versus weakly Gibbsian. Almost Gibbsian states have a version for their conditional distributions which is almost surely continuous. Weakly Gibbsian states allow for an absolutely convergent potential. Each of the two definitions can be further weakened or made stronger. In general however, almost Gibbsianness looks for a large set of 'good' configurations defined in terms of continuity points of the conditional distributions, while weak Gibbsianness asks for a large set of 'good' configurations on which the potential satisfies a certain summability. We are therefore reminded of the Kozlov-Sullivan theorems and we observe that almost Gibbs implies weakly Gibbs. The converse is not true in general. From the point of view of weak Gibbsianness, continuity requirements (almost sure quasi-locality) are rather bizarre. We will illustrate these definitions and their meaning via very simple examples. As soon becomes clear, in comparing the two definitions, we are confronted with certain incommensurabilities between topological and measure-theoretic largeness of the set of 'good' configurations.

# 2 Potentials and Specifications

In this section we introduce some basic stuff related to lattice spin systems, used throughout this paper. We refer to [van Enter *et al.*93] and [Fernandez *et al.*97] for further details.

Let $W := \{1, \ldots, q\}$ be a finite set (the single site state space). For simplicity of notation we work on the regular $d-$dimensional lattice $\mathbb{Z}^d$ and $\mathcal{L} := \{\Lambda, |\Lambda| < \infty\}$ denotes the set of finite subsets of $\mathbb{Z}^d$. The complement of a set $\Lambda \subset \mathbb{Z}^d$ is $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$. Infinite volume limits of a function $a : \mathcal{L} \to \mathbb{R}$

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} a(\Lambda) = a \tag{2.1}$$

must be understood as saying that for all $\epsilon > 0$ there is a $\Lambda(\epsilon) \in \mathcal{L}$ so that $|a(\Lambda) - a| < \epsilon$ whenever $\Lambda(\epsilon) \subset \Lambda \in \mathcal{L}$. This is applied for infinite sums which should be read as

$$\sum_M u(M) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \sum_{M \subset \Lambda} u(M) \tag{2.2}$$

and (2.1) must be applied to $a(\Lambda) = \sum_{M \subset \Lambda} u(M)$. Such sums can be convergent (i.e. having a definite finite limit) without being absolutely summable and there is no explicit fixing of a sequence of increasing volumes along which the limits ought to be taken.

The state space is $\Omega := W^{\mathbb{Z}^d}$ and its elements (= configurations) are denoted by greek letters $\eta, \omega, \sigma, \xi, \ldots$. The value of $\omega$ at a site $i \in \mathbb{Z}^d$ is written as $\omega(i) (\in W)$. On $\Omega$ we have the natural action of translations $\tau_a$, $a \in \mathbb{Z}^d$ defined by $\tau_a \eta(x) := \eta(x-a), x \in \mathbb{Z}^d$. The restriction of $\omega$ to a subset $M \subset \mathbb{Z}^d$ is $\omega_M$ and for $\omega, \eta \in \Omega$ we define $\omega_M^\eta \in \Omega$ by

$$\omega_M^\eta(k) := \begin{cases} \omega(k), & k \in M, \\ \eta(k), & k \in M^c. \end{cases} \tag{2.3}$$

For example, $\omega_M^1(k) = 1$ if $k \in M^c$ and $= \omega(k)$ for $k \in M$. We call two configurations $\omega$ and $\omega'$ equivalent, notation $\omega \sim \omega'$ if they differ only on a finite number of lattice sites, i.e., if $\exists \Lambda \in \mathcal{L}$ such that $\omega(i) = \omega'(i)$ $\forall i \in \Lambda^c$. The $\sigma$-algebra generated by the evaluation maps $X_i, X_i(\omega) := \omega(i), i \in M$ is written as $\mathcal{F}_M = \sigma\{X_i,\ i \in M\}$. For an event $F \in \mathcal{F}_M$ we write $1_F$ for the indicator function : $1_F(\omega) = 1_F(\omega_M) = 1$ if $\omega \in F$ and is zero otherwise. When $M = \mathbb{Z}^d$, we set $\mathcal{F} := \mathcal{F}_{\mathbb{Z}^d}$. The *tailfield* $\sigma$-algebra $\mathcal{T}^\infty$ is defined as

$$\mathcal{T}^\infty := \bigcap_{\Lambda \in \mathcal{L}} \mathcal{F}_{\Lambda^c}. \tag{2.4}$$

The configuration space $\Omega$ is a compact metric space in the product topology. We call a function $f$ on $\Omega$ local if it depends only on a finite number of coordinates, i.e., there is a $\Lambda \in \mathcal{L}$ such that $f(\eta) = f(\eta_\Lambda)$. For $f : \Omega \to \mathbb{R}$, $\theta \in \Omega$ we say that $f$ is *continuous in the direction* $\theta$ iff

$$\forall \omega \in \Omega : \ f(\omega) = \lim_{\Lambda \uparrow \mathbb{Z}^d} f(\omega_\Lambda^\theta). \tag{2.5}$$

A *continuous* function is continuous in every direction but a function can fail to be continuous even if continuous in every direction. If the function is uniformly continuous in a direction ((2.5) uniformly in $\omega$), then the function is actually continuous. Every local function is of course continuous. The set of local functions on $\Omega$ is dense in $C(\Omega)$, the set of real-valued continuous functions, in the topology of uniform convergence.

## 2.1 Specifications.

For completeness we repeat what can be found in many places ([Georgii88], [Fernandez *et al.*97], [van Enter *et al.*93]).

**Definition 2.1** *A specification $\Gamma$ on $\mathcal{L}$ is a family of probability kernels $\Gamma = \{\gamma_\Lambda,\ \Lambda \in \mathcal{L}\}$ on $(\Omega, \mathcal{F})$, such that the following hold:*

1. *$\gamma_\Lambda(\cdot|\omega)$ is a probability measure on $(\Omega, \mathcal{F})$ for all $\omega \in \Omega$;*

*2.* $\gamma_\Lambda(F|\cdot)$ *is* $\mathcal{F}_{\Lambda^c}$ *-measurable for all* $F \in \mathcal{F}$*;*

*3.* $\gamma_\Lambda(F|\omega) = 1_F(\omega)$ *if* $F \in \mathcal{F}_{\Lambda^c}$*;*

*4.* $\gamma_{\Lambda_2}\gamma_{\Lambda_1} = \gamma_{\Lambda_2}$ *if* $\Lambda_1 \subset \Lambda_2$

The last property (consistency) is most important in characterizing equilibrium. One should also remember that $\gamma_\Lambda(\sigma_\Lambda^\omega|\omega)$, i.e. the probability to find $\sigma$ in $\Lambda$ given $\omega$ outside, is a function defined (pointwise) for every $\sigma, \omega \in \Omega$ (which only depends on $\sigma_\Lambda$ and $\omega_{\Lambda^c}$ ).

In this article we only consider specifications that are uniformly nonnull. I.e. $\forall \Lambda \in \mathcal{L}$ $\exists$ a constant $m_\Lambda > 0$ such that $\forall \omega$

$$\gamma_\Lambda(\omega_\Lambda|\omega_{\Lambda^c}) \geq m_\Lambda. \tag{2.6}$$

A specification $\Gamma$ is said to be *translation invariant* iff $\forall a \in \mathbb{Z}^d$, $\forall \Lambda \in \mathcal{L}$, $\forall \omega \in \Omega$ and for all bounded functions $f$

$$\gamma_\Lambda(f \circ \tau_a|\omega) = \gamma_{\Lambda+a}(f|\tau_a\omega). \tag{2.7}$$

**Definition 2.2** *A probability measure* $\mu$ *on* $(\Omega, \mathcal{F})$ *is consistent with a specification* $\Gamma$ *(notation :* $\mu \in \mathcal{G}(\Gamma)$*) if* $\forall \Lambda \in \mathcal{L}$ *and for every continuous function* $f$ *on* $\Omega$*:*

$$\mu \circ \gamma_\Lambda(f) = \mu(f) \tag{2.8}$$

where $\mu \circ \gamma_\Lambda(f) = \int \mu(d\omega)\ \gamma_\Lambda(f|\omega)$.

<u>Remark:</u>

We can also say that a specification $\Gamma$ is consistent with the probability measure $\mu$. Notice that $\mathcal{G}(\Gamma)$ is a convex set. In general there can be many elements in that set. On the other hand, it is easy to find some specification $\Gamma$ consistent with a given $\mu$, see [Sokal81], [Goldstein78]. To make the concept of specification more useful, we must require some kind of locality.

We define (measurable) sets of continuity points for a specification $\Gamma$ as follows :

$$\Omega_\Gamma := \{\omega :\ \forall \Lambda \in \mathcal{L}\, \forall F \in \mathcal{F}_\Lambda\, \gamma_\Lambda(F|\cdot) \text{is continuous in } \omega\}, \tag{2.9}$$

$$\Omega_\Gamma^\theta := \{\omega :\ \forall \Lambda \in \mathcal{L}\, \forall F \in \mathcal{F}_\Lambda\, \gamma_\Lambda(F|\cdot) \text{is continuous in the direction } \theta\}. \tag{2.10}$$

Note that the continuity of $\gamma_\Lambda(F|\cdot)$ for all sets $\Lambda \in \mathcal{L}$ follows from the continuity of the single site probabilities $\gamma_{\{i\}}(\sigma(i)|\cdot)$. The same is true for $\theta$-continuity.

A probability measure $\mu$ is called a Gibbs measure if there exists a continuous (or quasi-local) specification $\Gamma$ consistent with $\mu$ (i.e. $\exists \Gamma, \mu \in \mathcal{G}(\Gamma), \Omega_\Gamma = \Omega$). We say that $\mu$ is *almost Gibbsian* if there is a specification $\Gamma$ with $\mu \in \mathcal{G}(\Gamma)$ and $\mu(\Omega_\Gamma) = 1$. If, for $\theta \in \Omega$, there is a $\Gamma, \mu \in \mathcal{G}(\Gamma)$ for which $\mu(\Omega_\Gamma^\theta) = 1$, then we say that $\mu$ is *almost Gibbsian in the direction* $\theta$.

## 2.2 Potentials.

**Definition 2.3** *A potential $U$ is a real valued function on $\mathcal{L} \times \Omega$*

$$U: \ \mathcal{L} \times \Omega \to \mathbb{R} \tag{2.11}$$

*such that $U(\Lambda, \cdot) \in \mathcal{F}_\Lambda$ for all $\Lambda \in \mathcal{L}$ (put $U(\emptyset, \cdot) = 0$).*

$U(A, \sigma) = U_A(\sigma)$ represents the interaction between spins in $A$ for a configuration $\sigma$. A potential $U$ is *translation invariant* if $\forall A \in \mathcal{L}$, $a \in \mathbb{Z}^d$, $\eta \in \Omega$

$$U(A, \eta) = U(A + a, \tau_a \eta). \tag{2.12}$$

We say that a potential $U$ is a *vacuum potential* with vacuum $\theta$ if $U(A, \omega) = 0$ whenever $\omega(x) = \theta(x)$ for some $x \in A$.

To be useful a potential has to be sufficiently local, i.e. the interaction of a finite piece of the lattice with the rest of the world must be finite in some sense. We are therefore led to certain convergence properties (remembering (2.2)).

**Definition 2.4** *A potential $U$ is called $\theta - \mathcal{L}$-convergent in $\omega \in \Omega$ if for all $\Lambda \in \mathcal{L}$ the conditional energy field*

$$E_\theta^U(\Lambda, \omega) := \sum_{A \bigcap \Lambda \neq \emptyset} \Big[ U(A, \omega) - U(A, \omega_\Lambda^\theta) \Big] < \infty \tag{2.13}$$

*is well-defined.*

**Definition 2.5** *A potential $U$ is convergent in $\omega$ if for all $\Lambda \in \mathcal{L}$ the local Hamiltonian*

$$H_\Lambda^U(\omega) := \sum_{A \bigcap \Lambda \neq \emptyset} U(A, \omega) < \infty \tag{2.14}$$

*is well-defined.*

If we have for $\omega$ that the local Hamiltonians $H_\Lambda^U(\eta_\Lambda^\omega) < \infty$ are well-defined for all $\eta \in \Omega$, then we can use (2.5) to define the finite volume Gibbs measures

$$\mu_\Lambda^U(\eta|\omega) := \frac{1_{[\eta = \omega \text{ on } \Lambda^c]}}{Z_\Lambda(\omega_{\Lambda^c})} \exp\left[ -H_\Lambda^U(\eta_\Lambda^\omega) \right], \eta \in \Omega, \tag{2.15}$$

with fixed boundary condition $\omega$. Here $Z_\Lambda(\omega_{\Lambda^c})$ is the normalisation. We did not include here the usual pre-factor $\beta$ (inverse temperature) nor did we add any specific a priori measure (other than the counting measure) on the single site state space. Both (temperature and a priori weights) are supposed to be contained in the potential. Under the same condition (2.5) we set

**Definition 2.6** *A potential $U$ is compatible in $\omega \in \Omega$ with a specification $\Gamma$ if for all $\Lambda \in \mathcal{L}$*

$$\mu_\Lambda^U(\eta|\omega) = \gamma_\Lambda(\eta|\omega) \tag{2.16}$$

Alternatively, if we only have (2.4), we put

**Definition 2.7** *A potential $U$ is $\theta$-compatible in $\omega$ with a specification $\Gamma$ if for all $\Lambda \in \mathcal{L}$*

$$-\log \frac{\gamma_\Lambda(\omega|\omega)}{\gamma_\Lambda(\theta_\Lambda^\omega|\omega)} = E_\theta^U(\Lambda, \omega). \tag{2.17}$$

Remark:
If there exists a vacuumpotential $V_\Gamma^\theta$ with vacuum $\theta$, compatible with a given specification $\Gamma$ then this $V_\Gamma^\theta$ is unique and given by

$$V_\Gamma^\theta(A,\omega) = -\sum_{\Lambda \subset A} (-1)^{|A\setminus\Lambda|} \log \frac{\gamma_\Lambda(\omega_\Lambda^\theta|\theta)}{\gamma_\Lambda(\theta|\theta)}. \tag{2.18}$$

Note however that

1. the potential constructed in expression (2.18) above doesn't need to be convergent in general and
2. even when it is convergent it is still possible that this $V_\Gamma^\theta$ is NOT compatible with $\Gamma$ (cf. Proposition 4.2).

The vacuum potential is necessarily translation invariant if $\Gamma$ is and $\theta$ is constant.

In the traditional theory of lattice spin systems one requires that the sums above are uniformly and absolutely summable. The strongest (but often used) is requiring that for all $\Lambda \in \mathcal{L}$

$$\sum_{A \bigcap \Lambda \neq \emptyset} \sup_\omega |U(A,\omega)| < \infty. \tag{2.19}$$

One can then define (2.15) for all possible boundary conditions $\omega$ and hence, construct the Gibbs specification $\Gamma^U$ according to (2.6). Clearly, $\Omega_{\Gamma^U} = \Omega$. Hence, if $\mu \in \mathcal{G}(\Gamma^U)$, then $\mu$ is a Gibbs measure (with respect to the potential $U$ satisfying (2.19). The theorems of Kozlov and Sullivan, [Kozlov74], [Sullivan73], deal with the converse : whether Gibbs measures (in the sense of Section 2.1) allow (what kind of) a potential (in the sense of (2.15)).

We still need the following

**Definition 2.8** *A potential $U$ is absolutely convergent in $\omega \in \Omega$ if for all $\Lambda \in \mathcal{L}$*

$$\sum_{A \bigcap \Lambda \neq \emptyset} |U(A, \omega)| < \infty. \tag{2.20}$$

**Definition 2.9** *$U$ is absolutely $\theta - \mathcal{L}$-convergent in $\omega$ if for all $\Lambda \in \mathcal{L}$*

$$\sum_{A \bigcap \Lambda \neq \emptyset} \left| U(A, \omega) - U(A, \omega^\theta_\Lambda) \right| < \infty. \tag{2.21}$$

# 3 Generalizations of Kozlov-Sullivan.

We give here our generalizations of the results of Sullivan [Sullivan73] and Kozlov [Kozlov74]. We present these results without proofs since the latter are straightforward extensions of the proofs in [Sullivan73] and [Kozlov74], replacing uniform by pointwise formulations. In what follows we always have a specification $\Gamma$ as in (Definition 2.1) with corresponding sets (2.9) and (2.10).

**Theorem 3.1** *(*generalized Sullivan-result*) There exists a potential $U$ so that for all $\omega \in \Omega^\theta_\Gamma$, $U$ is in $\omega$ $\theta - \mathcal{L}$-convergent and is $\theta$-compatible in $\omega$ with $\Gamma$.*

<u>Remark:</u>
If $\theta$ is constant and $\Gamma$ is a translation invariant specification then $U$ is also translation invariant. If moreover, $\Gamma$ is a quasi-local specification ($\Omega_\Gamma = \Omega$), then there exists a translation invariant potential $U$, everywhere compatible with $\Gamma$ and absolutely $\theta - \mathcal{L}$-convergent, uniformly in $\omega$ (as in Definition 2.9).

**Theorem 3.2** *(*generalized Kozlov-result*) Let $\Gamma$ be a specification. Then $\forall\ \theta \in \Omega$ there exists a potential $U$ which is absolutely convergent (as in (2.8)) and compatible with $\Gamma$ in every $\omega \in \Omega^\theta_\Gamma$. For all $\omega \in \Omega^\theta_\Gamma$, the vacuumpotential $V^\theta$ with vacuum $\theta$ (as in (2.18)) is convergent in $\omega$ (as in* Definition 2.5*) and is compatible in $\omega$ with $\Gamma$.*

<u>Remark:</u>
In the case of a translation invariant specification $\Gamma$, and $\theta$ constant, the potential $U$ of Theorem 3.2 is not necessarily translation invariant, whereas the vacuumpotential $V^\theta_\Gamma$ is translation invariant but not necessarily *absolutely* convergent. The best we can hope for in that case is to obtain a translation invariant potential which is absolutely convergent (as in Definition 2.8). We

first need some further notations. For $x \in \mathbb{Z}^d$ and real $r$, put $B_x(r) := \{y \in \mathbb{Z}^d \ : |x-y| \le r\}$ where $|x-y| := \max_{j=1,\ldots,d} |x^j - y^j|$. Define

$$\phi_x(r,\omega) := \left| \gamma_{\{x\}}(\omega|\omega) - \gamma_{\{x\}}(\omega^\theta_{B_x(r)}|\omega^\theta_{B_x(r)}) \right| \tag{3.22}$$

where we take $\theta$ a constant configuration. We say that $\omega$ is a 'good' configuration if $\omega \in K^\theta_\Gamma := \bigcap_x K^\theta_x$ where

$$K^\theta_x = \{\omega : \sum_{i=1}^{\infty} \sum_{y \in B_x(2^i)} \phi_y(2^i,\omega) < \infty\}. \tag{3.23}$$

**Theorem 3.3** (generalized Kozlov-result) *Let $\Gamma$ be a translation invariant specification for which the translation invariant set $K^\theta_\Gamma$ of 'good' configurations was defined in (3.23). There exists a translation invariant potential $U$ such that in every $\omega \in K^\theta_\Gamma$, $U$ is absolutely convergent (as in* Definition 2.8*) and compatible with $\Gamma$ (as in* Definition 2.6*).*

In the formulation of the above theorems we did not need to speak about a specific measure; these theorems deal with the relation between specifications and potentials. However, in applications, one wants to understand whether a specific measure allows (what kind of) a potential. In this case, the specification $\Gamma$ of the theorems above really stands for (a version of) the conditional probabilities of the measure and it acquires a potential via (2.15) and (2.6). We give here a first application of this. Consider the plus-phase $\nu$ of the low-temperature standard ferromagnetic Ising model on $\mathbb{Z}^{d+1}$. It was proven that the restriction $\mu$ of $\nu$ to a hyperplane (that is $\mathbb{Z}^d$) is non-Gibbsian, see [Schonmann89], [Fernandez *et al.*97]. From [Fernandez *et al.*97] it follows that $\mu \in \mathcal{G}(\Gamma)$ for some right-continuous translation invariant specification $\Gamma$ ( i.e. $\Omega^\theta_\Gamma = \Omega$ for $\theta \equiv 1$ ). Therefore, we can apply Theorem 3.2 to obtain

**Theorem 3.4** *There is a translation invariant vacuum potential $V^1$ so that for all $\omega \in \Omega$, $V^1$ is convergent in $\omega$ and $V^1$ is $\theta \equiv 1$-compatible in all $\omega$ with a $\Gamma$ for which $\mu \in \mathcal{G}(\Gamma)$.*

What is lacking is an absolutely convergent translation invariant potential compatible with $\mu$.

We know that for $d+1 = 2$ (the projection to a line ($= \mathbb{Z}$) of the two-dimensional Ising model) $\mu$ is weakly Gibbsian (for a translation invariant absolutely summable potential), see [Dobrushin95], [Dobrushin *et al.*97], [Maes *et al.*97]. The proof of this can in fact be obtained à la Theorem 3.3. We repeat here both the statement and (part of) the proof of this result in the present context.

The projection of the two-dimensional Ising model is a probability measure $\mu$ on Ising spins $\omega(x) = \pm 1, x \in \mathbb{Z}$ obtained from restricting the standard two-dimensional Ising measure $\nu$ to a line crossing the origin. Let $\Omega^+ = \bigcap_{x\in\mathbb{Z}} \Omega_x^+$ be defined via

$$\begin{aligned}\Omega_x^+ &= \{\omega \in \{+,-\}^{\mathbb{Z}}:\ \exists \ell^+(x,\omega), \ell^-(x,\omega) < \infty, \\ &\quad \forall k > \ell^+(x,\omega) : \forall l > \ell^-(x,\omega), \\ &\quad 1/k \sum_{y=0}^{k} \omega(y) > 8/9, 1/l \sum_{y=-l}^{-1} \omega(y) > 8/9\}. \end{aligned} \tag{3.24}$$

Clearly, $\Omega^+ \in \mathcal{T}^\infty$ and is translation invariant. It is also straightforward to prove that $\mu(\Omega^+) = 1$ for low enough temperatures, see [Maes *et al.*97]. The configurations in $\Omega^+$ are 'good' as shows the next theorem. For the proof, we refer to [Maes *et al.*97]

**Theorem 3.5** *(*see [Maes *et al.*97]*) There is a translation invariant potential $U$ which is absolutely summable on $\Omega^+$, i.e.*

$$\sum_{A \ni x} |U(A,\omega)| < \infty, \quad \omega \in \Omega^+, \tag{3.25}$$

*so that the conditional distributions of $\mu$ are $\mu-$almost surely given by* (2.15)*, i.e.*

$$\mu(X = \eta \text{ on } \Lambda | \mathcal{F}_{\Lambda^c})(\omega) = \frac{1}{Z_\Lambda(\omega)} \exp\left[-H_\Lambda^U(\eta_\Lambda^\omega)\right], \eta \in \{+,-\}^{\mathbb{Z}}, \Lambda \in \mathcal{L}, \tag{3.26}$$

*for $\omega \in \Omega^+$.*

Remark:
The condition (3.23) in terms of (3.22) is not quite optimal. From generalizing the proof of Theorem 3.3 in ([Kozlov74]) one learns that one can work directly with bounds on ratio's of conditional probabilities. For example, in $d = 1$ for Ising spins, we get a bound on the potential $U$ which is

$$|U(A,\omega)| \leq \left| \log \frac{\gamma_{\{j\}}(\omega^+_{[j,k]}|\omega^+_{[j,k]})}{\gamma_{\{j\}}(\omega^+_{]j,k]}|\omega^+_{]j,k]})} \frac{\gamma_{\{j\}}(\omega^+_{]j,k[}|\omega^+_{]j,k[})}{\gamma_{\{j\}}(\omega^+_{[j,k[}|\omega^+_{[j,k[})} \right| \tag{3.27}$$

for $A = \{j, j+1, \ldots, k\}, j < k$ and is zero otherwise. Most of the time these ratio's can be rewritten in terms of correlation functions for some non translation invariant measure.

In the following Sections we give more applications.

# 4 From almost to non-Gibbsian states.

**Definition 4.1** *A probability measure $\mu$ on $(\Omega, \mathcal{F})$ is* almost Gibbsian *if there exists a specification $\Gamma$ such that $\mu \in \mathcal{G}(\Gamma)$ and $\mu(\Omega_\Gamma) = 1$.*

This is equivalent with asking that some version of the conditional probabilities of $\mu$ is continuous $\mu$-almost surely.

Remarks:

1. Of course every Gibbs measure is almost Gibbsian since in that case there exists a specification for which $\Omega_\Gamma = \Omega$.
2. Since $\Omega_\Gamma$ is a set in the tail field (2.4), we always have that for extremal $\mu \in \mathcal{G}(\Gamma)$ $\mu(\Omega_\Gamma) = 0$ or $\mu(\Omega_\Gamma) = 1$, i.e. the specification is $\mu$-a.s. continuous or $\mu$-a.s. discontinuous.
3. The mere fact that $\mu(\Omega_\Gamma) = 1$ does not seem to imply that $\Omega_\Gamma$ is also large in a topological sense. Yet, in Appendix A, we prove that this is indeed the case : $\Omega_\Gamma$ is of second Baire category.
4. We can extend Definition (4.1), calling $\mu$ *almost Gibbsian in the direction* $\theta$ iff $\mu(\Omega^\theta_\Gamma) = 1$. The set of almost Gibbs meaures (in the direction $\theta$) is of course a convex set (but may be empty).

**Definition 4.2** *A probability measure $\mu$ on $(\Omega, \mathcal{F})$ is* weakly Gibbsian *if there exists a potential $U$ and a tail field set $\Omega_U$ such that*

1. *$U$ is absolutely convergent on $\Omega_U$ (see Definition 2.8) ;*
2. *$\mu(\Omega_U) = 1$ ;*
3. *For every $\mathcal{F}_\Lambda$-measurable function $f$, its expectation value is given by*

$$\mu(f) = \int d\mu(\omega_{\Lambda^c}) \frac{1}{Z_\Lambda(\omega_{\Lambda^c})} f(\omega_\Lambda) e^{-H^U_\Lambda}(\omega) \tag{4.28}$$

*where*

$$Z_\Lambda(\omega_{\Lambda^c}) = \sum_{\omega_\Lambda} \exp\{-H^U_\Lambda(\omega)\} \tag{4.29}$$

Remarks:

1. That $\Omega_U$ is a tail-field makes $Z_\Lambda(\omega_{\Lambda^c})$ well-defined for all $\omega \in \Omega_U$;

2. In the case of a translation invariant $\mu$ one would of course like to have $U$ and $\Omega_U$ translation invariant. However, even in the regular Gibbs case (i.e. quasi-locality everywhere) it is unknown whether this is always possible.

3. Definition 4.2 could be reformulated in terms of specifications as follows: $\mu$ is weakly Gibbsian if there exists a potential U, absolutely convergent on a tailfield set $\Omega_U$ and there exists a specification $\Gamma$ compatible with U on $\Omega_U$ such that $\mu \in \mathcal{G}(\Gamma)$ and $\mu(\Omega_U) = 1$.

4. Even when $H^U_\Lambda(\omega) = \sum_{A \bigcap \Lambda \neq \emptyset} U(A, \omega)$ exists on $\Omega_U$, we cannot conclude continuity of $H^U_\Lambda$ in $\omega$. Therefore Definition 4.2 is **much** weaker than definition 4.1, see further example 5.3. For a weakly Gibbsian measure $\mu$ we can have that for every specification $\Gamma$ with $\mu \in \mathcal{G}(\Gamma)$, $\mu(\Omega_\Gamma) = 0$. If however $\Omega_U = \Omega$, then there is a dense set of continuity points of $H^U_\Lambda$. We will discuss these points in detail in Appendix A.

5. One could still weaken Definition 4.2. We can refrain from asking the absolute convergence (convergence is enough) or we can only ask the weaker $\theta - \mathcal{L}-$ convergence (of Definition 2.4). Or we could ask more, like requiring that the potential be strictly local $\mu-$ almost surely. Or, requiring that $\mu \in \mathcal{G}(\Gamma)$ for a $\Gamma$ compatible with an (absolutely) convergent vacuum potential (see (2.18)). In the same way one can weaken or strengthen Definition 4.1. For example, one may be happy finding a $\theta \in \Omega$ for which $\Omega^\theta_\Gamma = \Omega$. Etc. etc. To prefer working with one specific definition must be motivated by examples and by illustrations of the power and the glory of one's preference. It seems to be too early for that.

From Section 3, Theorem 3.3, we get the following

**Proposition 4.1** *If $\mu$ is almost Gibbs then $\mu$ is weakly Gibbs and we can choose $\Omega_U = \Omega_\Gamma$ where $\Gamma$ is such that $\mu \in \mathcal{G}(\Gamma)$.*

About the converse, we have not much to say except that it is not true in general that weakly Gibbsian measures are also almost Gibbsian, see e.g. example 4.3, and that some topological information can be obtained on the set of continuity points of the associated specification, see Corollary 6.1. There is nevertheless some direct information : convergence of the vacuum potential $V^\theta_\Gamma$ does give continuity in the direction $\theta$ of the specification $\Gamma$, see (2.18). We fix a specification $\Gamma$ and a configuration $\theta$ :

**Proposition 4.2** *The vacuumpotential $V^\theta_\Gamma$ is convergent (as in Definition 2.5) in $\omega$ and compatible with $\Gamma$ in $\omega$ iff $\omega \in \Omega^\theta_\Gamma$. As a consequence, if $\mu$ is* *weakly Gibbsian for the vacuum potential $V_\Gamma^\theta$, then $\mu$ is almost Gibbsian in the direction $\theta$.*

**Proof :** From the identity (2.18) one obtains

$$\sum_{A\subset L, A\bigcap\Lambda\neq\emptyset} V_\Gamma^\theta(A,\omega) = -\log\frac{\gamma_\Lambda(\omega_\Lambda|\omega_{L\setminus\Lambda}\theta_{L^c})}{\gamma_\Lambda(\theta_\Lambda|\omega_{L\setminus\Lambda}\theta_{L^c})} \tag{4.30}$$

Proposition 4.2 easily follows from taking the limit $L\uparrow\mathbb{Z}^d$. ∎

Remark:
If $\mu$ is almost Gibbs and translation invariant, we still cannot conclude in general that $\mu$ is weakly Gibbsian for a translation invariant potential. In order to have this, the extra condition (3.22) of Theorem 3.3 must be met by a specification $\Gamma$ for which $\mu\in\mathcal{G}(\Gamma)$.

We will now introduce the concept of "bad" configurations for a probability measure $\mu$. This yields a necessary and sufficient condition on the *finite* conditional distributions of $\mu$ for (almost) Gibbsianness of $\mu$. It is in the spirit of Proposition 4.2 in [Fernandez *et al.*97].

**Definition 4.3** *A configuration $\xi\in\Omega$ is* "bad" *for $\mu$ if for some $i\in\mathbb{Z}^d,\epsilon>0$, no matter what or how big $\Lambda\in\mathcal{L}$ we choose, there are $\Lambda'\supset\Lambda,|\Lambda'|<\infty$ and $\omega,\omega'\in\Omega$ such that*

$$\left|\mu(\xi_i|\xi_{\Lambda\setminus i}\omega_{\Lambda'\setminus\Lambda})-\mu(\xi_i|\xi_{\Lambda\setminus i}\omega'_{\Lambda'\setminus\Lambda})\right|\geq\epsilon. \tag{4.31}$$

The notation in (4.31) is supposed to be self-explanatory : we ask for the variation in the finite conditional probabilities (of finding the value $\xi_i$ for the spin at site $i$ when the configuration is fixed and equal to $\xi$ in $\Lambda\setminus\{i\}$, while equal to $\omega$ or $\omega'$ in $\Lambda'\setminus\Lambda$).

**Proposition 4.3** *If $\xi$ is a "bad" configuration for $\mu$ and if $\Gamma$ is a specification consistent with $\mu$ ($\mu\in\mathcal{G}(\Gamma)$), then $\exists i\in\mathbb{Z}^d$ such that $\gamma_{\{i\}}(\omega|\cdot)$ is discontinuous in $\cdot=\xi$.*

**Proof**. For simplicity we take $i$ of Definition (4.3) to be the origin. By contradiction, suppose that there exists a specification $\Gamma$ such that $\gamma_{\{0\}}(\xi_0|\cdot)$ is continuous in $\xi$. Then $\forall\epsilon>0$ (and in particular for the $\epsilon$ of Definition 4.3) $\exists\Lambda\in\mathcal{L}\ \forall\omega\in\Omega$:

$$\left|\gamma_{\{0\}}(\xi|\xi)-\gamma_{\{0\}}(\xi|\xi_\Lambda^\omega)\right|\leq\frac{\epsilon}{2}. \tag{4.32}$$

We therefore have that $\forall\omega,\omega'\in\Omega$

$$\left|\gamma_{\{0\}}(\xi|\xi_\Lambda^\omega)-\gamma_{\{0\}}(\xi|\xi_\Lambda^{\omega'})\right|\leq\epsilon. \tag{4.33}$$

For every finite set $\Lambda' \supset \Lambda$ we have that

$$
\begin{aligned}
&\left|\mu(\xi_0|\xi_{\Lambda\backslash 0}\omega_{\Lambda'\backslash\Lambda}) - \mu(\xi_0|\xi_{\Lambda\backslash 0}\omega'_{\Lambda'\backslash\Lambda})\right| \\
&= \left|\int d\mu_{\Lambda'^c}(\eta_{\Lambda'^c})\left[\gamma_{\{0\}}(\xi|\xi_\Lambda\omega_{\Lambda'\backslash\Lambda}\eta_{\Lambda'^c}) - \gamma_{\{0\}}(\xi|\xi_\Lambda\omega'_{\Lambda'\backslash\Lambda}\eta_{\Lambda'^c})\right]\right| && (4.34)\\
&\le \int d\mu_{\Lambda'^c}(\eta_{\Lambda'^c})\left|\gamma_{\{0\}}(\xi|\xi_\Lambda\omega_{\Lambda'\backslash\Lambda}\eta_{\Lambda'^c}) - \gamma_{\{0\}}(\xi|\xi_\Lambda\omega'_{\Lambda'\backslash\Lambda}\eta_{\Lambda'^c})\right| && (4.35)\\
&\le \epsilon && (4.36)
\end{aligned}
$$

which contradicts that $\xi$ is a "bad" configuration for $\mu$. ■

**Proposition 4.4** *A probability measure $\mu$ without "bad" configurations is Gibbs.*

**Proof**. We show that there exists a specification $\Gamma$ such that $\mu \in \mathcal{G}(\Gamma)$ and $\Omega_\Gamma = \Omega$. Let $\Lambda_n := [-n, n]^d$. We show that the limit

$$\lim_{n\to\infty}\mu(\xi_0|\xi_{\Lambda_n}) \tag{4.37}$$

exists for every $\xi$. Indeed for $p \ge n$

$$\mu(\xi_0|\xi_{\Lambda_n}) = \int \mu(\xi_0|\xi_{\Lambda_n}\omega_{\Lambda_p\backslash\Lambda_n})\mu(d\omega_{\Lambda_p\backslash\Lambda_n}|\xi_{\Lambda_n}) \tag{4.38}$$

so that

$$
\begin{aligned}
&\left|\mu(\xi_0|\xi_{\Lambda_n}) - \mu(\xi_0|\xi_{\Lambda_p})\right| \\
&= \int \mu(d\omega_{\Lambda_p\backslash\Lambda_n}|\xi_{\Lambda_n})\left[\mu(\xi_0|\xi_{\Lambda_n}\omega_{\Lambda_p\backslash\Lambda_n}) - \mu(\xi_0|\xi_{\Lambda_p})\right]. && (4.39)
\end{aligned}
$$

Since $\xi$ is not a "bad" configuration

$$\lim_{n,p\to\infty}\left|\mu(\xi_0|\xi_{\Lambda_n}\omega_{\Lambda_p\backslash\Lambda_n}) - \mu(\xi_0|\xi_{\Lambda_p})\right| = 0 \tag{4.40}$$

Define

$$\gamma_{\{0\}}(\xi|\xi) := \lim_{n\to\infty}\mu(\xi_0|\xi_{\Lambda_n\backslash 0}) \tag{4.41}$$

and analogously

$$\gamma_{\{i\}}(\xi|\xi) := \lim_{n\to\infty}\mu(\xi_i|\xi_{\Lambda_n\backslash i}). \tag{4.42}$$

We prove that $\gamma_{\{0\}}(\xi|\cdot)$ is continuous. Choose $\xi \in \Omega$ and $\epsilon > 0$. Since $\xi$ is not "bad," there must be $\Lambda \subset \mathbb{Z}^d$ so that for all $\Lambda' \supset \Lambda$ and every $\omega, \omega'$

$$\left|\mu(\xi_0|\xi_\Lambda\omega_{\Lambda'\backslash\Lambda}) - \mu(\xi_0|\xi_\Lambda\omega'_{\Lambda'\backslash\Lambda})\right| \le \epsilon. \tag{4.43}$$

Choose now $n$ such that $\Lambda_n \supset \Lambda$, then for $\tilde{\xi} = \xi$ on $\Lambda$

$$\left|\mu(\xi_0|\xi_{\Lambda_n}) - \mu(\xi_0|\tilde{\xi}_{\Lambda_n})\right| \tag{4.44}$$

$$= \left|\mu(\xi_0|\xi_\Lambda\xi_{\Lambda_n\setminus\Lambda}) - \mu(\xi_0|\xi_\Lambda\tilde{\xi}_{\Lambda_n\setminus\Lambda})\right| \tag{4.45}$$

Therefore

$$\left|\gamma_{\{0\}}(\xi_0|\xi) - \gamma_{\{0\}}(\xi_0|\tilde{\xi})\right|$$

$$= \lim_{n\to\infty}\left|\mu(\xi_0|\xi_{\Lambda_n}) - \mu(\xi_0|\tilde{\xi}_{\Lambda_n})\right| \leq \epsilon \tag{4.46}$$

yielding the continuity of $\gamma_{\{0\}}(\xi|\cdot)$. We finally prove that $\mu \in \mathcal{G}(\Gamma)$. Let $f$ be $\mathcal{F}_\Lambda$-measurable, then

$$\int d\mu\ \gamma_\Lambda(f)$$

$$= \lim_{n\to\infty}\int \mu(d\omega)\int \mu(d\sigma_\Lambda|\omega_{\Lambda_n\setminus\Lambda})f(\sigma_\Lambda) \tag{4.47}$$

$$= \lim_{n\to\infty}\int d\mu\mathbb{E}_\mu[f|\mathcal{F}_{\Lambda_n\setminus\Lambda}] \tag{4.48}$$

$$= \int d\mu f = \mu(f). \tag{4.49}$$

■

For $\mu$ a probability measure on $(\Omega, \mathcal{F})$, put

$$S_\mu := \{\xi \in \Omega|\ \xi \text{ is "bad" for } \mu\}. \tag{4.50}$$

From Proposition 4.3 and the proof of Proposition 4.4 we get

**Proposition 4.5** *1.* $S_\mu = \emptyset \Leftrightarrow \mu$ *is Gibbs;*

*2.* $\mu(S_\mu) = 0 \Leftrightarrow \mu$ *is almost Gibbs;*

*3.* $\mu(S_\mu) > 0 \Leftrightarrow \mu$ *is not almost Gibbs.*

# 5 Examples.

In this section we discuss various examples of non-Gibbsian measures, mostly constructed as transformations of Bernoulli measures. The examples are set up to see what kind of objects we can have, and thus they are more serving as mathematical illustrations than real examples of physical interest.

## 5.1 A measure where every configuration is "bad".

Consider $0 < \rho' < \rho < 1$ and $\mu := \lambda\nu_\rho + (1-\lambda)\nu_{\rho'}$ a non-trivial convex combination of two Bernoulli measures with densities $\rho$ and $\rho'$ on $\Omega = \{0,1\}^{\mathbb{Z}}$.

Put $A_\rho = \{\eta \in \Omega : \liminf_{n\to\infty}(1/2n)\sum_{j=-n}^{n}\eta(j) = \rho\}$. For a local $\mathcal{F}_\Lambda$-measurable function $f$ the following holds, $\mu$ a.s.

$$\mu(f|\eta_{\Lambda^c}) = 1_{A_\rho}(\eta)\int \nu_\rho(d\eta_\Lambda)f(\eta_\Lambda) + 1_{A_{\rho'}}\int \nu_{\rho'}(d\eta_\Lambda)f(\eta_\Lambda). \tag{5.51}$$

These conditional probabilities are thus discontinuous everywhere. This well-known fact is valid for more general non-trivial convex combinations of Gibbs measures at different temperatures or belonging to physically non-equivalent potentials (see e.g. [van Enter *et al.*93]). Observe that $A_\rho$ and $A_{\rho'}$ are both dense subsets of $\Omega$ with $\mu(A_\rho) = \lambda$, $\mu(A_{\rho'}) = 1-\lambda$, so that clearly, we cannot modify the conditional probabilities on a set of $\mu$-measure zero obtaining a continuous version. From this, cf. Proposition 4.5, we conclude that $\mu$ is not almost Gibbsian. We will actually see that every configuration is "bad", i.e., $S_\mu = \Omega$

Indeed,

$$\begin{aligned}&\mu(\eta_0 = 1|\eta_1,\ldots,\eta_n,\eta_{-1},\ldots\eta_{-n})\\ &\quad= \frac{\lambda\rho + (1-\lambda)\rho'(\frac{\rho'}{\rho})^{N_n^1(\eta)}(\frac{1-\rho'}{1-\rho})^{2n-N_n^1(\eta)}}{\lambda + (1-\lambda)(\frac{\rho'}{\rho})^{N_n^1(\eta)}(\frac{1-\rho'}{1-\rho})^{2n-N_n^1(\eta)}}.\end{aligned} \tag{5.52}$$

where $N_n^1(\eta) = \sum_{j\in[-n,n]\setminus\{0\}}\eta(j)$. From this expression it is easy to verify that there are $\epsilon > 0$ and an integer $n_o > 0$ so that for all $n \geq n_o$, $\eta_1,\ldots,\eta_n,\eta_{-1},\ldots,\eta_{-n}$ there are $\eta'_{n+1},\ldots,\eta'_{2n}$, $\eta'_{-n-1},\ldots,\eta'_{-2n}$ so that

$$\begin{aligned}&|\mu(\eta_0|\eta_1,\ldots,\eta_n,\eta_{-1},\ldots,\eta_{-n}) -\\ &\quad\mu(\eta_0|\eta_1,\ldots,\eta_n\eta'_{n+1},\ldots,\eta'_{2n},\eta_{-1},\ldots,\eta_{-n},\eta'_{-n+1},\ldots,\eta'_{-2n})| \qquad (5.53)\\ &\quad > \epsilon. \qquad (5.54)\end{aligned}$$

Hence, every configuration is "bad."

Write $\phi(\eta) = \lim_{n\to\infty}(1/2n)\sum_{-n}^{n}\eta(x)$ on the set of configurations where the limit exists. For all $i \in \mathbb{Z}$

$$\mu(\eta(i)|\eta) = \eta(i)\phi(\eta) + (1-\eta(i))(1-\phi(\eta)) \quad \mu-\text{ a.s.}. \tag{5.55}$$

As a consequence, the one point Hamiltonian $H_{\{i\}}$ (= the logarithm of the expression above) is $\mu-$ a.s. a pointwise limit of continuous (even local) functions. Still, $H_{\{i\}}$ is not only discontinuous in every configuration, but also, given $\theta \in \Omega$, there does not exist a set of $\mu$ measure one on which $H_{\{i\}}$ is continuous in the direction $\theta$. From this it is clear that there does not exist a $\theta$ such that this measure is almost Gibbsian in the direction $\theta$, but we do not know whether it is weakly Gibbsian or not.

## 5.2 An ergodic translation invariant measure with $S_\mu = \Omega$ that fails to be weakly Gibbsian.

Put $\Omega = \{-1,+1\}^{\mathbb{Z}}$ and take $\nu_p$ the Bernoulli measure with density $p < 1/2$. Consider $\eta(x) = \sigma(x)\sigma(x+1)$ and let $\mu_p$ be the induced measure on the $\eta$ configuration. In what follows, we omit the subindex $p$. As was first observed by J. van den Berg ([van den Berg], [Lörinczi *et al.*97a]) every configuration is "bad" for this $\mu$. From this fact, it follows that $\mu$ is not almost Gibbsian, Proposition 4.5. We will actually show that $\mu$ is even not weakly Gibbsian. Put $\xi(p) = (p/(1-p))^{1/2}$ (note $\xi(p) < 1$ ) and for $\eta_1, \ldots, \eta_n, \eta_{-1}, \ldots, \eta_{-n} \in \{-1,+1\}$ write

$$\begin{aligned} C_j(\eta) &= \prod_{i=1}^{j-1} \eta_i \\ &\text{and} \\ C_{-j}(\eta) &= \prod_{i=1}^{j} \eta_{-i}. \end{aligned} \tag{5.56}$$

Next, abbreviate $S_n^+(\eta) := \sum_{j=1}^n C_j(\eta)$ and $S_n^-(\eta) := \sum_{j=1}^n C_{-j}(\eta)$. Then we compute

$$\begin{aligned} \mu & \quad (\eta(0) = +1 | \eta_1 \ldots \eta_n, \eta_{-1} \ldots \eta_{-n}) \\ = & \left(1 + (\frac{1-p}{p})^2 \xi(p)^{-2S_n^+(\eta) - 2S_n^-(\eta)} + \frac{1-p}{p}\left(\xi(p)^{-2S_n^+(\eta)} + \xi(p)^{-2S_n^-(\eta)}\right)\right)^{-1} \\ + & \left(1 + (\frac{p}{1-p})^2 \xi(p)^{2S_n^+(\eta) + 2S_n^-(\eta)} + \frac{p}{1-p}\left(\xi(p)^{2S_n^+(\eta)} + \xi(p)^{2S_n^-(\eta)}\right)\right)^{-1} \end{aligned} \tag{5.57}$$

It is clear that e.g. for $n > 4$ and for given $\eta_1, \ldots, \eta_n, \eta_{-1}, \ldots, \eta_{-n}$ we can choose $\zeta_{n+1}, \ldots, \zeta_{2n}, \zeta_{-n-1}, \ldots, \zeta_{-2n}$ such that

$$|\xi(p)^{\pm S_n^+(\eta)} - \xi(p)^{\pm S_{2n}^+(\eta\zeta)}| \geq |\xi(p)^{n/4} - \xi(p)|. \tag{5.58}$$

and

$$|\xi(p)^{\pm S_n^-(\eta)} - \xi(p)^{\pm S_{2n}^-(\eta\zeta)}| \geq |\xi(p)^{n/4} - \xi(p)|. \tag{5.59}$$

This implies that there is an $\epsilon > 0$ such that for all $n \in \mathbb{N}$ and $\eta_1, \ldots, \eta_n, \eta_{-1}, \ldots, \eta_{-n}$ there are $\zeta_{n+1}, \ldots, \zeta_{2n}, \zeta_{-n-1}, \ldots, \zeta_{-2n}$ so that

$$|\mu(\eta(0) = +1|\eta) - \mu(\eta(0) = +1|\eta\zeta)| \geq \epsilon. \tag{5.60}$$

implying that indeed every configuration is "bad." In order to determine the infinite conditional probabilities, we have to take the limit $n \to \infty$ in equation (5.57). By construction of $\mu$, the distribution of $S_n^+(\eta)$ under $\mu$ is the distribution of

$$\sigma(1) + \sigma(1)\sigma(2)\sigma(2)\sigma(3) + \ldots + \tag{5.61}$$

$$\sigma(1)\sigma(2)\sigma(2)\sigma(3)\ldots\sigma(n-1)\sigma(n-1)\sigma(n)$$

$$= \sigma(1)\sum_{j=2}^{n}\sigma(j) \tag{5.62}$$

under the Bernoulli measure $\nu_p$. Since $p < 1/2$, $\lim_{n\to\infty}\sum_{j=2}^{n-1}\sigma(j) = -\infty$ for $\nu_p$ almost all $\sigma$. Analogously, the distribution of $S_n^-(\eta)$ under $\mu_p$ is the distribution of $\sigma(0)\sum_{j=-n}^{-1}\sigma(j)$ which goes to $\sigma(0)(-\infty)$ as $n \to \infty$. Therefore, looking at the expression (5.57) for the conditional probabilities of $\mu$, we obtain that $\mu(\eta(0) = +1|\eta)$ is equal to 1 on a set of $\mu$-measure $p^2 + (1-p)^2$ and equal to 0 on a set of measure $1 - p^2 - (1-p)^2$. From this fact we conclude that the local Hamiltonian $H_{\{0\}}$ is $+\infty$ on a set of positive measure, and thus $\mu$ is not weakly Gibbsian. It is nice to know that not every measure is weakly Gibbsian.

We can slightly modify this example in order to obtain an almost Gibbsian measure. To do this, let $\Omega = \{-1,+1\}^{\mathbb{Z}}$ and let $\sigma$ and $\sigma'$ be independent Bernoulli configurations : $\sigma$ is distributed according to $\nu_p$, $p < 1/2$ and $\sigma'$ is distributed according to $\nu_{1/2}$. Let $\xi \in \{0,1\}^{\mathbb{Z}}$ and put

$$\eta_\xi(x) = \xi(x)\sigma(x)\sigma(x+1) + (1-\xi(x))\sigma'(x). \tag{5.63}$$

If $\xi(x) = 0$ then the $\sigma$-algebras $\sigma\{\eta_\xi(y) : y < x\}$ and $\sigma\{\eta_\xi(y) : y > x\}$ are independent. Therefore if $\xi(0) = 1$, then

$$\begin{aligned} &P(\eta_\xi(0) = +1|\eta_1, \ldots, \eta_n, \eta_{-1}, \ldots, \eta_{-n}) \\ &= \quad P(\eta_\xi(0) = +1|\eta_1, \ldots, \eta_{\tau_n^+(\xi)}, \eta_{-1}, \ldots, \eta_{\tau_n^-(\xi)}). \end{aligned} \tag{5.64}$$

where

$$\tau_n^+(\xi) = \inf\{k > 0; k \leq n : \xi(k) = 0\} \wedge n. \tag{5.65}$$

and

$$\tau_n^-(\xi) = -\inf\{k > 0; k \leq n : \xi(-k) = 0\} \vee (-n). \tag{5.66}$$

(Here we put $\inf\emptyset = -\infty$). Combining this fact with the previous example, we obtain that every $\eta$ is "bad" for the distribution of $\eta_\xi$ iff there exists $i \in \mathbb{Z}^d$

such that $\xi(x) = 1 \forall x \geq i$ or $\xi(x) = 1 \forall x \leq i$. Therefore the (minimal) set of discontinuities of a specification $\Gamma$ consistent with the joint distribution of $(\xi, \eta_\xi)$ is

$$\Omega_\Gamma^c = \bigcup_{i \in \mathbb{Z}} \{(\xi, \eta) : \xi(x) = 1 \forall x \leq i \text{ or } \xi(x) = 1 \forall x \geq i, \eta \in \{-1, +1\}^{\mathbb{Z}}\}. \tag{5.67}$$

One easily checks that this set is of first Baire category and is dense in $\{0,1\}^{\mathbb{Z}} \times \{-1,+1\}^{\mathbb{Z}}$. If we now choose e.g. a Bernoulli measure for the distribution of the $\xi$ configuration, then the set $\Omega_\Gamma^c$ has measure zero for the joint distribution of $(\eta, \xi)$.

## 5.3 A weakly Gibbsian measure which is not almost Gibbsian.

We consider $\Omega = \{0,1\}^{\mathbb{N}}$ and define for $0 < \rho < 1$ (fixed) a potential

$$U([0,2n], \xi) = \xi(0)\xi(2n)\rho^{n - N_{2n}/\xi} 1_{\{N_{2n}(\xi) \leq n\}} \tag{5.68}$$

where

$$N_{2n}(\xi) \;=\; \max\{j \geq 0 |\ \xi(2n)\xi(2n-1)\ldots\xi(2n-j) = 1\} \tag{5.69}$$

and $U(\Lambda, \xi) = 0$ when $\Lambda \notin \{[0,2n] :\ n \in \mathbb{N}\}$. To clarify these formulas, the interaction of the interval $[0,2n]$ is obtained as follows : count from the right endpoint the number of successive ones, say $N$. The interaction reaches its maximum for $N = n$ (it is then equal to 1), it becomes zero for $N > n$ and decreases in $0 \leq N \leq n$ to the minimal value $\rho^n$ in $N = 0$. We define

$$H(\eta) := \sum_{n \geq 0} U([0,2n], \eta) \tag{5.70}$$

on the set of $\eta$'s for which this sum converges. We will show in Lemma 5.1 below that $H(\eta)$ exists a.s. for the Bernoulli measure $\nu_{1/2}$ on $\{0,1\}^{\mathbb{N}}$. Henceforth we abbreviate $\nu = \nu_{1/2}$. We will then construct $\mu$ as the measure having density $\mathrm{e}^{-H(\eta)}$ with respect to $\nu$, and we show that for this $\mu$ the set of "bad" configurations has positive measure.

**Lemma 5.1**

$$\sum_{n \geq 0} U([0,2n], \eta) < \infty \qquad \nu - a.s.. \tag{5.71}$$

**Proof.** For $\epsilon > 0$ we have

$$\{\eta :\ \sum_n U([0,2n],\eta) = \infty\} \subset A_\epsilon$$

where

$$\begin{aligned} A_\epsilon &:= \{\xi \in \Omega :\ \forall N \in \mathbb{N}\ \exists p \geq N :\ \xi(2p)\xi(2p-1)\ldots\xi(p+(\log p)^{1+\epsilon}) = 1\} \\ &= \bigcap_{N\in\mathbb{N}} \bigcup_{p\geq N} A_{p,\epsilon} \end{aligned} \tag{5.72}$$

where $A_{p,\epsilon} = \{\xi :\ \xi(2p)\xi(2p-1)\ldots\xi(p+(\log p)^{1+\epsilon}) = 1\}$. Now we have

$$\nu(A_{p,\epsilon}) = (\frac{1}{2})^{p-(\log p)^{1+\epsilon}} \tag{5.73}$$

and thus

$$\sum_{p=0}^{\infty} \nu(A_{p,\epsilon}) < \infty. \tag{5.74}$$

Therefore $\nu(A_\epsilon) = 0$. ■

Define the measure $\mu$ on $\{0,1\}^{\mathbb{N}}$ by

$$\int \mu(d\eta)\ f(\eta) = \frac{\int \mathrm{e}^{-H(\eta)}\ f(\eta)\ \nu(d\eta)}{\int \mathrm{e}^{-H(\eta)}\ \nu(d\eta)}. \tag{5.75}$$

For the conditional probability $\mu(\sigma_0 = 1|\xi_1\ldots\xi_{2n})$ we obtain

$$\begin{aligned} \mu(\sigma_0 = 1|\xi_1\ldots\xi_{2n}) &= \frac{\int \mathrm{e}^{-H(1\xi_1\ldots\xi_{2n}\eta_{[0,2n]^c})}\ \nu(d\eta)}{1+\int \mathrm{e}^{-H(1\xi_1\ldots\xi_{2n}\eta_{[0,2n]^c})}\ \nu(d\eta)} \\ &= \frac{1}{1+[\int \mathrm{e}^{-H(1\xi\eta)}\ \nu(d\eta)]^{-1}}. \end{aligned} \tag{5.76}$$

We first show that for $n$ large enough and for all $\xi_1\ldots\xi_{2n}$ there exist $\xi^1_{2n+1}\cdots\xi^1_{4n+1}$ and $\exists\xi^2_{2n+1}\ldots\xi^2_{4n+1}$ such that $|H(1\xi\xi^1\eta)\ -\ H(1\xi\xi^2\eta)| > \frac{\rho}{2}$. Indeed, choose $\xi^1_{2n+1} = 0$, $\xi^1_{2n+2} = \ldots = \xi^1_{4n} = 1$, $\xi^1_{4n+1} = 0$ and $\xi^2_{2n+1} = \ldots = \xi^2_{4n+1} = 0$. Then

$$H(1\xi\xi^1\eta)\ -\ H(1\xi\xi^2\eta) = \sum_{k=1}^{n}\rho^k > \frac{\rho}{2} \tag{5.77}$$

Therefore, combining this with (5.76)

$$\begin{aligned} |\mu\ \ &(\sigma(0) = 1|\xi_1\ldots\xi_{2n}\xi^1_{2n+1}\cdots\xi^1_{4n+1}) - \mu(\sigma(0) = 1|\xi_1\ldots\xi_{2n}\xi^2_{2n+1}\cdots\xi^2_{4n+1})| \\ \geq\ \ &\frac{1}{4}\int \nu(d\eta)\exp[-H(1\xi\xi^1\eta](e^{\rho/2}-1). \end{aligned} \tag{5.78}$$

If we now restrict our $\xi$'s to the set $\{\xi : H(\xi) \leq a\}$ which has strict positive measure for $a$ large enough, then, for these $\xi$'s, using Jensen's inequality

$$\int \nu(d\eta) \exp(-H(1\xi\xi^1\eta)) \geq \exp(-(a+b+1/\rho)). \tag{5.79}$$

where $b = \int \nu(d\eta)H(1\eta) < \infty$. Thus, there is a set of "bad" configurations of strict positive $\mu$-measure. Now it is clear that $\mu$ is weakly Gibbsian with potential $U$, since $\mu-$ a.s.

$$\begin{aligned}
\mu(\sigma_\Lambda|\omega_{\Lambda^c}) &= \frac{\exp(-\sum_{A\subset \mathbb{Z}} U(A, \sigma_\Lambda\omega_{\Lambda^c})}{\sum_{\sigma'_\Lambda} \exp(-\sum_{A\subset \mathbb{Z}} U(A, \sigma'_\Lambda\omega_{\Lambda^c})} \\
&= \frac{\exp(-\sum_{A\bigcap\Lambda\neq\emptyset} U(A, \sigma_\Lambda\omega_{\Lambda^c})}{\sum_{\sigma'_\Lambda} \exp(-\sum_{A\bigcap\Lambda\neq\emptyset} U(A, \sigma'_\Lambda\omega_{\Lambda^c})}
\end{aligned} \tag{5.80}$$

Since there exists a set of "bad" configurations of strict positive measure, $\mu$ is not almost Gibbsian but since by Lemma 5.1 the potential converges (absolutely) on a set of $\mu-$ measure 1, $\mu$ is weakly Gibbsian.

# 6 Appendix: topological properties of $\Omega_\Gamma$.

## 6.1 Topological size of $\Omega_\Gamma$.

In this appendix we show that the set of continuity points of a specification is topologically larger than the set of its discontinuity points. To introduce the notion of "topologically large" we use the concept of Baire category. A set is said to be of first Baire category iff it is a countable union of nowhere dense sets ( i.e. sets of which the interior of the closure is empty ). A set is of second Baire category iff it is not of first category. E.g. the set of rational numbers is of first category whereas the set of irrational numbers is of second category. In particular every complete metric space is of second category. We will see that $\Omega_\Gamma$ is of second Baire category whereas $\Omega_\Gamma^c$ is of first Baire category. From the consistency of the specification, we can easily see that if $\omega \in \Omega_\Gamma$ and $\omega \sim \omega'$, then $\omega' \in \Omega_\Gamma$, which implies that $\Omega_\Gamma$ is in the tail field. Now it is proved in [Fernandez *et al.*97] that every non empty set in the tail field is dense in $\Omega$. Therefore if $\Omega_\Gamma \notin \{\emptyset, \Omega\}$ then both $\Omega_\Gamma$ and $\Omega_\Gamma^c$ are dense. Every function $\gamma_\Lambda(F|\cdot)$ is continuous on $\Omega_\Gamma$. We will see in Lemma 6.1 below that a function which is continuous on a dense set is discontinuous on a set of first category. Therefore $\Omega_\Gamma^c$ is a countable union of sets of first category, and thus of first category itself. Since $\Omega$ is a complete metric space, $\Omega$ is of second category, and therefore $\Omega_\Gamma = \Omega \setminus \Omega_\Gamma^c$ is of second category. If we try to imagine this situation on an interval ( $\Omega = [0,1]$ ), then we can have e.g.

$\Omega_\Gamma = [0,1]\backslash\mathbb{Q}$ and $\Omega_\Gamma^c = [0,1]\bigcap\mathbb{Q}$ but not the converse situation. In particular we can have that $\Omega_\Gamma^c$ is countable whereas $\Omega_\Gamma$ is always uncountable (when not empty). Of course these facts do not give information on the measure of $\Omega_\Gamma$ for a given $\mu \in \mathcal{G}(\Gamma)$. It is still possible that $\mu(\Omega_\Gamma) = 0$ and $\mu(\Omega_\Gamma^c) = 1$. We now proceed to the proof of the announced lemma.

**Lemma 6.1** *Let $f : \Omega \to \mathbb{R}$ and suppose $f$ is continuous on a dense subset of $\Omega$. Then $f$ is discontinuous on a set of first category.*

**Proof:** (cf. [Boas] p. 101-102) Put

$$E_n = \{\omega \in \Omega : \exists(\omega_k)_k \text{ with } \omega_k \to \omega \text{ and } f(\omega_k) - f(\omega) \geq 1/n\} \tag{6.81}$$

and let $D_f$ be the set of points of discontinuity of $f$. We have

$$D_f = \bigcup_{n\in\mathbb{N}} E_n \tag{6.82}$$

so in order to prove that $D_f$ is of first category, we have to show that each $E_n$ is nowhere dense. Suppose on the contrary that some $E_n$ is not nowhere dense. Then, using that the continuity points of $f$ are dense, we can find $\omega$, continuity point of $f$, which is a limit point of $E_n$. Since $f$ is continuous in $\omega$, we can take $\delta > 0$ so that $d(\omega,\eta) < \delta$ implies $\mid f(\omega) - f(\eta) \mid \leq 1/2n$. Pick $\xi \in E_n$ with $d(\xi,\omega) < \delta/2$ and $(\xi_k)_k$ the sequence appearing in the deinition of $E_n$, to obtain

$$1/n <\mid f(\xi_k) - f(\xi) \mid \leq \mid f(\xi_k) - f(\omega) \mid + \mid f(\xi) - f(\omega) \mid \tag{6.83}$$

For $k$ large enough $d(\omega,\xi_k) < \delta/2$ and thus the right hand side is not larger than $1/n$, which gives a contradiction. ∎

From this lemma, and the preceding discussion it follows that the set $\Omega_\Gamma^c$ is a countable union of sets of first category and so we have:

**Proposition 6.1** *Let $\Gamma$ be a specification, then the set $\Omega_\Gamma$ of its continuity points is a set of second category and its complement is of first category.*

## 6.2 Continuity of specifications consistent with a potential

If a probability measure $\mu$ is almost Gibbs, then we conclude from the results of Section 3 (4.1) that on $\Omega_\Gamma$ we can find an absolutely convergent potential. This potential then defines local Hamiltonians $H_\Lambda^U$ which are automatically continuous on $\Omega_\Gamma$ with the restricted topology. If we now start from the information that there exists an absolutely convergent potential $U$ consistent with $\mu$, then we can in general not conclude that the associated Hamiltonians $H_\Lambda^U$ are continuous on $\Omega_U$. As we saw in example 5.3, we can have that $\mu(\Omega_U) = 1$ and every specification is discontinuous on the whole of $\Omega_U$. Since the potential will only be (absolutely) convergent on the set $\Omega_U$, it is more appropiate to study continuity of a consistent specification on this set $\Omega_U$ with the restricted topology. Therefore we do not consider the set $\Omega_\Gamma$ (which will in general be empty since the sum of potentials diverges on the dense set $\Omega_U^c$) but the set $\tilde{\Omega}_\Gamma$ defined as the maximal subset of $\Omega_U$ on which $\gamma_{\{0\}}(\sigma_0|\cdot)$ is continuous in the restricted topology of $\Omega_U$. Note that we only consider $\gamma_{\{0\}}(\sigma_0|\cdot)$ because we will study here only translation invariant specifications consistent with $U$.

We suppose that the potential $U$ is consistent with a translation invariant specification. Since $\Gamma$ is consistent with a potential $U$, the function $\gamma_{\{0\}}(\sigma_0|\cdot)$ is a pointwise limit of continuous functions on $\Omega_U$ (with the restricted topology) and thus, if the set $\Omega_U$ is not too small (in a topological sense) we can conclude that $\tilde{\Omega}_\Gamma$ is not empty and moreover, is dense in $\Omega_U$. This is a (rather obvious) extension of the result for pointwise limits of continuous functions on an interval. More precisely we have the following result:

**Proposition 6.2** 1. *Let $f$ be the limit of continuous functions $f : [0,1] \to$ $\mathbb{R}$ and $f = \lim_{n\to\infty} f_n$ where these $f_n$ are continuous . Then $f$ is continuous on a dense subset of* $[0,1]$.

2. *Let $f$ be a function $f : K \to \mathbb{R}$ where $K$ has the following properties*

   - *$K$ is a metric space*
   - *$\forall x \in K$ $\forall \epsilon > 0$: $B_x(\epsilon)$ is a set of second category.*

   *Then $f$ is continuous on dense subset $K' \subset K$ of second category.*

For the proof we refer to [Boas60] p99-102.

**Corollary 6.1** *If $\mu$ is weakly Gibbs and consistent with a potential $U$ such that $\Omega_U$ satisfies the conditions of Proposition 6.2 above, then $\tilde{\Omega}_\Gamma$ is a dense subset of $\Omega_U$ of second category.*

<u>Remark:</u>
In the special case $\Omega_U = \Omega$, i.e. the potential is everywhere convergent, we can conclude more. We have that $\tilde{\Omega}_\Gamma$ is dense in any nonempty perfect subset of $\Omega$ (cf. [Boas60]p.101), i.e. in every closed set $A \subset \Omega$ such that every environment of $x \in A$ contains a point $y \in A, y \neq x$. This property of being dense in every perfect subset is much stronger than being dense, since one can construct nowhere dense perfect subsets of $\Omega$.

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