Gauge symmetries and percolation in $\pm J$ Ising spin glasses

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December 11, 1997

Abstract

We consider the $\pm$ J spin glass on a finite graph $G = (V, E)$, with i.i.d. couplings. Our approach considers the $\mathbf{Z}_2$ local gauge invariance of the system. We see the gauge group as a graph theoretic linear code $\mathcal{C}$ over $GF(2)$. The gauge is fixed by choosing a convenient linear supplement of $\mathcal{C}$. Assuming some relation between the disorder parameter p and the inverse temperature of the thermal bath β_{p_b}, we study percolation in the random interaction random cluster model, and extend the results to dilute spin glasses.

Keywords: Spin glasses, Hopfield neural networks, graph theoretic codes, local gauge invariance, percolation, random interaction random cluster measure.

Running title: Gauge symmetries in Ising spin glasses

1 Introduction

Let $G = (V, E)$ be a finite connected graph with node set V and edge set E, and consider a spin glass with Hamiltonian

$$H_J(\sigma) = \sum_{(ij)\in E} J_{(ij)}\sigma_i\sigma_j, \quad \sigma \in \Omega := \{-1,+1\}^V, \tag{1.1}$$

where $J_{(ij)} \in \mathbb{R}$, $\forall (ij) \in E$. The disorder is modeled through the random fluctuations of the $J_{(ij)}$, and can be thus defined in various ways. The $J_{(ij)}$ are generally statistically dependent, like the Hebbian weights of the Hopfield neural network or the weights of the Mattis model.

Here, we will consider the $\pm J$ Ising spin glass, where the $J_{(ij)}$s are i.i.d. with distribution given by $P(J_{(ij)} = -1) = p$ and $P(J_{(ij)} = 1) = 1 - p$, for some $0 \leq p \leq 1$. $p = 0$ corresponds to the ferromagnetic Ising model, the pure spin glass is defined by choosing $p = 1/2$, and $p = 1$ gives the antiferromagnetic Ising model.

In a different setting, one can construct Hamiltonians on the basis of linear and nonlinear codes, which are close to Derrida's random energy model (Sourlas (1989,1994)). Bruck and Blaum(1989) proposed a Hopfield neural network storing the cut-code $\mathcal{C} \subset GF(2)^{|E|}$ of G, with Hamiltonian given by

$$H_y(\sigma) = \sum_{(ij)\in E} (-1)^{y_{(ij)}}\sigma_i\sigma_j, \quad y \in GF(2)^{|E|}, \tag{1.2}$$

where the sum is taken over the edges $e = (ij) = (ji) \in E$, and partition function $Z_{p_b}(y) := \sum_\sigma \exp(\beta_{p_b} H_y(\sigma))$, for some inverse temperature $\beta_{p_b} \geq 0$, which is equivalent to the random $\pm J$ model if the weights are given by i.i.d. random variables $y_{(ij)}$ with $P(y_{(ij)} = 1) = p$ and $P(y_{(ij)} = 0) = 1 - p$. The advantage of such a representation is to allow the reduction of the disorder by using gauge symmetries (sections 3 and 4). The relevant disorder is given by a linear supplement $\mathcal{T}$ of $\mathcal{C}$. We illustrate this reduction by giving some properties of the ground states of (1.2).

We next consider percolation in the random interaction random cluster model, by using gauge symmetries. We follow essentially Kasai and Okiji(1988) and Gandolfi, Keane and Newman(1992) (see also Cataudella *et altri*(1994)). Let $0 \leq p_b \leq 1$ be the bond probability, which is linked to the inverse temperature $\beta_{p_b} \geq 0$ by the relation

$$p_b = 1 - \exp(-\beta_{p_b}).$$

Given a realization of the disorder variable $y \in GF(2)^{|E|}$, the FK probability measure μ_n^y is defined on the set of bonds E associated with G. Let $\{n_b;\ b \in E\}$ be a family of binary random variables having the interpretation $n_b = 1$ if and only if bond b is occupied. μ_n^y and μ_S^y are defined to be the marginals of a probability μ^y defined on the product $\tilde{\Omega} = \Omega \text{ x } \{0,1\}^E$, where $\Omega := \{-1,+1\}^V$. First, consider μ', the joint distribution on $\tilde{\Omega}$ of mutually i.i.d. random variables $\{S_i', (n)_b';\ i \in V,\ b \in E\}$ with $P(S_i' = 1) = P(S_i' = -1) = 1/2$ and $P(n_b' = 1) = p_b$. Next, consider the event in $\tilde{\Omega}$

$$\tilde{U}_y \ := \ \{\forall (ij) \in E,\ J_{(ij)} n_{(ij)}' S_i' S_j' \geq 0\}, \quad J_{(ij)} \equiv (-1)^{y_{(ij)}}. \tag{1.3}$$

Then μ^y is μ' conditioned on $\tilde{U}_y$, that is

$$\mu^y \ := \ \mu'(\tilde{U}_y)^{-1} \mu' \ I_{\tilde{U}_y}, \tag{1.4}$$

where $I_{\tilde{U}_y}$ is the indicator function of the event $\tilde{U}_y$. The spin-measure μ_S^y is defined by

$$\mu_S^y(\sigma) \ = \ Z_{\hat{p}_b}(y)^{-1} \exp(\frac{\beta_{p_b}}{2} \sum_{(ij)\in E} J_{(ij)} \sigma_i \sigma_j), \tag{1.5}$$

$\hat{p}_b := 1 - \sqrt{1-p_b}$, $\beta_{\hat{p}_b} = \beta_{p_b}/2$, and the bond measure is given by

$$\mu_n^y((n)) \ = \ Z_{n,p_b}(y)^{-1} 2^{|(n)|} \mu_{n,p_b}^{\text{ind}}((n)) I_{U_y}((n)), \tag{1.6}$$

where $Z_{\hat{p}_b}(y)$ is the spin glass partition function (Definition 4), $Z_{n,p_b}(y) = \exp(-|E|\beta_{\hat{p}_b}) Z_{\hat{p}_b}(y)$ is a normalizing partition function, $|(n)|$ denotes the number of occupied clusters (including one site clusters) in the bond configuration (n), μ_{n,p_b}^{ind} is the Bernoulli product measure with parameter p_b, and U_y is the event in $\{0,1\}^E$

$$U_y \ := \ \{(n);\ \text{ there exists some choice of } \sigma \in \Omega \ \text{ with } \ (\sigma,(n)) \in \tilde{U}_y\}. \tag{1.7}$$

Let

$$\eta_{ij}((n)) = I(i \rightsquigarrow j \text{ in } (n)) \prod_{e\in\gamma} (-1)^{y_e},$$

where $i \rightsquigarrow j$ means that i and j are linked in (n) by a path of occupied edges γ. By construction, $\prod_{e\in\gamma}(-1)^{y_e}$ does not depend on γ when i and j are in the same unfrustrated cluster. It is possible to see that

$$\mathrm{E}_{\mu_S^y}(\sigma_i\sigma_j) \ = \ \mathrm{E}_{\mu_n^y}(\eta_{ij}),$$

with

$$\mu_n^y(i \rightsquigarrow j) \ = \ \mu_n^y(\eta_{ij} = +1) + \mu_n^y(\eta_{ij} = -1),$$

and

$$\mathrm{E}_{\mu_S^y}(\sigma_i\sigma_j) \ = \ \mu_n^y(\eta_{ij} = +1) - \mu_n^y(\eta_{ij} = -1),$$

(see Newman(1994)). In the ferromagnetic case $y = 0$, the above identities give the well known Fortuin-Kasteleyn relation $\mathrm{E}_{\mu_S^0}(\sigma_i\sigma_j) = \mu_n^0(i \rightsquigarrow j)$, which is a basic tool for studying phase transition in the ferromagnetic Ising model. In the disordered regime, we have only the inequality $|\mathrm{E}_{\mu^y_S}(\sigma_i\sigma_j)| \leq \mu^y_n(i \rightsquigarrow j)$.

Let $Q_{p_b,p,G}$ be the joint distribution of y and (n), which is called the *random interaction random cluster measure*, and let $Q^{\text{edge}}_{p_b,p,G}$ be its marginal on the bond variables. We consider the associated dependent percolation problem on the *Nishimori* curve $\hat{p}_b = p^* := 1 - \sqrt{p/(1-p)}$, $p < 1/2$, or $\tanh(\beta_{\hat{p}_b}) = 1 - 2p$. We prove in Lemma 3 that $Q^{\text{edge}}_{p_b,p,G}$ dominates a product Bernoulli measure with parameter $p_b/(2-p_b)$. In the antiferromagnetic regime $p > 1/2$, we consider the same problem on the curve $\hat{p}_b = (1-p)^*$, and obtain the same result under the additional hypothesis that G is bipartite.

Finally, we restrict our study to $\mathbf{Z}^d$, $d \geq 2$, with the usual nearest neighbors structure. Consider the finite box $B = \{x \in \mathbf{Z}^d;\ ||x|| \leq N\}$, with edge set E_B. Let $p_c(\mathrm{Z}^d)$ be the critical probability associated with bond percolation on $\mathbf{Z}^d$. Assuming periodic boundary conditions and that $\hat{p}_b = p^*$, $p < (1/2)(1-p_c(\mathbf{Z}^d))$ or that $\hat{p}_b = (1-p)^*$, $p > (1/2)(1+p_c(\mathbf{Z}^d))$, we get in Corollary 2 that

$$\mu^y_n(\text{ the origin is in an infinite cluster of occupied bonds}) > 0,$$

for almost all realizations of the disorder $y \in GF(2)^{|E|}$. We also extend these results to diluted spin glasses.

2 Spin glasses and neural networks

Bruck and Blaum(1989) and Sourlas(1989,1994) established connections between Hopfield neural networks, error correcting codes and maximization of polynomials over the $|E|$-cube.

Definition 1 *Let $G = (V, E)$ be a finite undirected weighted graph, with node set V and edge set E. Let W be the symmetric matrix giving the weights of the undirected edges. Let V_1 be a subset of V, and set $V_2 := V \setminus V_1$. The set of edges $c \subset E$ each of which is incident at a node in V_1 and at a node in V_2 is called a cut in G. The set of cuts will be denoted by $\mathcal{C}$. The weight $w(c)$ of a cut $c \in \mathcal{C}$ is the sum of its edge weights. A minimum cut is a cut of minimum weight, and the set of minimum weight cuts is denoted by $C_{\min}$, that is $C_{\min} := \{c \in \mathcal{C};\ w(c) = \min_{c' \in \mathcal{C}} w(c')\}$.*

Definition 2 *Let $G = (V, E)$ be a finite undirected graph, with $E = \{e_k,\ k = 1,..,|E|\}$. To every subset $S \subset E$ we associate a characteristic vector f_S defined by $f_S(e_k) = 1$ if $e_k \in S$ and $f_S(e_k) = 0$ otherwise. The incidence matrix of the graph is the $|V|$ x $|E|$ matrix D_G in which row i, $i = 1,..,|V|$ is the characteristic vector of the set of edges incident to nodes $i \in V$. In what follows, $GF(2)$ denotes the field of two elements. The set of characteristic vectors corresponding to the cuts in G forms the linear space $\mathcal{C}$ of $GF(2)^{|E|}$ of dimension $|V| - |G|$, where $|G|$ denotes the number of connected components, which is called the cut* *code associated with G (see e.g. van Lint(1992) or Biggs(1974)). In the remaining, we assume that $|G| = 1$. The incidence matrix has rank $|V|-1$. Every row $c(i)$, $i = 1,..,|V|$, $S = \{i\}$, of the incidence matrix is a characteristic vector of an elementary cut, and every $|V|-1$ subset of rows of D_G forms a basis of the cut-code.*

Assume now that a code word $c \in \mathcal{C}$ is sent through a binary symmetric channel with crossover probability $0 \leq p \leq 1$ and output $c' \in \{0,1\}^E$. Maximum likelihood decoding is obtained by choosing the closest word $c'' \in \mathcal{C}$ in Hamming distance.

Proposition 1 *[Bruck-Blaum(1989)] Let G be a connected graph with cut code $\mathcal{C}$. Let $c' \in GF(2)^{|E|}$. Consider the weighted graph with weights given by*

$$J_{ij} \;=\; (-1)^{c'_{(ij)}}.$$

Then, maximum likelihood decoding with respect to the linear code $\mathcal{C}$ is equivalent to finding

$$H_{J,\max} \;:=\; \{\sigma \in \Omega;\; H_J(\sigma) \;=\; \max_{\sigma' \in \Omega} H_J(\sigma')\}.$$

In fact, the correspondence runs as follows: to every configuration of spins σ we associate a partition (V_1, V_2) of V by setting $V_1 := \{i \in V;\; \sigma_i = 1\}$ and $V_2 := \{i \in V;\; \sigma_i = -1\}$, which gives a cut in a natural way. Let us denote this mapping by Ψ. Using the relation

$$\sum_{(ij)\in E} \sigma_i \sigma_j (-1)^{y_{(ij)}} \;=\; \sum_{(ij)\in E} (-1)^{y_{(ij)}} - 2 \sum_{(ij)\in E, \sigma_i\sigma_j = -1} (-1)^{y_{(ij)}}, \tag{2.1}$$

$\forall y \in GF_2^{|E|}$, $\sigma \in \Omega$, we get the well known result

$$C_{min} \;=\; \Psi(H_{J,\max}), \tag{2.2}$$

(see e.g. Welsh(1993)).

Assume that the sent code word is the neutral element $\underline{0} \in \mathcal{C}$, and that the received word is $c' \in \{0,1\}^E$, which has i.i.d. components $c'_{(ij)}$, $(ij) \in E$ with distribution given by $P(c'_{(ij)} = 1) = p$ and $P(c'_{(ij)} = 0) = 1 - p$. Let us rewrite the mapping (2.2): we have

$$\Psi(\sigma) \;=\; \sum_{(ij)\in E} e_{(ij)} \frac{1}{2}(1 - \sigma_i \sigma_j), \tag{2.3}$$

where $\{e_{(ij)}, (ij) \in E\}$ forms the canonical basis of $GF(2)^{|E|}$. The number of decoding errors N_e when sending $\underline{0}$ is then given by

$$N_e \;=\; |\Psi(\sigma^*)| = \sum_{(ij)\in E} \frac{1}{2}(1 - \sigma_i^* \sigma_j^*) \;=\; \mathrm{Pe}(\sigma^*), \tag{2.4}$$

where $|\Psi(\sigma^*)|$ is the Hamming weight of $\Psi(\sigma^*) \in \mathcal{C}$, σ^* is a spin configuration maximizing the Hamiltonian, and Pe denotes the perimeter of this configuration, that is the number of pairs $(ij) \in E$ with $\sigma_i^* \sigma_j^* = -1$.

Remark 1 *Proposition 1 is given in Sourlas(1989,1994), but at the level of the code generators. The number of decoding errors is then linked to the magnetization of the ground states.*

Remark 2 *Graph theoretic codes have poor minimum distance properties. Let $d(G)$ be the minimum distance associated with the linear code $\mathcal{C}$. Then $d(G) \leq 2|E|/|V|$. It can be proved that $d(G) \leq 5$ for a large family of planar graphs G (see Peterson and Weldon(1972), p. 136-138). Theorem 4.2.6 and Corollary 4.2.7 in Roman (1992) give the obvious result $P(\mathrm{Pe}(\sigma^*) = 0) = P(N_e = 0) \longrightarrow 0$ as $|E| \longrightarrow \infty$, $\forall p > 0$, for every family of increasing graphs, since in this case $\delta_{|E|} := d(G)/|E| \longrightarrow 0$. It would be interesting to check the behavior of probabilities like $P(\mathrm{Ne} < \alpha|E|)$, for some positive constant $0 < \alpha < 1$, by using tools of coding theory.*

Example 1 *Consider a one dimensional spin glass with free boundaries. The associated linear code $\mathcal{C}$ has dimension $d = |V| - 1 = |E|$, showing that in this case $\mathcal{C} = GF(2)^{|E|}$. Maximum likelihood decoding is thus simple: the received code word c' is decoded as c' itself. By Euler's formula, the same result is valid for any tree.*

3 Gauge symmetries and disorder reduction

Toulouse(1977) introduced the $\mathbf{Z}_2$ lattice gauge group in the context of disordered magnets when considering the *frustration network*. The basic Hamiltonian is invariant under the transformation

$$\sigma_i \longrightarrow -\sigma_i, \quad J_{(ij)}(j \text{ adjacent to } i) \longrightarrow -J_{(ij)} \tag{3.1}$$

$i, j \in V$, $\sigma \in \{-1,+1\}^V$, $J \in \{-1,+1\}^E$. This gauge invariance was then used by several authors (Fradkin, Hubermann and Shenker(1978) and Nishimori(1980)). We translate $J_{(ij)}$ (j adjacent to i) $\longrightarrow -J_{(ij)}$ by writing $y \longrightarrow y + c(i)$, $J_{(ij)} = (-1)^{y_{(ij)}}$, where $y \in GF(2)^{|E|}$ and $c(i) \in \mathcal{C}$ is the elementary cut centered at $i \in V$ (see Definition 2).

Definition 3 *Let $\mathcal{T}$ be a linear supplement of $\mathcal{C}$, that is, $\mathcal{T}$ is a vector space over $GF(2)$ such that $\mathcal{C} \oplus \mathcal{T} = GF(2)^{|E|}$ and $\dim(\mathcal{T}) = |E| - |V| + 1$. Every $y \in GF(2)^{|E|}$ decomposes in a unique way as a sum of the form $y = c + t$ mod 2, where $c \in \mathcal{C}$ and $t \in \mathcal{T}$. A function $f : GF(2)^{|E|} \longrightarrow \mathbb{R}$ is said to be gauge invariant when $f(y + c) \equiv f(y)$, $\forall y \in GF(2)^{|E|}$, $c \in \mathcal{C}$. f is then constant on the cosets $\{\{t + c;\ c \in \mathcal{C}\},\ t \in \mathcal{T}\}$ of $GF(2)^{|E|}/\mathcal{C}$, which form a partition of $GF(2)^{|E|}$.*

Using the local gauge invariance, it is easy to see that $H_y(\sigma^*(y)) = H_{y'}(\sigma^*(y'))$, where $\sigma^*(y)$ is a spin configuration realizing the maximum of H_y, and $y' = y+c$ for some $c \in \mathcal{C}$. This property shows that $H_y(\sigma^*(y))$, taken as function of y, is constant on the cosets $\{c+t;\ c \in \mathcal{C}\}$, and is gauge invariant. Let H_t^* be the value of $\max_{\sigma\in\Omega} H_y(\sigma)$ on the coset $\{t+c;\ c \in \mathcal{C}\}$.

Let $\mu_{n,p}^{\rm ind}$ be the product Bernoulli probability measure on $\{0,1\}^E$ with parameter p. Then

$$\bar{H}_p \ := \ \sum_y H_y(\sigma^*(y))\mu_{n,p}^{\rm ind}(y) \ = \ \sum_{t\in\mathcal{T}} H_t^* \sum_{c\in\mathcal{C}} \mu_{n,p}^{\rm ind}(c+t). \qquad (3.2)$$

When $p = 1/2$, we arrive at $\bar{H}_{1/2} \ = \ |\mathcal{T}|^{-1}\sum_{t\in\mathcal{T}} H_t^*$.

Let $\bar{\mu}_{n,p}^{\rm ind}$ be the marginal of $\mu_{n,p}^{\rm ind}$ on the linear space $\mathcal{T}$, that is the probability measure given by

$$\bar{\mu}_{n,p}^{\rm ind}(t) \ := \ \sum_{c\in\mathcal{C}} \mu_{n,p}^{\rm ind}(c+t). \qquad (3.3)$$

Let

$$z_p \ := \ \frac{p}{1-p}, \ p < 1. \qquad (3.4)$$

We have

$$\bar{\mu}_{n,p}^{\rm ind}(t) \ = \ (1-p)^{|E|}\sum_{c\in\mathcal{C}} z_p^{|c+t|},$$

where $|y|$ denotes the Hamming weight of $y \in \{0,1\}^E$.

Definition 4 *Given $y \in \{0,1\}^E$ and $q \le 1$, the spin glass partition function associated with the disorder y is defined by*

$$Z_q(y) \ = \ \sum_{\sigma\in\Omega} \exp(\beta_q \sum_{(ij)\in E} (-1)^{y(ij)}\sigma_i\sigma_j),$$

where $\beta_q \in \mathbb{R}$ is such that

$$q \ = \ 1 - \exp(-\beta_q).$$

Given $p \in [0,1[$, let

$$p^* \ := \ 1 - \sqrt{\frac{p}{1-p}}, \quad \tanh(\beta_{p^*}) \ = \ 1-2p. \qquad (3.5)$$

Fradkin *et altri*(1978) showed the following identity

$$Z_{p^*}(t) \ \equiv \ \cosh(\beta_{p^*})^{|E|}2^{|E|+1}\bar{\mu}_{n,p}^{\rm ind}(t), \quad \forall t \in \mathcal{T}, \qquad (3.6)$$

which can be seen as follows: Let $t \in \mathcal{T}$, $c \in \mathcal{C}$, and choose some $\sigma \in \Psi^{-1}(c)$ (see (2.3)). Then, using (2.1), we arrive at

$$\begin{aligned}\sum_{(ij)\in E} \sigma_j \sigma_i (-1)^{t(ij)} &= \sum_{(ij)\in E} (-1)^{t(ij)} - 2 \sum_{(ji)\in E, \sigma_i\sigma_j=-1} (-1)^{t(ji)} \\ &= \sum_{(ij)\in E} (-1)^{t(ij)+c(ij)},\end{aligned}$$

and (3.6) follows by choosing convenient normalizing constants.

The same approach was considered later by Nishimori(1980): On the curve $\tanh(\beta_{p_b}/2) = 1-2p$, $p < 1/2$, it is possible to give the quenched averages of the internal energy and gauge invariant correlation function, and bounds for the specific heat. We will also consider this curve when studying percolation in the random cluster model (section 5).

Remark 3 *Using (3.6), we get that*

$$\bar{\mu}^{\mathrm{ind}}_{n,p}(t=\underline{0}) = Z_{p^*}(t=\underline{0}) 2^{-|E|-1} \cosh(\beta_{p^*})^{-|E|},$$

where $Z_{p^*}(t=\underline{0})$ is the Ising partition function on the graph G. When $G = \mathbf{Z}^2$, $\bar{\mu}^{\mathrm{ind}}_{n,p}(t=\underline{0})$ has a singularity in the thermodynamic limit when $\tanh(\beta_{p^*}) = 1-2p = \sqrt{2}-1$, that is when the concentration of anti-ferromagnetic bonds p is given by $p = 1-1/\sqrt{2}$, which corresponds to Schuster's critical antiferromagnetic bond concentration associated with the pair dissociation phase transition (see Schuster(1979), Kolan and Palmer(1981) or Liebmann and Schuster(1981)).

4 On the Geometry of $\mathcal{T}$

As we have seen, simplifications occur and the study of gauge invariant functions can be limited to the gauge $\mathcal{T}$. In this paragraph, we give some geometrical results on the shape of $\mathcal{T}$ which will be useful for the study of the percolation problem.

Lemma 1 *Let $G = (V, E)$ be a finite connected graph, and let g be a spanning tree of G. Let $\mathcal{T}_g$ be the linear space over $GF(2)$ generated by the characteristic vectors associated with the edges of $E \setminus E(g)$, where $E(g)$ denotes the edge set of g. Then $\mathcal{C} \oplus \mathcal{T}_g = GF(2)^{|E|}$.*

Proof: By Euler's formula, $|E(g)| = |V|-1$, and thus $|E \setminus E(g)| = |E|-|V|+1 = |E| - \dim(\mathcal{C})$. It remains to check that $\mathcal{T}_g + \mathcal{C} = GF(2)^{|E|}$. We show that every characteristic vector $f_{\{e\}}$ associated with $e \in E(g)$ can be written as a mod 2 sum of cuts of $\mathcal{C}$ and characteristic vectors associated with edges of $E \setminus E(g)$.

Given some $i \in V$, let $c(i)$ be the elementary cut of $\mathcal{C}$ centered at i, which can be reduced to a cut of the cut code $\mathcal{C}_g$ associated with the spanning tree by adding modulo 2 the characteristic vectors associated with the edges e' of $E \setminus E(g)$ incident to i. We can concentrate our proof to the spanning tree g, by showing that every edge of g can be written as a mod 2 sum of cuts of $\mathcal{C}_g$. But the cut code of any tree is equal to the full binary space $GF(2)^{|E(g)|}$, and the result follows.

An edge e of a finite connected graph is said to be an isthmus when the removal of e from E increases the number of connected components of G. In what follows $< \cdot, \cdot >_{\mathbb{R}}$ denotes the usual scalar product. When x and y are binary vectors, the meaning of $< x, y >_{\mathbb{R}} = 0$ is that x and y have disjoint supports.

Corollary 1 *Let $G = (V, E)$ be a finite connected graph, and assume that $e \in E$ is not an isthmus. Then, there is a linear space $\mathcal{T}'(e)$ with $\mathcal{T}'(e) \oplus [f_{\{e\}}] \oplus \mathcal{C} = GF(2)^{|E|}$, where $[f_{\{e\}}]$ is the linear space generated by the singleton $f_{\{e\}}$, and $< f_{\{e\}}, t' >_{\mathbb{R}} = 0$, $\forall t' \in \mathcal{T}'(e)$.*

Proof: By assumption e is not an isthmus, and there is a spanning tree g_e such that $e \notin E(g_e)$. By Lemma 1, we can choose $\mathcal{T}'(e)$ to be the linear space over $GF(2)$ generated by the characteristic vectors associated with the edges $e' \in E \setminus E(g_e) \setminus \{e\}$, which proves the result.

Lemma 2 *Let $G = (V, E)$ be a finite connected graph, and assume that $e \in E$ is an isthmus. Then $f_{\{e\}} \in \mathcal{C}$, and there is a linear supplement $\mathcal{T}(e)$ of $\mathcal{C}$ with $< f_{\{e\}}, t >_{\mathbb{R}} = 0$, $\forall t \in \mathcal{T}(e)$.*

Proof: Consider the first assertion. Let $e = (e_-, e_+)$ be an isthmus of G. The removal of e decomposes G in two connected components, say $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with $e_- \in V_1$, $e_+ \in V_2$, $E_1 \cap E_2 = \emptyset$, $V_1 \cap V_2 = \emptyset$, $E_1 \cup E_2 = E \setminus \{e\}$, and $V_1 \cup V_2 = V$. Let $\mathcal{C}^1 \subset GF(2)^{|E_1|}$ and $\mathcal{C}^2 \subset GF(2)^{|E_2|}$ be the associated cut codes. We have $f_{\{e\}} = c(e_+) + c^2(e_+)$, where $c(e_+)$ is the elementary cut word of $\mathcal{C}$ centered at e_+, and $c^2(e_+)$ is the same cut in $\mathcal{C}^2$. But $\sum_{i \in V_2} c^2(i) = \underline{0}$, and it follows that $c^2(e_+) = \sum_{i \in V_2 \setminus \{e_+\}} c^2(i)$. The deletion of e from G modifies only the cuts $c(e_-)$ and $c(e_+)$ of $\mathcal{C}$, and thus $c^2(i) = c(i)$, $\forall i \in V_2 \setminus \{e_+\}$. It follows that

$$f_{\{e\}} \;=\; c(e_+) + \sum_{i \in V_2 \setminus \{e_+\}} c(i) \in \mathcal{C}.$$

Next, e is an isthmus of G, and every spanning tree g of G must contain e in its edge set. Let g be an arbitrary spanning tree, and let $\mathcal{T}_g$ be the linear supplement of $\mathcal{C}$ given by Lemma 1. Of course, $f_{\{e\}} \notin \mathcal{T}_g$, and, by construction, $< t, f_{\{e\}} >_{\mathbb{R}} = 0$, $\forall t \in \mathcal{T}_g$.

Example 2 *Let G be a finite grid of $\mathbf{Z}^2$. Given $e \in E$, it is easy to find a spanning tree $\gamma(e)$ not containing e, as shown in the following picture.*

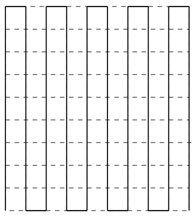

Fig 1. The spanning tree is given by the continuous line. $\mathcal{T}$ is generated by the characteristic vectors associated with the dashed edges.

We then see that there exists a linear space $\mathcal{T}(e)$ generated by a subset of the set of the horizontal edges, the vertical edges being not involved in $\mathcal{T}(e)$. From the point of view of physics, this means that the edges of $\gamma(e)$ remain ferromagnetic, and only edges of $E \setminus E(\gamma(e))$ can be both. Note that this particular choice is similar to the axial or temporal gauge of $\mathbf{Z}_2$ lattice gauge theory (see e.g. Itzykson and Drouffe(1989), vol.1, p. 333).

Example 3 *Let G be a finite connected graph. When $p = 1$, $\mu_{n,1}^{\mathrm{ind}} = \delta_{\underline{1}}$, the dirac probability measure with mode $\underline{1} = (1, .., 1)^{\mathrm{tr}}$. Then $\bar{\mu}_{n,1}^{\mathrm{ind}}(t) = \sum_{c \in \mathcal{C}} \delta_{\underline{1}}(c+t)$, and thus, using the unique decomposition of $\underline{1} = c_1 + t_1$, $c_1 \in \mathcal{C}$, $t_1 \in \mathcal{T}$, we obtain that $\bar{\mu}_{n,1}^{\mathrm{ind}} = \delta_{t_1}$. The average of local gauge invariant functions $f(y)$ like the partition function $Z_{\hat{p}_b}(y)$, $\max_{\sigma \in \Omega} H_y(\sigma)$, Wilson loops or the random cluster mass $\mu_n^y((n))$ on a fixed graph (n) (see section 5) are such that $f(\underline{1}) = f(t_1)$. Assume that G is bipartite, and let (V_1, V_2) be a partition of V associated with a two-colouring. Then $\underline{1} = \sum_{i \in V_1} c(i)$, where $c(i)$ is the elementary cut centered at $i \in V_1$. It follows that $\underline{1} \in \mathcal{C}$, and thus $\bar{\mu}_{n,1}^{\mathrm{ind}} = \delta_{\underline{0}}$ with $f(\underline{1}) = f(\underline{0})$. Note that in general $t_1 \neq \underline{0}$ (e.g. the triangular lattice).*

5 Percolation in the random cluster model

We consider the random cluster measure defined in section 1. Using the local gauge invariance, we obtain that

$$(\sigma, (n)) \in \tilde{U}_y \ \text{ if and only if } \ (\theta_i(\sigma), (n)) \in \tilde{U}_{y'}, \tag{5.7}$$

where $\theta_i : \Omega \longrightarrow \Omega$ is the mapping which reverses the value of the spin located at site $i \in V$, and $y' = y + c(i) \bmod 2$.

Given a bond configuration $(n) \in \{0, 1\}^E$, we see that

$$(n) \in U_y \ \text{ if and only if } \ (n) \in U_{y'}, \tag{5.8}$$

showing that the sets U_y are preserved on the cosets $\{t+c;\ c \in \mathcal{C}\}$.

The bond measure μ_n^y depends on the interaction y only through the indicator function I_{U_y}, and its properties are also preserved on the cosets $\{t+c;\ c \in \mathcal{C}\}$. We can thus reduce the study of $\mu_n^y(i \rightsquigarrow j)$ where $\mathcal{L}(y) = \mu_{n,p}^{\text{ind}}$ to the probability mass

$$\mu_n^t(i \text{ is connected to } j \text{ by a path of occupied edges}), \quad t \in \mathcal{T}, \tag{5.9}$$

when the disorder $\{t \in \mathcal{T}\}$ is distributed according to the probability measure $\bar{\mu}_{n,p}^{\text{ind}}$ for some antiferromagnetic bond concentration p.

Let $Q_{p,p_b,G}$ be the joint distribution of y and (n), which is called the *random interaction random cluster measure* (see Gandolfi, Keane and Newman(1992)). Let $Q_{p,p_b,G}^{\text{edge}}$ be its marginal on the bond variables. Let $\mu_1,\ \mu_2 \in \mathcal{M}(\{0,1\}^E)$. μ_1 is dominated by μ_2, written $\mu_1 << \mu_2$, when $\mu_1(h) \leq \mu_2(h)$ for any componentwise increasing function h.

Lemma 3 *Let G be a finite connected graph. Let*

$$p^* := 1 - \sqrt{\frac{p}{1-p}}, \quad \hat{p}_b := 1 - \sqrt{1-p_b}, \quad \textit{and assume that} \quad \hat{p}_b = p^*,\ p < 1/2.$$

Then, $\forall e \in E$,

$$\frac{Q_{p,p_b,G}^{\text{edge}}(n_e = 1, \bar{n}_e)}{Q_{p,p_b,G}^{\text{edge}}(n_e = 0, \bar{n}_e)} \geq \frac{p_b}{2(1-p_b)}, \quad Q_{p,p_b,G}^{\text{edge}}(n_e = 1 | \bar{n}_e) \geq \frac{p_b}{2-p_b}, \tag{5.10}$$

where $\bar{n}_e$ denotes the bond configuration $\{n_{e'},\ e' \neq e\}$, and thus $Q_{p,p_b,G}^{\text{edge}} >> \mu_{n,h(p_b)}^{\text{ind}}$, where $\mu_{n,h(p_b)}^{\text{ind}}$ is the Bernoulli product measure on $\{0,1\}^E$ with parameter $h(p_b) := p_b/(2-p_b)$.

Proof: Assume that $e \in E$ is not an isthmus. By Corollary 1, we can find a linear space $\mathcal{T}'(e)$ with $\mathcal{T}'(e) \oplus [f_{\{e\}}] \oplus \mathcal{C} = GF(2)^{|E|}$, and $< t', f_{\{e\}} >_{\mathbb{R}} = 0$, $\forall t' \in \mathcal{T}'(e)$. Let $\mathcal{T}(e) := \mathcal{T}'(e) \oplus [f_{\{e\}}]$. The local gauge invariance gives that

$$\begin{aligned} Q_{p,p_b,G}^{\text{edge}}(n_e = 1, \bar{n}_e) &= \sum_y \mu_{n,p}^{\text{ind}}(y) Q_{p,p_b,G}(n_e = 1, \bar{n}_e | y) \\ &= \sum_{t \in \mathcal{T}(e)} \bar{\mu}_{n,p}^{\text{ind}}(t) Q_{p,p_b,G}(n_e = 1, \bar{n}_e | t) \end{aligned}$$

We must then consider the ratio

$$\frac{\sum_{t \in \mathcal{T}(e)} \bar{\mu}_{n,p}^{\text{ind}}(t) Q_{p,p_b,G}(n_e = 1, \bar{n}_e | t)}{\sum_{t \in \mathcal{T}(e)} \bar{\mu}_{n,p}^{\text{ind}}(t) Q_{p,p_b,G}(n_e = 0, \bar{n}_e | t)}.$$

Relation (3.6) shows that $\bar{\mu}^{\rm ind}_{n,p}(t)$ is proportional to the spin glass partition function $Z_{p^*}(t)$. This leads to consider the ratio R_1/R_2 where

$$R_1 = \sum_{t\in\mathcal{T}(e)} Z_{p^*}(t) p_b^{N_1(n_e=1,\bar{n}_e)}(1-p_b)^{N_0(n_e=1,\bar{n}_e)} 2^{|(n_e=1,\bar{n}_e)|} I_{U_t}(n_e=1,\bar{n}_e) Z^{-1}_{n,p_b}(t),$$

and

$$R_2 = \sum_{t\in\mathcal{T}(e)} Z_{p^*}(t) p_b^{N_1(n_e=0,\bar{n}_e)}(1-p_b)^{N_0(n_e=0,\bar{n}_e)} 2^{|(n_e=0,\bar{n}_e)|} I_{U_t}(n_e=0,\bar{n}_e) Z^{-1}_{n,p_b}(t),$$

with $Z_{n,p_b}(y) \equiv \exp(-|E|\beta_{\hat{p}_b}) Z_{\hat{p}_b}(y)$. Next $Z_{p^*}(t)/Z_{\hat{p}_b}(t) \equiv 1$, since $p^* = \hat{p}_b$. Every element $t \in \mathcal{T}(e)$ can be written in the form $t = \alpha f_{\{e\}} + t'$, $\alpha \in \{0,1\}$, with $< t', f_{\{e\}} >_{\rm I\!R} = 0$. The ratio becomes

$$\frac{\sum_{t',\alpha} p_b^{N_1(n_e=1,\bar{n}_e)}(1-p_b)^{N_0(n_e=1,\bar{n}_e)} 2^{|(n_e=1,\bar{n}_e)|} I_{U_t}(n_e=1,\bar{n}_e)}{\sum_{t',\alpha} p_b^{N_1(n_e=0,\bar{n}_e)}(1-p_b)^{N_0(n_e=0,\bar{n}_e)} 2^{|(n_e=0,\bar{n}_e)|} I_{U_t}(n_e=0,\bar{n}_e)}, \tag{5.11}$$

which is larger than

$$\min_{t'} \frac{\sum_{\alpha} p_b^{N_1(n_e=1,\bar{n}_e)}(1-p_b)^{N_0(n_e=1,\bar{n}_e)} 2^{|(n_e=1,\bar{n}_e)|} I_{U_t}(n_e=1,\bar{n}_e)}{\sum_{\alpha} p_b^{N_1(n_e=0,\bar{n}_e)}(1-p_b)^{N_0(n_e=0,\bar{n}_e)} 2^{|(n_e=0,\bar{n}_e)|} I_{U_t}(n_e=0,\bar{n}_e)},$$

where t' is taken as the graph-theoretic restriction of t to $E \setminus \{e\}$. The end of the proof follows the lines of the proof of Theorem 3 of Gandolfi *et altri*(1992) or Newman(1994). One finally obtains that the ratio is larger that $p_b/(2(1-p_b))$.

Assume now that $e \in E$ is an isthmus. By Lemma 2, $f_{\{e\}} \in \mathcal{C}$, and we can find a linear space $\mathcal{T}(e)$ with $\mathcal{T}(e) \oplus \mathcal{C} = GF(e)^{|E|}$, with $< f_{\{e\}}, t >_{\rm I\!R} = 0$, $\forall t \in \mathcal{T}(e)$. In this representation, $f_{\{e\}} \in \mathcal{C}$ means that the coupling associated with e is always ferromagnetic. The ratio (5.11) becomes

$$\frac{\sum_{t} p_b^{N_1(n_e=1,\bar{n}_e)}(1-p_b)^{N_0(n_e=1,\bar{n}_e)} 2^{|(n_e=1,\bar{n}_e)|} I_{U_t}(n_e=1,\bar{n}_e)}{\sum_{t} p_b^{N_1(n_e=0,\bar{n}_e)}(1-p_b)^{N_0(n_e=0,\bar{n}_e)} 2^{|(n_e=0,\bar{n}_e)|} I_{U_t}(n_e=0,\bar{n}_e)},$$

where $t_e = 0$. By assumption, $e = (e_-, e_+)$ is an isthmus, and thus e_- and e_+ are not connected by any path of edges in $\bar{n}_e$. The ratio is then larger than $p_b/(2(1-p_b))$, which shows the result (see e.g. Gandolfi *et altri* for more details).

Lemma 4 *Let G be a finite bipartite connected graph. Let $1 > p > 1/2$ and choose p_b in such a way that $\hat{p}_b = (1-p)^*$. Then (5.10) holds.*

Proof: According to the proof of Lemma 3, we have only to modify the simplification $Z_{p^*}(t)/Z_{\hat{p}_b}(t) \equiv 1$. When $p > 1/2$, we have that $\bar{\mu}^{\rm ind}_{n,p}(t) \equiv \cosh(\beta_{p^*})^{-|E|}$ $2^{-1-|E|}$ $Z_{p^*}(t)$, with $p^* = 1 - \sqrt{p/(1-p)} < 0$ and $\exp(-\beta_{p^*}) = \sqrt{p/(1-p)} > 1$, that is $\beta_{p^*} < 0$. The associated partition function takes then the form $Z_{p^*}(t) = \sum_{\sigma\in\Omega}\exp(-|\beta_{p^*}|\sum_{(ij)\in E}(-1)^{t(ij)}\sigma_i\sigma_j)$. However, when G is bipartite, $Z_{p^*}(t) = \sum_{\sigma\in\Omega}\exp(|\beta_{p^*}|\sum_{(ij)\in E}(-1)^{t(ij)}\sigma_i\sigma_j)$, with $|\beta_{p^*}| = \log(\sqrt{p/(1-p)}) = \beta_{(1-p)^*}$. We obtain thus the same kind of simplification by choosing $\hat{p}_b = (1-p)^*$.

Remark 4 *The arguments given in the proof of Lemma 3 show that*

$$Q^{\rm edge}_{p,p_b,G}((n)) \;=\; \frac{1}{(2-p_b)^{|E|}}\mu^{\rm ind}_{n,p_b}((n))2^{|(n)|-1}\sum_{t\in\mathcal{T}} I_{U_t}((n)), \quad \hat{p}_b \;=\; p^*.$$

When G is a tree, $\mathcal{T} = \{\underline{0}\}$, there is no frustration and thus $\mu^y_n = \mu^{y=\underline{0}}_n$, $Q^{\rm edge}_{p,p_b,G} = \mu^{y=\underline{0}}_n$, $\forall p$. Moreover, $|(n)| = 1 + |E| - N_1((n))$, where $N_1((n))$ denotes the number of occupied edges in the subgraph (n) (see e.g. Ioffe(1995)), and it follows that

$$\mu^{y=\underline{0}}_n((n)) \;=\; \Big(\frac{2(1-p_b)}{2-p_b}\Big)^{N_0((n))}\Big(\frac{p_b}{2-p_b}\Big)^{N_1((n))} \;=\; \mu^{\rm ind}_{n,h(p_b)}((n)), \quad \forall (n),$$

showing that the lower bound obtained in Lemma 3 is realized on trees.

The Lattice $\mathbf{Z^d}$

We focus on $\mathbf{Z}^d$, $d \geq 2$, with the usual nearest neighbors structure. Consider a finite box $B \;=\; \{x \in \mathbf{Z}^d;\ ||x|| \leq N\}$, with edge set E_B. Let $\mu^{\rm F}_{n,p_b} = \mu^{y=\underline{0}}_n$ be the ferromagnetic bond measure. The following inequalities hold (see Newman(1994))

$$\mu^{\rm ind}_{n,h(p_b)} << \mu^{\rm F}_{n,p_b} << \mu^{\rm ind}_{n,p_b}, \tag{5.12}$$

and

$$\mu^y_n << \mu^{\rm ind}_{n,p_b},\ \forall y.$$

Lemma 3 extends thus the left side of (5.12) to the regime $\hat{p}_b = p^*$, $p < 1/2$. Let $Q_{p,p_b,\mathbf{Z}^d}$ be a weak limit of Q_{p,p_b,G_B} obtained by letting tend the box $B \subset \mathbf{Z}^d$ to $\mathbf{Z}^d$ along a convenient sequence, and let $Q^{\rm edge}_{p,p_b,\mathbf{Z}^d}$ be its marginal on the bond variables.

Corollary 2 *Take $G = \mathbf{Z}^d$, and assume periodic boundary conditions. Suppose that*

$$\hat{p}_b \;=\; p^*, \quad p < \frac{1-p_c(\mathbf{Z}^d)}{2},$$

or that

$$\hat{p}_b \;=\; (1-p)^*, \quad p > \frac{1+p_c(\mathbf{Z}^d)}{2},$$

where $p_c(\mathbf{Z}^d)$ is the critical probability associated with bond percolation on $\mathbf{Z}^d$, $d \geq 2$. Then

$$Q^{\text{edge}}_{p,p_b,\mathbf{Z}^d}(n_e = 1|\bar{n}_e) > p_c(\mathbf{Z}^d),$$

and

$$\mu^y_n(\text{ the origin is in an infinite cluster of occupied bonds}) > 0,$$

for almost all realizations of the disorder y, where μ^y_n is the weak limit of μ^y_{n,G_B} along a sequence of boxes converging to $\mathbf{Z^d}$ for which Q_{p,p_b,G_B} converges weakly.

Proof: Follows essentially the lines of the proof of Corollary 1 in Gandolfi *et altri* (1992), and uses mainly the notion of finite energy.

Aizenman, Chayes, Chayes and Newman(1987) used the FK representation when investigating percolation in diluted ferromagnets (see also Newman(1994)). We use their approach in the context of the $\pm J$ dilute spin glass, where the couplings are i.i.d. with distribution given by $P(J_{(ij)} = 0) = \epsilon$, $P(J_{(ij)} = -1) = p$ and $P(J_{(ij)} = 1) = 1 - \epsilon - p$, $\epsilon \geq 0$, $p \geq 0$ and $1 - \epsilon - p \geq 0$. Let B be a finite box of $\mathbf{Z}^d$, with associated graph G_B. Let Q_{p,ϵ,p_b,G_B} be the joint distribution of (n) and J, with edge marginal $Q^{\text{edge}}_{p,\epsilon,p_b,G_B}$. Let μ_n be a weak limit of $Q^{\text{edge}}_{p,\epsilon,p_b,G_B}$ along a convenient sequence of boxes B converging to $\mathbf{Z}^d$, and let μ^J_n be the weak limit of the FK measure μ^J_{n,G_B}.

Theorem 1 *Let $G = \mathbf{Z}^d$, $d \geq 2$, and assume periodic boundary conditions. Consider the diluted $\pm J$ Ising spin glass with i.i.d. random couplings distributed as follows: $P(J_{(ij)} = 0) = \epsilon$, $P(J_{(ij)} = -1) = p$ and $P(J_{(ij)} = 1) = 1-p-\epsilon \geq 0$. Suppose that $P(J_{(ij)} \neq 0) = 1 - \epsilon > p_c(\mathbf{Z}^d)$, where $p_c(\mathbf{Z}^d)$ is the critical bond percolation probability on $\mathbf{Z}^d$. Let $p' := p/(1-\epsilon)$, and assume that $\hat{p}_b = (p')^*$ or that $\hat{p}_b = (1-p')^*$. Let p and ϵ be such that*

$$p < \frac{1}{2}(1 - \epsilon - p_c(\mathbf{Z}^d)), \quad \text{or} \quad p > \frac{1}{2}(1 - \epsilon + p_c(\mathbf{Z}^d)).$$

Let μ^J_n be the FK bond measure associated with a realization of the random coupling J. Then

$$\mu^J_n(\text{ there exists a unique infinite cluster of occupied edges}) \;=\; 1,$$

for almost all J.

Proof: The associated FK measure can be constructed in two steps: first, consider a bond percolation model on G with occupation probability $1-\epsilon$; given a realization of this random graph $G(\omega)$, construct the random cluster measure $\mu^y_{n,G(\omega)}$ on $G(\omega)$ with probability $p' = p(y_{(ij)} = 1) = p/(1-\epsilon) = P(J_{(ij)} = -1)/(P(J_{(ij)} = -1) + P(J_{(ij)} = 1))$, and $P(y_{(ij)} = 0) = 1 - p'$. Of course, $G(\omega)$ is not connected in general; let $\otimes_c \mu^y_{n,G_c(\omega)}$ be the product random cluster measure associated with the decomposition of $G(\omega)$ in connected components $G(\omega)_c$. The bond measure becomes $\otimes_c \mu^y_{n,G_c(\omega)} \otimes_{e \notin E(\omega)} \mu^{\text{ind}}_{n,0}$, where $E(\omega)$ denotes the edge set of $G(\omega)$. But y is distributed according to a product Bernoulli measure on $\{0,1\}^E$, and we can factorize the joint $(y,(n))$ distribution $Q_{p',p_b,G(\omega)}$ as $\otimes_c Q_{p',p_b,G_c(\omega)} \otimes_{e \notin E(\omega)} \mu^{\text{ind}}_{n,0}$, with edge-marginal $Q^{\text{edge}}_{p',p_b,\omega} := \otimes_c Q^{\text{edge}}_{p',p_b,G_c(\omega)} \otimes_{e \notin E(\omega)} \mu^{\text{ind}}_{n,0}$. When G is bipartite, every subgraph of G is bipartite, and we can thus envisage the antiferromagnetic regime $p' > 1/2$. By Lemma 3, choosing $\hat{p}_b = p'^*$ when $p' < 1/2$ or $\hat{p}_b = (1-p')^*$, $p' > 1/2$, we get that the random measure $Q^{\text{edge}}_{p',p_b,\omega}$ dominates the product Bernoulli measure which has parameter $h(p_b) = p_b/(2-p_b)$ on the connected components of $G(\omega)$ and parameter 0 on the remaining of E, which can be written as a random Bernoulli measure with random parameter $p_b(\omega)$ distributed as $P(p_b(\omega) = 1 - \exp(-\beta_{p_b}) \neq 0) = 1-\epsilon$. Let $Q^{\text{edge}}_{p',p_b,G}$ be its marginal on the bond variables, which is given by $Q^{\text{edge}}_{p',p_b,G}((n)) = \sum_J P(J)\mu^J_{n,G}((n))$. Then

$$Q^{\text{edge}}_{p',p_b,G} >> \mathrm{E}(\otimes_{e\in E}\mu^{\text{ind}}_{n,h(p_b(\omega))}) \;=\; \otimes_{e\in E}\mu^{\text{ind}}_{n,(1-\epsilon)h(p_b)}, \tag{5.13}$$

where $\mathrm{E}(\cdot)$ denotes the expectation with respect to J.

Finally, assume that $G = G_B$ is associated with a finite box B of $\mathbf{Z}^d$, and let $\mu_{n,\mathbf{Z}^d}$ be a weak limit of Q_{p,ϵ,p_b,G_B} obtained by letting tend B to $\mathbf{Z}^d$ along a convenient sequence, and let $Q^{\text{edge}}_{p',p_b,\mathbf{Z}^d}$ be its marginal on the bond variables. $Q^{\text{edge}}_{p',p_b,\mathbf{Z}^d}$ is translation invariant and, by (5.13), has finite energy. It follows that (see Gandolfi *et altri*(1992))

$$Q^{\text{edge}}_{p',p_b,\mathbf{Z}^d}(\text{ there exists a unique infinite cluster of occupied edges}) \;=\; 1,$$

when $(1-\epsilon)h(p_b) > p_c(\mathbf{Z}^d)$, that is when

$$p < \frac{1}{2}(1-\epsilon-p_c(\mathbf{Z}^d)) \text{ or when } p > \frac{1}{2}(1-\epsilon+p_c(\mathbf{Z}^d)),$$

with

$$P(J_{(ij)} \neq 0) \;=\; 1-\epsilon > p_c(\mathbf{Z}^d).$$

6 Concluding Remark

Fig 2: Phase diagram of the Ising spin glass with periodic boundary conditions. The vertical axis indicates the temperature $T^b = \beta_{p_b}^{-1}$, and the horizontal axis corresponds to the antiferromagnetic bond concentration. The Nishimori curve $p^* = \hat{p}_b$ is given by $N(p) \equiv 1/(\log((1-p)/p))$.

Let $G = \mathbf{Z}^d$. We restrict our discussion to the case $0 \leq p \leq 1/2$. The ferromagnetic critical temperature is denoted by $T_c^b(p = 0)$ $(d = 2)$, and the curve $(p, T_c^b(p))$ gives the hypothetical critical temperatures as function of $0 \leq p \leq 1/2$. The domain limited by $(p, T_c^b(p))$ containing 0 is called the ferromagnetic (F) regime. The zero p_f of T_c^b is called the *critical ferromagnetic concentration*, under which ferromagnetism is supposed to exist, at $T^b = 0$. When $d = 2$, numerical simulations indicate that p_f is situated in the interval $I = [0.1, 0.15]$, $T^b = 0$ (see Binder and Young(1986), p. 917-919), and perturbative studies shed some light on the volume of (F) (see Griffiths-Lebowitz(1968), Avron-Roepstorff-Schulman(1981), Roepstorff(1981) or Chayes, Chayes and Fröhlich(1985)). The remaining of the diagram is composed of a paramagnetic phase (P) which is similar to the high temperature regime associated with the two dimensional ferromagnetic Ising model. On $\mathbf{Z}^d$, $d \geq 3$, the spin glass phase (S) is supposed to exist in a neighborhood of the point $(1/2, 0)$.

Our result shows that percolation occurs on the portion of the curve $\hat{p}_b = p^*$ given by $p < (1 - p_c(\mathbf{Z}^d))/2$, which might cross the curve $(p, T_c^b(p))$ separating (F) and (P); this would confirm that percolation in the random interaction random cluster model is not necessarily a precise indicator of phase transition (see e.g. Cataudella(1991) or Cataudella *et altri*(1994)). Unfortunately, the geometry of (F) is unknown. Let $(\rightsquigarrow)$ be the domain of the (p, T^b) plane ensuring percolation of $Q^{\text{edge}}_{p,p_b,\mathbf{Z}^d}$. Then $(\rightsquigarrow)$ contains the segment $\{(0, T^b);\ T^b < T_c^b(0)\}$, the Nishimori segment $\{(p, N(p));\ p < (1/2)(1 - p_c(\mathbf{Z}^d))\}$. When $d \geq 3$, Gandolfi *et altri*(1992) proved that $(\rightsquigarrow)$ also contains a segment $\{(1/2, T^b);\ T^b < T_0^b\}$, for some small enough T_0^b. It seems plausible that $(\rightsquigarrow)$ contains the domain

$\{(p, T^b);\ T^b \leq N(p),\ p < (1/2)(1 - p_c(\mathbf{Z}^d))\}$.

Following Kesten(1989) (see e.g. Grimmett(1989)), $p_c(\mathbf{Z}^d) \sim (2d)^{-1}$, $d >> 1$, and thus percolation occurs when $p < (1 - p_c(\mathbf{Z}^d))/2 \sim 1/2$, the whole Nishimori curve. This poses the problem of unicity of the limiting Gibbs measures: Using Dobrushin-Shlosman techniques, Newman(1994) proved that uniqueness in the ferromagnetic model at inverse temperature $\beta_{\hat{p}_b}$ implies uniqueness in the disordered system at the same inverse temperature. When $d >> 1$, the critical ferromagnetic inverse temperature collapses with that of the mean field Curie-Weiss model (see Bricmont and Fontaine(1982) or Ellis(1989)), which is asymptotically given by $\beta_c^{\mathrm{CW}}(d) \sim (2d)^{-1}$.

When $p^* = \hat{p}_b$ and $p \sim (1 - p_c(\mathbf{Z}^d))/2 \sim (1 - (2d)^{-1})/2$, we get that $\beta_{\hat{p}_b} \sim (2d)^{-1} \sim \beta_c^{\mathrm{CW}}(d)$.

Finally, let us give some possible extensions. Assume that $G = \mathbf{Z}^d$, $d \geq 2$, and consider the $\mathbf{Z}_2$ action on $\{0,1\}^E$ (gauge+matter field)

$$A(y) \ = \ \kappa \sum_{\Box} \prod_{e \in \Box} (-1)^{y_e} + \lambda \sum_{e=(ij) \in E} S_i S_j (-1)^{y_e},$$

where the matter variables are such that $S_i \in \{-1, +1\}$, $\forall i \in V$, and $\Box$ denotes an elementary plaquette of the lattice. The associated Yang Mills measure is then given by $\nu(y) = Z_{YM}^{-1} \exp(A(y))$, where Z_{YM} is the partition function of the system. Note that the product Bernoulli measure $\mu_{n,p}^{\mathrm{ind}}(y)$ can be realized by choosing $\kappa = 0$ and quenched matter variables $S_i \equiv 1$. Such models have been developped in the setting of particle physics; Wilson(1974) proposed the gauge invariant observable

$$W_\gamma(y) \ := \ \prod_{e \in \gamma} (-1)^{y_e},$$

where γ is a loop of the lattice, which can be used to investigate the problem of confinement of quarks. When $\lambda = 0$, the average of Wilson loops $< W_\gamma >_\nu$ exhibits a critical behavior for large loops: at small κ, $< W_\gamma >_\nu \asymp \exp(-\mathrm{Area}(\gamma))$, and for large values of κ, $< W_\gamma >_\nu \asymp \exp(-\mathrm{Perimeter}(\gamma))$ (see Baliant, Drouffe and Itzykson(1975)). The phase diagram of the gauge+matter field model has been investigated (see e.g. Fradkin and Shenker(1979)), and exhibits similar behaviors.

In our setting, we have considered the F-K gauge invariant observable

$$\begin{aligned}
\mu_n^y((n)) \ &= \ Z_{\hat{p}_b}^{-1}(y) \mu_{n,p_b}^{\mathrm{ind}}((n)) 2^{|(n)|} I_{U_y}((n)) \\
&= \ Z_{\hat{p}_b}^{-1}(y) (p_b/(1-p_b))^{\mathrm{Perimeter}((n))} (1-p_b)^{|E|} 2^{|(n)|} \prod_{\gamma \in C_{(n)}} \frac{1 + W_\gamma(y)}{2},
\end{aligned}$$

where $C_{(n)}$ denotes the set of loops contained in the subgraph (n). $\mu_n^y((n))$ is thus expressed in terms of Wilson loops; it would be interesting to check its critical behavior in this framework.

Acknowledgment: The author is indebted to Luc Rey-Bellet and Michel Droz for numerous instructive discussions on spin glasses, and to Emmanuel Nanchen and Fabrice Liardet for valuable comments.

7 References

Aizenman, M., Chayes, J., Chayes, L. and Newman C.(1987). The phase boundary in dilute and random Ising and Potts ferromagnets. *Journal of Physics A: Mathematical and General* 20, L313-L318.

Avron J.E., Roepstorff G. and Schulman L.S.(1981). Ground state degeneracy and ferromagnetism in a spin glass. *J. Stat. Phys.* 26, 25.

Balliant R., Drouffe J.M. and Itzykson C.(1975). Gauge fields on a lattice. Gauge invariant Ising model. *Phys. Rev. D.*, Vol. 11, no. 8, 2098-2119.

Biggs N. *Algebraic graph theory.* Cambridge. 1992.

Binder K. and Young A.P.(1986). Spin glasses: Experimental facts, theoretical concepts, and open problems. *Reviews of Modern Physics*, Vol. 58, No. 4.

Bricmont J. and Fontaine J.(1982). Perturbation about the Mean Field Critical Point, *Commun. Math. Phys.* 86, 337-362.

Bruck J. and Blaum M.(1989). Neural networks, error-correcting codes, and polynomials over the binary n-cube. *IEEE Transactions on information theory* vol. 35, no.5, 976-987.

Cataudella V.(1991). Percolation transition in systems with frustration. *Physica A*, 183, 249-254.

Cataudella V., Franzese G., Nicodemi M. and Scola A.(1994). Critical clusters and efficient dynamics for frustrated spin models. *Phys. Rev. Letters* 72, 10, 1541-1544.

Chayes J.T., Chayes L. and Fröhlich J. (1985). The Low-Temperature Behavior of Disordered Magnets. *Commun. Math. Phys.*, 100, 399-437.

Ellis R. *Entropy, Large Deviations and Statistical Mechanics.* Springer. 1985.

Fradkin E., Huberman B.A. and Shenker S.H.(1978). Gauge symmetries in random magnetic systems. *Physical Review B*,vol. 18. no. 9, 4789-4814.

Fradkin E. and Shenker S.H.(1979). Phase diagrams of lattice gauge theories with Higgs fields. *Phys. Rev. D.*, Vol. 19, no. 12, 3692-3697.

Gandolfi A., Keane M.S and Newman C.M.(1992). Uniqueness of the infinite component in a random graph with applications to percolation and spin glasses. *Probability theory and related fields* 92, 511-527.

Griffiths, R.B. and Lebowitz, J.L. (1968). Random spin systems: some rigourous results. *J. Math. Phys.*, 9, 1284.

Grimmett, G. *Percolation.* Springer. 1989.

Ioffe D.(1995). Extremality of the disordered state for the Ising model on general trees. In *Trees, Workshop in Versailles, June 14-16*, Progress in Probability, Birkhaüser.

Itzykson C. and Drouffe J. *Statistical field theory, Vol.1: From Brownian motion to renormalization and lattice gauge theory.* Cambridge monographs on mathematical physics. 1989.

Kasai J. and Okiji A.(1988). Percolation problem describing $\pm J$ Ising spin glasses system. *Progress of theoretical physics* vol. 79. no.5.

Kesten H. Asymptotics in High dimensions for Percolation. In *Disorder in Physical Systems, a Volume in Honour of John. M. Hammersley*, Ed. G.R. Grimmett and D.J.A. Welsh, Oxford. 1989.

Kolan A.J. and Palmer R.G.(1980). Distribution of frustrated plaquettes in the random bond square lattice . *J.Phys. C: Solid St. Phys.* 13(1980) L575-80.

Liebman and Schuster(1981). Correlation between frustrated plaquettes in the random bond square lattice. *J. Phys. C: Solid St. Phys.* 14(1981). 709-716,

Newman C.M.(1990). Ising models and dependent percolation, in *Topics in Statistical dependence*, *Institute of Mathematical statistics*, Lecture notes no. 16.

Newman C.M.(1994). Disordered Ising systems and random cluster representation, in *Probability and phase transition*, Ed. G. Grimmett, *NATO ASI Series* vol. 420, Serie C.

Nishimori H.(1980). Exact results and critical properties of the Ising model with competing interactions, *J.Phys. C: Solid St. Phys.* 13(1980), 4071-6.

Peterson W.W. and Weldon E.J. *Error-Correcting Codes.* MIT Press, Cambridge. (1972).

Roepstorff G.(1981). The Peierls-Griffiths argument for disordered Ising systems. *J. Math. Phys.*, 22, 3002.

Roman S. *Coding and Information Theory*, Springer 1992.

Schuster H.G.(1979). Phase transitions in the frustration network. *Z. Physik* B35, 163-170.

Sourlas N.(1989). *Nature* 339, p. 693.

Sourlas N.(1994). Statistical mechanics and error-correcting codes, in *From statistical mechanics to statistical inference and back*, Ed. Grassberger and Nadal, *NATO ASI Serie C.*

Toulouse G.(1977). Theory of the frustration effect in spin glasses, *Comm. Phys.* 2(1977) 115-119.

van Lint J.H. and Wilson R.M. *A course in combinatorics*, Cambridge, 1992.

Welsh D.J.A. *Complexity: Knots, Colourings and Counting*, London Mathematical Society, Lecture Notes Series 186, Cambridge University Press. 1993.

Wilson K.G.(1974). Confinement of quarks. *Phys. Rev. D*, vol. 10. no. 8. 2445-2459.