Exponential Bounds on Curvature-Induced Resonances in a Two-Dimensional Dirichlet Tube

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Abstract. We consider curvature–induced resonances in a planar two–dimensional Dirichlet tube of a width d. It is shown that the distances of the corresponding resonance poles from the real axis are exponentially small as $d \to 0+$, provided the curvature of the strip axis satisfies certain analyticity and decay requirements.

1 Introduction

Spectral and scattering properties of Dirichlet Laplacians in curved tubes have attracted a wave of physical interest attention recently, because they provide models of some quantum

systems which new experimental techniques made possible to construct, such as semiconductor quantum wires — see, *e.g.*, [ABGM, Ba, CLMM, DE, Sa, SRW, VOK1] and references therein — or hollow–fiber atomic waveguides [SMZ], and because they exhibit some unexpected mathematical properties leading to new physical effects.

The key observation is that a nonzero curvature gives rise to an effective interaction which produces localized solutions of the corresponding Helmholtz (or stationary Schrödinger) equation — *cf.* [EŠ, GJ] and the review paper [DE] — with eigenvalues below the bottom of the continuous spectrum. The same mechanism is responsible for a nontrivial structure of the scattering matrix manifested by resonances in the vicinity of all the higher thresholds. These resonances modify substantially transport properties of such a "quantum waveguide"; they have been observed in numerically solved examples, for instance, in [SM, VKO, VOK1, VOK2, WS].

On the mathematical side, it was shown in [DEŠ] that if a curved planar strip has a constant width d which is small enough, and if the strip–axis curvature satisfies certain regularity and analyticity assumptions, there is a finite number of resonances in the vicinity of the higher thresholds (which coincides with the number of isolated eigenvalues below the bottom of the continuous spectrum). Moreover, an expansion of the resonance–pole positions in terms of d was derived and the imaginary part of the first non–real term given by the "Fermi golden rule" was shown to be exponentially small as $d \to 0+$.

The present paper addresses the question whether also the *total resonance widths* are exponentially small as $d \to 0+$. We give an affirmative answer under essentially the same assumptions as used in [DEŠ] and obtain the same expression for the exponential factor in the bound. Furthermore the exponential factor we obtain coincides with the heuristic semi-classical prediction, *cf.* [LL, DEŠ].

As explained in [DEŠ] and in section 2.3 below, the mechanism behind the formation of these resonances is a tunneling effect, however, in the "momentum direction". To estimate this effect we therefore need exponential bounds on eigensolutions in the Fourier representation. This is a novelty and a difficulty, since the Schrödinger operator becomes nonlocal in this representation. To deal with such nonlocal operators we have developed an appropriate functional calculus based on the Dunford–Taylor integral. This made possible, in particular, to extend to such a situation the complex deformations of operators and the Agmon method [Ag]. These new techniques has already been announced in [DM1]. Here they are presented in detail for the case of bounded nonlocal operators; an extension to some unbounded cases will be given in [DM2].

Rigorous analysis of tunneling in phase space is a rather new field of interest. Some recent works on this topic based on micro-local analysis and pseudo-differential techniques can be found in [HeSj, Ma, N]. In particular L.Nedelec [Ne] has recently obtained our Theorem 2.2 with such methods.

2 The results

2.1 Preliminaries

Let us recall briefly the problem; for more details we refer to [DEŠ]. The object of our interest is the Dirichlet Laplacian $-\Delta_D^\Omega$ for a curved strip $\Omega \subset \mathbb{R}^2$ of a fixed width d. We exclude the trivial case and make a global restriction:

(a0) Ω is not straight and does not intersect itself.

If the boundary of Ω is sufficiently smooth — which will be the case with the assumptions mentioned below — one can check using natural curvilinear coordinates that $-\Delta_D^\Omega$ is unitarily equivalent to the Schrödinger type operator

$$H := -\partial_s b \partial_s - \partial_u^2 + V \tag{2.1}$$

on the Hilbert state space on the "straightened" strip, $\mathcal{H} := L^2(\mathbb{R}\times(0,d), ds\,du)$, with the Dirichlet condition at the boundary, $u = 0, d$, where b, V are operators of multiplication by the functions

$$b := (1+u\gamma)^{-2}, \tag{2.2}$$

$$V := -\frac{1}{4} b\gamma^2 + \frac{1}{2} b^{3/2} u\gamma'' - \frac{5}{4} b^2 u^2 \gamma'^2, \tag{2.3}$$

respectively, and the function $\gamma : \mathbb{R} \to \mathbb{R}$ in these relations characterizes the strip boundary $u = 0$ through its signed curvature $\gamma(s)$ at the point tagged by the longitudinal coordinate s — *cf.* [EŠ, DE].

Let us list now the used hypotheses. In addition to the assumption (a0), we shall suppose that

(a1) γ extends to an analytic function in $\Sigma_{\alpha_0,\eta_0} := \{ z \in \mathbb{C} \; : \; |\arg(\pm z)| < \alpha_0$ or $|\mathrm{Im}\, z| < \eta_0 \}$ with $\alpha_0 < \frac{\pi}{2}$ and $0 < \eta_0$; for the sake of simplicity we denote it by the same symbol.

(a2) For all $\alpha < \alpha_0$ and all $\eta < \eta_0$ there are positive constants $c_{\alpha,\eta}$ and ε such that $|\gamma(z)| < c_{\alpha,\eta}(1+|z|)^{-1-\varepsilon}$ holds in $\Sigma_{\alpha,\eta}$.

By an easy application of the Cauchy formula, the assumptions (a1) and (a2) imply that the derivatives of γ satisfy

$$|\gamma^{(r)}(z)| < c_{r,\alpha,\eta}(1+|z|)^{-1-r-\varepsilon}$$

in $\Sigma_{\alpha,\eta}$ for any $\alpha < \alpha_0$ and any $\eta < \eta_0$. This yields for the potential (2.3) the bound

$$|V(z,u)| < c'_{\alpha,\eta}(1+|z|)^{-2-\varepsilon} \tag{2.4}$$

with some $c'_{\alpha,\eta} > 0$ for all d small enough.

We are interested in resonances of H which are understood in the standard way [AC, RS, Hu]: suppose that the function $z \mapsto F_\psi(z) := ((H-z)^{-1}\psi, \psi)$ admits a meromorphic continuation from the open upper complex half-plane to a domain in the lower half-plane for ψ from a dense subset $\mathcal{A} \subset \mathcal{H}$. If F_ψ has a pole for some $\psi \in \mathcal{A}$, we call the former a *resonance*.

Resonances are often obtained as perturbations of an operator with eigenvalues embedded in the continuous spectrum. This is also the case in our present situation; the corresponding comparison operator is

$$H^0 := A - \partial_u^2, \qquad A := -\partial_s^2 + V^0, \tag{2.5}$$

with $V^0(s) := V(s,0) = -\frac{1}{4}\gamma(s)^2$ and domain $\mathcal{D}(H^0) := \mathcal{H}^2(\mathbb{R}) \otimes (\mathcal{H}_0^1 \cap \mathcal{H}^2)((0,d))$, where $\mathcal{H}^n$ and $\mathcal{H}_0^n$ are the usual Sobolev spaces. The perturbation is defined by

$$W := H - H^0. \tag{2.6}$$

The spectrum of the operator H^0 is of the form

$$\sigma(H^0) = \left\{ \lambda + E \, : \, \lambda \in \sigma(A), \; E \in \sigma(-\partial_u^2) \right\},$$

where

$$\sigma(A) = \{ \lambda_n \}_{n=1}^N \cup [0,\infty), \qquad \sigma(-\partial_u^2) = \{ E_j \}_{n=1}^\infty$$

with $E_j := \left(\frac{\pi j}{d}\right)^2$. Since $\int_{\mathbb{R}} V^0(s)\,ds < 0$ due to (a0), the discrete spectrum of A is nonempty. The eigenvalues λ_n are negative, simple, and their number N is finite in view of the bound (2.4) — *cf.* [RS, Sec.XIII.3]. Then the eigenvalues

$$E^0_{j,n} := \lambda_n + E_j$$

above E_1 are embedded in the continuous spectrum of H^0; for small enough d this occurs for all $j \geq 2$ and $n = 1, \dots, N$.

An alternative way to express the operator H^0 and functions of it, is through the transverse–mode decomposition. Denote by

$$\chi_j(u) := \sqrt{\frac{2}{d}} \sin\left(\frac{\pi j u}{d}\right), \qquad j = 1, 2, \dots, \tag{2.7}$$

the eigenfunctions of $-\partial_u^2$ corresponding to the eigenvalues E_j. Let $\mathcal{J}_j$ be the embedding $L^2(\mathbb{R}, ds) \to L^2(\mathbb{R}, ds) \otimes \chi_j \subset \mathcal{H}$; the adjoint of this operator is $\mathcal{J}_j^* : \mathcal{H} \to L^2(\mathbb{R}, ds)$ acting as $(\mathcal{J}_j^* f)(s) = \int_0^d f(s,u)\chi_j(u)\,du$. Given $j \in \mathbb{N}$, we denote by P_j the projection onto the mode with index j, $P_j := \mathcal{J}_j \mathcal{J}_j^*$, and set $Q_j := I_{\mathcal{H}} - P_j$.

The perturbation W consists of operators which couple different transverse modes. As a result the embedded eigenvalues turn into resonances. With our assumptions the result of [DEŠ] holds:

Theorem 2.1 *Assume (a0)–(a2). For all sufficiently small* d *each eigenvalue* $E^0_{j,n}$ *of* H^0 , $j \ge 2\,, n = 1, \dots, N$, *gives rise to a resonance* $E_{j,n}(d)$ *of* H, *the position of which is given by a convergent series*

$$E_{j,n}(d) \;=\; E^0_{j,n} + \sum_{m=1}^{\infty} e^{(j,n)}_m(d)\,, \tag{2.8}$$

where $e^{(j,n)}_m(d) = \mathcal{O}(d^m)$ *as* $d \to 0+$. *The first term of the series is real–valued, and the second satisfies the bound*

$$0 \;\le\; -\mathrm{Im}\, e^{(j,n)}_2(d) \;\le\; c_{\eta,j}\, e^{-2\pi\eta\sqrt{2j-1}/d} \tag{2.9}$$

for any $\eta \in (0,\eta_0)$, *the constant* $c_{\eta,j}$ *depending on* η *and* j .

2.2 Main theorems

Our aim in this paper is to show that similar bounds can be proven for the total resonance width. This is the contents of the following two theorems:

Theorem 2.2 *Assume (a0)–(a2). Then for any* $\eta \in (0,\eta_0)$, $j \ge 2$ *and* $n = 1, \dots, N$ *there is* $C_{\eta,j} > 0$ *such that*

$$0 \;\le\; -\mathrm{Im}\, E_{j,n}(d) \;\le\; C_{\eta,j}\, e^{-2\pi\eta\sqrt{2j-1}/d} \tag{2.10}$$

holds for all d *small enough.*

Theorem 2.3 *In addition, assume that* γ *extends to a meromorphic function in* Σ_{α_0,η_1} *with* $\eta_1 > \eta_0$. *Let* $\eta_p < \eta_1$ *be the minimal distance to the real axis of the poles and assume that the maximal order of the poles at this distance is* $1 \le m < \infty$ *; then there are positive constants* $C^{(1)}_j$ *and* $C^{(2)}_j$ *such that*

$$0 \;\le\; -\mathrm{Im}\, E_{j,n}(d) \le C^{(1)}_j \,\exp\left\{ -\frac{2\pi\eta_p}{d}\, \sqrt{2j-1}\, \left(1 - C^{(2)}_j\, d^{1/(m+1)}\right)\right\} \tag{2.11}$$

holds for all d *small enough.*

Remarks 2.4 (i) There is an heuristic prediction for the value of $\mathrm{Im}\, E_{j,n}(d)$ based on a formal semi-classical analysis where the role of the semi-classical parameter is played by d as d tends to zero. What one expects according to this prediction (for the details we refer the reader to [DEŠ], in particular Remark 4.2e therein and also to the scheme of the proof below) is that $\mathrm{Im}\, E_{j,n}(d)$ should behave like $C_j(d)\exp\left(-\frac{2\pi\eta_0}{d}\sqrt{2j-1}\right)$ where $C_j(d)$ is polynomially bounded in d^{-1}. However there is no chance to get such a precise behaviour without knowing the type of singularity that the curvature γ exhibits at distance η_0 from the real axis. This is why in Theorem 2.2 we lose an arbitrary small part of the exponential

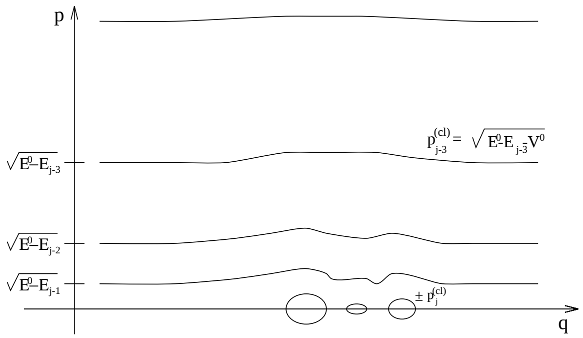

Figure 1: A schematic phase-space portrait of a bent waveguide

decay rate and get a prefactor $C_{j,\eta}$ which may eventually diverge as η tends to η_0 . This kind of bound is typical in such a semi-classical context, see e.g [Ag].

(ii) The merit of Theorem 2.3 is to show that with a precise assumption on the type of singularity of γ we are able to produce a bound which has a leading behaviour in accordance with the heuristic prediction. We would like to stress that, to our knowledge, this is the most precise bound obtained so far on the total resonance width in such a situation.

Since we shall deal in the following mostly with a single resonance, we drop the subscripts j, n as well as the argument d whenever they are clear from the context.

2.3 A sketch of the proofs

Consider first the system described by the decoupled Hamiltonian H^0. Each state ϕ of this system can be decomposed into the sum of its transverse modes, $\phi^j \otimes \chi_j, j = 1, 2, \dots,$ and this decomposition is invariant under the dynamics generated by H^0. For each channel j the dynamics of ϕ^j is governed by the "longitudinal" Hamiltonian $A + E_j = -\partial_s^2 + V^0 + E_j$ which, due to the nonzero curvature of the guide and its decay at infinity (by (a0) and (a2)), possesses either bound states for energies below E_j or scattering states otherwise. Fix now a $j \ge 2$ and suppose that d is small enough so that a given bound–state energy $E^0 = E_j + \lambda_n$ of $A + E_j$ is embedded in the continuous spectrum of the lower modes. The only possible solutions to the equation $H^0\phi = E^0\phi$ are then this bound state in the j–th mode and $j-1$ scattering states in the modes below. This structure is reflected in the classical phase space portrait of H^0 at energy E^0 (see Figure 1; we recall that for a matrix Hamiltonian $H^{(cl)}(q,p)$ the energy shell at E^0 is given by $\det(H^{(cl)}(q,p) - E^0) = 0$); the energy shell of H^0 is the union of the curves $p^{(cl)}_k(s) := \pm\sqrt{E^0 - E_k - V^0(s)},\; k = 1, \dots, j$. As expected only the j–th curve is compact.

Let now ϕ be a bound state of H^0 in the j–th channel and consider its evolution under the full dynamics given by $H = H^0 + W$. In general, various channels of H^0 are now coupled by W and ϕ will undergo transitions to all other energetically allowed channels. For d small enough there will be no significant changes of the classical phase space portrait by the addition of W. Thus for the classical dynamics no transition is possible between different channels. Hence the transitions in the quantum dynamics are of the tunneling type, but in contrast to the usual situation the tunneling takes place in the p (*i.e.* momentum) direction. More precisely, the projection of the energy shell on the p–axis consists of intervals of classically allowed momenta, one for the j–th mode situated at the origin and two for all other modes with index $k < j$, situated around $\pm\sqrt{E^0 - E^k}$. They are separated by "gaps", *i.e.*, classically forbidden (momentum) regions which have a size of order d^{-1} as d tends to zero. The existence of such gaps suggests that the solutions of $H\phi = E\phi$, with E close to E^0, decay exponentially as functions of p in these gaps, a key property in the sequel. The main contribution to this tunneling process should come from the transitions from $p_j^{(cl)}$ to $p_{j-1}^{(cl)}$, since it is the first gap that the quantum state has to cross. This motivates our decomposition of the momentum space into $\overline{\Omega}_i \cup \Omega_e$, with $\Omega_i \cap \Omega_e = \emptyset$ and $\Omega_i := (-\omega, \omega)$, where ω is approximately equal to $\sqrt{E_j - E_{j-1}}$, the size of the first gap.

Let $H_\theta \phi_\theta = E\phi_\theta$ be the eigenvalue equation for a resonance E and the corresponding resonance function ϕ_θ and assume that this resonance is associated to $E^0_{j,n} = E^0$ as in Theorem 2.1. The complex deformation, denoted by θ, is chosen as a scaling of the momentum exterior to Ω_i, *cf.* (3.1). In the transverse mode decomposition the eigenvalue equation becomes an infinite system of coupled differential equations in $L^2(I\!R)$ which can be solved for the j–th component $\phi_\theta^j := \mathcal{J}_j^* \phi_\theta$ of ϕ_θ leading to the following equation

$$E\phi_\theta^j \;=\; (H_\theta^j - B_\theta^j(E))\phi_\theta^j \tag{2.12}$$

in $L^2(I\!R)$, where $H_\theta^j := \mathcal{J}_j^* H_\theta \mathcal{J}_j$, $B_\theta^j(E) := \mathcal{J}_j^* W_\theta \widehat{R}_\theta^j(E) W_\theta \mathcal{J}_j$ and $\widehat{R}_\theta^j(E) := Q_j(Q_j H_\theta Q_j - E)^{-1} Q_j$. In section 4, with the help of (2.12) we prove stability of the spectral value E^0 of H_θ^0 under the perturbation by W_θ. Then we are able to show that for d small enough the tunneling picture given in the previous paragraph is correct. Indeed we obtain the following exponential bound on the j–th component of ϕ_θ: let ρ be a function obeying (3.2), then

$$\left\| -i\partial_s\, e^{\rho(-i\partial_s)} \phi_\theta^j \right\|^2 + \left\| e^{\rho(-i\partial_s)} \phi_\theta^j \right\|^2 < \infty\,, \tag{2.13}$$

the bound being uniform as d tends to zero. This is one of the main ingredients of this paper. We turn now to the explanation of how one can use (2.13) to derive our exponential estimate on $\operatorname{Im} E$, which is the purpose of Section 6.

From (2.12) we obtain by straightforward algebraic computations the following equation for $\operatorname{Im} E$:

$$\begin{aligned} \operatorname{Im} E \;&=\; ((\operatorname{Im} H_\theta^j - Z_\theta(E))\phi_\theta^j, \phi_\theta^j) \\ Z_\theta(E) \;&:=\; \mathcal{J}_j^* \left\{ 2\mathrm{Re}\,[\operatorname{Im}(W_\theta)\widehat{R}_\theta^j W_\theta] - (\widehat{R}_\theta^j W_\theta)^* \operatorname{Im} \widehat{H}_\theta^j\, \widehat{R}_\theta^j W_\theta \right\} \mathcal{J}_j \end{aligned} \tag{2.14}$$

with $\widehat{H}_\theta^j := Q_j H_\theta Q_j$ provided ϕ_θ is chosen with the unit norm. The merit of this cumbersome formula is that it indicates that the operator $\operatorname{Im} H_\theta^j - Z_\theta(E)$ should acts as a localization on

Ω_e in the momentum space. This is due to the fact that each of its three terms contains an imaginary part of a scaled operator which is expected to vanish on Ω_i where the deformation does not operate. If this localization property would be true then (2.14) combined with (2.13) would give immediately the desired exponential estimate on $\operatorname{Im} E$:

$$|\operatorname{Im} E| \le \text{const}\; e^{-2\rho(\omega)}\,.$$

Unfortunately, since most of the operators involved here are non-local in the momentum variable, this simple reasoning does not work. However we are able to show directly that this localization property is valid in the following weaker sense

$$e^{-\rho}|\operatorname{Im} H_\theta^j - Z_\theta(E)|e^{-\rho} \le \text{const}\; e^{-2\rho(\omega)}(-\partial_s^2 + 1)$$

which is all what we need.

Let us finish the survey of the paper contents. To deal with the Schrödinger operator in the momentum representation, and in particular, with its image under an exterior scaling in the momentum variable, we have developed in Section 3 a functional calculus based on the Dunford-Taylor integral. All the necessary material for the exterior scaling is presented in Section 3. Finally the extension of our method to the case where the nearest singularity of the curvature in the complex plane is a pole is done in Section 7.

3 Complex scaling and functional calculus

From this moment on we pass to the unitary equivalent situation by performing the inverse Fourier transformation in the s variable, denoted by F_s^{-1}. We introduce the notation:

$$p \;:=\; F_s^{-1}\, i\partial_s\, F_s \qquad \text{and} \qquad D \;:=\; -i\partial_p \;=\; F_s^{-1}\, s\, F_s$$

For all other transformed operators we shall use the same symbols as before:

$$H = p\, b(D,u)\, p - \partial_u^2 + V(D,u).$$

Note that now $\mathcal{D}(H^0) := \mathcal{D}(p^2) \otimes (\mathcal{H}_0^1 \cap \mathcal{H}^2)((0,d))$.

3.1 Exterior scaling in momentum representation

Complex dilations represent a useful tool to reveal resonances in systems with Hamiltonians having certain analytic properties. In the present case, we use the exterior dilation defined as follows:

$$p_\theta(t) \;:=\; \begin{cases} t & \text{if} \quad t \in \Omega_i := (-\omega, \omega) \\[2ex] \pm\omega + e^\theta(t \mp \omega) & \text{if} \quad t \in \Omega_e := I\!R \setminus \overline{\Omega}_i \end{cases} \tag{3.1}$$

where ω is a positive number to be determined later. The parameter θ takes complex values in a strip around the real axis; defining the sets $\mathcal{S}_\alpha := \{\theta \in \mathbb{C},\ |\mathrm{Im}\,\theta| < \alpha\}\,, \alpha > 0$ we have $\theta \in \mathcal{S}_{\alpha_0}$. The function p_θ is for real θ a piecewise differentiable homeomorphism of $\mathbb{R}$ whose unitary implementation on $L^2(\mathbb{R})$ is defined by

$$U_\theta \varphi := {p'_\theta}^{1/2} \varphi \circ p_\theta .$$

U_θ and p_θ are both called (exterior) dilation. In general, to denote the image under this dilation we use the index θ.

Recall how one uses U_θ to get a complex deformation of operators. With a given closed operator T, one constructs the family of operators for $\theta \in \mathbb{R}$:

$$\theta \to T_\theta := U_\theta T\, U_\theta^{-1}.$$

If this function has an analytic extension to some $\mathcal{S}_\alpha$ (in a suitable sense — *cf.* [Ka, Ch.VII]), the resulting family is what is usually called a complex (family of) deformation(s) of T.

We begin by considering the complex deformation of p and D:

Proposition 3.1 *(i)* $\{\, p_\theta^2 :\ \theta \in \mathbb{C}\,\}$ *is a self–adjoint family of type A in the sense of [Ka] with common domain* $\mathcal{D}(p_\theta^2) = \mathcal{D}(p^2)$.
(ii) $\sigma(p_\theta^2) = [0, \omega^2] \cup p_\theta^2(\Omega_e)$.

We would like to remark here that since we are scaling in the Fourier image, we will have to use θ's with a negative imaginary part to make the essential spectrum turn into the lower complex half-plane.

Proposition 3.2 *(i)* $\{\, D_\theta \,:\, \theta \in \mathbb{C}\,\}$ *is a self–adjoint analytic family. A vector* u *belongs to* $\mathcal{D}(D_\theta)$ iff $u \in \mathcal{H}^1(\Omega_i) \oplus \mathcal{H}^1(\Omega_e)$ *and* $u(\pm\omega \pm 0) = e^{\theta/2} u(\pm\omega \mp 0)$ *, the action of the operator being given by*

$$(D_\theta u)(t) \;=\; (p'_\theta)^{-1}(D\,u)(t) \;=\; \begin{cases} -iu'(t) & \text{if} \quad t \in \Omega_i \\[2ex] -ie^{-\theta} u'(t) & \text{if} \quad t \in \Omega_e \end{cases}$$

(ii) $\sigma(D_\theta) = e^{-\theta}\, \mathbb{R}$.

Proof: (i) By the standard argument — *cf.* [CDKS] for the case of the Laplacian.
(ii) This is a straightforward calculation using the explicit expression of the resolvent kernel of D_θ and bounding it by the Schur-Holmgren norm. Recall that the Schur-Holmgren norm of an integral operator B is defined through its kernel as

$$\|B\|_{SH} \;:=\; \max\left\{ \sup_x \int |B(x,y)|\, dy\,,\ \sup_y \int |B(x,y)|\, dx \right\} .$$

(*cf.* [Ka, Example III.2.4]) and majorizes the operator norm. ∎
To study a complex deformation of operators of the form $f(D)$ we need to develop a functional calculus for D_θ .

3.2 Functional calculus for D_θ

Since the operator under consideration contains the metric term (2.2) and the potential (2.3), we have to define the corresponding operator functions. The standard functional calculus is not applicable here, because D_θ is not even normal for complex θ; instead we use the Dunford-Taylor integral. The original theory for unbounded operators is exposed in [DS, Sec.VII.9]. But since analytic families of operators are not treated there and since it is necessary for our estimates to modify the original definition, we present the adapted theory in detail.

Definition 3.3 *Let a function* $f : \mathbb{C} \to \mathbb{C}$ *satisfy the same requirements as* γ *in (a1) and (a2). Suppose that* T *is a closed operator in* $L^2(\mathbb{R})$ *and there is an open set* $\mathcal{V}$ *which obeys strict inclusions* $\sigma(T) \subset \mathcal{V} \subset \Sigma_{\alpha_0,\eta_0}$ *and whose boundary* $\partial\mathcal{V}$ *consists of a finite number of rectifiable Jordan curves with a positive orientation. Suppose also that* $(T - z)^{-1}$ *is uniformly bounded on* $\partial\mathcal{V}$ *. Then we define*

$$f(T) \;:=\; \frac{i}{2\pi} \int_{\partial\mathcal{V}} f(z)(T - z)^{-1}\, dz\,.$$

The operators defined this way will be called Dunford-Taylor operators.

Lemma 3.4 *(i)* $f(T)$ *is a well–defined bounded operator on* $L^2(\mathbb{R})$.
(ii) If T *is self–adjoint,* $f(T)$ *coincides with the operator obtained by the usual functional calculus.*
(iii) $Bf(T)B^{-1} = f(BTB^{-1})$ *holds for any bounded operator* B *with bounded inverse.*
(iv) For $\theta \in \mathbb{R}$ *, let* $T_\theta := U_\theta T U_\theta^{-1}$ *such that* $\{T_\theta, \theta \in \mathcal{S}_\alpha\}\,, 0 < \alpha \le \alpha_0$ *is an analytic family of operators. Assume that there is an open set* $\mathcal{V}$ *with* $\bigcup_{\theta \in \mathcal{S}_\alpha} \sigma(T_\theta) \subset \mathcal{V} \subset \Sigma_{\alpha_0,\eta_0}$ *, that* $(T_\theta - z)^{-1}$ *is uniformly bounded on* $\partial\mathcal{V}$ *for all* $\theta \in \mathcal{S}_\alpha$ *and that* $\partial\mathcal{V}$ *obeys the conditions of the definition. Then for all* $\theta \in \mathcal{S}_\alpha$

$$\Big(f(T)\Big)_\theta \;=\; f(T_\theta)$$

and these operators form a bounded analytic family on $L^2(\mathbb{R})$.

Proof: We prove this lemma in appendix A.

In our case $T := D_\theta$; since $\sigma(D_\theta) = e^{-\theta}\mathbb{R}$, it is clear that for any θ with $|\mathrm{Im}\,\theta| < \alpha_0$ there is a domain $\mathcal{V}_\theta$ satisfying strict inclusions $\sigma(D_\theta) \subset \mathcal{V}_\theta \subset \Sigma_{\alpha_0,0}$. But we still need to control the resolvent of D_θ .

Before doing that we want to introduce another operator deformation we shall need later:

$$T_\rho \;:=\; e^\rho\, T\, e^{-\rho},$$

is usually called the image of T under the *boost* $-i\rho$, where ρ is an absolutely continuous function. In particular, $D_\rho = D + i\rho'$, suggesting the origin of this terminology. We shall only consider real functions for the boosting, *i.e.* only purely imaginary boosts. Furthermore it will be sufficient for our purpose to use only boosts being constant on Ω_e. Then, of course, the boost commutes with the exterior scaling and there should be no confusion concerning our notation, $T_{\theta,\rho}$, for the indication of the two deformations of T.

If ρ' is supported on Ω_i, one has $D_{\theta,\rho} = D_\theta + i\rho'$. Note that we write $D^*_{\theta,\rho}$ for $(D_{\theta,\rho})^*$. Finally since ρ, and therefore also $e^{\pm\rho}$ are bounded, we have $e^\rho f(T) e^{-\rho} = f(T_\rho)$ by Lemma 3.4 (iii).

Due to group property of the exterior dilation in θ it is sufficient to perform the estimates for purely imaginary θ; we shall write and prove the corresponding bounds for $\theta = i\beta$, $\beta \in I\!R$ only.

We will use the symbol χ_A to denote the characteristic function of a set A.

Proposition 3.5 *Let ρ be a real, bounded, absolutely continuous function on $I\!R$ which is constant on Ω_e. Furthermore define $\mathcal{G}_{\beta,\rho} := \{z \in \mathbb{C} : |\arg\left(\pm z e^{-i\beta/2}\right)| \le |\beta|/2$ or $|\mathrm{Im}\, z| \le \|\rho'\|_\infty\}$ — cf. Figure 2. Then for all β with $|\beta| < \frac{\pi}{2}$ and all $z \in \mathbb{C} \setminus \mathcal{G}_{\beta,\rho}$*

$$\left\| (D_{i\beta,\rho} - z)^{-1} \right\| \le \operatorname{dist}(z, \mathcal{G}_{\beta,\rho})^{-1}$$

Proof: Let $v \in \mathcal{D}(D_{i\beta})$ and $w := p'_{-i\beta} v$. Then $\|w\| \le \|v\|$; we have

$$\begin{aligned} \| (D_{i\beta} + i\rho' - z)v\| \, \|w\| \;&\ge\; |(p'_{i\beta}(z - i\rho' - D_{i\beta})v, v)| \;\ge\; |\mathrm{Im}\, (p'_{i\beta}(z - i\rho' - D_{i\beta})v, v)| \\ &= \; |((\mathrm{Im}\, p'_{i\beta} z - \rho')v, v)|\,; \end{aligned}$$

The last equality is due to:

$$\begin{aligned} \mathrm{Im}\,(p'_{i\beta}\, D_{i\beta} v, v) \;&=\; \frac{1}{2}\left\{(\chi_{\Omega_i} v', v) + (\chi_{\Omega_i} v, v') + (\chi_{\Omega_e} v', v) + (\chi_{\Omega_e} v, v')\right\} \\ &=\; \frac{1}{2}\left\{ |v|^2\Big|_{-\omega+0}^{+\omega-0} + |v|^2\Big|_{+\omega+0}^{-\omega-0} \right\} \\ &=\; 0\,, \end{aligned}$$

using that for $v \in \mathcal{D}(D_{i\beta})$ the discontinuity at $\pm\omega$ is just the phase $e^{i\beta/2}$. ∎

For the sake of brevity we shall use the shorthand $f_\vartheta := f(D_\vartheta)$ for the Dunford–Taylor operators under consideration, where $\vartheta = \theta, \rho$, *etc.;* the superscript c will denote the complement of a set. The last proposition implies, in particular, that

$$\max\left\{ \left\| (D_{i\beta,\rho} - z)^{-1} \right\|, \left\| (D^*_{i\beta,\rho} - z)^{-1} \right\| \right\} \le \operatorname{dist}(z, \Sigma_{|\beta|,\|\rho'\|_\infty})^{-1}$$

holds for $z \in (\overline{\Sigma}_{|\beta|,\|\rho'\|_\infty})^c$; the involved domains are sketched on Figure 2. Furthermore we want to state the general conditions on ρ which will be imposed up to the end of Section 6:

$$\left\{ \begin{array}{ll} \text{(i)} & \rho \text{ is a real, absolutely continuous function on } I\!R \text{ which is constant on } \Omega_e\,, \\ \text{(ii)} & \|\rho'\|_\infty \le \eta < \eta_0. \end{array} \right. \qquad (3.2)$$

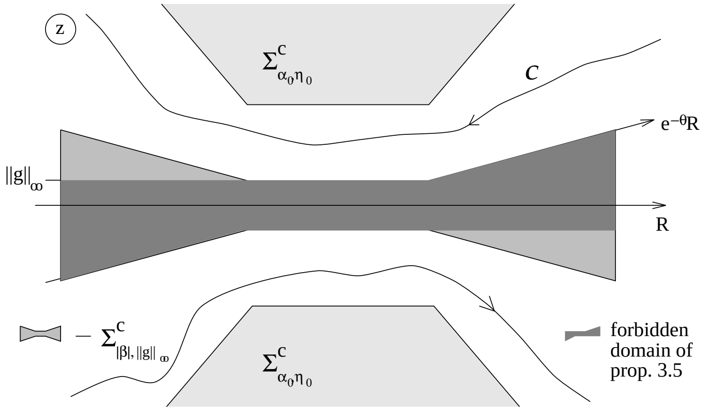

Figure 2: To the definition of the Dunford-Taylor integral (the case $\mathrm{Im}\,\theta < 0$)

The functions b and V — *cf.* (2.2) and (2.3) — are understood as rational functions of $u\gamma$, $u\gamma'$ and $u\gamma''$. Notice that their structure is particularly simple; there are only powers of $1 + u\gamma$ appearing in the denominator.

Proposition 3.6 *Let ρ satisfy (3.2).*
(i) $\gamma_{\theta,\rho}$ and therefore $V^0_{\theta,\rho}$, as well as $\gamma'_{\theta,\rho}$ and $\gamma''_{\theta,\rho}$ are bounded self–adjoint analytic families of operators in $L^2(I\!R)$ in θ provided $\theta \in \mathcal{S}_{\alpha_0}$.
(ii) Let $\alpha_1 < \alpha_0$. Then for d small enough, depending only on α_1 and η, the operators $V(D_{\theta,\rho}, u)$ and $b(D_{\theta,\rho}, u)$ in $\mathcal{H}$ form bounded self-adjoint analytic families in θ provided $\theta \in \mathcal{S}_{\alpha_1}$.

Proof: The first assertion is an immediate consequence of Lemma 3.4, Proposition 3.5 and (a1)–(a2). For (ii) notice that we have for a function f obeying the same requirements as γ in (a1) and (a2) and $\mathrm{Im}\,\theta = \beta$ the bound

$$\| f(D_{i\beta,\rho}) \| \le \left(\operatorname{dist}(\partial\mathcal{V}, \Sigma_{|\beta|,\|\rho'\|_\infty})\right)^{-1} \int_{\partial\mathcal{V}} |f(z)|\,|dz|\,, \qquad (3.3)$$

for any integration path $\partial\mathcal{V} \subset \Sigma_{\alpha_0,\eta_0} \setminus \overline{\Sigma}_{|\beta|,\|\rho'\|_\infty}$ verifying the conditions in Definition 3.3. Thus we see that the operators $\gamma_{\theta,\rho}$ for $\theta \in \mathcal{S}_{\alpha_1}$ can be bounded by a constant depending only on α_1 and η. Choosing d small enough, this immediately implies that $\|u\gamma_{\theta,\rho}\|$ can be made smaller than one. Thus $(1 + u\gamma_{\theta,\rho})^{-1}$ exists and is bounded, which is all we need in view of the structure of b and V and (i). ∎

When there is no possibility of confusion, we will use for the operators $h(D_\vartheta, u)$ the symbol h_ϑ, too.
Even though formula (3.3) will be useful later on, it is not sufficient. In particular we will

need more information about the dependence of the norm on β which is provided by the following proposition.

Proposition 3.7 *Let f obey (a1) and (a2). Then for any compact subset I of $(-\alpha_0,\alpha_0)$ there exists a constant C such that for all $\alpha,\beta\in I$,*

$$\| f(D_{i\beta})-f(D_{i\alpha})\| \le C \sin\Big|\frac{\beta-\alpha}{2}\Big| .$$

Proof: We have

$$f_{i\beta}-f_{i\alpha} = \frac{i}{2\pi}\int_{\partial\mathcal{V}} f(z)\,(r_{i\beta}(z)-r_{i\alpha}(z))\,dz ,$$

where $r_\bullet(z):=(D_\bullet-z)^{-1}$. For $v,w\in L^2(I\!R)$ let $\hat v:=r_{i\beta}(z)v$ and $\hat w:=r_{-i\alpha}(\overline{z})w$. Then

$$\begin{aligned}((r_{i\beta}(z)-r_{i\alpha}(z))v,w) &= (\hat v, D_{-i\alpha}\hat w)-(D_{i\beta}\hat v,\hat w)\\ &= i\left\{\hat v\overline{\hat w}\Big|_{-\omega+0}^{+\omega-0}+e^{-i\alpha}\hat v\overline{\hat w}\Big|_{+\omega+0}^{-\omega-0}\right\}+\left(e^{i(\beta-\alpha)}-1\right)(\chi_{\Omega_e}D_{i\beta}\hat v,\hat w)\\ &= ie^{-i\alpha}\left(1-e^{-i(\beta-\alpha)/2}\right)\hat v\overline{\hat w}\Big|_{+\omega+0}^{-\omega-0}+\left(e^{i(\beta-\alpha)}-1\right)(\chi_{\Omega_e}D_{i\beta}\hat v,\hat w),\end{aligned}$$

where we used that $\hat v\in\mathcal{D}(D_{i\beta})$ and that $\hat w\in\mathcal{D}(D_{-i\alpha})$. We can now rewrite the boundary term as $i\hat v\overline{\hat w}|_{+\omega+0}^{-\omega-0}=e^{i\alpha}(\chi_{\Omega_e}\hat v, D_{-i\alpha}\hat w)-e^{i\beta}(\chi_{\Omega_e}D_{i\beta}\hat v,\hat w)$, so that

$$((r_{i\beta}(z)-r_{i\alpha}(z))v,w) = \left(1-e^{-i(\beta-\alpha)/2}\right)(\chi_{\Omega_e}\hat v, D_{-i\alpha}\hat w)+\left(e^{i(\beta-\alpha)/2}-1\right)(\chi_{\Omega_e}D_{i\beta}\hat v,\hat w).$$

This can be written, dropping the argument z in the resolvents, as

$$\frac{r_{i\beta}-r_{i\alpha}}{2i} = z\,\sin\tfrac{\beta-\alpha}{2}r_{i\alpha}\chi_{\Omega_e}r_{i\beta}+\sin\tfrac{\beta-\alpha}{4}\left(e^{-i(\beta-\alpha)/4}\chi_{\Omega_e}r_{i\beta}+e^{i(\beta-\alpha)/4}r_{i\alpha}\chi_{\Omega_e}\right) \tag{3.4}$$

implying by proposition 3.5

$$\begin{aligned}\|r_{i\beta}(z)-r_{i\alpha}(z)\| &\le 2\left|\sin\tfrac{\beta-\alpha}{2}\right|\left(\|r_{i\beta}(z)\|+\|r_{i\alpha}(z)\|+|z|\,\|r_{i\beta}(z)\|\,\|r_{i\alpha}(z)\|\right)\\ &\le 2\left|\sin\tfrac{\beta-\alpha}{2}\right|\frac{\mathrm{dist}(z,\Sigma_{|\beta|,0})+|z|+\mathrm{dist}(z,\Sigma_{|\alpha|,0})}{\mathrm{dist}(z,\Sigma_{|\beta|,0})\,\mathrm{dist}(z,\Sigma_{|\alpha|,0})}.\end{aligned}$$

We also used that since $|\beta-\alpha|<\pi$, we can bound $|\sin\frac{\beta-\alpha}{4}|$ by $|\sin\frac{\beta-\alpha}{2}|$. Furthermore we can suppose without restriction of generality that $|\beta|\ge|\alpha|$. Then we can choose $\partial\mathcal{V}$ in $\Sigma_{\alpha_0,\eta_0/2}\setminus\Sigma_{|\beta|,0}$ such that the last factor on the left side is bounded by a constant depending only on I; due to (a2) the statement follows. ∎

3.3 Some estimates on $p_{i\beta}$ and $W_{i\beta,\rho}$

The leading longitudinal part of the dilated Hamiltonian is the operator of multiplication by $p_{i\beta}^2$. Here we collect some simple bounds which we shall need in the following.

Proposition 3.8 *(i) For all $p\in I\!R$, any positive integer n, any $|\beta|\le\frac{\pi}{2}$ and $\omega\ge0$ we have $p^{2n}\ge|p_{i\beta}^n|^2\ge p^{2n}\cos^n\beta$. (ii) The function $p\mapsto p_{i\beta}(p)$, satisfies on Ω_e for any $\omega\ge0$ the bounds: $\mathrm{Re}\,p_{i\beta}^2\ge p^2\cos2\beta+2\omega^2\sin^2\beta$ if $|\beta|\le\frac{2\pi}{3}$.*

Proof: (i) It is sufficient to consider $n=1$. For every real p we have

$$|p_{i\beta}|^2 = p^2\chi_{\Omega_i}+\Big(\omega^2+(|p|-\omega)^2+2\omega(|p|-\omega)\cos\beta\Big)\chi_{\Omega_e}.$$

The part on the right side restricted to Ω_e satisfies

$$p^2+2(\omega^2-\omega|p|)(1-\cos\beta)\ge p^2+(\omega^2-p^2)(1-\cos\beta) = p^2\cos\beta+\omega^2(1-\cos\beta);$$

since the very last term is nonnegative, we obtain the second inequality. Furthermore, $\omega^2-\omega|p|<0$ on Ω_e, so the same expression may be estimated from above by p^2 and thus the first inequality follows.

The identity

$$\mathrm{Re}\,p_{i\beta}^2 = p^2\cos2\beta+\omega^2(1-\cos2\beta)+2\omega(|p|-\omega)(\cos\beta-\cos2\beta)$$

yields (ii) on Ω_e, because $\cos\beta-\cos2\beta\ge0$ for $|\beta|\le\frac{2\pi}{3}$. ∎

Let us fix an α_1 in $(0,\alpha_0)$ and define the weight

$$\langle p\rangle := \left(p^2+\tau\right)^{1/2},\quad \tau:=\sup\{\|V^0_{i\beta,\rho}\| : |\beta|\le\alpha_1,\ \rho\text{ verifying (3.2)}\}, \tag{3.5}$$

The supremum exists by (3.3) and is strictly positive by (a0).

The motivation for the choice of this weight is that it will simplify the statements and permit us to obtain particularly simple constants in the subsequent resolvent estimates. Notice that it depends only on the fixed parameters α_1 and η, but not on d.

Furthermore we fix d_0 such that $b_{i\beta,\rho}$ and $V_{i\beta,\rho}$ exist and are bounded for all $|\beta|\le\alpha_1$ and all ρ verifying (3.2), if $d\le d_0$.

Proposition 3.9 *Let $|\beta|<\alpha_1$, and $d\le d_0$. Then there exists a constant $c^W_{i\beta}>0$ for all ρ satisfying (3.2) such that $\|\langle p\rangle^{-1}W_{i\beta,\rho}\langle p\rangle^{-1}\|\le c^W_{i\beta}d$.*

Proof: Since $W_{i\beta,\rho} = p_{i\beta}(b-1)_{i\beta,\rho}p_{i\beta}+(V-V^0)_{i\beta,\rho}$ we get

$$c^W_{i\beta} = \max_{0\le d\le d_0}\left\{\frac{1}{d}\left(\|(b-1)_{i\beta,\rho}\|+\frac{1}{\tau}\|(V-V^0)_{i\beta,\rho}\|\right)\right\}$$

which does exist because $|(b-1)(\cdot,u)|d^{-1}$ and $|(V-V^0)(\cdot,u)|d^{-1}$ obey (a2) for all $u\in[0,d]$, $d\in(0,d_0]$. Notice that we also used $|p_{i\beta}/p|\le1$ as proven in Proposition 3.8. ∎

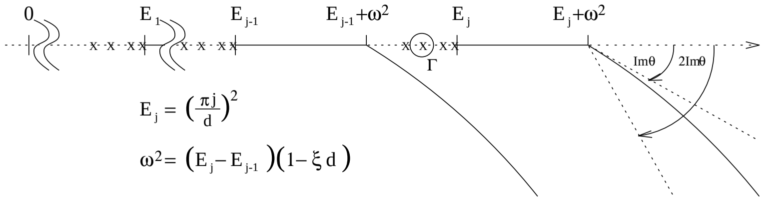

Figure 3: The spectrum of H^0_θ — Stability of the eigenvalues

3.4 The operators H^0_θ and H_θ

Let us finally collect some basic properties of the operators H^0_θ and H_θ, images under the exterior scaling of the "free" and the full Hamiltonian, H^0 and H, respectively.

Theorem 3.10 *(i) The operators H^0_θ for $|\mathrm{Im}\,\theta| < \alpha_0$ form a self–adjoint analytic family of type A with the common domain $\mathcal{D}(H^0) = \mathcal{D}(p^2) \otimes (\mathcal{H}^1_0 \cap \mathcal{H}^2)((0,d))$. Moreover (cf. Figure 3),*

$$\sigma(H^0_\theta) \;=\; \Big\{\lambda + E_j \,:\, \lambda \in \Big(\{\lambda_n\}_{n=1}^N \cup \nu \cup \sigma(p^2_\theta)\Big)\;,\; j = 1,2,\ldots\;\Big\}\;,$$

where ν denotes the set of resonances of the operators $A_\theta = p^2_\theta + V^0_\theta$ (which may be empty).

(ii) For all sufficiently small d, the operators H_θ with $|\mathrm{Im}\,\theta| < \alpha_1$ form a self-adjoint analytic family of type A with the common domain $\mathcal{D}(H^0)$.

Proof: (i) Analyticity of H^0_θ follows from the boundedness and the analyticity of γ_θ, see Proposition 3.6. The form of the spectrum is due to the analyticity, the structure of the operator, $H^0_\theta = A_\theta \otimes I + I \otimes (-\partial^2_u)$, and the p^2-compactness of γ^2_θ in view of (a2).
(ii) Due to Proposition 3.9, $W_{i\beta}$ is $H^0_{i\beta}$–bounded with a bound smaller than one if d is small enough and $|\beta| < \alpha_1$. This bound extends by the group property of the exterior dilation to all $\theta \in \mathcal{S}_{\alpha_1}$. Thus the analyticity of H_θ follows. ∎

Remark 3.11 (i) The set ν of resonances of A_θ cannot contain embedded eigenvalues due to the decay assumption (a2) (*cf.* [RS, Sect. XIII.13]).
(ii) If ω is chosen large enough no resonances of A_θ will be disclosed at all; the continuous spectrum is deformed only in a sector with vertex ω^2, whereas the resonances lie inside a disc around the origin with a fixed radius of order $\|V^0\|$.

4 Stability of the discrete spectrum

Our next goal is to estimate the effect the perturbation $W_{i\beta}$ has on the spectrum of the operator $H^0_{i\beta}$.

4.1 Estimates on $R^0_{i\beta}$ and $R^0_{i\beta,\rho}$

Let $E^0_{j,n} = \lambda_n + E_j\,, j \geq 2$ be a fixed eigenvalue of H^0_θ. We choose:

$$\Gamma := \{z \in \mathbb{C} \,:\, |z - E^0_{j,n}| = r\}, \quad \text{with } r := \frac{1}{2}\mathrm{dist}(\lambda_n, \sigma(A) \setminus \{\lambda_n\}) \tag{4.1}$$

to be a circular contour around $E^0_{j,n}$ such that no other eigenvalue of $H^0_{i\beta}$ is within Γ, and denote $\mathcal{D}_\Gamma := \{z \in \mathbb{C} \,:\, |z - E^0_{j,n}| \leq r\}$.

Having the intention to prove stability by perturbation we have to control $R^0_{i\beta}(z)$ on Γ. For the estimate it is advantageous to pass to the transverse mode decomposition, $H^0_{i\beta}$ being diagonal in this decomposition:

$$H^0_{i\beta} = \sum_{k\geq 1} \mathcal{J}_k H^{0,k}_{i\beta} \mathcal{J}^*_k, \qquad H^{0,k}_{i\beta} := \mathcal{J}^*_k H^0_{i\beta} \mathcal{J}_k \;\text{ on }\; L^2(\mathbb{R}, dp)\,.$$

Since $H^{0,k}_{i\beta}$ is not self-adjoint (for $\beta \neq 0$) and a part of Γ lies in the numerical range of $H^{0,k}_{i\beta}$ for every $k \leq j$ (*cf.* Figure 3), simple estimates in terms of the distance to the numerical range do not work here.

The difficulties for the j-th mode result from our desire to chose the ω of the exterior dilation (see eq.(3.1)) as $\omega = \mathcal{O}(\sqrt{E_j - E_{j-1}})$. This d dependence of ω implies a d dependence of the deformed operators $H^{0,k}_{i\beta}$ so that the usual argument using the compactness of Γ to assure uniform boundedness of $R^{0,j}_{i\beta}(z)$ are not applicable here. Instead we chose to perturbate around $\beta = 0$; it turns out that then the estimate is independent of ω, and thus of d:

Lemma 4.1 *Let $c^{(2)} := \frac{2}{r}\max\{1, 3\tau\}$. There exists $0 < \beta_1 \leq \min\{\frac{\pi}{4}, \alpha_1\}$ so that for $|\beta| \leq \beta_1$ one has*

$$\sup_{z\in\Gamma} \|\langle p\rangle^\ell R^{0,j}_{i\beta}(z)\langle p\rangle\| \leq c^{(2)},\; l = 0,1 \quad \text{and} \quad \sup_{z\in\Gamma} \|\langle p\rangle^\ell R^{0,j}_{i\beta}(z)\| \leq c^{(2)},\; l = 0,1,2.$$

Proof: We prove the claim using that $R^{0,j}_{i\beta}(z) \to R^{0,j}(z)$ in the operator norm as $\beta \to 0$ uniformly for $z \in \Gamma$. By the resolvent identity,

$$R^{0,j}_{i\beta}(z) - R^{0,j}(z) \;=\; R^{0,j}_{i\beta}(z)\,\Big(p^2 - p^2_{i\beta} + V^0 - V^0_{i\beta}\Big)\,R^{0,j}(z)\,.$$

One has for every β the inequality $|p^2 - p_{i\beta}^2| \le 6|\sin\frac{\beta}{2}|p^2$. Let $0 < \beta_1 < \alpha_0$. Using Proposition 3.7 implies that there is a constant C for all $|\beta| \le \beta_1$ such that $\|V^0 - V_{i\beta}^0\| \le C|\sin\frac{\beta}{2}|$. Thus we get the estimate for every $z \in \Gamma$:

$$\begin{aligned}
&\left\| \left(p^2 - p_{i\beta}^2 + V^0 - V_{i\beta}^0\right) R^{0,j}(z) \right\| \\
&= \left\| \left\{ \left(p^2 - p_{i\beta}^2\right)(p^2+1)^{-1} + (V^0 - V_{i\beta}^0)(p^2+1)^{-1} \right\} (p^2+1)\, R^{0,j}(z) \right\| \\
&\le \max\{6, C\} |\sin\frac{\beta}{2}| \left(1 + \left(1+r+\|V^0-\lambda_n\|\right) \frac{1}{r} \right).
\end{aligned}$$

Taking β_1 small enough one can certainly ensure that for all β with $|\beta| \le \beta_1$ and all $z \in \Gamma$ one has $\| \left(p^2 - p_{i\beta}^2 + V^0 - V_{i\beta}^0\right) R^{0,j}(z)\| \le \frac{1}{2}$. Solving the resolvent identity gives $\|R_{i\beta}^{0,j}(z)\| \le \frac{2}{r}$. Thus the use of $\langle p\rangle^2 R^{0,j}(z) = 1 + (\tau + z - E_j - V^0)\, R^{0,j}(z)$, and of the facts that in our situation $\|V^0 - \lambda_n\| \le \|V^0\|$ and $r \le \|V^0\|/2$, yields the claim for $\ell = 2$ in the second formula. The statement in the case of only one weight present is then obtained by an obvious quadratic estimate. The symmetric case with one weight on each side of the resolvent is handled by following estimate:

$$\|\langle p\rangle R^{0,j}(z) \langle p\rangle\|^2 \le \|\langle p\rangle R^{0,j}(z) \langle p\rangle\| + (r + 2\tau) \|\langle p\rangle R^{0,j}(z)\|^2. \quad \blacksquare$$

The restriction $|\beta| < \frac{\pi}{4}$ is not necessary in this lemma, but for later convenience we prefer having it stated. Indeed to simplify the statements we work from now on with a *fixed* β:

$$\beta \in [-\beta_1, 0), \text{ with } \beta_1 \text{ given by Lemma 4.1.} \tag{4.2}$$

Consequently, the dependence of the constants on β will be no longer specified.

For $k \ne j$ the resolvents $R_{i\beta}^{0,k}(z)$ are estimated considering $V_{i\beta}^0$ as a perturbation. We use the fact that the distance between $\mathcal{D}_\Gamma$ and the spectrum of $H_{i\beta}^{00,k} := E_k + p_{i\beta}^2$ tends to infinity as d tends to zero whereas $V_{i\beta}^0$ is bounded, independently of d. We choose ω to be

$$\omega := \frac{\pi}{d} \sqrt{(2j-1)(1-\xi d)}, \tag{4.3}$$

where ξ is a supplementary positive parameter specified below to govern the distance of the spectrum of $H_{i\beta}^{00,j-1}$ to the contour Γ — *cf.* Figure 3.

Lemma 4.2 *Let ρ verify condition (3.2) and $c^{(1)} := 8\sqrt{3}$. Then for all (d, ξ) verifying*

$$1 \ge \xi d \ge \frac{c^{(1)}\tau}{|\sin\beta|} d^2 \tag{4.4}$$

one has for $\ell = 0, 1$

(a) $\sup_{z\in\mathcal{D}_\Gamma} \|\langle p\rangle^\ell R_{i\beta,\rho}^{0,j-1}(z) \langle p\rangle\| \le \frac{c^{(1)}}{|\sin\beta|\xi} d^{-\ell}$,

(b) $\sup_{z\in\mathcal{D}_\Gamma} \|\langle p\rangle^\ell R_{i\beta,\rho}^{0,k}(z) \langle p\rangle\| \le \frac{c^{(1)}}{|\sin\beta|} d^{1-\ell}$ *for all $k \ne j,\ j-1$.*

Proof of the lemma is given in appendix B.

4.2 Stability of the resolvent set of $H_{i\beta}^0$

Lemma 4.3 *Let Γ and β be fixed by (4.1) and (4.2). Then for all sufficiently small $d \le d_0$ such that the condition (4.4) is verified for $\xi \ge 2c^W \max\{c^{(2)}, \frac{c^{(1)}}{|\sin\beta|}\}$ ($c^W \equiv c_{i\beta}^W$), the contour Γ belongs to the resolvent set $\rho(H_{i\beta})$.*

Proof: If we can show that

$$R_{i\beta}(z) = R_{i\beta}^0(z)\langle p\rangle \left(1 + \langle p\rangle^{-1} W_{i\beta} R_{i\beta}^0(z) \langle p\rangle\right)^{-1} \langle p\rangle^{-1}$$

makes sense for all $z \in \Gamma$, we are done. The boundedness of $R_{i\beta}^0(z)\langle p\rangle$ has already been proven in the two preceding lemmas. Thus showing that $\|\langle p\rangle^{-1} W_{i\beta} R_{i\beta}^0(z)\langle p\rangle\| < 1$ will conclude the proof. First of all one has $R_{i\beta}^0(z) = \sum_{k\ge1} \mathcal{J}_k R_{i\beta,\rho}^{0,k}(E) \mathcal{J}_k^*$. Secondly, the operators $\mathcal{J}_k$, $\mathcal{J}_k^*$ commute with $\langle p\rangle$ and the map $\mathcal{J} := \sum_{k\ge1} \mathcal{J}_k : \bigoplus_{k\ge1} L^2(I\!R, dp) \to \mathcal{H}$ is an isometry. So we can employ Lemmas 4.1, 4.2 and Proposition 3.9 to get

$$\begin{aligned}
\|\langle p\rangle^{-1} W_{i\beta} R_{i\beta}^0(z) \langle p\rangle\| &\le \|\langle p\rangle^{-1} W_{i\beta} \langle p\rangle^{-1}\| \left\| \bigoplus_{k=1}^{\infty} \langle p\rangle R_{i\beta}^{0,k}(z) \langle p\rangle \right\| \\
&\le c^W d \max_k \left\| \langle p\rangle R_{i\beta}^{0,k}(z) \langle p\rangle \right\| \\
&= c^W d \max\left\{ c^{(2)}, \frac{c^{(1)}}{|\sin\beta|\xi d} \right\} \le \frac{1}{2};
\end{aligned}$$

recall that $0 < \xi d \le 1$. $\blacksquare$

Corollary 4.4 *Under the same assumptions, the eigenvalue $E_{j,n}^0$ of $H_{i\beta}^0$ gives rise to a single perturbed eigenvalue of $H_{i\beta}$ of the same multiplicity.*

Proof: By standard interpolation between the respective projections,

$$P_{i\beta}^0 := \frac{i}{2\pi} \int_\Gamma R_{i\beta}^0(z)\, dz \quad \text{and} \quad P_{i\beta} := \frac{i}{2\pi} \int_\Gamma R_{i\beta}^0(z)(1 + W_{i\beta} R_{i\beta}^0(z))\, dz. \quad \blacksquare$$

5 Exponential decay estimates

Please keep in mind that β is now considered to be a fixed parameter, *cf.* (4.2) and that, up to the end of the proof of Theorem 2.2, ρ obeys the condition (3.2); these facts might not always be stated explicitely. Let $E \equiv E_{j,n}$ be the resonance associated with $E_{j,n}^0$. Under the conditions of the last section the corresponding complex eigenvalue equation

$$H_{i\beta}\phi_{i\beta} = E\phi_{i\beta} \tag{5.1}$$

can be easily demonstrated to be equivalent to the system

$$\left(P_jH_{i\beta}P_j - P_jW_{i\beta}\widehat{R}^j_{i\beta}(E)W_{i\beta}P_j\right)\phi_{i\beta} \;=\; EP_j\phi_{i\beta}\,, \tag{5.2}$$

$$Q_j\phi_{i\beta} \;=\; -\widehat{R}^j_{i\beta}(E)W_{i\beta}P_j\phi_{i\beta} \tag{5.3}$$

for a given $j = 2, 3\ldots$, as pointed out in Section 2.3. Recall that there we defined $\widehat{R}^j_{i\beta}(E) = Q_j(Q_j(H_{i\beta}-E)Q_j)^{-1}Q_j$. We shall introduce the analogous notation, $\widehat{A}^j$, also for a general closed operator A: we define $\widehat{A}^j := Q_jAQ_j$ meaning that the operator is restricted to the orthogonal complement of the subspace associated to the mode $H^{0,j}$. In the case of resolvents the hat designates the resolvent on $Q_j\mathcal{H}$, that is $\widehat{(A-z)}^{-1} := Q_j(Q_j(A-z)Q_j)^{-1}Q_j$.

Using the embedding operators (see Section 2.1), we find that (5.2) is further equivalent to

$$\left(H^j_{i\beta} - B^j_{i\beta}(E)\right)\phi^j_{i\beta} \;=\; E\phi^j_{i\beta}\,, \quad B^j_{i\beta}(E) \;:=\; \mathcal{J}^*_jW_{i\beta}\widehat{R}^j_{i\beta}(E)W_{i\beta}\mathcal{J}_j \tag{5.4}$$

on $L^2(I\!R)$. First we have to establish that these equations make indeed sense.

Proposition 5.1 *Under the conditions of Lemma 4.3 on d and ξ and the condition (3.2) on ρ the following bounds are valid*
(i) $\|\langle p\rangle\widehat{R}^j_{i\beta,\rho}(E)\langle p\rangle\| \le \frac{2c^{(1)}}{|\sin\beta|\xi d}$,
(ii) $\|\langle p\rangle^{-1}W_{i\beta,\rho}\langle p\rangle^{-1}\|\,\|\langle p\rangle\widehat{R}^j_{i\beta,\rho}(E)\langle p\rangle\| \le 1$ *and,*
(iii) $\|\langle p\rangle^{-1}B^j_{i\beta,\rho}(E)\langle p\rangle^{-1}\| \le c^Wd$.

Proof: We can write

$$\widehat{R}^j_{i\beta,\rho}(E) \;=\; \left(Q_jH^0_{i\beta,\rho}Q_j + Q_jW_{i\beta,\rho}Q_j - E\right)^{-1} \;=\; \widehat{R}^{0,j}_{i\beta,\rho}(E)\left(I + \widehat{W}_{i\beta,\rho}\widehat{R}^{0,j}_{i\beta,\rho}(E)\right)^{-1}Q_j\,. \tag{5.5}$$

Now since $\widehat{R}^{0,j}_{i\beta,\rho}(E) = \sum_{k\neq j}\mathcal{J}_kR^{0,k}_{i\beta,\rho}(E)\mathcal{J}^*_k$ one has by the argument in the proof of Lemma 4.3 and by Lemma 4.2

$$\left\|\langle p\rangle^{-1}\widehat{W}_{i\beta,\rho}\widehat{R}^{0,j}_{i\beta,\rho}(E)\langle p\rangle\right\| \;\le\; c^Wd\,\max_{k\neq j}\left\|\langle p\rangle R^{0,k}_{i\beta,\rho}(E)\langle p\rangle\right\| \;\le\; \frac{1}{2}\,.$$

Hence $\|\langle p\rangle\widehat{R}^j_{i\beta,\rho}(E)\langle p\rangle\| \le \frac{2c^{(1)}}{|\sin\beta|\xi d}$. The condition on ξ ensures then that $\frac{2c^{(1)}c^W}{|\sin\beta|\xi} \le 1$ and thus (ii) by Proposition 3.9. The last assertion is due to the estimate

$$\left\|\langle p\rangle^{-1}B^j_{i\beta,\rho}(E)\langle p\rangle^{-1}\right\| \le \left\|\langle p\rangle^{-1}W_{i\beta,\rho}\langle p\rangle^{-1}\right\|^2\left\|\langle p\rangle\widehat{R}^j_{i\beta,\rho}(E)\langle p\rangle\right\| \le c^Wd. \;\blacksquare$$

Let $\phi_{i\beta}$ be a normalized solution of the above complex eigenvalue equation (5.1). Denote the boosted eigenfunction $e^\rho\phi_{i\beta}$ by $\phi_{i\beta,\rho}$, where ρ obeys (3.2). Then equation (5.4) implies

$$e^\rho(H^j_{i\beta} - B^j_{i\beta}(E) - E)e^{-\rho}\phi^j_{i\beta,\rho} \;=\; 0\,,$$

which in turn gives the relation

$$\mathrm{Re}\,\left(\left(H^j_{i\beta,\rho} - B^j_{i\beta,\rho}(E) - E\right)\phi^j_{i\beta,\rho}, \phi^j_{i\beta,\rho}\right) \;=\; 0. \tag{5.6}$$

To be able to apply the usual Agmon technique [Ag], we need the following

Proposition 5.2 *Let d be small enough so that $\cos 2\beta - 2c^Wd > 0$. Then $p_\star$ defined by*

$$\frac{1}{2}p_\star^2 \;:=\; \frac{\|V^0_{i\beta,\rho}\| + |\mathrm{Re}\,E - E_j| + 2c^W\,\tau\,d}{(\cos 2\beta - 2c^Wd)}, \tag{5.7}$$

is uniformly bounded in d. Under the conditions of Proposition 5.1 the following inequality holds in the form sense on $\mathcal{D}(p^2\otimes I)$:

$$\mathrm{Re}\,\left(H^j_{i\beta,\rho} - B^j_{i\beta,\rho}(E) - E\right) \;\ge\; (\cos 2\beta - 2c^Wd)\left(p^2 - \frac{1}{2}p_\star^2\right)\,.$$

Proof: The statement on $p_\star$ is trivial. Then using the estimates on $p_{i\beta}, W_{i\beta,\rho}$ and $B^j_{i\beta,\rho}$ of Propositions 3.8, 3.9, and 5.1 we get

$$\begin{aligned}\mathrm{Re}\left(H^j_{i\beta,\rho} - B^j_{i\beta,\rho}(E) - E\right) \;&\ge\; \mathrm{Re}\,p^2_{i\beta} + \mathrm{Re}\left(W^j_{i\beta,\rho} - B^j_{i\beta,\rho}(E)\right) - \|V^0_{i\beta,\rho}\| - \mathrm{Re}\,(E - E_j)\\ &\ge\; \cos(2\beta)p^2 - 2c^Wd\langle p\rangle^2 - \|V^0_{i\beta,\rho}\| - |\mathrm{Re}\,E - E_j|\,. \;\blacksquare\end{aligned}$$

We conclude this section with the main ingredient for the proof of the estimate on the resonance width: the exponential decay as d tends to zero of the resonance wave function $\phi^j_{i\beta}$ in the $\mathcal{H}^1$ sense.

Theorem 5.3 *Denote $\Omega_\star := (-p_\star, p_\star)$, where $p_\star$ is defined by (5.7). Then for any $\beta \in [-\beta_1, 0)$ and any $\eta \in (0, \eta_0)$ there is a $d_\eta \le d_0$, such that for $d \in (0, d_\eta)$ one has $\cos 2\beta - 2c^Wd > 0$ and $\omega > p_\star$, with ξ as in Lemma 4.3, and for*

$$\rho(p) \;:=\; \eta\int_{\min\{0,p\}}^{\max\{0,p\}}\chi_{\Omega_i\setminus\Omega_\star}(t)\,dt. \tag{5.8}$$

we have

$$\left\|\phi^j_{i\beta,\rho}\right\|^2 \;\le\; 2 \qquad\text{and}\qquad \left\|p\phi^j_{i\beta,\rho}\right\|^2 \;\le\; 2p_\star^2\,. \tag{5.9}$$

Proof: The first statement is evident, since β and η are fixed parameters and $p_\star$ remains bounded as d tends to zero. For the proof of the second statement note that ρ satisfies (3.2). At the same time, ρ' is by definition zero on $\Omega_\star \subset \Omega_i$. So the use of the preceding proposition with $\cos 2\beta - 2c^Wd > 0$ and the relation (5.6) yields

$$\begin{aligned}\left(\left(p^2 - \frac{1}{2}p_\star^2\right)\chi_{\Omega_\star^c}\phi^j_{i\beta,\rho}, \phi^j_{i\beta,\rho}\right) \;&\le\; \left(\left(\frac{1}{2}p_\star^2 - p^2\right)\chi_{\Omega_\star}\phi^j_{i\beta,\rho}, \phi^j_{i\beta,\rho}\right)\\ &\le\; \frac{1}{2}p_\star^2\left\|\chi_{\Omega_\star}\phi^j_{i\beta}\right\|^2 \le \frac{1}{2}p_\star^2\end{aligned}$$

Evidently we have

$$(p^2 - \frac{1}{2}p_\star^2)\,\chi_{\Omega_\star^c} \;\ge\; \frac{1}{2}p_\star^2\,\chi_{\Omega_\star^c}\,.$$

Inserting this into the above inequality, we first find $\|\chi_{\Omega_\star^c}\phi^j_{i\beta,\rho}\|^2 \le 1$, and using the same inequality for the second time, we arrive at the estimate

$$\|p\chi_{\Omega_\star^c}\phi^j_{i\beta,\rho}\|^2 \;\le\; p_\star^2\,.$$

The observation that $\left\|\phi^j_{i\beta,\rho}\chi_{\Omega_\star}\right\| = \left\|\phi^j_{i\beta}\chi_{\Omega_\star}\right\| \le 1$ and that $\left\|p\phi^j_{i\beta,\rho}\chi_{\Omega_\star}\right\| \le p_\star$ finishes the proof. $\blacksquare$

6 Concluding the proof of Theorem 2.2

Up to now we have employed the real part of equation (5.4). The imaginary part yields

$$\mathrm{Im}\, E\|\phi^j_{i\beta}\|^2 = (\mathrm{Im}\,(H^j_{i\beta} - B^j_{i\beta}(E))\phi^j_{i\beta}, \phi^j_{i\beta})\;; \tag{6.1}$$

for the moment we do not need the complex boosts. Using the following simple identity,

$$\mathrm{Im}\,(ABA) = 2\mathrm{Re}\,[\,\mathrm{Im}\,(A)BA\,] + A^*\,\mathrm{Im}\,(B)A\,, \tag{6.2}$$

together with resolvent equation, we can express $\mathrm{Im}\, B^j_{i\beta}$, as

$$\mathrm{Im}\, B^j_{i\beta} \;=\; Z_{i\beta} + \mathrm{Im}\, E\,|\widehat{R}^j_{i\beta} W_{i\beta}\mathcal{J}_j|^2,$$

$$Z_{i\beta} \;:=\; \mathcal{J}^*_j\left\{2\mathrm{Re}\,\left[\mathrm{Im}\,(W_{i\beta})\widehat{R}^j_{i\beta}W_{i\beta}\right] - W^*_{i\beta}\widehat{R}^{j\,*}_{i\beta}\mathrm{Im}\,(\widehat{H}^j_{i\beta})\widehat{R}^j_{i\beta}W_{i\beta}\right\}\mathcal{J}_j,$$

where we have already ceased denoting the explicit dependence of the resolvents on E. Inserting this into (6.1) we get

$$\mathrm{Im}\, E\left(\|\phi^j_{i\beta}\|^2 + \|\widehat{R}^j_{i\beta}W_{i\beta}\mathcal{J}_j\phi^j_{i\beta}\|^2\right) \;=\; ((\mathrm{Im}\, H^j_{i\beta} - Z_{i\beta})\phi^j_{i\beta}, \phi^j_{i\beta})\,.$$

Now the equation (5.3) together with $\mathcal{J}_j\mathcal{J}^*_j = I_{L^2(I\!R,dp)}$ yields

$$\|\phi^j_{i\beta}\|^2 + \|\widehat{R}^j_{i\beta}W_{i\beta}\mathcal{J}_j\phi^j_{i\beta}\|^2 \;=\; \|P_j\phi_{i\beta}\|^2_{\mathcal{H}} + \|Q_j\phi_{i\beta}\|^2_{\mathcal{H}} \;=\; \|\phi_{i\beta}\|^2_{\mathcal{H}}\;;$$

hence if the complex-scaled eigenvector $\phi_{i\beta}$ is normalized, equation (6.1) is equivalent to

$$\mathrm{Im}\, E \;=\; ((\mathrm{Im}\, H^j_{i\beta} - Z_{i\beta})\phi^j_{i\beta}, \phi^j_{i\beta})\,. \tag{6.3}$$

The following proposition shows that considering the imaginary part means in a sense a localization of the Dunford-Taylor operators on Ω_e, which is naturally the case for local operators, *i.e.* $p_{i\beta}$. Together with the estimates on the exponential decay of the resonance function, this will yield the sought estimate on $\mathrm{Im}\, E$.

Proposition 6.1 *Assume the conditions of theorem 5.3 and put* $\rho_\star := \rho(\omega)$. *Then there exists a number* c_η *such that*
(i) $\|\, e^{-\rho}\,\mathrm{Im}\, V^0_{i\beta}\, e^{-\rho}\,\| \le c_\eta\, e^{-2\rho_\star}$,

(ii) $\|\,\langle p\rangle^{-1}e^{-\rho}\,\mathrm{Im}\, W_{i\beta}e^{-\rho}\langle p\rangle^{-1}\| \le dc_\eta\, e^{-2\rho_\star}$ *and*

(iii) there is a number C_η *such that* $\|\langle p\rangle^{-1}e^{-\rho}\,(\mathrm{Im}\, H^j_{i\beta} - Z_{i\beta})\,e^{-\rho}\langle p\rangle^{-1}\| \;\le\; C_\eta e^{-2\rho_\star}$.

Proof: Since Σ_{α_0,η_0} and $\Sigma_{\beta,\eta}$ are symmetric with respect to the real axis we can choose the integration path $\partial\mathcal{V}$ in $\Sigma_{\alpha_0,\eta_0}\setminus\Sigma_{\beta,\eta}$ invariant under complex conjugation. Using then the Schwarz reflection principle, it is straightforward to compute for a function f obeying (a1)-(a2)

$$\mathrm{Im}\, f(D_{i\beta}) \;=\; \frac{1}{2i}\,\frac{i}{2\pi}\int_{\partial\mathcal{V}} f(z)\,(r_{i\beta}(z) - r_{-i\beta}(z))\,dz\,.$$

Now again by the norm convergence of the integral we have

$$e^{-\rho}\,\mathrm{Im}\, f(D_{i\beta})\, e^{-\rho} \;=\; \frac{i}{2\pi}\int_{\partial\mathcal{V}} f(z)\, e^{-\rho}\left(\frac{r_{i\beta}(z) - r_{-i\beta}(z)}{2i}\right) e^{-\rho}\,dz\,.$$

Using equation (3.4) with $\alpha = -\beta$ yields, again omitting the argument z for the resolvents, we get

$$e^{-\rho}\left(\frac{r_{i\beta} - r_{-i\beta}}{2i}\right)e^{-\rho}$$
$$= e^{-2\rho_\star}\left(\sin\beta\, z\, r_{-i\beta,-\rho}\chi_{\Omega_e}r_{i\beta,\rho} + \sin\tfrac{\beta}{2}\left(e^{-i\beta/2}\chi_{\Omega_e}r_{i\beta,\rho} + e^{i\beta/2}r_{-i\beta,-\rho}\chi_{\Omega_e}\right)\right)\,.$$

Thus we obtain by Proposition 3.5

$$\left\|e^{-\rho}\left(\frac{r_{i\beta}(z) - r_{-i\beta}(z)}{2i}\right)e^{-\rho}\right\| \le e^{-2\rho_\star}\,|\sin\beta|\,\frac{2\mathrm{dist}(z,\Sigma_{\beta,\eta}) + |z|}{\mathrm{dist}(z,\Sigma_{\beta,\eta})^2}\,. \tag{6.4}$$

This justifies the statement for V^0 as in Proposition 3.7.
For (ii) we have

$$\mathrm{Im}\,\left(p(b-1)p\right)_{i\beta} \;=\; 2\mathrm{Re}\left(\mathrm{Im}\,(p_{i\beta})\,(b-1)_{i\beta}p_{i\beta}\right) + p_{-i\beta}\,\mathrm{Im}\,(b-1)_{i\beta}\,p_{i\beta}$$
$$=\; p\left(2\mathrm{Re}\left(\frac{\mathrm{Im}\,(p_{i\beta})}{p}\,(b-1)_{i\beta}\frac{p_{i\beta}}{p}\right) + \frac{p_{-i\beta}}{p}\,\mathrm{Im}\,(b-1)_{i\beta}\,\frac{p_{i\beta}}{p}\right)p\,,$$

yielding

$$\left\|\langle p\rangle^{-1}e^{-\rho}\,\mathrm{Im}\,\left(p(b-1)p\right)_{i\beta}e^{-\rho}\langle p\rangle^{-1}\right\| \;\le\; e^{-2\rho_\star}\,2\left\|(b-1)_{i\beta,\rho}\right\| + \left\|e^{-\rho}\,\mathrm{Im}\, b_{i\beta}\,e^{-\rho}\right\|$$
$$\le\; c'_\eta\, d\, e^{-2\rho_\star}\,,$$

for some number c'_η. We have used here the fact that the imaginary part of $p_{i\beta}$ is zero on Ω_i and that $b-1 = uf$, with f obeying (a1)–(a2) uniformly for $u\in[0,d]$, $d\in(0,d_0)$. Thus we can apply the calculation used in (i) above. This is also possible for $\mathrm{Im}\,(V - V^0)_{i\beta}$. Now (iii) is easy, since $\mathrm{Im}\, H^j_{i\beta} = \mathrm{Im}\,(p^2_{i\beta} + \mathcal{J}_jW_{i\beta}\mathcal{J}^*_j + V^0_{i\beta})$. Evidently we have

$$\left\|\langle p\rangle^{-1}e^{-\rho}\,\mathrm{Im}\, p^2_{i\beta}\, e^{-\rho}\langle p\rangle^{-1}\right\| \;\le\; e^{-2\rho_\star}$$

The term $e^{-\rho}Z_{i\beta}e^{-\rho}$ is handled by noting that $\mathrm{Im}\,\widehat{H}^j_{i\beta} = Q_j\mathrm{Im}\,\left(p^2_{i\beta} + W_{i\beta} + V^0_{i\beta}\right)Q_j$ and that $\|\langle p\rangle\widehat{R}^j_{i\beta,\rho}W_{i\beta\rho}\langle p\rangle^{-1}\| \le 1$ by Propositions 5.1(ii):

$$\left\|\langle p\rangle^{-1}e^{-\rho}Z_{i\beta}e^{-\rho}\langle p\rangle^{-1}\right\| \le 2\left\|\langle p\rangle^{-1}e^{-\rho}\mathrm{Im}\, W_{i\beta}e^{-\rho}\langle p\rangle^{-1}\right\| + \left\|\langle p\rangle^{-1}e^{-\rho}\mathrm{Im}\, H_{i\beta}e^{-\rho}\langle p\rangle^{-1}\right\|$$
$$\le (3c_\eta d + c_\eta + 1)e^{-2\rho_\star}\,. \quad \blacksquare$$

Returning to $\operatorname{Im} E$ we know by general arguments that it cannot be positive — *cf.* [RS, Sec.XII.6], so equation (6.3), the above estimate, and Theorem 5.3 yield

$$0 \le -\operatorname{Im} E \le C_\eta \, e^{-2\rho_\star} \left\{ \|p\phi^j_{i\beta,\rho}\|^2 + \tau \|\phi^j_{i\beta,\rho}\|^2 \right\} \le \frac{1}{2} C_\eta \, (p_\star^2 + \tau) \, e^{-2\rho_\star} \, . \tag{6.5}$$

The assertion of Theorem 2.2 now follows from the observation that τ and $p_\star$ are bounded as d tends to zero and that

$$\exp\{-2\rho_\star\} = \exp\left\{ -\frac{2\pi\eta}{d} \sqrt{2j-1} \, (1 + \mathcal{O}(\xi d)) \right\} .$$

7 Proof of Theorem 2.3

The proof uses the same ideas as the proof of Theorem 2.2 except that due to the strengthened assumptions on the function γ, we can allow now a boost function ρ with $\|\rho'\|_\infty$ exploiting asymptotically the full width of the analyticity strip, *i.e.* $\|\rho'\|_\infty$ tending to η_p as d approaches zero.

The key to this is the representation of $f(D_{i\beta})$ below when f is a meromorphic function. For the sake of simplicity, assume that f has a single pair of complex conjugated poles in $\Sigma_{\alpha_0,\eta_1} \setminus \Sigma_{\alpha_0,\eta_0}$; an extension to any finite number is straightforward. Let the order of these poles be N; for the proof of Theorem 2.3 we will have to consider several meromorphic functions made out of γ with poles varying in order, not necessarily equal to m. Without loss of generality, we also may suppose that the poles lie on the imaginary axis at $z_p = i\eta_p$ and $\overline{z}_p$. In view of the Schwarz reflection principle, it is sufficient to discuss the behaviour of f around the pole in the upper half-plane and to translate the results by mirror transformation to its counterpart; in particular, the integration contour $\partial\mathcal{V}$ in the Dunford-Taylor integrals will always supposed to be symmetric with respect to the real axis, *i.e.* of the form $\partial\mathcal{V} := \mathcal{K} \cup \overline{\mathcal{K}}$ with a suitable upper branch $\mathcal{K}$. By assumption, f can be expanded into its singular and regular part in a pierced neighbourhood of z_p,

$$f(p) = \sum_{k=1}^{N} \frac{f_{-k}}{(p - z_p)^k} + f_{reg}(p) \, , \qquad 0 < |p - z_p| < \varepsilon \, .$$

for some $\varepsilon > 0$. Let $\mathcal{K}$ now be passing above the pole z_p, but lying entirely inside Σ_{α_0,η_1} Then the residue theorem yields the following

Proposition 7.1 *Let f obey the same requirements as γ in (a1)–(a2) and let f and $\partial\mathcal{V}$ be as above. Then*

$$f(D_{i\beta}) = \sum_{k=1}^{N} \left(f_{-k} \, (D_{i\beta} - z_p)^{-k} + \overline{f}_{-k} \, (D_{i\beta} - \overline{z}_p)^{-k} \right) + \frac{i}{2\pi} \int_{\partial\mathcal{V}} f(z) (D_{i\beta} - z)^{-1} dz \, .$$

This proposition together with Proposition 3.5 yields immediately a bound on the boosted operator:

$$\|f(D_{i\beta,\rho})\| \le \frac{C}{(\eta_p - \|\rho'\|_\infty)^N} \, ; \tag{7.1}$$

note that the integral part can be uniformly bounded, since the integration path $\mathcal{K}$ can be kept at a finite distance, independent of d, from the horizontal $z = i\|\rho'\|_\infty$, so that this formula holds for all $\|\rho'\|_\infty < \eta_p$ with an appropriate constant C. In view of the basic decomposition (2.6) we thus have to investigate how we can apply this formula to b–1, V–V^0, and V^0 and how this conditions the maximal $\|\rho'\|_\infty$ to be chosen. For this recall that we can interpret b and b–1 as a simple rational function of $u\gamma$. Choosing $\|\rho'\|_\infty = \eta_p - d^{1/(m+1)}$ implies $\|u\gamma_{i\beta,\rho}\| = \mathcal{O}(d^{1/(m+1)})$ and $\left\|V^0_{i\beta,\rho}\right\| = \mathcal{O}(d^{-2m/m+1})$, the order of the pole of γ being m. Since $b-1 = -u\gamma(2+u\gamma)b$ and $V - V^0 = V^0(b-1) + \frac{1}{2}u\gamma'' b^{3/2} - \frac{5}{4}u^2\gamma'^2 b^2$, we obtain the bounds applying the above proposition and inequality (7.1) to the various powers and powers of derivatives of γ observing that $\|b_{i\beta,\rho}\|$ is uniformly bounded.

But before stating all the necessary bounds in a proposition let us be more precise about the choice of ρ. It shall be defined by formula (5.8) with η replaced by $\eta_p - d^{1/(m+1)}$ and $p_\star$ by $p'_\star d^{-m/m+1}$ where $p'_\star$ is a quantity uniformly bounded with respect to d to be fixed later. Note that $\rho_\star$ still denotes $\rho(\omega)$. We also have to be precise concerning the weight $\langle p \rangle$:

$$\langle p \rangle^2 := p^2 + \tau, \quad \tau := c_\gamma d^{-2m/m+1} \, .$$

Recall that τ had been chosen to be a uniform bound on $\left\|V^0_{i\beta,\rho}\right\|$. As is confirmed in the next proposition this is again the case. All previous estimates involving τ used only this property and remain thus valid. Notice that $\left\|V^0_{i\beta,\rho}\right\|$ does not depend on $p'_\star$, *cf.* (7.1).

Proposition 7.2 *With the definitions above and for d small enough*
(i) there exist numbers c_γ and c_b such that

$$\sup_{0 \le -\beta \le \beta_1} \left\|V^0_{i\beta,\rho}\right\| \le c_\gamma d^{\frac{-2m}{m+1}} \qquad \textit{and} \qquad \|\langle p \rangle^{-1} W_{i\beta,\rho} \langle p \rangle^{-1}\| \le c_b d^{\frac{1}{m+1}} \, .$$

(ii) There exists a constant c such that

$$\left\| e^{-\rho} \operatorname{Im} V^0_{i\beta} \, e^{-\rho} \right\| \le c \, d^{-\frac{2m+1}{m+1}} \, e^{-2\rho_\star} \qquad \textit{and} \qquad \|\langle p \rangle^{-1} e^{-\rho} \operatorname{Im} W_{i\beta} \, e^{-\rho} \langle p \rangle^{-1}\| \le c \, e^{-2\rho_\star} \, .$$

Proof: (i) The first statement is clear, the second statement is obtained as in Proposition 3.9. We have here

$$c_b = \max_{0 \le d \le d_0} \left\{ d^{-\frac{1}{m+1}} \left(\|(b-1)_{i\beta,\rho}\| + \tau^{-1} \|(V - V^0)_{i\beta,\rho}\| \right) \right\} , \tau^{-1} = \frac{d^{\frac{2m}{m+1}}}{c_\gamma} \, .$$

By the above discussion it is easy to see that c_b is uniformly bounded in d. For (ii) we can use the same algebra as in Proposition 6.1(i)–(ii). It remains only to prove the proper localization of the residual parts. We have, switching back to f as in Proposition 7.1 and using the notation of Proposition 3.7, for some $1 \le k \le N$

$$\operatorname{Im}\,\Big(f_{-k}\, r_{i\beta}(z_p)^k + \overline{f_{-k}}\, r_{i\beta}(\overline{z_p})^k\Big) \;=$$

$$= \operatorname{Re} f_{-k} \operatorname{Im}\Big(r_{i\beta}(z_p)^k + r_{-i\beta}(z_p)^{k*}\Big) + \operatorname{Im} f_{-k} \operatorname{Re}\Big(r_{i\beta}(z_p)^k - r_{-i\beta}(z_p)^{k*}\Big)$$

$$= \operatorname{Re} f_{-k} \operatorname{Im}\Big(r_{i\beta}(z_p)^k - r_{-i\beta}(z_p)^k\Big) + \operatorname{Im} f_{-k} \operatorname{Re}\Big(r_{i\beta}(z_p)^k - r_{-i\beta}(z_p)^k\Big).$$

The trivial identity $A^k - B^k = (A-B)A^{k-1} + B(A^{k-1} - B^{k-1})$ implies $e^{-\rho}(A^k - B^k)e^{-\rho} = \sum_{\ell=0}^{k-1} B_\rho^\ell e^{-\rho}(A-B)e^{-\rho}A_\rho^{k-1-\ell}$. We obtain by Proposition 3.5

$$\begin{aligned}\Big\| e^{-\rho}\Big(r_{i\beta}(z_p)^k - r_{-i\beta}(z_p)^k\Big)e^{-\rho}\Big\| \;&\le\; k\, d^{-\frac{k-1}{m+1}} \Big\| e^{-\rho}\Big(r_{i\beta}(z_p) - r_{-i\beta}(z_p)\Big)e^{-\rho}\Big\| \\ &\le\; c'\, N\, d^{-\frac{N+1}{m+1}} e^{-2\rho_\star};\end{aligned}$$

for the second inequality use (6.4) and majorize k by N. We explicitly have $c' := |\sin\beta|(\eta_p + 2d^{1/(m+1)})$. Taking the appropriate N for each of the functions concerned yields the result. ∎

Recall that the crucial equations to be justified are (5.6) and (6.3) which means the justification of the existence of $B^j_{i\beta,\rho}(E)$ and thus of $\widehat{R}^j_{i\beta,\rho}(E)$. Of course we still would like to use the resolvent equation (5.5) of Proposition 5.1, so we need $\|\langle p\rangle^{-1}\widehat{W}_{i\beta,\rho}\widehat{R}^{0,j}_{i\beta,\rho}(E)\langle p\rangle\| < 1$, which is impossible unless we make ω smaller. We modify (4.3) by choosing

$$\omega \;:=\; \frac{\pi}{d}\sqrt{(2j-1)\,(1-\xi\, d^{1/(m+1)})}\,. \tag{7.2}$$

We follow the proof of Lemma 4.2 in Appendix B. Applying Proposition B.1 with $\kappa = \xi\, d^{1/(m+1)}$ we get:

$$\forall\, k \ne j \quad \|V^0_{i\beta,\rho} R^{00,k}_{i\beta}(z)\| \;\le\; \frac{c^{(1)}\tau}{2|\sin\beta|}\frac{d^2}{\xi d^{1/(m+1)}} \;=\; \frac{c^{(1)}c_\gamma}{2|\sin\beta|}\frac{d^{1/(m+1)}}{\xi}\,.$$

Thus for d small enough choosing $1 \ge \xi\, d^{1/(m+1)} \ge c^{(1)}c_\gamma|\sin\beta|^{-1} d^{2/(m+1)}$ implies $\|V^0_{i\beta,\rho} R^{00,k}_{i\beta}(z)\| \le 1/2$ and we obtain by the resolvent identity (B.4) and (B.5)

$$\Big\|\,\langle p\rangle \widehat{R}^{0,j}_{i\beta,\rho}(E)\langle p\rangle\Big\| \;\le\; \frac{c^{(1)}}{|\sin\beta|\xi}\, d^{-1/(m+1)}\,. \tag{7.3}$$

Proceeding as in Proposition 5.1 we need that $\|\langle p\rangle^{-1}\widehat{W}_{i\beta,\rho}\widehat{R}^{0,j}_{i\beta,\rho}(E)\langle p\rangle\| \le 1/2$, which is the case if $\xi \ge 2c^{(1)}c_b|\sin\beta|^{-1}$. Consequently

$$\Big\|\,\langle p\rangle \widehat{R}^{j}_{i\beta,\rho}\langle p\rangle\,\Big\| \;\le\; \frac{2c^{(1)}}{|\sin\beta|\xi}\, d^{-1/(m+1)} \qquad\text{and}\qquad \|\langle p\rangle^{-1} B^j_{i\beta,\rho}(E)\langle p\rangle^{-1}\| \;\le\; c_b\, d^{1/(m+1)}\,.$$

The inequality of Proposition 5.2 can be formulated now as

$$\operatorname{Re}\,\Big(H^j_{i\beta,\rho} - B^j_{i\beta,\rho}(E) - E\Big) \;\ge\; \Big(\cos 2\beta - 2c_b\, d^{1/(m+1)}\Big)\Big(p^2 - \frac{{p'_\star}^2}{2} d^{-2m/(m+1)}\Big)$$

with the new, but nevertheless uniformly bounded (in d)

$$\frac{{p'_\star}^2}{2} \;:=\; \frac{c_\gamma + 2c_b\, c_\gamma d^{1/(m+1)} + d^{2m/(m+1)}|\operatorname{Re} E - E_j|}{\cos 2\beta - 2c_b\, d^{1/(m+1)}}\,;$$

the inequality is, of course, to be interpreted in the form sense on $\mathcal{D}(p^2 \otimes I)$. Thus for all sufficiently small d the formula (5.9) remains valid with $p_\star = p'_\star d^{-m/(m+1)}$ when η is changed to $\eta_p - d^{1/(m+1)}$ in the definition (5.8) of ρ.

Since also the algebra used for Proposition 6.1(iii) can be applied without change, we just need to substitute corresponding constants to arrive at the following inequality replacing (6.5)

$$0 \;\ge\; \operatorname{Im} E \;\ge\; -C\,({p'}^2_\star + c_\gamma)\, d^{-4+3/(m+1)}\, e^{-2\rho_\star}\,,$$

for some constant C. To conclude the proof, it remains to expand $\rho_\star$:

$$\begin{aligned}\rho_\star \;&=\; (\eta_p - d^{1/(m+1)})\Big(\frac{\pi}{d}\sqrt{(2j-1)\,(1-\xi\, d^{1/(m+1)})} \;-\; p'_\star d^{-m/(m+1)}\Big) \\ &=\; \frac{\pi\eta_p}{d}\sqrt{2j-1}\,\Big(1 + \mathcal{O}(d^{1/(m+1)})\Big)\,,\end{aligned}$$

and to notice that negative powers of d in the prefactor have no significance and can be absorbed in the error term of the exponential decay rate.

A Proof of Lemma 3.4

Proof of Lemma 3.4: (i) By hypothesis $(T-z)^{-1}$ is bounded on the integration path and f decays rapidly enough to make the integral converge in operator norm. Furthermore, the integral does not depend on the path, since both the resolvent of T as a function of z and f are analytic in the considered region. (ii) Since $f(T)$ is bounded, it suffices to show that $(f(T)u,v) = (f_{sp}(T)u,v)$ holds for all $u,v \in L^2(I\!R)$, where $f_{sp}(T)$ denotes the operator defined by the spectral theorem. One has

$$\begin{aligned}(f_{sp}(T)u,v) \;&=\; \int_{I\!R} f(\lambda)\, d(E_\lambda u,v) \\ &=\; \int_{I\!R} d(E_\lambda u,v)\frac{i}{2\pi}\int_{\partial\mathcal{V}} \frac{f(z)}{\lambda - z}\, dz \\ &=\; \frac{i}{2\pi}\int_{\partial\mathcal{V}} dz \int_{I\!R}\frac{f(z)}{\lambda - z}\, d(E_\lambda u,v) \\ &=\; \frac{i}{2\pi}\int_{\partial\mathcal{V}} f(z)((T-z)^{-1}u,v)\, dz \;=\; (f(T)u,v)\,,\end{aligned}$$

where in the third and the last step we have employed the Fubini theorem.
(iii) It follows from the norm convergence of the integral that

$$B\,\frac{i}{2\pi}\int_{\partial\mathcal{V}} f(z)(T-z)^{-1}\,dz\,B^{-1} \;=\; \frac{i}{2\pi}\int_{\partial\mathcal{V}} f(z)\,B(T-z)^{-1}B^{-1}\,dz \;=\; f(BTB^{-1})\,.$$

(iv) The operators U_θ are unitary for $\theta \in I\!R$, and therefore $\big(f(T)\big)_\theta = f(T_\theta)$ by (iii). Furthermore, the resolvent $(T_\theta - z)^{-1}$ is by hypothesis uniformly bounded on $\partial\mathcal{V}$ for all $\theta \in \mathcal{S}_\alpha$, and analytic in θ. Thus the analyticity follows by the convergence in operator norm of the integral, since the limit function of a uniformly convergent sequence of analytic functions is analytic (see *e.g.* [Di, Thm. 9.12.1]). ∎

B Proof of Lemma 4.2

We need the following

Proposition B.1 *Let* $R^{00,k}_{i\beta}(z) := (p^2_{i\beta} + E_k - z)^{-1}$, *where* $0 < |\beta| \le \min\{\frac{\pi}{4}, \alpha\}$. *Let* $\omega \ge 0$ *be defined by* $\omega^2 = (1-\kappa)(E_j - E_{j-1})$, *where* $\kappa \in (0,1]$ *and* $c^{(1)} := 8\sqrt{3}$. *Then for all* (κ, d) *such that*

$$1 \ge \kappa \ge \frac{\tau}{\pi^2|\sin\beta|}\,d^2\,, \tag{B.1}$$

all $z \in \mathcal{D}_\Gamma$ *and* $\ell = 0,1,2$ *one has*

(i) $\|\langle p\rangle^\ell R^{00,j-1}_{i\beta}(z)\| \le \dfrac{c^{(1)}}{2|\sin\beta|\kappa}\,d^{2-\ell}$ *and*

(ii) $\|\langle p\rangle^\ell R^{00,k}_{i\beta}(z)\| \le \dfrac{c^{(1)}}{2|\sin\beta|}\,d^{2-\ell} \quad \forall\, k \ne j,\ j-1\,.$

Proof : We first estimate $R^{00,k}_{i\beta}(z)$, $k \ne j$. Define $\zeta := z - E_j$ and $\Delta_{j,k} := E_j - E_k$. If

$$-\frac{1}{2}\kappa\Delta_{j,j-1} \le \mathrm{Re}\,\zeta \le 0 \quad \text{and} \quad \mathrm{Im}\,e^{-i\beta}\zeta \ge \frac{1}{2}\kappa\Delta_{j,j-1}\sin\beta, \tag{B.2}$$

then one obtains by simple geometric considerations

$$\left\|R^{00,k}_{i\beta}(z)\right\| \le \frac{2}{|\sin\beta|\Delta_{j,k}},\ k < j-1, \qquad \left\|R^{00,k}_{i\beta}(z)\right\| \le \frac{1}{\Delta_{k,j}},\ k > j,$$

and

$$\left\|R^{00,j-1}_{i\beta}(z)\right\| \le \frac{2}{|\sin\beta|\kappa\Delta_{j,j-1}}.$$

The condition $\kappa \ge (\pi^2|\sin\beta|)^{-1}\|V^0\|d^2$ is sufficient to ensure that $\mathcal{D}_\Gamma$ is for all $j \ge 2$ contained in the domain described by (B.2) and we have, of course, $\|V^0\| \le \tau$. Thus the case $\ell = 0$ is proven noticing that $|\Delta_{k,j}|^{-1}, k \ne j$, is uniformly bounded by $(3\pi^2)^{-1}d^2$.

To treat the case $\ell = 2$ we write

$$\langle p\rangle R^{00,k}_{i\beta}(z)\langle p\rangle = \frac{(p^2+\tau)}{(p^2_{i\beta}+\tau)}\Big(1 + (\tau - E_k + z)R^{00,k}_{i\beta}(z)\Big)\,.$$

The first factor is uniformly bounded by $\sqrt{3}$ for $|\beta| \le \frac{\pi}{4}$. Again simple geometric considerations suffice to bound the term $|E_k - z|\|R^{00,k}_{i\beta}(z)\|$. Note that the overall constant is made d independent by condition (B.1).

The remaining case $\ell = 1$ is handled by the inequality

$$\left\|\langle p\rangle R^{00,k}_{i\beta}(z)\right\| \le \left\|R^{00,k}_{i\beta}(z)\right\|^{1/2}\left\|\langle p\rangle^2 R^{00,k}_{i\beta}(z)\right\|^{1/2}. \;\blacksquare$$

Proof of Lemma 4.2: For $\ell = 0$ observe that by replacing κ with ξd one has

$$\|V^0_{i\beta,\rho}R^{00,k}_{i\beta}(z)\| \le \|V^0_{i\beta,\rho}\|\,\|R^{00,k}_{i\beta}(z)\| \le \frac{c^{(1)}\tau}{2|\sin\beta|}\frac{d}{\xi} \le \frac{1}{2} \tag{B.3}$$

by the condition on ξ, uniformly for all $z \in \mathcal{D}_\Gamma$ and $k \ne j$, so that by the above estimates the bounds follow immediately in this case. To prove the estimates in the case $\ell = 1$ we use the resolvent identity

$$\langle p\rangle R^{0,k}_{i\beta,\rho}\langle p\rangle = \langle p\rangle R^{00,k}_{i\beta}\langle p\rangle + \langle p\rangle R^{00,k}_{i\beta}V^0_{i\beta,\rho}\Big(1 + R^{00,k}_{i\beta}V^0_{i\beta,\rho}\Big)^{-1}R^{00,k}_{i\beta}\langle p\rangle\,. \tag{B.4}$$

This yields

$$\begin{aligned}\left\|\langle p\rangle R^{0,k}_{i\beta,\rho}(z)\langle p\rangle\right\| &\le \left\|\langle p\rangle^2 R^{00,k}_{i\beta}(z)\right\| + \left\|\langle p\rangle R^{00,k}_{i\beta}(z)\right\|^2 \frac{\|V^0_{i\beta,\rho}\|}{1 - \|V^0_{i\beta,\rho}\|\,\|R^{00,k}_{i\beta}(z)\|}\\ &\le \left\|\langle p\rangle^2 R^{00,k}_{i\beta}(z)\right\| + \left\|\langle p\rangle^2 R^{00,k}_{i\beta}(z)\right\| \frac{\|V^0_{i\beta,\rho}\|\,\|R^{00,k}_{i\beta}(z)\|}{1 - \|V^0_{i\beta,\rho}\|\,\|R^{00,k}_{i\beta}(z)\|}\\ &\le 2\left\|\langle p\rangle^2 R^{00,k}_{i\beta,\rho}(z)\right\| \end{aligned} \tag{B.5}$$

using in the last step (B.3), and in the second to last step the fact that $R^{00,k}_{i\beta}(z)$ is a multiplication operator. So the bounds follow again easily. ∎

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