Abnormal escape rates from nonuniformly hyperbolic sets

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Abstract. Consider a $C^{1+\epsilon}$ diffeomorphism f having a uniformly hyperbolic compact invariant set Ω, maximal invariant in some small neighbourhood of itself. The asymptotic exponential rate of escape from any small enough neighbourhood of Ω is given by the topological pressure of $-\log |J^u f|$ on Ω (Bowen–Ruelle [1975]). It has been conjectured (Eckmann–Ruelle [1985]) that this property, formulated in terms of escape from the support Ω of a (generalized SRB) measure, using its entropy and positive Lyapunov exponents, holds more generally. We present a simple C^∞ two-dimensional counterexample, constructed by a surgery using a Bowen-type hyperbolic saddle attractor as the basic plug.

1. Introduction

In the rigorous theory of dynamical systems, a great deal of effort has been devoted to the study of strange attractors. Understanding chaotic transitive attractors (in a measure-theoretical sense) allows us to statistically predict the asymptotic fate of "many" initial conditions. A widely shared feeling is that transient behaviour (i.e., what happens before the system settles to equilibrium) is in some sense irrelevant.

Although this point of view is very much justified for a variety of reasons, the study of non-attracting invariant sets also deserves our attention, if only because the time needed for a system to settle to equilibrium can be very long, much longer than what is observable in real life, laboratory experiments, or computer simulations (see Kantz–Grassberger [1985], Crutchfield–Kaneko [1988], and references therein). As a consequence, the dynamics which is actually observed at realistic time-scales can merely be a succession of more or less chaotic transients. Another motivation is the desire to rigorously describe diffusive phenomena, such as scattering or leaking (particles escaping from a "box"), which are by nature nonequilibrium processes (see e.g. the survey by Gaspard and Dorfman [1995], or the more general monograph by Cvitanović et al.

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[1997]). Finally (and in the same direction), the recent renewal of interest for nonequilibrium statistical mechanics (Gallavotti-Cohen [1995], Ruelle [1996a, 1996b]) has created among mathematical physicists a demand for mathematical results in the ergodic theory of dynamical systems, specifically in a non-attracting situation: The dearth of supply, when uniform hyperbolicity is not assumed, has become quite apparent.

In this article we concentrate on a specific conjecture of Eckmann and Ruelle [1985] that we describe now. We first recall a result from the uniformly hyperbolic setting. Let f be a smooth (say $C^{1+\epsilon}$) Axiom A diffeomorphism on a compact manifold (see e.g. Bowen [1975] for definitions), and consider a topologically transitive basic piece Ω for f. Then, writing h_ν for the Kolmogorov entropy of an ergodic f-invariant measure ν (all our measures are Borel) and $\chi^+(\nu)$ for the sum of the nonnegative Lyapunov exponents $\lambda_i(\nu)$ of ν (see Walters [1982] for definitions), it is known that there is a unique measure μ which realizes the maximum

$$P = \sup\{h_\nu - \chi^+(\nu) \mid \nu \text{ ergodic } f-\text{invariant probability measure, supp}\,(\nu) \subset \Omega\}\,.$$

This maximum $P \le 0$ is the topological pressure of $-\log J^u(f)$ where $J^u f$ is the Jacobian of f restricted to unstable manifolds; moreover P vanishes if and only if Ω is an attractor, and if and only if μ is absolutely continuous with respect to Lebesgue along unstable foliations (we refer again to Bowen [1975]). When $P = 0$ the measure μ is called an SRB measure for $f|_\Omega$. When $P < 0$, the measure μ, which is related to a conditionally invariant measure which is absolutely continuous with respect to Lebesgue along unstable foliations (see e.g. Ruelle [1996a]), can be called a *generalized SRB* measure for $f|_\Omega$, as suggested by Ruelle [1996b].

The property we are concerned with is the escape rate from neighbourhoods of Ω. In the Axiom A case, it has been proved (Bowen-Ruelle [1975], see Ruelle [1996a, Proposition 3.1] for an explicit statement) that for any small enough open neighbourhood U of Ω, the Lebesgue volume of the set U_m of points x such that $f^k(x) \in U$ for all $0 \le k \le m$ satisfies:

$$\lim_{m\to\infty} \frac{1}{m} \log \mathrm{Vol}\, U_m = P\,. \tag{ER}$$

(We call $E(U) = \lim_{m\to\infty} \frac{1}{m} \log \mathrm{Vol}\, U_m$ the *escape rate* from U.)

Numerical evidence and heuristic arguments (Kantz–Grassberger [1985], Grebogi et al. [1983], Kadanoff–Tang [1984]) indicate that the escape rate property (ER) holds in a more general setting and have prompted the following conjecture:

Conjecture (Eckmann and Ruelle [1985]). *Write $P_\nu = h_\nu - \chi^+(\nu)$ for ν an ergodic f-invariant probability measure. Assume that there exists an ergodic f-invariant probability μ such that*

$$P_\mu \ge P_\nu$$

for all ergodic f-invariant probability measures ν with support contained in the support Ω of μ. Then P_μ is the escape rate $E(U)$ from any sufficiently small open neighbourhood U of Ω.

We now list existing partial results. In one-dimension, Contreras-Barandiarán [1990] proved the Eckmann–Ruelle conjecture in the special case of escape from Julia sets of certain rational maps (with preperiodic critical points), and for certain C^∞ endomorphisms of the interval.

In arbitrary finite dimension, Young [1990] has proved the *lower bound*

$$E(U) \geq \sup\{h_\nu - \chi^+(\nu) \mid \nu \text{ ergodic } f-\text{invariant probability measure, supp}\,(\nu) \subset K\}$$

where K is the compact maximal f-invariant set in an open set U. She assumes that f is a C^2 diffeomorphism but does not require K to be the support of any equilibrium measure. Young [1990] also proves the *upper bound* $E(U) \leq \sup\{h_\nu - \chi^+(\nu)\}$ but only in the case where f is a *uniformly partially hyperbolic diffeomorphism.* (This is a relatively strong assumption, which is in particular not satisfied by the persistently transitive examples constructed recently by Bonatti–Díaz [1996], following the footsteps of Shub [1971] and Mañé [1978].)

The Eckmann–Ruelle conjecture has also been proved for a class of billiards by Lopes and Markarian [1996]. More recently, Chernov et al. [1996] (see also references therein) have studied Anosov maps with suitable small "holes" (the points mapped into the holes never return), and constructed the generalized SRB measures and the corresponding conditionally invariant measure, showing in particular the conjectured escape rate formula. (Concerning related literature on conditionally invariant measures, we mention the pioneering work of Pianigiani and Yorke [1979], and, more recently, the article of Collet et al. [1996] who consider stochastic perturbations of certain expanding maps of the interval where the noise can produce leaks outside of the interval.)

To summarize, the lower bound for $E(U)$ in the conjecture has been proved (Young [1990]), but the upper bound is known only for a relatively restricted class of examples. Eckmann and Ruelle [1985, p. 644] had pointed out that "it is unknown how generally the conjecture holds." Candidates for reasonably general sufficient conditions ensuring the upper bound still do not seem to exist.

The main result of this paper (Theorem 2 in Section 2) says that there exists a C^∞ diffeomorphism on a surface with an ergodic invariant measure μ which is a unique generalized SRB measure on its support, with strictly negative P_μ, but such that the escape rate from arbitrarily small isolating neighbourhoods of this support is zero. This is to our knowledge the first counterexample to the Eckmann–Ruelle conjecture. We point that our counterexample is smooth, low-dimensional, and such that the compact maximal invariant set considered is the support of a *unique* "natural" (i.e., generalized SRB) measure μ, which does not have zero Lyapunov exponents. At the end of the paper we mention a variant of our counterexample with non zero escape rate.

We very briefly describe the construction. The basic idea is to insert by smooth surgery a Bowen "eye-like" attractor (composed of the two separatrices, or saddle connections, joining two hyperbolic saddle fixed-points, see Figure 1) in a uniformly hyperbolic quasi-attractor $F_0|_{\Omega_0}$. The Bowen example itself has many abnormal features, in particular the Birkhoff sums for initial points in a neighbourhood of the two separatrices do not converge (see Takens [1994], Takahashi [1984, pp. 457–458]). However, the maximal invariant set formed by the two saddle connections is not the support of any invariant measure, so that the Bowen example per se does not contradict the conjecture. To perform the smooth surgery, we first embed the Bowen example as part of a slightly more sophisticated plug satisfying the properties stated in Lemma 1 in Section 2. (The actual construction of this *local model* is done in Section 3.) In Section 2, we also explain how to replace a well-chosen hyperbolic saddle fixed point of $F_0|_{\Omega_0}$ by the plug, in such a way as to obtain a counterexample to the Eckmann–Ruelle conjecture.

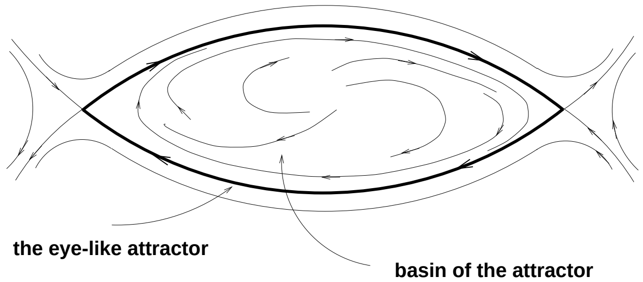

Figure 1 The Bowen example

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2. A counterexample to the Eckmann-Ruelle conjecture

2.A. The local model.

Our counterexample is obtained by a simple surgery, where a saddle of a uniformly hyperbolic repellor is replaced by a nonuniformly hyperbolic set with zero escape rate (see Figure 2). In Lemma 1 we state the properties of the nonuniformly hyperbolic plug that we shall need in our construction.

For $R \geq 0$, we write $B_R \subset \mathbb{R}^2$ for the ball of radius R centered at 0.

Lemma 1. *For fixed $\sigma > 1$ and $0 < \lambda < 1$, let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map $\begin{pmatrix} \lambda & 0 \\ 0 & \sigma \end{pmatrix}$. There exist a C^∞ diffeomorphism $f_A : \mathbb{R}^2 \to \mathbb{R}^2$, an f_A-invariant compact set $K_A \subset B_1$, and an open neighbourhood $U \subset B_1$ of ∂K_A with the following properties:*

1. *$f_A|_{\mathbb{R}^2 \setminus K_A}$ is conjugated to $A|_{\mathbb{R}^2 \setminus \{0\}}$ by a C^∞ diffeomorphism $h_A : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 - K_A$, with h_A the identity map outside of B_1;*
2. *∂K_A is the maximal f_A-invariant set in U, i.e., $\cap_{n \in \mathbb{Z}} f_A^n(\overline{U}) = \partial K_A$;*
3. *there is a finite set $\{q_1, \dots, q_N\} \subset \partial K_A$ of hyperbolic fixed saddles for f_A, with the unstable eigenvalue of each q_i satisfying $\sigma(q_i) > \sigma$, and such that the only ergodic f_A-invariant measures supported in U are Dirac masses at one of the q_i;*
4. *the Lebesgue volume of $\cap_{n \in \mathbb{N}} f_A^{-n}(U)$ is strictly positive.*

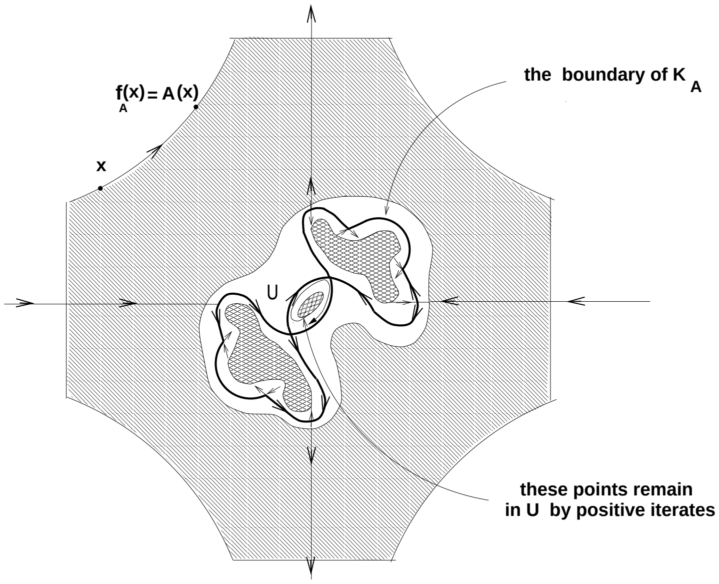

Figure 2 A plug with zero escape rate from the (white) isolating neighbourhood U (Lemma 1)

The proof of Lemma 1 is postponed to Section 3, and we move on to the actual construction of the counterexample.

2.B The global model.

If $F : M \to M$ is a transformation of a Riemann manifold M and $U \subset M$ is an open set, we set for any $m \geq 0$

$$U_m := \{x \in U \mid F^k(x) \in U\,, 0 \leq k \leq m\}\,, \tag{2.1}$$

and, writing Vol for Lebesgue volume on M, we define the *escape rate* from U with $\mathrm{Vol}\, U < \infty$ by

$$E(U) := \limsup_{m\to\infty} \frac{1}{m} \log \mathrm{Vol}\, U_m\,. \tag{2.2}$$

Assume that F is differentiable. For an F-invariant Borel probability measure ν on M we note

$$P_\nu = h_\nu - \int \chi^+(x)\, d\nu\,, \tag{2.3}$$

where $\chi^+(x)$ is the sum of the positive Lyapunov exponents of a regular point x (if ν is ergodic, $\chi^+ \equiv \chi^+(\nu)$ is ν-almost everywhere constant) and h_ν is the Kolmogorov entropy of ν. (We refer to Walters [1982] for a definition of Lyapunov exponents, a statement of the Oseledec theorem, as well as more on measure-theoretical entropy. We just recall that a point is called regular if its Lyapunov exponents are well-defined, and that the Oseledec theorem guarantees that the set of regular points has measure 1 for any invariant measure ν.) We say that an ergodic F-invariant probability measure μ with support equal to $\Omega \subset M$ is a *generalized SRB measure* (Ruelle [1996b]) in Ω if it realizes the following variational principle

$$P_\mu = \max\{h_\nu - \chi^+(\nu) \mid \nu \text{ ergodic } F - \text{invariant probability measure, } \mathrm{supp}\,(\nu) \subset \Omega\}\,. \tag{2.4}$$

When $P_\mu = 0$ the generalized SRB measure is an SRB measure in the usual sense (see e.g. Eckmann–Ruelle [1985]). We say that an ergodic F-invariant probability measure is *hyperbolic* if all its Lyapunov exponents are nonzero.

We can now formally state our main result:

Theorem 2. *On any surface S there exists a C^∞ diffeomorphism F and an ergodic hyperbolic F-invariant probability measure μ, such that μ is the unique generalized SRB measure in $\Omega = \operatorname{supp}\mu$, and $P_\mu < 0$ but for any open neighbourhood W of Ω, the escape rate $E(W)$ from W is zero. Furthermore, if the neighbourhood W is sufficiently small, then $\Omega = \operatorname{supp}\mu$ is the maximal F-invariant subset of F in W.*

Remark 1.

(1) It follows from the proof of Theorem 2 that there is a neighbourhood W of $\operatorname{supp}\mu$ such that any ergodic F-invariant measure ν with support included in W is a hyperbolic measure. The counterexample is thus hyperbolic in a measure-theoretical sense.

(2) It can also be easily seen from the proof of Theorem 2 that the diffeomorphism F^{-1} has a unique SRB measure (in the usual sense of the word). In the notation used below this measure is just the pull-back of the usual SRB measure of the uniformly hyperbolic attractor F_0^{-1} on Ω_0, where $F_0|_{\Omega_0}$ is the repellor starting point of the construction.

(3) The very slightly modified construction described in Remark 2 of Section 3 (where the eye-like component of the plug is repelling) has the property that the corresponding inverse map F^{-1} admits a unique SRB measure, the basin of which is a strict subset of the topological basin.

We now start the construction giving Theorem 2.

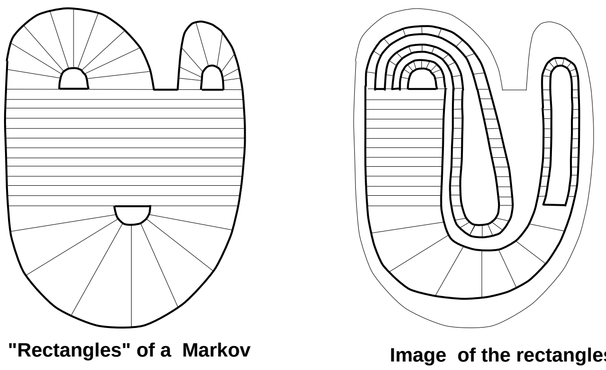

Figure 3 A Plykin attractor

Let S be a surface and let $F_0 : S \to S$ be a C^∞ diffeomorphism with a hyperbolic repellor Ω_0. Hyperbolicity means that there is a constant $0 < \lambda < 1$ such that the tangent bundle over Ω_0 decomposes as a Whitney sum of TF_0-invariant subbundles $T_{\Omega_0} = E^u \oplus E^s$ with $\|TF_0(x)|_{E^s}\| \le \lambda$ and $\|TF_0^{-1}(x)|_{E^u}\| \le \lambda$ for all $x \in \Omega_0$. The repelling property means that there is an open neighbourhood $V \subset S$ of Ω_0 with

$F_0^{-1}(\overline{V}) \subset V$ and such that $\Omega_0 = \cap_{n\in\mathbb{N}} F_0^{-n}(\overline{V})$. (This implies in particular that $\Omega_0 = \cup_{x\in\Omega_0} W^s(x)$, where $W^s(x) = \{y \in S \mid \lim_{m\to\infty} d(F_0^m(x), F_0^m(y)) = 0\}$ is the stable manifold of $x \in S$.) We refer e.g. to Bowen [1975], Newhouse's section in the book Guckenheimer et al. [1980], and Shub [1987] for the theory of hyperbolic sets. Assume further that Ω_0 is nontrivial and F_0 is topologically transitive on Ω_0, i.e., that F_0 has a nonperiodic dense orbit in Ω_0. Finally, we shall work with an open neighbourhood V of Ω_0 which is small enough so that Ω_0 is the maximal invariant set in V (as in Lemma 1 (2)). A specific example of hyperbolic repellor would be the inverse of the Plykin attractor described e.g. in Arnold [1980, pp. 262–263] or Devaney [1987, pp. 205–208], see Figure 3.

We say that a periodic point p of $F_0|_{\Omega_0}$ is *s-lateral* if there is a connected subset C of $W^u(p)$, the closure of which contains $\{p\}$ as a strict subset, and which does not intersect $\cup_{x\in\Omega_0} W^s(x)$ (the definition makes sense for a general uniformly hyperbolic set, note that in our repellor example we have $\cup_{x\in\Omega_0} W^s(x) = \Omega_0$). When choosing which saddle point of the repellor to blow up in our surgery, we want to avoid such "boundary" points because their local stable manifolds are not accumulated on both sides by local stable manifolds of points in Ω_0: Newhouse and Palis [1973] have shown that, except for a finite number of exceptions, the periodic points of a hyperbolic repellor are *not* s-lateral (see also Bonatti et al. [1996, Proposition 3.3.1] for a detailed statement). We may thus pick such a non s-lateral periodic point $p_0 \in \Omega_0$. Replacing F_0 by the corresponding iterate if necessary, we reduce to the case where p_0 is a fixed point. We *assume* further that F_0 is C^∞ linearizable in a neighbourhood $O_{p_0} \subset V$ of p_0 (this is the case if the eigenvalues $\sigma(p_0) > 1$, $0 < \lambda(p_0) < 1$ of $TF_0(p_0)$ satisfy a nonresonance condition, see e.g. Ruelle [1989, Section 1.5.8] for a statement and references).

Now that the conditions on $p_0 \in \Omega_0$ have been made precise, we may perform the surgery, using the local model f_A (with $A = TF_0(p_0)$) and $K_A \subset U$ given by Lemma 1. Let $1 < r < R$ be such that $B_1 \subset A(B_r) \subset B_R$ and $f_A(B_r) \subset B_R$. We denote a C^∞ linearising chart for F_0 at p_0 by $h_{p_0} : O_{p_0} \to B_R$. We construct a new C^∞ diffeomorphism F on S by setting:

$$F(x) = \begin{cases} F_0(x) & \text{for } x \in S \setminus h_{p_0}^{-1}(B_r) \\ h_{p_0}^{-1}(f_A(h_{p_0}(x))) & \text{for } x \in h_{p_0}^{-1}(B_r)\,. \end{cases} \tag{2.5}$$

Set $K = h_{p_0}^{-1}(K_A)$. Since $f_A|_{B_1\setminus K_A}$ is conjugated to $A|_{B_1\setminus\{0\}}$ by h_A, the new diffeomorphism $F|_{S\setminus K}$ is conjugated to $F_0|_{S\setminus\{p_0\}}$ by the C^∞ diffeomorphism $H : S\setminus\{p_0\} \to S\setminus K$ defined by

$$H(x) = \begin{cases} x & \text{for } x \in S \setminus h_{p_0}^{-1}(B_r) \\ h_{p_0}^{-1}(h_A(h_{p_0}(x)) & \text{for } x \in h_{p_0}^{-1}(B_r) \setminus \{p_0\}\,. \end{cases} \tag{2.6}$$

(The reader is invited to draw a three stories high commutative diagram.) Note that H^{-1}, which is a priori defined only on on $S\setminus K$, extends by continuity to a transformation on the whole surface S, still noted H^{-1}, if we map K to $\{p_0\}$.

We first list in Lemma 3 useful topological properties of our global map F, and describe next in Lemmas 4 and 5 ergodic properties relevant to prove Theorem 2.

Lemma 3. *Define a compact set $\Omega \subset S$ and an open set $\Omega \subset W \subset S$ by*

$$\Omega = H(\Omega_0 \setminus \{p_0\}) \cup h_{p_0}^{-1}(\partial K_A), \quad W = (H(V) \setminus h_{p_0}^{-1}(K_A)) \cup h_{p_0}^{-1}(U).$$

Then

(1) *Ω is the maximal F-invariant set in W, i.e., $\Omega = \cap_{n\in\mathbb{Z}} F^n(\overline{W})$;*
(2) *Ω is the closure of $H(\Omega_0 \setminus \{p_0\})$.*

Proof of Lemma 3. Because of our construction, the only nonobvious fact in the first claim is the inclusion $\cap_{n\in\mathbb{Z}} F^n(\overline{W}) \subset \Omega$. But this is in fact a consequence of Lemma 1 (2).

To prove $\Omega \subset \overline{H(\Omega_0 \setminus \{p_0\})}$ in the second claim (the other inclusion is obvious), take a connected neighbourhood N_x of an arbitrary point $x \in \partial\Omega$; then, by continuity, $C = H^{-1}(N_x)$ is a connected set containing $\{p_0\}$ as a strict subset. Since the point p_0 is not s-lateral, we have $C \cap (\Omega_0 \setminus \{p_0\}) \neq \emptyset$. To show this, we use that Ω_0 is a lamination by stable manifolds: choose a foliated chart of this lamination at p_0, and project C to $W^u(p_0)$ along the stable direction. If the projection D is reduced to p_0, then $C \subset W^s(p_0)$ and thus $C \subset \Omega_0$. Otherwise the connected set D contains (but is not reduced to) p_0, therefore, since p_0 is non s-lateral $D \cap (\Omega_0 \setminus \{p_0\}) \neq \emptyset$. Finally, by definition of a foliated chart, the set of points of C projecting to any $x \in D \cap (\Omega_0 \setminus \{p_0\})$ lie in $C \cap (\Omega_0 \setminus \{p_0\})$. □

We make a few remarks before stating Lemmas 4 and 5.

Since F_0 is uniformly hyperbolic and topologically transitive on Ω_0, there is a unique generalized SRB measure μ_0 in Ω_0, and its support coincides with Ω_0. Since F_0 is smooth and Ω_0 is not an attractor, $P_{\mu_0} < 0$ (i.e., μ_0 is not an SRB measure in the strict sense). Since μ_0 is the unique generalized SRB measure we have $P_{\mu_0} > -\log \sigma(p_0)$ (the entropy of a Dirac mass is zero). Finally, μ_0 is atomless so that $\mu_0(\{p_0\}) = 0$. (See e.g. Bowen [1975] for the above statements.)

If ν_0 is an F_0-invariant probability measure supported in Ω_0 such that $\nu_0(\{p_0\}) = 0$, then the push-forward $H_*(\nu_0)$ (given by $H_*(\nu_0)(E) = \nu_0(H^{-1}(E)))$ is a well-defined F-invariant probability measure, with support contained in Ω by Lemma 3 (2). In the other direction, if ν is an F-invariant probability measure with support included in W, then, by continuity, the pull-back $H_*^{-1}(\nu)$ is a well defined F_0-invariant probability measure, with support inside Ω_0 by our choice of V.

Lemma 4 below will be used in the proof of Lemma 5:

Lemma 4. *Let ν be an ergodic F-invariant probability measure ν supported in W. Then either ν is the Dirac mass at one of the F-fixed points $\{p_1, \dots, p_N\} \subset \partial K$, where $p_i = h_{p_0}^{-1}(q_i)$ and the f_A-fixed points $q_i \in \partial K_A$ are as in Lemma 1 (3); or $\nu = H_*(\nu_0)$ where ν_0 is an ergodic F_0-invariant measure supported in Ω_0, different from the Dirac mass at p_0.*

Proof of Lemma 4. Note that ν is supported inside Ω, by Lemma 3 (1). If ν is not the Dirac mass at one of the fixed points $\{p_1, \dots, p_N\}$ then, by ergodicity and Lemma 1 (3), we have $\nu(K) = 0$ (recall that K is F-invariant). It follows that the (ergodic) F_0-invariant probability measure $H_*^{-1}(\nu)$ has no atom at p_0 so that $\nu = H_* H_*^{-1}(\nu)$. □

Lemma 5. *The measure $H_*(\mu_0)$ is the unique generalized SRB measure for F with support included in W.*

Proof of Lemma 5. We first show that for any ergodic F_0-invariant measure ν_0 with support so that $\{p_0\} \neq \mathrm{supp}\,(\nu_0) \subset \Omega_0$, we have $P_{H_*(\nu_0)} = P_{\nu_0}$. For this, we first remark that $h_{H_*(\nu_0)} = h_{\nu_0}$ because H is a bijection on the set of total $H_*(\nu_0)$-mass $S \setminus \{p_0\}$, and thus a measure-theoretical isomorphism (see e.g. Walters [1982]). We next recall that a point is called *generic* for an ergodic invariant measure if its Lyapunov exponents are well-defined and coincide with those of the measure. (The Oseledec theorem guarantees that the set of generic points has measure 1.) To prove the claimed equality it suffices to check that for any ν_0-generic point $x \in \Omega_0$ we have $\chi^+(x) = \chi^+(H(x))$, so that $\chi^+(\nu_0) = \chi^+(H_*\nu_0)$ ($H^{-1}(y)$ is ν_0-generic for $H_*(\nu_0)$-almost each $H_*(\nu_0)$-generic point y). For this, note that we may assume that the F_0-orbit of x is dense in the support of ν_0 (a compact invariant set different from $\{p_0\}$), by the Birkhoff ergodic theorem and using that the intersection of two sets of full measure has full measure. Therefore there is a sequence n_i of iterates tending to infinity such that $F_0^{n_i}(x)$ is not in $O'_{p_0} := h_{p_0}^{-1}(h_A^{-1}(U))$ (use Lemma 1 (2) and the definition of K). We may assume that x itself is not in O'_{p_0}. Since $F^{n_i}(x) = H^{-1}(F_0^{n_i}(H(x)))$ and $\|TH\|$, respectively $\|TH^{-1}\|$, is bounded outside of the neighbourhood $O'_{p_0} \supset \{p_0\}$, respectively $h_{p_0}^{-1}(U) \supset K$, we get the claimed equality between Lyapunov exponents.

Let then ν be an ergodic F-invariant probability measure supported inside W. If ν is generalized SRB, it cannot be a Dirac mass at one of the p_is, because then the entropy of ν would vanish so that

$$P_\nu = -\log \sigma(p_i) \leq -\log \sigma(p_0) < P_{\mu_0} = P_{H_*(\mu_0)}\,, \tag{2.7}$$

a contradiction (we used Lemma 1 (3) and the equality proved in the first paragraph of the present proof). Lemma 4 therefore implies that $\nu = H_*(\nu_0)$ with ν_0 an ergodic F_0-invariant measure supported in Ω_0, and not the Dirac mass at p_0. Since $P_\nu = P_{\nu_0}$ (use again the equality proved above) and μ_0 is the unique generalized SRB measure for F_0 in Ω_0, we are done. $\square$

Proof of Theorem 2. Consider the C^∞ diffeomorphism $F : S \to S$ and the neighbourhood W of the compact F-invariant set Ω that we have constructed above. (By Lemma 3 (1), Ω is the maximal invariant set in W.) We have shown in Lemma 5 that there is a unique ergodic generalized SRB measure μ supported in W. Since we have seen that $\mu = H_*(\mu_0)$ (where μ_0 is the unique generalized SRB measure for the uniformly hyperbolic map $F_0 : \Omega_0 \to \Omega_0$), we find $P_\mu = P_{\mu_0}$ as in the proof of Lemma 5, which also shows that μ is hyperbolic, because its Lyapunov exponents coincide with those of μ_0. We already noted that $P(\mu_0)$ is strictly negative. Observe next that the support of $H_*(\mu_0)$ coincides with Ω (use again the fact that p_0 is not s-lateral to show that ∂K is accumulated by points $H(x)$ with $x \in \Omega_0$). Finally, the escape rate from W is zero because of Lemma 1 (4). $\square$

3. The local model

This section contains a constructive proof of Lemma 1.

We proceed in two steps, constructing first a diffeomorphism $\tilde{f}_A$ with a compact invariant set $\tilde{K}_A$ and an open neighbourhood $\tilde{U}$ of $\partial\tilde{K}_A$ satisfying (1–3) in the statement of Lemma 1 (Figure 4). We then perform an identification producing (Figure 5) a Bowen-example attractor as in Figure 1.

To perform the first step it is more convenient to view our diffeomorphisms as the time-one maps of suitable flows. The flow giving rise to the linear map A is just the linear flow $A^t = \begin{pmatrix} \lambda^t & 0 \\ 0 & \sigma^t \end{pmatrix}$. Note for further use that a point $(1, y)$ is mapped by A^t to $(y^{\frac{-\log\lambda}{\log\sigma}}, 1)$ after a time $t(y) = (-\log y)/\log\sigma$. We will in fact proceed by glueing such linear flows in flow-boxes (any such linear flows are conjugated to the trivial flow away from the origin) in a neighbourhood of a suitable transversals that we identify.

As a warm-up exercise we shall first glue two successive linear flows

$$A_i^t = \begin{pmatrix} \lambda_i^t & 0 \\ 0 & \sigma_i^t \end{pmatrix}, \quad 0 < \lambda_i < 1\,, \sigma_i > 1\,, \quad i = 1, 2\,. \tag{3.1}$$

We see now how to choose the multipliers of A_1^t and A_2^t in order to obtain a flow conjugated to A^t by a C^∞ diffeomorphism defined in the intersection of the quadrant $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ with a neighbourhood of the origin. The flow A_1^t is considered in the closure of a *cross-shaped neighbourhood* D_1 of the origin obtained by the union of segments of orbits $A_1^t(\pm 1, y)$ with $y \in]-a_1, a_1[$, and $0 \le t \le t(y)$, where $t(0) = \infty$ and $t(y)$ is defined by the condition $A_1^{t(y)}(\pm 1, y) = (x(y), \pm y/|y|)$ for $y \ne 0$, together with the unstable separatrices $\{0\} \times [-1, 1]$. In particular the two transversals $T_1^{\text{in}} = \{1\} \times [-a_1, a_1]$ and $T_1^{\text{out}} = [-b_1, b_1] \times \{1\}$, with $b_1 = a_1^{\frac{-\log\lambda_1}{\log\sigma_1}}$, are contained in ∂D_1. The flow A_2^t is considered in a similar domain, its boundary contains the transversals $T_2^{\text{in}} = \{1\} \times [-a_2, a_2]$ with $a_2 = b_1$ and $T_2^{\text{out}} = [-b_2(a_2), b_2(a_2)] \times \{1\}$. We identify the transversals T_1^{out} and T_2^{in} with the map $(x, 1) \mapsto (1, x)$. There is a unique way to glue the two flows in a neighbourhood of the identified transversals via flow-boxes, producing a C^∞ flow on a compact surface with boundary, which can clearly be identified with a neighbourhood of the origin in $\mathbb{R}^2$. The holonomy (input-output) map sends $(1, y)$ to $(y^{\frac{\log\lambda_1}{\log\sigma_1}\frac{\log\lambda_2}{\log\sigma_2}}, 1)$ after a time

$$t(y) = \frac{-\log y}{\log\sigma_1} + \frac{-\log\left(y^{\frac{-\log\lambda_1}{\log\sigma_1}}\right)}{\log\sigma_2}\,. \tag{3.2}$$

Necessary and sufficient conditions to permit the desired conjugacy with A^t are therefore

$$\begin{cases} -\frac{\log\lambda}{\log\sigma} &= \frac{\log\lambda_1}{\log\sigma_1}\frac{\log\lambda_2}{\log\sigma_2} \\ \frac{1}{\log\sigma} &= \frac{1}{\log\sigma_1}\left(1 - \frac{\log\lambda_1}{\log\sigma_2}\right). \end{cases} \tag{3.3}$$

Indeed, consider the intersection D_+ of a cross-shaped domain for A^t with the first quadrant. The conjugacy is obtained by identifying the input transversal $T_+^{\rm in}$ of the domain D_+ for A^t with the positive half of $T_1^{\rm in}$, identifying the output transversal $T_+^{\rm out}$ of D_+ with the positive half of $T_2^{\rm out}$, and propagating by the flow. This ends our exercise.

In order to make the identification necessary to produce the Bowen example we need to glue four successive linear flows A_i^t, $i = 1, \ldots, 4$, producing a flow B^t on an abstract planar surface with boundary. It will be convenient to choose $A_2^t = A_3^t$ and we write $\tilde{A}^t$ for the common value. The conditions guaranteeing the existence of the smooth conjugacy become

$$\begin{cases} -\frac{\log \lambda}{\log \sigma} & = \frac{\log \lambda_1}{\log \sigma_1} \left(\frac{\log \tilde{\lambda}}{\log \tilde{\sigma}}\right)^2 \frac{\log \lambda_4}{\log \sigma_4} \\ \frac{1}{\log \sigma} & = \frac{1}{\log \sigma_1}\left(1 - \frac{\log \lambda_1}{\log \tilde{\sigma}} + \frac{\log \lambda_1 \log \tilde{\lambda}}{\log^2 \tilde{\sigma}} - \frac{\log \lambda_1 \log^2 \tilde{\lambda}}{\log^2 \tilde{\sigma} \log \sigma_4}\right). \end{cases} \tag{3.4}$$

In order to satisfy Lemma 1 (3) we add the condition

$$\min(\sigma_1, \sigma_4, \tilde{\sigma}) > \sigma\,. \tag{3.5}$$

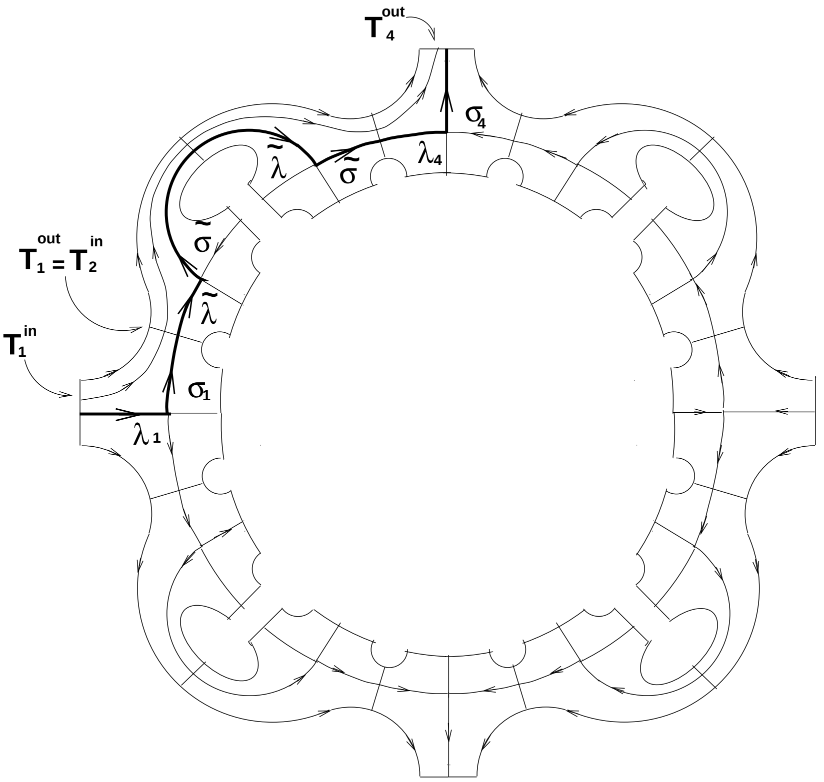

Figure 4 Local model before identification

Finally, to get an attracting Bowen example, i.e., to ensure Lemma 1 (4) we require

$$\frac{\log^2 \tilde{\lambda}}{\log^2 \tilde{\sigma}} > 1 \quad \text{i.e.,} \quad \tilde{\sigma}\tilde{\lambda} < 1 \tag{3.6}$$

(use the formula for the input-output map of $\tilde{A}^t$ glued with itself). Conditions (3.4–3.6) define a nonempty open subset $\mathcal{L}$ of points $(\lambda_1, \sigma_1, \tilde{\lambda}, \tilde{\sigma}, \lambda_4, \sigma_4)$ in a submanifold in $\mathbb{R}^6$.

We now glue four copies B_i^t of the flows B^t and identify the abstract planar surface thus obtained with an annulus contained in the unit ball of $\mathbb{R}^2$, as described in Figure 4 (each quadrant contains a copy of the flow B^t). By construction, the time-one map of the flow thus obtained is C^∞ conjugated to A outside of the origin and satisfies properties (1–3) of Lemma 1, ending the first step of the construction of the local model. (The hyperbolic saddles mentioned in Lemma 1 (3) come from the fixed points of the linear flows used in the construction, their eigenvalues are λ_i, σ_i; $i = 1, 4$, and $\tilde{\lambda}$, $\tilde{\sigma}$.)

Let q_2^j, respectively q_3^j denote the fixed point originating from the fixed point of A_2^t, respectively A_3^t in the j^{th} quadrant. Since we were careful to choose $A_2^t = A_3^t = \tilde{A}^t$ we may identify a suitably small neighbourhood of q_2^1 (respectively q_3^1) with a corresponding neighbourhood of q_3^3 (respectively q_2^3) via minus the identity map. Let us denote by q_2, q_3 the identified saddle fixed points (the eigenvalues at both fixed points are $\tilde{\sigma} > 1$, $0 < \tilde{\lambda} < 1$). We obtain in this way a new flow, on a new planar surface which can be embedded in a neighbourhood of the origin in $\mathbb{R}^2$ (see Figure 5).

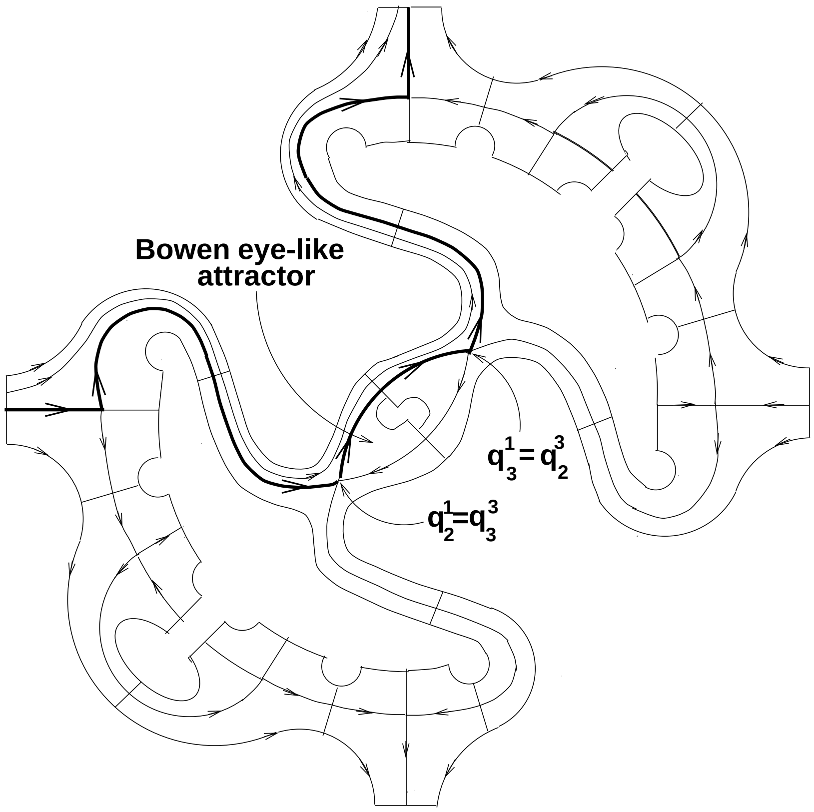

Figure 5 Local model after identification

The time-one map of this flow is the diffeomorphism f_A needed for Lemma 1. The set ∂K_A is formed of separatrices corresponding to the stable and unstable manifolds of the glued flows. There are $N = 10$ saddles q_i. Property (4) is satisfied because of the presence of the Bowen invariant set formed by the two separatrices joining the fixed points q_2 and q_3 (recall (3.6)).

Remark 2. By choosing $\tilde\lambda$ and $\tilde\sigma$ in such a way that $\tilde\lambda\tilde\sigma > 1$, so that the Bowen example of Figure 1 loses its attracting property, it is easy to see that we obtain a plug (f_A, K_A) satisfying the properties stated in Lemma 1 except that (4) is replaced by

(4') $\limsup_{m\to\infty} \frac{1}{m} \log \mathrm{Vol}\ U_m = P$,

where $-\log\sigma < P \le 0$ is arbitrary. In this way, we obtain a counterexample like in Theorem 2, but where the escape rate $E(W) = P < 0$ for an arbitrarily chosen $P > P_\mu$.

A variant of the counterexample:
Starting from a saddle hyperbolic set instead of a repellor

Instead of starting with a uniformly hyperbolic repellor we could take a general non-trivial topologically transitive uniformly hyperbolic set $F_0 : \Omega_0 \to \Omega_0$. The construction is the same as above except that we must be a little bit more careful to guarantee the property that ∂K is contained in the suppport of the pull-back of the generalized SRB measure μ_0 of F_0. Again, we must avoid "boundary" points when choosing the fixed point p_0 to be blowed up (this time we use also that there are only finitely many u-lateral periodic points, see Newhouse–Palis [1973]), but since Ω_0 is not a union of stable manifolds any more, this is not enough.

We will now show that the claimed property of the support is true, however, if we consider values of the multipliers in a suitable subset of $\mathcal{L}$, when performing the local construction. To prove the claim, we shall use the following property of linear hyperbolic flows (recall that a residual subset of a nonempty topological space is a countable intersection of dense open subsets, and that a residual set is never empty):

Lemma 6. *For $0 < \lambda_1 < 1$ and $\sigma_1 > 1$, write $A_1^t = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$. Let Σ_1 be an open subset of $]1,\infty[$, Then for any $0 < \lambda_1 < 1$ and any sequence of points r_i in the first quadrant of B_1 accumulating on $(1,0)$, there exists a residual subset of Σ_1 such that for any $\sigma_1 \in \Sigma_1$, every point in the half unstable separatrix $\{0\}\times]0,1]$ is accumulated by iterates $A_1^{n_i}(r_i)$, $n_i \in \mathbb{N}$, of the r_is by the time-one map.*

Proof of Lemma 6. Let $D_{1,+}$ be the intersection of a cross-shaped domain for A_1^t with the first quadrant. For each i, let $\tilde r_i = A^{\tilde t_i}(r_i)$ be the point in the orbit of r_i on the input boundary transversal of $D_{1,+}$ i.e., such that $\tilde r_i = (1, y_i)$. Note that $\lim_{i\to\infty} \tilde t_i = 0$. Let $t(y_i) = \frac{-\log y_i}{\log \sigma_1}$ be the output time of $\tilde r_i$. Denote by $[t]$ the integer part of t. It is not difficult to check, on the one hand, that the set $\{t(y_i) - [t(y_i)] \mid i \in \mathbb{N}\}$ is dense in $[0,1]$ for a residual set of σ_1s, and on the other, that this implies the claimed accumulation property. Details are left to the reader. $\square$

Lemma 6 implies that $(0,1)$ is accumulated by a sequence $s_i = A_1^{n_i}(r_i)$ for a residual set of values of σ_1. Note $\tilde s_i = (1, z_i)$ the corresponding points on the output transversal. Consider now the setting of the warm-up exercise above, i.e., a fixed flow A^t with eigenvalues λ, σ, and two linear hyperbolic flows A_1^t, A_2^t, with $\sigma_1 > \sigma$, $\sigma_2 > \sigma$, and λ_1, λ_2 chosen so as to satisfy (3.3) (write $\mathcal{L}' \subset \mathbb{R}^4$ for the corresponding set of $(\lambda_1, \sigma_1, \lambda_2, \sigma_2)$). We claim that for any A_2^t and any sequence r_i accumulating on a point of the input stable separatrix of A_1^t, there is a residual subset of σ_1 in $\Sigma =]\sigma, \infty[$ such that unstable separatrices of both A_1^t and A_2^ts are accumulated by iterates of the r_i by the time-one map of the glued flow. This is because the intersection of two residual sets is residual, and because the times used to check that the first separatrix is accumulated are as in the proof of Lemma 6, and those used to check that the second separatrix is accumulated are given by

$$t(y_i) = \frac{-\log y_i}{\log \sigma_1} + \frac{-\log\left(y_i^{\frac{-\log \lambda_1}{\log \sigma_1}}\right)}{\log \sigma_2}. \tag{3.7}$$

(Fix $0 < \lambda_1 < 1$ and let $\mathcal{R}_1$ be the residual subset of $\sigma_1 \in \Sigma$ given by the proof of Lemma 6, consider then $\sigma_2 > \sigma$ in the projection of $\mathcal{L}'$ corresponding to λ_1 and $\mathcal{R}_1$, and define a second residual set $\mathcal{R}_2$ of $\sigma_1 \in \Sigma$ according to (3.7). Finally, take $\sigma_1 \in \mathcal{R}_1 \cap \mathcal{R}_2$ and $0 < \lambda_2 < 1$ so as to satisfy (3.3).) Consider now a succession of four linear flows as in the first step giving the local model and an arbitrary sequence of points r_i (or more generally a finite number of sequences r_i^j) accumulating to a point in the stable separatrix of A_1^t. The procedure just described can be modified to produce points $(\lambda_1, \sigma_1, \tilde{\lambda}, \tilde{\sigma}, \lambda_4, \sigma_4)$ in $\mathcal{L}$ (i.e., satisfying (3.4–3.6)), so that the three successive output separatrices (of A_1^t, $A_2^t = \tilde{A}^t$, and $A_3^t = \tilde{A}^t$) are accumulated by suitable iterates n_i of the r_i (more generally, suitable iterates $n_{i,j}$ of the r_i^j) by the time-one map of the glued flow.

We now see how to reduce to the setting of four successive glued hyperbolic linear flows in order to apply the above consequences of the proof of Lemma 6. For this, we start by choosing four suitable sequences in a neighbourhood of p_0. Take a neighbourhood $\mathcal{N}$ of p_0 in $\Omega_0 = \operatorname{supp} \mu_0$, such that the connected component $W^s_{\mathcal{N}}(p_0)$ of $W^s(p_0) \cap \mathcal{N}$ containing p_0 separates $\mathcal{N}$ in two connected components, $\mathcal{N}_1$ and $\mathcal{N}_2$. Consider first two sequence of points in $\mathcal{N}$ converging to an arbitrary point $q \neq p_0$ in $W^s_{\mathcal{N}}(p_0)$, with the property that the first sequence is in $\mathcal{N}_1$ and the second in $\mathcal{N}_2$. Fixing a point q' in $\mathcal{N}$ in the connected component of $W^s_{\mathcal{N}}(p_0) \setminus \{p_0\}$ which does not contain q, choose next two sequences accumulating on q', again on opposite sides of the stable separatrix. Finally, lift all four sequences to Ω, via the conjugacy used in our construction. Since the lifted situation has been treated above, we have proved that the construction can be made in such a way as to ensure that the support of $H_*(\mu_0)$ contains ∂K_A, ending the description of the variant of our counterexample.

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