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# Non-Gibbsianness of the reduced SOS-measure

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## Abstract

It is shown that the Gibbs probability measure of the SOS model transformed by reducing the level-variables under the sign-map provides an example of a non-Gibbsian measure. The persistence of non-Gibbsianness under some transformations is discussed.

KEYWORDS: Gibbs measures, non-Gibbsian measures, renormalization transformations

# 1 Introduction

As R.L. Dobrushin [2] and H. van Beijeren [1] have shown, the standard Ising interaction on the three dimensional regular lattice has the peculiarity of giving rise to a non-translation invariant low-temperature state. This state is obtained by choosing + boundary condition on one side, and − boundary condition on the opposite side of, for instance, a cubic volume. Midway between the two sides then an interface appears, with a fixed position, showing a particular magnetization profile. At low temperatures, the width of the interface is uniformly bounded, and it drops to zero exponentially fast as the temperature goes to zero. This phenomenon is related to the general situation in which a so called roughening point appears, separating two different thermodynamic behaviours of the interface in a neighbourhood below and above a temperature $T_R \leq T_c$. As already pointed out, at temperatures lower than $T_R$ the phase separation surface is localized (or rigid) in the lattice, meaning that typically the two phases do not penetrate arbitrarily deep inside one another by local fluctuations. At temperatures higher than $T_R$ the interface becomes delocalized. In this region the Dobrushin state becomes the equiweighted mixture of the + and − phases in the thermodynamic limit. Although it is rigorously proven that the critical temperature $T_c$ is larger or equal than the roughening temperature $T_R$, it is a long standing open question whether for the three dimensional system they actually coincide. Numerical evidence suggests that there is a gap between the two values [11], and thus the phase diagram of the three dimensional Ising model is expected to be more complex than that for the two dimensional case, where $T_R = 0$.

A more detailed study on the properties of the phase separation surface and its vicinity requires a certain amount of simplification to be made on the structure of the interface. Such an approximation is proposed by the SOS (solid-on-solid) model in which the surface is given by a single valued function of the lattice sites (and therefore certain complexities of the interface are not taken into account). In the SOS approximation the three dimensional Ising system actually shows a roughening transition. The equilibrium properties of the interface are governed by a Hamiltonian which in the SOS model is taken to be proportional to the sum of the differences of nearest neighbour levels. This choice makes the interface tend not to develop large nearest neighbour level differences. However, a competing entropic effect pulls the levels off for obtaining more freedom for fluctuations.

These effects involve some form of long range order and persistence of strong correlations over large distances. As a consequence, some interesting transformations such as renormalisations or lower dimensional projections of the relevant Gibbs states may become non-Gibbsian [8, 3, 6, 10, 5]. This possibility has been so far explored mainly for interfaces described by a massless Gaussian model, showing that the states of the associated system obtained in reduced variables cannot be described as Gibbs measures. The associated system was simply constructed by taking the sign of the level-variables instead of their exact value. In this note we want to show that likewise for the SOS model, the measure obtained under this reduction cannot be a Gibbs measure. (For an up-to-date review on the occurrence of non-Gibbsian states in lattice spin systems and examples we refer to [4].)

## 2 Notations and main result

We shall consider spins placed at the sites of the lattice $\mathbb{Z}^d$ with $d \geq 2$. Two sites, $j$ and $k$ are nearest neighbours when $|j-k| = 1$; this will be denoted $j \sim k$. Each spin is assumed to take its values from the same set $\mathbb{Z}$, thus the infinite volume configuration space is $\Omega = \mathbb{Z}^{\mathbb{Z}^d}$. The notation $\omega_\Lambda$ stands for the projection of $\omega \in \Omega$ to the finite volume configuration space $\mathbb{Z}^\Lambda$; in particular we put $\omega_j$ when $\Lambda = \{j\}$. We write $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$, and put $\underline{\partial}\Lambda$ and $\overline{\partial}\Lambda$ for the inner and the outer nearest neighbour boundaries of $\Lambda$, respectively. The configuration space will be provided with its Borel $\sigma$-field $\mathcal{F}$.

The finite volume Hamiltonians with boundary condition $\xi_{\Lambda^c}$ in the SOS-model are defined by

$$\mathcal{H}^J_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}) = J \sum_{\substack{j,k\in\Lambda \\ j\sim k}} |\omega_j - \omega_k| + J \sum_{\substack{j\in\underline{\partial}\Lambda, k\in\overline{\partial}\Lambda \\ j\sim k}} |\omega_j - \xi_k| \tag{2.1}$$

We assume throughout translation invariance of the summand (i.e., the interaction).

The finite volume Hamiltonians can be used for constructing Gibbs measures. A Gibbs measure for the SOS model is any probability measure $\varrho$ whose finite volume conditional probabilities can be written $\varrho$-almost surely as

$$\varrho_\Lambda(\eta \in \Omega : \eta_\Lambda = \omega_\Lambda | \eta_{\Lambda^c} = \xi_{\Lambda^c}) = \frac{\exp\left(-\mathcal{H}^J_\Lambda(\omega_\Lambda|\xi_{\Lambda^c})\right)}{Z_\Lambda(\xi_{\Lambda^c})} \tag{2.2}$$

(The inverse temperature $\beta$ is included to the interaction.) These conditional probabilities satisfy the DLR equations

$$\frac{\varrho_\Lambda(\omega_\Lambda|\xi_{\Lambda^c})}{\varrho_\Lambda(\tau_\Lambda|\xi_{\Lambda^c})} = \exp\left(-\mathcal{H}^J_{rel}(\omega_\Lambda, \tau_\Lambda|\xi_{\Lambda^c})\right) \tag{2.3}$$

for every finite volume $\Lambda$, where

$$\mathcal{H}^J_{rel}(\omega_\Lambda, \tau_\Lambda|\xi_{\Lambda^c}) = \mathcal{H}^J_\Lambda(\omega_\Lambda|\xi_{\Lambda^c}) - \mathcal{H}^J_\Lambda(\tau_\Lambda|\xi_{\Lambda^c}) \tag{2.4}$$

is the corresponding relative Hamiltonian.

One can introduce a partial order on the configuration space by putting $\omega \leq \tau$ whenever $\omega_j \leq \tau_j$, for every $j \in \mathbb{Z}^d$. This induces a partial order also on the set of functions of the configuration space, and another on the set of probability measures. Accordingly, we call any real valued measurable function $f$ on $\Omega$ increasing whenever $\omega \leq \tau$ implies $f(\omega) \leq f(\tau)$, and for measures we write $\varrho_1 \leq \varrho_2$, whenever $\varrho_1(f) \leq \varrho_2(f)$ holds for increasing measurable functions. (The notation $\varrho(f)$ stands for the expectation of $f$ taken with respect to $\varrho$.) The Gibbs measures for the SOS interaction satisfy the FKG property, which says that for every increasing real valued measurable function $f$ on $\Omega$ and for every $\xi_{\Lambda^c} \leq \zeta_{\Lambda^c}$

$$\varrho_\Lambda(f|\xi_{\Lambda^c}) \leq \varrho_\Lambda(f|\zeta_{\Lambda^c}) \tag{2.5}$$

Consider now two translation invariant probability measures $\mu$ and $\nu$ on a space $(X, \mathcal{X})$ having a product structure $X = W^{\mathbb{Z}^d}$, with a given finite set $W$. Suppose the Radon-Nikodym derivative $h_{\mu;\nu}$ of $\mu$ with respect to $\nu$ exists, and $\log h_{\mu;\nu} \in L^1(\mu)$. The functional

$$I(\mu|\nu) = \begin{cases} \int_X h_{\mu;\nu}(\xi) \log h_{\mu;\nu}(\xi) \nu(d\xi) & \text{if } \mu \ll \nu \\ \infty & \text{otherwise} \end{cases} \tag{2.6}$$

is the relative entropy of $\mu$ with respect to $\nu$. Denote by $\mu_\Lambda$ the restriction of $\mu$ to $\mathcal{X}_\Lambda$, the product Borel $\sigma$-field for $W^\Lambda$, and consider the similar object for $\nu$. The limit

$$i(\mu|\nu) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} I(\mu_\Lambda|\nu_\Lambda) \tag{2.7}$$

defined in van Hove sense, is the relative entropy density for $\mu$ with respect to $\nu$. It describes the large deviations of $\mu$ with respect to $\nu$. When the reference measure $\nu$ is a Gibbs measure (or has some other regularity properties), the above limit is known to exist. Since $X$ is compact, the following important property holds: $i(\mu|\nu) > 0$ if and only if $\mu$ and $\nu$ are Gibbs measures for two different interactions, and $i(\mu|\nu) = 0$ if and only if $\mu$ and $\nu$ are Gibbs measures for the same interaction. Also, $i(\mu|\nu) \geq 0$. (For more details and proofs see [7] section 15.3, and [6] section 2.6.6.).

Now we turn to formulating our main result. We will consider a model associated with the SOS model which will be given in the reduced variables $\{-1, +1\}$ instead of the original level-variables. That is, we only ask the question (or are given the information) whether at each site the interface is above or below a designated zero-level. We do this by considering a map $F : \Omega \to \{-1, +1\}^{\mathbb{Z}^d}$ that by definition almost surely coincides with the sign-function in the sense that $(F(\omega))_j = \operatorname{sgn} \omega_j$, for all $j \in \mathbb{Z}^d$. Our main question will concern the properties of the probability measure on $\{-1, +1\}^{\mathbb{Z}^d}$, induced under $F$.

**Theorem 1** *In the system of reduced variables we have that $i(\delta^+|\varrho \circ F) = 0$. It follows in particular that there is no translation invariant absolutely summable interaction on $\{-1, +1\}^{\mathbb{Z}^d}$ for which the probability measure $\varrho \circ F$ is a Gibbs measure.*

(The notation $\delta^+$ corresponds to the Dirac measure concentrated on the configuration $\omega_j = +1$, for all $j \in \mathbb{Z}^d$.)

*Proof*: We proceed by the 'algorithm' in [6], Section 4.4.1, based on arguments in [8]. Consider an arbitrary finite set of sites $\Lambda$, and denote by $I_\Lambda$ its characteristic function. By the DLR equations it is straightforward to obtain the identity

$$\int_\Omega f(\omega) \varrho(d\omega) = \int_\Omega f(\omega + hI_\Lambda) \exp\left(-\mathcal{H}^J_{rel}(\omega + hI_\Lambda, \omega|\xi)\right) \varrho(d\omega) \tag{2.8}$$

for any real valued function $f$ on the configuration space, and where it is understood that $\omega$ agrees with $\xi$ on $\Lambda^c$. Here $h$ is the configuration for which $h_j = h$ at every $j \in \mathbb{Z}^d$. The relative energy appearing here is dominated by a boundary term since

$$\mathcal{H}^J_{rel}(\omega + hI_\Lambda, \omega|\xi) = \sum_{\substack{j \in \underline{\partial}\Lambda, k \in \overline{\partial}\Lambda \\ j \sim k}} \left(|\omega_j + hI_\Lambda - \xi_k| - |\omega_j - \xi_k|\right) \leq 2|h||\overline{\partial}\Lambda| \tag{2.9}$$

Note that this inequality does not survive when instead of the SOS model one is working with the massless Gaussian model.

Introduce the event $E_\Lambda^k = \{\omega \in \Omega : \omega_\Lambda > k\}$. We choose now in particular $f(\omega) = I_{E_\Lambda^0}(\omega)$ and $h > 0$ in (2.8). Hence we get

$$\varrho(E_\Lambda^0) \geq e^{-2h|\overline{\partial}\Lambda|}\varrho(E_\Lambda^{-h}) \tag{2.10}$$

uniformly in $h$. On the other hand, by sitewisely repeated application of the FKG inequality and by using translation invariance, we obtain

$$\varrho(E_\Lambda^{-h}) \geq \prod_{j\in\Lambda} \varrho(E_j^{-h}) = [\varrho(E_0^{-h})]^{|\Lambda|} \tag{2.11}$$

(where we supposed without constraining generality that $0 \in \Lambda$). From here it then follows that in van Hove sense

$$\lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|} \log \varrho(E_\Lambda^0) \geq \log \varrho(E_0^{-h}) \tag{2.12}$$

By taking the limit $h \to \infty$, the right hand side can be seen to be going to zero. After taking the sign this then implies by (2.6) and (2.7) that $\delta^+$ has zero relative entropy density with respect to $\varrho \circ F$, and thus $\varrho \circ F$ cannot be a Gibbs measure. □

**Remark 1** Consider a regular sublattice $b\mathbb{Z}^d$, with some $b = 2, 3, ...$, and denote by $T_b$ the operation of taking the marginal of a measure on the subfield $\{-1, +1\}^{b\mathbb{Z}^d}$ (i.e. a decimation transformation). Denote by $T_b^{n+1}$ the $n$th iterate of the map $T_b$, and convene on $T_b^1 = T_b$. Then it is easy to show that for all integers $b \geq 2$ and $n = 0, 1, 2, ...$ the measure $T_b^n(\varrho \circ F)$ is non-Gibbsian. Thus no decimation on whatever self-similar sublattice of $\mathbb{Z}^d$ will remove non-Gibbsianness. The proof in [10, 5] extends easily to this case and we do not reproduce it here.

**Remark 2** In fact, not only decimations of $\varrho \circ F$ yield further non-Gibbsian measures, but also other sorts of transformations. In particular, Theorem 3.1 of [5] applies also to the reduced SOS model, and thereby it follows that any Kadanoff transformation applied to $\varrho \circ F$ leads to a non-quasilocal, hence non-Gibbsian, image measure.

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