AN ALGEBRAIC INVARIANT FOR SUBSTITUTION TILING SYSTEMS

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Abstract

We consider tilings of Euclidean spaces by polygons or polyhedra, in particular, tilings made by a substitution process, such as the Penrose tilings of the plane. We define an isomorphism invariant related to a subgroup of rotations and compute it for various examples. We also extend our analysis to more general dynamical systems.

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1. Introduction

This paper concerns tilings of Euclidean spaces by polygons or polyhedra, more specifically, tilings made by a "substitution process". An example we will use is a "chair" tiling of the plane [Sen] made as follows. Consider the chair polygon of Fig. 1 with vertices given in rectangular coordinates by: $(3,-1),(3,1),(1,1),(1,3),$ $(-1,3),(-1,-1)$. Divide the chair into four small chairs as in Fig. 2 and expand the figure about the origin by a linear factor of 2, producing 4 chairs of the original size.

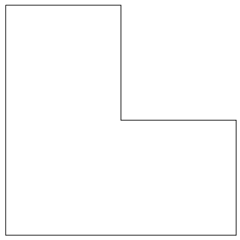

Figure 1. The chair polygon

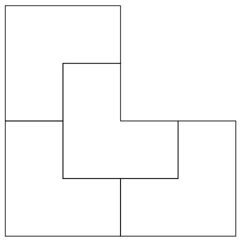

Figure 2. The substitution for chair tilings

Repeat this two-step procedure ϕ simultaneously for all the chairs of the figure, then again, an infinite number of times, producing a (chair) tiling C of the plane, a portion of which appears in Fig. 3. Such tilings have a hierarchical structure which is of interest for various reasons; in particular it leads to interesting behavior of the relative orientations of tiles within a tiling [Ra3].

We are concerned here with the question of isomorphism of such tilings. Based on their applications we use a notion of isomorphism between substitution tilings x and x' which does not require that x and x' be comprised of tiles of the same congruence classes, or even that the (finite) number of congruence classes of tiles in x be the same as for x'. We do this in two stages. First we embed x and x' in classes X and X', respectively, and then define a notion of isomorphism between classes of tilings. We say two tilings are isomorphic if and only if their classes are isomorphic.

We now define the class of all chair tilings. A tiling x of the plane by polygons is a chair tiling if for every closed ball $B_n(P)$, centered at any point P and of any radius $n > 0$, there is another ball $B_n(P')$ of the same radius such that $B_n(P) \cap x$ is congruent to $B_n(P') \cap C$. We note that there are uncountably many chair tilings which are pairwise noncongruent. In general, the class of *any* tiling y is the set of all tilings y' such that every bounded region of y' is congruent to a bounded region of y. (We will give a slightly different, although equivalent, formulation of the class of a substitution tiling system in §2.)

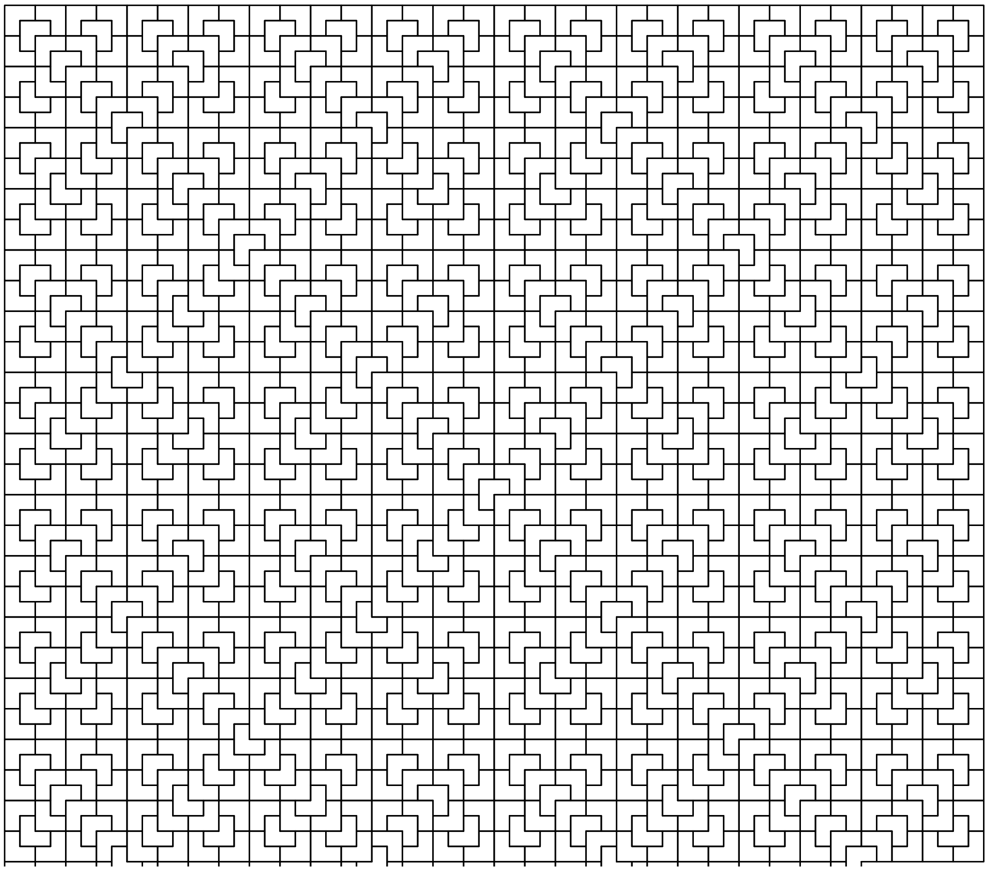

Figure 3. Part of a chair tiling

In later sections we will give examples of other tilings, made by a substitution process like the chair tilings, and we will be interested in determining when such tilings are isomorphic. We will define a metric on any class or space of substitution tilings (making it compact) such that two tilings are close if they are uniformly close in a large ball about the origin, and then (roughly speaking) declare two spaces X

and X' of substitution tilings to be *isomorphic* if there is a homeomorphism between them which intertwines the natural actions of the (connected) Euclidean group on X and X'. For background on recent work see [AnP, CEP, DwS, G-S, Kel, Ken, LaW, Min, Moz, Ra3, Rob, Sad, Sen, Sol, Tha] and references therein.

The objective of this paper is to define an isomorphism invariant related to a geometric feature of substitution tilings which is important in applications, namely the relative orientations of the tiles in a tiling. The difficulty of course is to capture this quantity in an invariant way, in particular without referring to individual tiles. We will compute the invariant for several examples, and extend our analysis to more general systems carrying an action of the Euclidean group.

2. Substitution Tiling Systems

Let $\mathcal{A}$ be a nonempty finite collection of polyhedra in m (typically 2 or 3) dimensions. Let $X(\mathcal{A})$ be the set of all tilings of Euclidean space by congruent copies, which we will call tiles, of the elements of (the "alphabet") $\mathcal{A}$. We label the "types" of tiles by the elements of $\mathcal{A}$. We endow $X(\mathcal{A})$ with the metric

$$d(x,y) \equiv \sup_n \frac{1}{n} m_H[B_n(\partial x), B_n(\partial x')], \qquad 1)$$

where $B_n(\partial x)$ denotes the intersection of two sets: the closed ball B_n of radius n centered at the origin of the Euclidean space and the union ∂x of the boundaries ∂a of all tiles a in x. m_H is the Hausdorff metric on compact sets defined as follows. Given two compact subsets A and B of $\mathbb{R}^m$, $m_H[A,B] = \max\{\tilde{d}(A,B), \tilde{d}(B,A)\}$, where

$$\tilde{d}(A,B) = \sup_{a\in A} \inf_{b\in B} ||a-b||, \qquad 2)$$

with $||w||$ denoting the usual Euclidean norm of w. Although the metric d depends on the location of the origin, the topology induced by d is translation invariant. A sequence of tilings converges in the metric d if and only if its restriction to every compact subset of $\mathbb{R}^m$ converges in m_H.

It is not hard to show [RaW] that $X(\mathcal{A})$ (which we assume is nonempty) is compact and that the natural action of the connected Euclidean group G_E on $X(\mathcal{A})$, $(g,x) \in G_E \times X(\mathcal{A}) \longrightarrow T^g x \in X(\mathcal{A})$, is continuous. A space of the form $X(\mathcal{A})$, carrying the natural action of G_E, will be called a "tiling dynamical system".

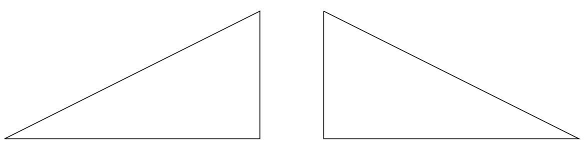

Figure 4. The alphabet for pinwheel tilings: 2 tiles

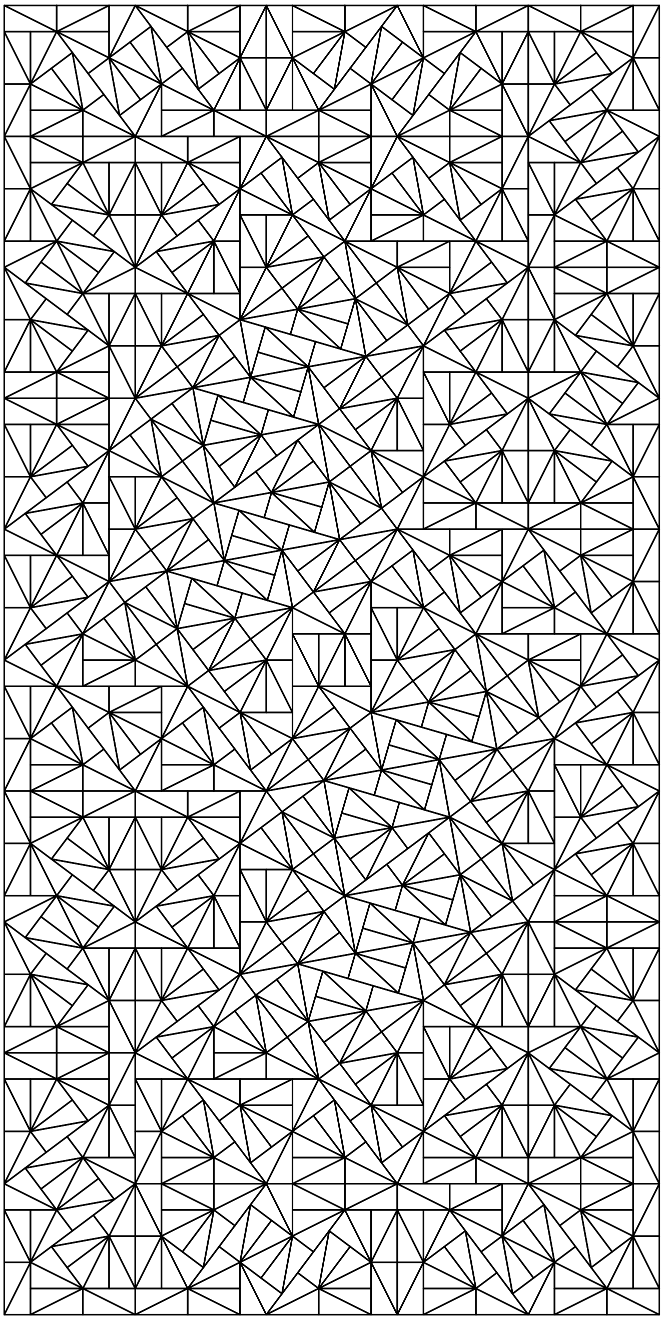

Figure 5. Part of a pinwheel tiling

We are interested primarily in certain closed subcollections $X_\phi \subset X(\mathcal{A})$ which we call substitution tiling systems. Two planar examples relevant to this paper are the pinwheel [Ra1] and chair [Sen] systems. The chair tilings were discussed in the introduction; see Figs. 4,5 for the pinwheel alphabet $\mathcal{A}_p$ and a portion of a pinwheel tiling. To define these systems we first need the notion of "patches". A patch is a (finite or infinite) subset of an element $x \in X(\mathcal{A})$; the set of all patches for a given alphabet will be denoted by W.

For substitution tiling systems one has, as for chairs, an auxiliary "substitution function" ϕ, a map from W to W, with the following properties:

i) There is some constant $|\phi| > 1$ such that, for any $g \in G_E$ and $x \in X_\phi$, $\phi[T^g x] = T^{\phi(g)}x$, where $\phi(g)$ is the conjugate of g by the similarity of Euclidean space consisting of stretching about the origin by $|\phi|$.
ii) For each tile $a \in \mathcal{A}$ and for each $n \geq 1$, the union of the tiles in $\phi^n a$ is congruent to $|\phi|^n a$, and these tiles meet full face to full face.
iii) For each tile $a \in \mathcal{A}$, ϕa contains at least one tile of each type.

Condition ii) is quite strong. It is satisfied by the chair tilings only if we imagine additional vertices at (1,-1) and (-1,1), creating a boundary of 8 edges of equal length. A similar adjustment is needed for the pinwheel and the other examples in this paper. Even with such adjustments, condition ii) is not satisfied by the kite & dart tilings [Gar], or those which mimic substitution tilings using so-called edge markings [G-S, Moz, Ra3]. To handle such examples we will use the general development of §3.

Now given ϕ, the corresponding substitution tiling system is defined as the set of tilings $x \in X(\mathcal{A})$ with the following property: every finite subpatch of x is congruent to a subpatch of $\phi^n a$ for some $n > 0$ and some $a \in \mathcal{A}$. With the convention that patches of the form $\phi^n a$ are called "supertiles" of "level" n and "type" a, it is easy to show by a diagonal argument that, for each $n \geq 0$, each tiling x is tiled by supertiles of level n [Ra3].

The pinwheel substitution tiling system $X_{\phi_p} \subset X(\mathcal{A}_p)$ is associated with the map, ϕ_p, defined on each element $a \in \mathcal{A}_p$ by partitioning the 1-2-$\sqrt{5}$ triangle a into 5 parts as in Fig. 6, followed by expansion of the result about the origin by $\sqrt{5}$. A patch $\phi_p^5 a$ appears in Fig. 7. The similar process for the chair system $X_{\phi_c} \subset X(\mathcal{A}_c)$ was discussed in the introduction, using the substitution function of Fig. 2.

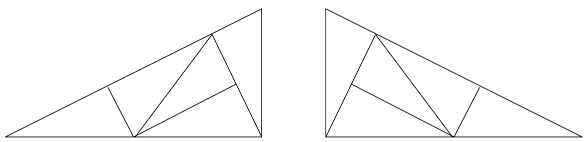

Figure 6. The substitution for pinwheel tilings

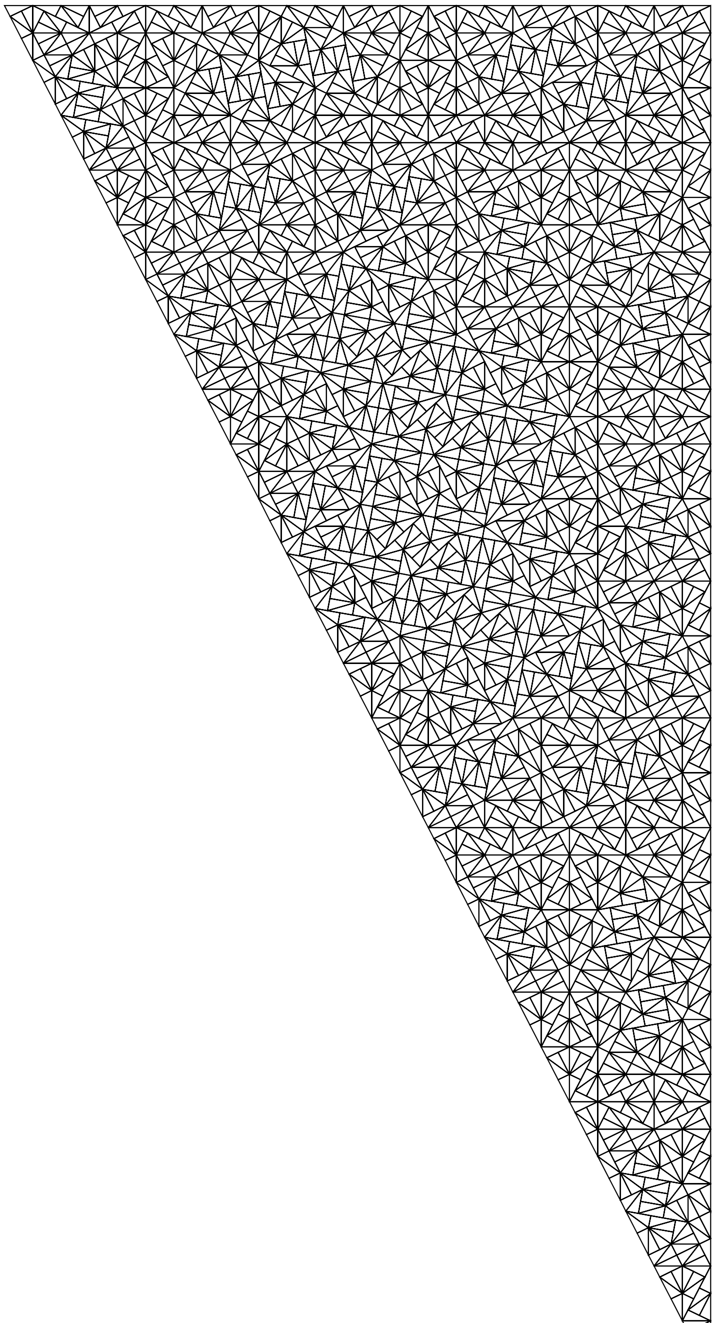

Figure 7. Pinwheel patch $\phi_p^5 a$

Historically, substitution tiling systems $X_\phi \subset X(\mathcal{A})$ have been of interest primarily when there are alphabets $\mathcal{A}'$ such that $X_\phi \simeq X(\mathcal{A}')$ (for some notion of isomorphism), as is the case for the pinwheel and chair systems. (For a discussion of the alphabets $\mathcal{A}'$ for the pinwheels and chairs see [Ra1] and [Sen].) This aspect of substitution tilings has only indirect relevance to this paper, although it is partial justification for our notion of isomorphism.

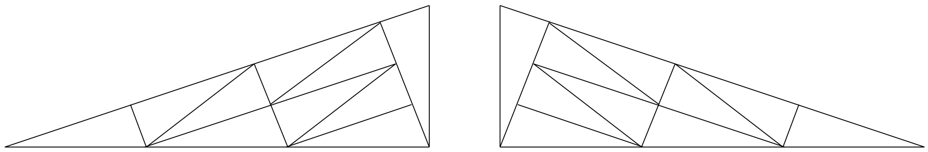

Figure 8. The substitution for pinwheel variant tilings

We are interested in classification of substitution systems. Basically, we view two classes of tilings as isomorphic if there is a homeomorphism intertwining the action of the Euclidean group of the two systems; a precise definition is given in §3. But there is a lot of extra structure involved in a substitution tiling system and it is appropriate to minimize the features necessary for the analysis. In particular, we are interested in using the geometry of the polyhedra to help classify tiling substitution systems. Consider the pinwheel and chair systems. It is not hard to show [Ra2] that for the pinwheel there is only one translation invariant (Borel probability) measure, which is therefore invariant under all rotations, while the only rotations under which all translation invariant measures for the chair system are invariant are the multiples of $2\pi/4$. This difference in symmetry should imply that the two systems are not isomorphic. (Note that this argument makes no use of the substitutions.)

A slight variant of the pinwheel is defined by the 1-3-$\sqrt{10}$ right triangle and its reflection, and the substitution of Fig. 8. The above technique would not differentiate between the pinwheel and this variant since in both cases there is only one translation invariant measure, with the same (full) rotational symmetry.

We will define a finer invariant, a subgroup of $SO(2)$, which *can* distinguish between the pinwheel and the variant. The fact that the pinwheel makes use of rotation by a different irrational angle than the variant, namely $\arctan(1/2)$ rather than $\arctan(1/3)$, gives rise to a different dense subgroup of $SO(2)$. The symmetry of translation invariant measures, which only senses the closure of this group, cannot tell the difference, but we will see that the difference in groups is sufficient to disprove isomorphism of the systems.

Given a tiling x and some tile a of type j in it, let $R_j(a, x) \subset SO(m)$ be the set of relative orientations with respect to a of the tiles of type j in x; that is, $R_j(a, x)$ is the set of rotations of x which bring a tile of type j parallel to (the fixed)

a. The group generated by $R_j(a, x)$ is easily seen to be generated by the relative orientations between *all* pairs of tiles of type j in x; in particular it is independent of a, and we denote it by $\mathcal{O}_j(x)$. Furthermore,

Lemma 1. If x' has a tile a' of type j parallel to the tile a in x, then $\mathcal{O}_j(x) = \mathcal{O}_j(x')$. For any $\tilde{x}$, $\mathcal{O}_j(\tilde{x})$ is conjugate to $\mathcal{O}_j(x)$.

Proof. First note that $\mathcal{O}_j(x)$ is generated by the relative orientations between a and all other tiles of type j in x. So consider the generator g of $\mathcal{O}_j(x)$ which is the relative orientation of a tile c with respect to a in x. We will show that $g \in \mathcal{O}_j(x')$, from which it follows that $\mathcal{O}_j(x) \subset \mathcal{O}_j(x')$. By symmetry, we would then have $\mathcal{O}_j(x') \subset \mathcal{O}_j(x)$, and hence $\mathcal{O}_j(x) = \mathcal{O}_j(x')$.

From the definition of substitution tilings, the tiles a and c can be thought of as belonging to some supertile A of level n (although not all of A need exist in x). Since x' is tiled by supertiles of level n, there is a supertile A' of level n in x' containing a pair of tiles, a'' and c'', which have the same positions relative to A' as do a and c relative to A. See Fig. 9.

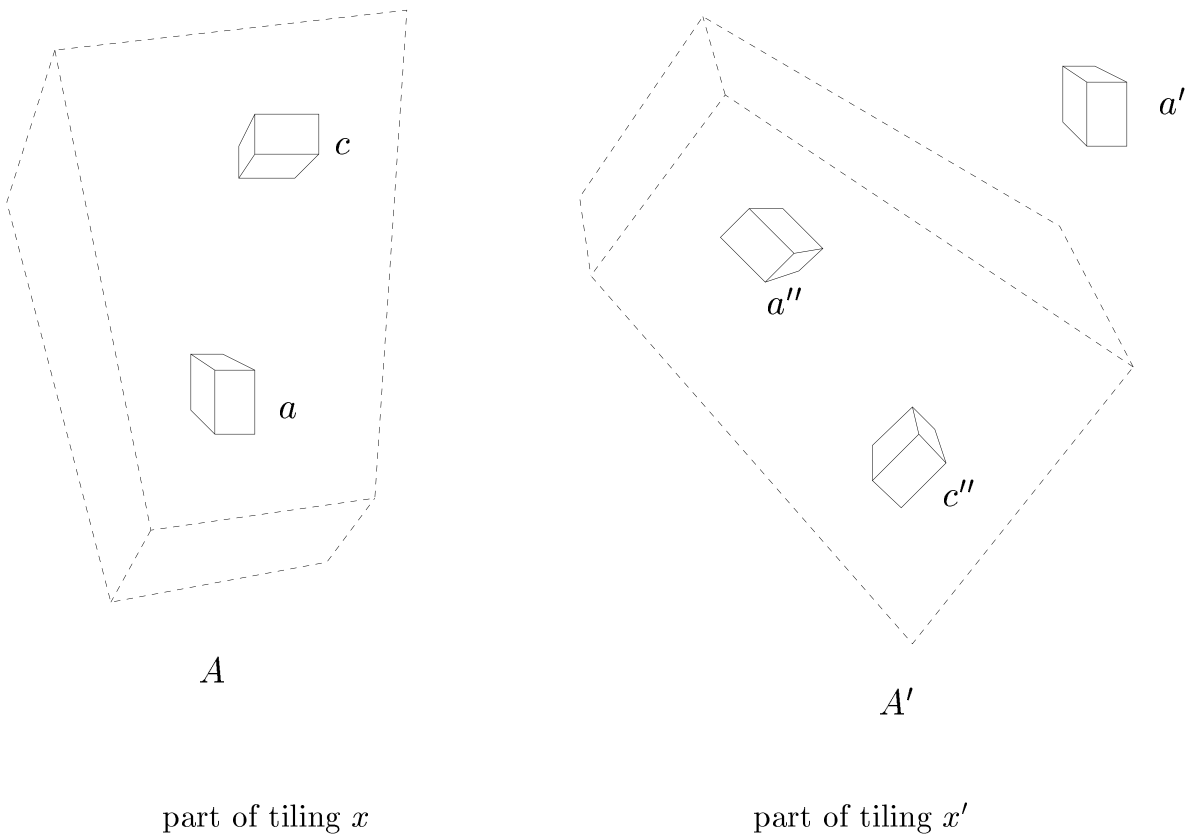

Figure 9

Let g' be the relative orientation of c'' with respect to a''. Then $g = R^{-1} g' R$

where R is the relative orientation of A' with respect to A. But R is then also the relative orientation of a'' with respect to a, which is the same as that of a'' with respect to a', and R is thus an element of $\mathcal{O}_j(x')$. But then g is an element of $\mathcal{O}_j(x')$, as claimed.

If $\tilde{x}$ is any tiling at all, $\mathcal{O}_j(x)$ and $\mathcal{O}_j(\tilde{x})$ are conjugate by an element of $SO(m)$, namely a rotation which makes a tile of type j in $\tilde{x}$ parallel to one in x. ∎

Finally we consider the dependence of $\mathcal{O}_j(x)$ on j. We call two subgroups of $SO(m)$ "c-equivalent" if each is conjugate (in $SO(m)$) to a subgroup of the other. Note that in $SO(2)$ c-equivalence is the same as identity.

Lemma 2. All $\mathcal{O}_j(x)$ are c-equivalent.

Proof. Consider any two tiles a and b of type j in x, and let g be the relative orientation of b with respect to a. Let k be a different tile type. After one substitution a and b give rise to tiles a' and b' of type k in the tiling ϕx. The relative orientation of b' with respect to a' is again g, since g takes each part of b onto the corresponding part of a. Applying this construction to all the generators of $\mathcal{O}_j(x)$, we see that $\mathcal{O}_j(x)$ is a subgroup of $\mathcal{O}_k(\phi x)$. Similarly, $\mathcal{O}_k(x)$ is a subgroup of $\mathcal{O}_j(\phi x)$. But $\mathcal{O}_j(x)$ and $\mathcal{O}_k(x)$ are conjugate to $\mathcal{O}_j(\phi x)$ and $\mathcal{O}_k(\phi x)$, respectively, so $\mathcal{O}_j(x)$ and $\mathcal{O}_k(x)$ are conjugate to subgroups of each other. ∎

In substitution tiling systems with the following property (including all the examples considered in this paper), the analysis simplifies considerably.

Property P: For every $a \in \mathcal{A}$, there exists $n_a > 0$ such that $\phi^{n_a} a$ contains a tile a', of the same type as a, that is parallel to a.

Lemma 3. Assume conditions i)-iii) and Property P. Then the groups $\mathcal{O}_j(x)$, for fixed x, are all the same.

Proof. Let $n = \Pi_a n_a$. Since the tilings defined by ϕ^n are the same as those defined by ϕ we can, without loss of generality, assume $n = 1$, so that ϕx contains tiles parallel to every tile of x. Then, by Lemma 1, $\mathcal{O}_j(x) = \mathcal{O}_j(\phi x)$ and $\mathcal{O}_k(x) = \mathcal{O}_k(\phi x)$. But we have shown that $\mathcal{O}_k(x) \subset \mathcal{O}_j(\phi x)$ and $\mathcal{O}_j(x) \subset \mathcal{O}_k(\phi x)$, so $\mathcal{O}_j(x) = \mathcal{O}_k(x)$. ∎

The conjugacy class of the groups $\mathcal{O}_j(x)$ is one key to the geometry of substitution tiling systems. But before we can use these groups for systems that are not associated with tilings, we must refer to the relative orientations in a more fundamental way. The essential observation is that, if tilings $x \neq y$ agree in some neighborhood of the origin in Euclidean space, then ϕx and ϕy will agree in a larger neighborhood of the origin, so we typically expect $d(\phi x, \phi y) < d(x, y)$. Thus ϕ defines a (local) contracting direction at each point in X_ϕ. Formally we define, for each x in the substitution tiling system X_ϕ and for each $\epsilon > 0$, the following subset of the ϵ-ball at x:

$$S^\epsilon(x) = \{y \in X_\phi \,:\, d(\phi^n y, \phi^n x) < \epsilon \text{ for all } n \geq 0, \text{ and } d(\phi^n y, \phi^n x) \underset{n\to\infty}{\longrightarrow} 0\}. \quad 3)$$

Theorem 1. Assume a substitution tiling system X_ϕ satisfying i)-iii).

a) Given any $\epsilon > 0$ there exists $N > 0$ such that $B_N(\partial y) = B_N(\partial x)$ implies $y \in S^\epsilon(x)$.

b) There exists $\epsilon > 0$ such that, for every $x \in X_\phi$, $y \in S^\epsilon(x)$ and every tile $a \in x$ that meets the origin, there is a tile $a' \in y$ that exactly coincides with a.

Proof. a) is immediate from the form of the metric. The proof of b) requires the following two lemmas.

Lemma 4. For every $N > 0$ and every neighborhood U of the identity in G_E there exists $\epsilon > 0$ with the following property: Let $x, x' \in X(\mathcal{A})$ be any two tilings with $d(x, x') < \epsilon$, and let a be a tile of x that is contained in B_N. Then x' contains a tile a' of the form $T^g a$ where $g \in U$.

Proof. Let $z > 0$ be such that for each $b \in \mathcal{A}$ some ball of diameter z lies in the interior of b. Fix some $\delta \in (0, z/3)$ and define the heart $h_\delta(b)$ of $b \in \mathcal{A}$ as $\{p \in b \,:\, ||p - q|| > \delta \text{ for all } q \in \partial b\}$. By the corridor $C_\delta(x)$ of a tiling x we mean the complement of $\cup_{b\in x} h_\delta(b)$. Let D be the largest of the diameters of all $b \in \mathcal{A}$. Without loss of generality, we can assume $N > D$.

With this notation we note that if $d(x, x') < \delta/(N + D)$ we have $B_N(\partial x) \subset C_\delta(x')$ and $B_N(\partial x') \subset C_\delta(x)$. So if $\delta \sim 0$ each tile in x in B_N is closely approximated by some tile in x' and vice versa. In particular it now follows that for small enough ϵ, if $d(x, x') < \epsilon$ then the tiles $a' \in x'$ must be of the same type as the tiles $a \in x$ they approximate, and in fact satisfy $a' = T^g a$ with $g \in U$. ■

Lemma 5. For each $N > 0$ there is a neighborhood U_N of the identity in G_E with the following property: If a is a tile in B_N and g_1 and g_2 are distinct elements of U_N, then $T^{g_1} a$ and $T^{g_2} a$ overlap but are distinct. In particular, it is impossible for $T^{g_1} a$ and $T^{g_2} a$ to both be tiles in the same tiling.

Proof. This follows from the continuity of the action of G_E on tiles, and the fact that polyhedra do not admit infinitesimal symmetries. ■

We now return to the proof of Theorem 1. We will occasionally use the notation $G_E = SO(m) \times \mathbb{R}^m$, with $g = (r, t)$ representing a rotation r about the origin followed by a translation t. Pick $N > D$ and let U_N be as in Lemma 5. Pick a smaller bounded neighborhood $U \subset U_N$ of the identity of G_E with the property that $(r, t) \in U$ implies $(r, |\phi| t) \in U_N$. Then pick ϵ small enough that Lemma 4 applies.

Let a be a tile of x containing the origin. By Lemma 4 there is a tile a' in y of the form $T^g a$ with $g = (r, t) \in U \subset G_E$. We will show that $t \neq 0$ implies that, for some n, $d(\phi^n x, \phi^n y) > \epsilon$, while $t = 0,\ r \neq 0$ implies that $\lim_{n\to\infty} d(\phi^n x, \phi^n y) \neq 0$. This will complete the proof.

Note that $\phi^n a' = T^{(r,|\phi|^n t)}\phi^n a$. If $t \neq 0$, pick n such that $(r, |\phi|^n t)$ is outside the neighborhood U but in U_N. Let $\tilde{a}$ be a tile of $\phi^n a$ containing the origin. Then there is a tile $\tilde{a}' = T^{(r,|\phi|^n t)}\tilde{a}$ in $\phi^n y$. But by Lemma 5 this means there cannot be a tile of the form $T^g\tilde{a}$ in y with $g \in U$. By Lemma 4 this means that $d(\phi^n x, \phi^n y) \geq \epsilon$.

If $t = 0$ then $\phi^n a' = T^{(r,0)}\phi^n a$. If $r \neq 0$, for every tile $\tilde{a} \in \phi^n a$ containing the origin there is a tile $\tilde{a}' \in \phi^n a'$ overlapping it and with relative orientation r, which implies that the distance between $\phi^n x$ and $\phi^n y$ will not go to zero. ∎

We now define the metric analog of our relative orientation groups $\mathcal{O}_j(x)$. As before, we describe an element of G_E as a rotation r followed by a translation t. Let

$$R^\epsilon(x) = \{r \in SO(m) : \text{there exists } t \text{ such that } T^{(r,t)}x \in S^\epsilon(x)\}. \qquad 4)$$

Let $\mathcal{O}^\epsilon(x)$ be the group generated by $R^\epsilon(x)$. Assuming ϵ small enough for Theorem 1b, we see that every $r \in R^\epsilon(x)$ is the relative orientation of a tile of x with respect to a corresponding tile of x near the origin. By Theorem 1a, if C is a region of x containing B_N, and if C' is any region of x congruent to C, then $R^\epsilon(x)$ includes the relative orientation of C' to C.

Before we proceed we will make the following technical assumption about the substitution tiling systems we consider:

Property F. The subset of tilings x, for which every fixed finite ball B of Euclidean space is contained in some supertile of finite level in x, is of full measure for every translation invariant measure on X_ϕ.

We will prove that Property F holds for a large class of interesting systems, at least those satisfying the following restriction:

iv) A tiling can only be tiled in one way by supertiles of level n, for any $n \geq 1$.

This assumption, which implies that ϕ is a homeomorphism on X_ϕ, is satisfied by all known nonperiodic examples. In fact it is automatically true for a system (such as the chair) that contains nonperiodic tilings and in which the tiles only appear in finitely many orientations in any tiling [Sol].

If a tiling contains two or more regions each tiled by supertiles of level n for all $n \geq 0$, we call these regions supertiles of infinite level. Recall that any tiling is tiled by supertiles of any finite level n. If a ball in a tiling x fails to lie in *any* supertile of any level n, then x is tiled by two or more supertiles of infinite level, with the offending ball straddling a boundary. (One can construct a pinwheel tiling with two supertiles of infinite level as follows. Consider the rectangle consisting of two supertiles of level $n-1$ in the middle of a supertile of level n. For each $n \geq 1$ orient such a rectangle with its center at the origin and its diagonal on the x-axis, and fill out the rest of a (non-pinwheel) tiling x_n by periodic extension. By compactness this sequence has a convergent subsequence, which will be a pinwheel tiling and which will consist of two supertiles of infinite level.)

We now use the above to prove:

Lemma 6. For a substitution tiling system satisfying i) – iv), let S be the set of tilings in which some ball does not lie within a supertile of any level n. S has zero measure with respect to any translation invariant measure on X_ϕ.

Proof. We only give the proof for dimension $m = 2$. Note first that the boundary of a supertile of infinite level must be either a line, or have a single vertex, since it is tiled by supertiles of all levels and therefore cannot contain a finite edge. Furthermore, for a given substitution system there is a constant K such that no tiling in it contains more than K vertices of supertiles of infinite level; specifically, one can take $K = 2\pi/p$ where p is the smallest angle of any of the vertices of the tiles.

Next we fix some orthogonal coordinate system in the plane and decompose S into disjoint subsets as follows. Let $C = [0,1) \times [0,1)$ be the "half open" unit edge square in $\mathbb{R}^2$. Let C_t be the translate of C by the vector t. Let S' be the subset of S consisting of tilings containing vertices of supertiles of infinite level. For $x \in S'$ we choose a vertex $V(x)$ by lexicographic order: we choose that vertex which in the given coordinate system has the largest first coordinate; if there is more than one with that coordinate we choose the one with the largest second coordinate. Then we decompose $S' = \cup_{t\in\mathbb{Z}^2} S'(t)$, where $x \in S' \cap S'(t)$ if $V(x) \in C_t$. It is easy to see that each $S'(t)$ is measurable, and that they are translates of one another so they must have zero measure with respect to any translation invariant measure.

The tilings $x \in S/S'$ contain two supertiles of infinite level, each occupying a half plane. Next we decompose $S/S' = \cup_{j,k\in\mathbb{Z}}\, \sigma_j \cup \sigma'_k$ where $x \in S/S' \cap \sigma_j$ if the boundary between the supertiles of infinite level crosses the first axis in $[j, j+1)$, and $x \in S/S' \cap \sigma'_k$ if the boundary between the supertiles of infinite level is parallel to the first axis and crosses the second axis in $[k, k+1)$. Note that all sets σ_j are translates of one another, and all sets σ'_k are translates of one another, so S/S' has zero measure with respect to any translation invariant measure. ■

Theorem 2. For any substitution tiling system X_ϕ satisfying i) - iv), there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, and for almost every tiling $x \in X_\phi$, $\mathcal{O}^\epsilon(x)$ is c-equivalent to $\mathcal{O}_j(x)$ for some (and therefore any) j. Up to conjugacy, $\mathcal{O}_j(x)$ is independent of x. Furthermore, if ϕ satisfies Property P then $\mathcal{O}^\epsilon(x) = \mathcal{O}_j(x)$ for some (and therefore any) j.

Proof. From Theorem 1b it follows that, for small ϵ, $\mathcal{O}^\epsilon(x)$ is contained in $\mathcal{O}_j(x)$, where j is the type of any of the tiles of x which meet the origin. On the other hand, let N correspond to ϵ in Theorem 1a. By Lemma 6, for almost every x there is some n such that the tiles which intersect B_N are contained in some supertile b of level n in x. Let k be the type of b. It follows from Theorem 1a that $\mathcal{O}_k(x')$ is a subgroup of $\mathcal{O}^\epsilon(x)$, where $x' = \phi^{-n}x$. But $\mathcal{O}_k(x')$ and $\mathcal{O}_j(x)$ are c-equivalent, so $\mathcal{O}_j(x)$ is conjugate to a subgroup of $\mathcal{O}_k(x')$, and therefore is conjugate to a subgroup of $\mathcal{O}^\epsilon(x)$. So $\mathcal{O}_j(x)$ and $\mathcal{O}^\epsilon(x)$ are c-equivalent. By Lemma 1, $\mathcal{O}_j(x)$ is, up to conjugacy, independent of x. If ϕ satisfies Property P, then $\mathcal{O}_k(x') = \mathcal{O}_k(x) = \mathcal{O}_j(x)$. Since $\mathcal{O}_j(x) = \mathcal{O}_k(x') \subset \mathcal{O}^\epsilon(x) \subset \mathcal{O}_j(x)$, $\mathcal{O}^\epsilon(x) = \mathcal{O}_j(x)$. ∎

The point of this theorem is to represent the geometric information contained in the relative orientations by an isomorphism invariant quantity; in particular to describe the information without referring directly to tiles.

3. Abstract Substitution Systems

In going from Lemma 2 to Theorem 2 we see that we can associate with each substitution tiling system a c-equivalence class of subgroups of $SO(m)$ in a reasonably fundamental way. We are now ready to address the abstract setting.

We define a substitution (dynamical) system as a compact metric space X_ϕ on which there is a continuous action $(g,x) \in G_E \times X_\phi \longrightarrow T^g x \in X_\phi$ of the connected Euclidean group G_E and a map $\phi : X_\phi \to X_\phi$ such that $\phi[T^g x] = T^{\phi(g)}x$ for all x, where $\phi(g)$ is the conjugate of g by the similarity of Euclidean space consisting of stretching about the origin by $|\phi| > 1$.

The map ϕ is not intrinsic to the substitution system X_ϕ since, for tiling systems, ϕ and ϕ^k lead to the same set of tilings. The objects $S^\epsilon(x)$, $R^\epsilon(x)$ and $\mathcal{O}^\epsilon(x)$ are well defined in our abstract setting. Motivated by the last section, we say that the substitution systems X_ϕ and X'_ϕ, with substitution functions ϕ and ϕ', are *isomorphic* if there are subsets $\hat{X}_\phi \subset X_\phi$ and $\hat{X}'_{\phi'} \subset X'_{\phi'}$, each invariant under G_E and of full measure for all translation invariant measures, and a homeomorphism τ between $\hat{X}_\phi$ and $\hat{X}'_{\phi'}$ which respects the action of G_E and the "local contracting directions" $S^\epsilon(x)$. Respecting the local contracting directions means that, for each $x \in \hat{X}_\phi$, $\epsilon > 0$ and $\epsilon' > 0$, there exist $\tilde{\epsilon} > 0$ and $\tilde{\epsilon}' > 0$ such that $\tau[S^{\tilde{\epsilon}}(x)] \subset S^\epsilon(\tau x)$, and $\tau^{-1}[S^{\tilde{\epsilon}'}(\tau x)] \subset S^{\epsilon'}(x)$. (This is a form of "almost topological" isomorphism.) We note that this allows us to generalize our discussion of substitution tiling systems to include tiling systems which do not quite fit the conditions of §2. In particular, our analysis will be generalized to include the various versions of Penrose tilings of the plane, such as the kite & dart tilings, both the substitution version and the version with edge markings [Gar, Ra3], and the various tilings discussed in [G-S, Moz].

We will need to introduce a few more definitions. Given two subgroups G_1 and G_2 of $SO(m)$ we write $G_1 \prec G_2$ if G_1 is conjugate (by an element of $SO(m)$) to a subgroup of G_2. The binary relation $\prec$ lifts in an obvious way to a partial ordering on the set of c-equivalence classes. We denote by "lower bound" to a set $\tilde{S}$

of subgroups of $SO(m)$ any c-equivalence class of groups G each of which satisfies $G \prec S$ for all $S \in \tilde{S}$. It is almost immediate that $\mathcal{O}^{\epsilon}(x) \prec \mathcal{O}^{\epsilon'}(x)$ if $\epsilon < \epsilon'$. For each $x \in X_\phi$ we define $\hat{\mathcal{O}}(x)$ as the set of all lower bounds of the family $\{\mathcal{O}^{\epsilon}(x) : \epsilon > 0\}$; it is nonempty since it contains $\{e\}$. Note that the set $\hat{\mathcal{O}}(x)$ is an invariant for substitution systems – if τ is an isomorphism then $\hat{\mathcal{O}}(\tau x) = \hat{\mathcal{O}}(x)$ for almost every x. For substitution tiling systems, the sets $\hat{\mathcal{O}}(x)$ have unique greatest elements which are constant for almost every x with respect to every translation invariant measure. In the latter case, where $\hat{\mathcal{O}}(x)$ has an almost everywhere constant greatest element, we denote this greatest element by $[\mathcal{O}](X_\phi)$. Note that $[\mathcal{O}](X_\phi)$ is a c-equivalence class, unlike $\mathcal{O}^{\epsilon}(x)$, which is a specific group. We have thus generalized the analysis of substitution tiling systems to the more general setting.

As with substitution tiling systems, we can avoid the use of c-equivalence classes for systems with a special property.

Property P$'$: For almost every x there exists an $\epsilon > 0$ such that, if $0 < \epsilon' < \epsilon$, then $\mathcal{O}^{\epsilon'}(x) = \mathcal{O}^{\epsilon}(x)$.

Note that, by Theorem 2, any tiling substitution system that satisfies Property P also satisfies Property P$'$. If a substitution system satisfies Property P$'$, we can define $\mathcal{O}(x)$ to be $\mathcal{O}^{\epsilon}(x)$ for ϵ sufficiently small. If the conjugacy class of $\mathcal{O}(x)$ is almost everywhere constant, we define $[\mathcal{O}]_0(X_\phi)$ to be that conjugacy class. The previously defined $[\mathcal{O}](X_\phi)$ is, of course, the c-equivalence class of $[\mathcal{O}]_0(X_\phi)$.

Lemma 7. If $\tau : (X, \phi) \to (X', \phi')$ is an isomorphism of substitution systems, and if (X, ϕ) satisfies Property P$'$, then (X', ϕ') satisfies Property P$'$. Furthermore, for almost every $x \in X$, $\mathcal{O}(\tau x) = \mathcal{O}(x)$. In particular, if $\mathcal{O}(x)$ is almost everywhere constant up to conjugacy then $\mathcal{O}(\tau x)$ is almost everywhere constant up to conjugacy and $[\mathcal{O}]_0(X'_{\phi'}) = [\mathcal{O}]_0(X_\phi)$.

Proof. Let x be a generic point of X. Since (X, ϕ) has Property P$'$ we can find $\epsilon_0 > 0$ such that, for $0 < \epsilon < \epsilon_0$, $\mathcal{O}^{\epsilon}(x) = \mathcal{O}^{\epsilon_0}(x) = \mathcal{O}(x)$. Since τ is an isomorphism we can find $\tilde{\epsilon}$ such that $\mathcal{O}^{\tilde{\epsilon}}(\tau x) \subset \mathcal{O}^{\epsilon_0}(x)$. Now let $\epsilon'_0 = \tilde{\epsilon}$. We will show that, for any $\epsilon' \leq \epsilon'_0$, $\mathcal{O}^{\epsilon'}(\tau x) = \mathcal{O}^{\epsilon'_0}(\tau x) = \mathcal{O}(x)$. From this it will follow that (X', ϕ') has Property P$'$ and that $\mathcal{O}(\tau x) = \mathcal{O}(x)$.

Fix any $0 < \epsilon' \leq \epsilon'_0$. Since τ is an isomorphism there exists $\tilde{\epsilon}' > 0$ such that $\mathcal{O}^{\tilde{\epsilon}'}(x) \subset \mathcal{O}^{\epsilon'}(\tau x)$. But $\mathcal{O}^{\epsilon'}(\tau x) \subset \mathcal{O}^{\epsilon'_0}(\tau x) \subset \mathcal{O}^{\epsilon_0}(x)$. If $\mathcal{O}^{\tilde{\epsilon}'}(x) = \mathcal{O}^{\epsilon_0}(x)$ then all the inclusions must be equalities, and we are done. So it suffices to show $\mathcal{O}^{\tilde{\epsilon}'}(x) = \mathcal{O}^{\epsilon_0}(x)$. If $\tilde{\epsilon}' \leq \epsilon_0$ this follows from the definition of ϵ_0. But if $\tilde{\epsilon}' > \epsilon_0$ then $O^{\epsilon_0}(x) \subset O^{\tilde{\epsilon}'}(x)$, so $\mathcal{O}^{\tilde{\epsilon}'}(x) = \mathcal{O}^{\epsilon_0}(x)$. ∎

4. Examples and Analysis of the Invariant

For the chair tilings, $[\mathcal{O}]_0(X_{\phi_c})$ is the cyclic subgroup of $SO(2)$ of order 4; for the pinwheel $[\mathcal{O}]_0(X_{\phi_p})$ is the group generated by rotations by $\pi/2$ and $2\arctan(1/2)$; for the variant of the pinwheel $[\mathcal{O}]_0(X_{\phi_v})$ is the group generated by $\pi/2$ and $2\arctan(1/3)$ [RaS]. It is clear that these are distinct.

Rotations appear in more interesting ways in 3 dimensional tilings, for example the quaquaversal and dite & kart substitution tiling systems, defined in [CoR] and [RaS] respectively. These systems both satisfy Property P and therefore Property P′. Let R_x^θ be a rotation about the x axis by an angle θ, with similar notation for other axes. If we denote by $G(p,q)$ the subgroup of $SO(3)$ generated by $R_x^{2\pi/p}$ and $R_y^{2\pi/q}$, it can be shown [CoR, RaS] that $[\mathcal{O}]_0(X_{\phi_q})$ is the conjugacy class of $G(6,4)$ for the quaquaversal tilings and $[\mathcal{O}]_0(X_{\phi_{d\&k}})$ is the conjugacy class of $G(10,4)$ for the dite & kart tilings. We shall see that $G(6,4)$ and $G(10,4)$ are not conjugate (indeed not even c-equivalent) by using the following obvious fact: if the groups G and G' are conjugate (or c-equivalent) and one of them has an element of order m (finite or infinite) then the other must have an element of order m.

Structure Theorem for G(p,q) [RaS]

a) If $p,q \geq 3$ are odd, then $G(p,q)$ is isomorphic to the free product

$$\mathbb{Z}_p * \mathbb{Z}_q = < \alpha, \beta \,:\, \alpha^p,\, \beta^q > . \tag{5}$$

b) If $p \geq 4$ is even and $q \geq 3$ is odd, then $G(p,q)$ has the presentation

$$< \alpha, \beta \,:\, \alpha^p,\, \beta^q,\, (\alpha^{p/2}\beta)^2 > . \tag{6}$$

b) If $p \geq 4$ is even and $q = 2s$, $s \geq 3$ odd, then $G(p,q)$ has the presentation

$$< \alpha, \beta \,:\, \alpha^p,\, \beta^q,\, (\alpha^{p/2}\beta)^2,\, (\alpha\beta^s)^2 > . \tag{7}$$

d) If 4 divides both p and q, then $G(p,q) = G([p,q],4)$, where $[p,q]$ denotes the least common multiple of p and q.

e) If 4 divides m, then $G(m,4)$ has the presentation

$$< \alpha, \beta, : \alpha^m,\, \beta^4,\, (\alpha^{m/2}\beta)^2,\, (\alpha\beta^2)^2, (\alpha^{m/4}b)^3 > . \tag{8}$$

In cases a), b) and c), the isomorphism between the abstract presentation and $G(p,q)$ is given by $\alpha \mapsto R_x^{2\pi/p}$, $\beta \mapsto R_y^{2\pi/q}$. In case e) the isomorphism is similar.

Theorem 3. a) If 4 does not divide both p and q then the orders of elements of finite order in $G(p,q)$ are $\{\text{factors of } p\} \cup \{\text{factors of } q\}$; b) If 4 divides both p and q then the orders of elements of finite order in $G(p,q)$ are $\{\text{factors of } [p,q]\} \cup \{3\}$.

Corollary 1. If p and q are not both divisible by 4, and p' is not a factor of p or q, then $G(p,q)$ and $G(p',q')$ are not c-equivalent.

Corollary 2. The quaquaversal and dite & kart systems are not isomorphic.

Proof of the theorem.

a) Assume $g \in G(p,q)$ has finite order $\neq 1$. We know g can be expressed in one of the forms $g = A^{a_1}B^{b_1}A^{a_2}\cdots A^{a_{n+1}}$, $g = B^{b_1}A^{a_1}B^{b_2}\cdots B^{b_{n+1}}$, $g = B^{b_1}A^{a_1}B^{b_2}\cdots A^{a_n}$ or $g = A^{a_1}B^{b_1}A^{a_2}\cdots B^{b_n}$, with all $0 < a_j < p$, $0 < b_j < q$ and $n \geq 1$. Within the class of $g' \in G(p,q)$ which are conjugate to g with respect to $G(p,q)$, we assume that n is minimal. Assume $n \geq 2$; we will obtain a contradiction. Since one could conjugate with A^{a_1}, the form $A^{a_1}B^{b_1}A^{a_2}\cdots A^{a_n}$ can be exchanged for $B^{b_1}A^{a_2}\cdots A^{a_n}$ and since one could conjugate with B^{b_1}, the form $B^{b_1}A^{a_1}B^{b_2}\cdots B^{b_n}$ can be exchanged for $A^{a_1}B^{b_2}\cdots B^{b_n}$, and the form $B^{b_1}A^{a_1}B^{b_2}\cdots A^{a_n}$ for $A^{a_1}B^{b_1}A^{a_2}\cdots B^{b_n}$. So we assume that the form is $A^{a_1}B^{b_1}A^{a_2}\cdots B^{b_n}$. If $a_j = p/2$ (resp. $b_j = q/2$) for any j then we could use the relation $A^{p/2}B^b = B^{-b}A^{p/2}$ (resp. $B^{q/2}A^a = A^{-a}B^{q/2}$) to reduce the value of n; thus these values of a_j (or b_j) cannot occur. But then, by the structure theorem, g has infinite order, which is a contradiction. Thus n must equal 1, and g can be assumed to be of the form A^{a_1}, B^{b_1} or $A^{a_1}B^{b_1}$. Considering $A^{a_1}B^{b_1}A^{a_1}B^{b_1}\cdots A^{a_1}B^{b_1}$, the only way $A^{a_1}B^{b_1}$ could have finite order is if $a_1 = p/2$ or $b_1 = q/2$, in which case g has order 2, and 2 is a factor of p or q. Finally, the elements A^{a_1} can have as orders any factor of p and the elements B^{b_1} can have as orders any factor of q.

b) If p and q are divisible by 4 then $G(p,q) = G([p,q],4)$, so we consider $G(m,4)$ with m divisible by 4. Using the presentation 8), we can put any $g \in G(m,4)$ in the form $WST^{a_1}ST^{a_2}\cdots ST^{a_n}E$ with $S = R_y^{2\pi/4}$, $T = R_x^{2\pi/m}$, $n \geq 0$, $a_j \neq km/4$ and with both W and E in the cube group $G(4,4)$. Assume g has finite order $\neq 1$ and that in its conjugacy class (which of course all have the same order), the smallest value of n in the above representation is ≥ 2. (We will obtain a contradiction to this.) By conjugation we eliminate W from g.

Now $G(4,4)$ can be partitioned: $G(4,4) = H_1 \cup H_1S \cup H_1SU$, where $U = R_x^{2\pi/4}$ and H_1 is the 8 element subgroup generated by S^2 and U. In detail,

$$H_1 = \{1, U, U^2, U^3, S^2, S^2U, S^2U^2, S^2U^3\}. \tag{9}$$

Some power of g equals the identity element:

$$(ST^{a_1}ST^{a_2}\cdots ST^{a_n}E)(ST^{a_1}ST^{a_2}\cdots ST^{a_n}E)(\cdots) = e. \tag{10}$$

We consider the three cases: i) $E \in H_1$; ii) $E \in H_1SU$; iii) $E \in H_1S$.

i) The factor E in 10) is of the form $(S^2)^aU^b$ with $a = 0,1$ and $b = 0,1,2,3$. We alter 10) to

$$[ST^{a_1}ST^{a_2}\cdots ST^{a_n}U^{(-1)^ab}(S^2)^a][ST^{a_1}ST^{a_2}\cdots ST^{a_n}U^{(-1)^ab}(S^2)^a][\cdots] = e, \tag{11}$$

or

$$\begin{split}[ST^{a_1}ST^{a_2}\cdots ST^{[a_n+(-1)^abm/4]}(S^{2a})][ST^{a_1}ST^{a_2}\cdots \\ \cdots ST^{[a_n+(-1)^abm/4]}(S^{2a})]\cdots = e,\end{split} \tag{12}$$

and we know [RaS] this cannot be the case. So we cannot have $E \in H_1$.

ii) E is now of the form $(S^2)^a U^b SU$ with $a = 0, 1$ and $b = 0, 1, 2, 3$. We now alter 10) to

$$ST^{a_1}ST^{a_2}\cdots ST^{[a_n+(-1)^a bm/4]}(S^2)^a SUST^{a_1}ST^{a_2}\cdots ST^{a_n}E\cdots = e. \qquad 13)$$

Using $SUS = USU$, 13) becomes

$$\begin{aligned} &ST^{a_1}ST^{a_2}\cdots ST^{[a_n+(-1)^a (b+1)m/4]}S^{2a}ST^{a_1+m/4}ST^{a_2}\cdots \\ &\cdots ST^{[a_n+(-1)^a (b+1)m/4]}S^{2a}SU\cdots = e. \end{aligned} \qquad 14)$$

Again, we know [RaS] this cannot be the case. So we cannot have $E \in H_1SU$.

iii) We cannot have $E \in H_1S$ and $n \geq 2$. For if we represent conjugacy by $\cong$,

$$\begin{aligned} g \cong ST^{a_1}ST^{a_2}\cdots ST^{a_n}E = ST^{a_1}ST^{a_2}\cdots ST^{a_n}(S^2)^a U^b S \\ \cong T^{a_1}ST^{a_2}\cdots ST^{[a_n+(-1)^a bm/4]}(S^2)^{a+1} \\ \cong ST^{a_2}\cdots ST^{[a_n+(-1)^a bm/4+(-1)^{a+1}a_1]}(S^2)^{a+1}, \end{aligned} \qquad 15)$$

and g is conjugate to a word with smaller n.

Thus $n = 0$ or $n = 1$. $n = 0$ means $g \in G(4,4)$, and these have orders $1, 2, 3, 4$. $n = 1$ means g is of the form $ST^{a_1}E$ where $a_1 \neq km/4$ and $E \in G(4,4)$. We again consider the three cosets to which E may belong. As before we see that cases i) and ii) lead to infinite order for g. But in case iii) g is conjugate to $T^{a_1}(S^2)^a U^b S^2 \cong T^c(S^2)^d$, which can have as orders the factors of m. ■

5. Conclusion

We have been concerned with substitution tilings of Euclidean spaces, and have defined an isomorphism invariant related to the group generated by the relative orientations of the tiles in a tiling. This feature is captured in an invariant way by means of a contractive behavior of the substitution. It is unrelated to other features of tiling systems, such as their topology, and we introduce the notion of substitution dynamical system to emphasize the features associated with the invariant.

To distinguish examples, for instance to distinguish the quaquaversal tilings from the dite & kart tilings, requires consideration of 2-generator subgroups of SO(3), in particular the orders of elements of such subgroups, which we analyze.

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