Dyonic Sectors and Intertwiner Connections in 2+1-dimensional Lattice $\mathbb{Z}_N$-Higgs Models.

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Abstract: We construct dyonic states ω_ρ in 2+1-dimensional lattice $\mathbb{Z}_N$-Higgs models, i.e. states which are both, electrically and magnetically charged. These states are parametrized by $\rho = (\varepsilon, \mu)$, where ε and μ are $\mathbb{Z}_N$-valued electric and magnetic charge distributions, respectively, living on the spatial lattice $\mathbb{Z}^2$. The associated Hilbert spaces $\mathcal{H}_\rho$ carry charged representations π_ρ of the observable algebra $\mathfrak{A}$, the global transfer matrix $\mathbf{t}$ and a unitary implementation of the group $\mathbb{Z}^2$ of spatial lattice translations. We prove that for coinciding total charges $q_\rho = \left(\sum_x \varepsilon(x), \sum_p \mu(p)\right) \in \mathbb{Z}_N \times \mathbb{Z}_N$ these representations are dynamically equivalent and we construct a local intertwiner connection $U(\Gamma) : \mathcal{H}_\rho \to \mathcal{H}_{\rho'}$, where $\Gamma : \rho \to \rho'$ is a path in the space of charge distributions $\mathcal{D}_q = \{\rho : q_\rho = q\}$. The holonomy of this connection is given by $\mathbb{Z}_N$-valued phases. This will be the starting point for a construction of scattering states with anyon statistics in a subsequent paper.

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Contents

1 Introduction

In this paper we continue our project initiated in [1] of a constructive analysis of states with anyonic statistics in a 2+1 dimensional lattice gauge theory. We investigate a general class of models with the discrete Abelian gauge group $\mathbb{Z}_N$ for arbitrary $N \in \mathbb{N}$, $N \geq 2$, and with discrete Higgs fields. The vacuum expectations in this theory are represented by classical expectations of an (euclidean) statistical mechanics model given by the thermodynamic limit of finite volume expectations of the form

$$\langle B_{cl} \rangle_\Lambda := \frac{\int d\alpha d\varphi \; B_{cl}(\alpha, \varphi) \, e^{-S_\Lambda}}{\int d\alpha d\varphi \; e^{-S_\Lambda}}, \tag{1.1}$$

with a generalized Wilson action:

$$S_\Lambda(\varphi, \, \alpha) := \sum_p S_g(d\alpha(p)) + \sum_b S_h(d\varphi(b) - \alpha(b)) \; . \tag{1.2}$$

Here φ and α are $\mathbb{Z}_N$-valued Higgs and gauge fields, respectively, on a euclidean space-time lattice $\mathbb{Z}^3$. Hence φ lives on sites, α lives on bonds, d denotes the lattice exterior derivative and the above sums go over all elementary positively oriented bonds b and plaquettes p in a finite space-time volume Λ of our lattice. B_{cl} is some classical observable, i.e. a gauge invariant function of the gauge field α and of the Higgs field φ with finite support. Here, gauge invariance means invariance under the simultaneous transformations $\varphi \to \varphi + \lambda$ and $\alpha \to \alpha + d\lambda$ for arbitrary $\lambda : \mathbb{Z}^3 \mapsto \mathbb{Z}_N$, with finite support. Above, the integrations over α and φ are actually finite sums, since these variables are discrete (to be precise, the discrete Haar measure on $\mathbb{Z}_N$ is employed).

In order to have charge conjugation symmetry and reflection positivity (i.e. a positive transfer matrix) the actions S_g and S_h will be chosen as even functions on $\mathbb{Z}_N \equiv \{0, \ldots, N-1\}$ taking their minimal value at $0 \in \mathbb{Z}_N$. Thus we have a general Fourier expansion

$$S_{g/h}(n) = -\frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \beta_{g/h}(m) \cos\left(\frac{2\pi mn}{N}\right), \tag{1.3}$$

$n \in \mathbb{Z}_N$, where we call $\beta_g(m)$ and $\beta_h(m)$, $m \in \mathbb{Z}_N$, the gauge and Higgs coupling constants, respectively[3]. We also require $\beta_{g/h}(m) = \beta_{g/h}(N-m)$, i.e. $\beta_{g/h}$ and $-S_{g/h}$ are in fact Fourier transforms of each other. This model describes a $\mathbb{Z}_N$-Higgs model where the radial degree of freedom of the Higgs field is frozen, i.e. $|\phi| = 1$ and $\phi(x) = e^{\frac{2\pi i}{N}\varphi(x)}$.

We will be interested in the so-called "free charge phase" of this model, which, roughly speaking, is obtained whenever, for all $0 \neq n \in \mathbb{Z}_N$,

$$S_g(n) - S_g(0) \quad \geq c_g \quad \gg 0, \tag{1.4}$$

$$S_h(n) - S_h(0) \quad \leq c_h^{-1} \quad \ll 1, \tag{1.5}$$

for large enough positive constants c_g and c_h. In this region convergent polymer and cluster expansions are available and have been analyzed in detail in [1], see also Appendix A for a short review.

[3] The values of $\beta_g(0)$ and $\beta_h(0)$ only determine additive constants to the action and will be fixed by convenient normalization conditions.

The first analysis of the structure of the charged states in this phase had been performed for the case of the group $\mathbb{Z}_2$ by Fredenhagen and Marcu in [2]. In that work it had been shown that electrically charged states exist in $d+1$ dimensions, $d \geq 2$ in the "free charge phase" of the model. The ideas employed by the authors involved a wide combination of methods from Algebraic Quantum Field Theory and Classical Statistical Mechanics. Later on the existence of electrically charged particles in the same model had been shown in [3] and the existence of multi-particle scattering states of these particles had been proven in [4], combining methods and results of [2] and of [5].

In [1] we extended some of these results to the $\mathbb{Z}_N$-Higgs model mentioned above and showed, after previous results of Gaebler on the $\mathbb{Z}_2$ case [6], the existence of magnetically charged states in $2+1$ dimensions. In [1] we also proved the existence of electrically and of magnetically charged particles in this model.

Our intention here is to show the existence of dyonic states ω_ρ in the "free charge phase" of our $\mathbb{Z}_N$-Higgs model, i.e. states carrying simultaneously electric and magnetic charges, $\rho = (\varepsilon, \mu)$. This had been performed in the $\mathbb{Z}_2$ case in [6]. We construct the associated charged representations of the observable algebra $\mathfrak{A}$ as the GNS-triples $(\pi_\rho, \mathcal{H}_\rho, \Omega_\rho)$ obtained from ω_ρ. These representations fall into equivalence classes labeled by the total charges $q_\rho = \left(\sum_x \varepsilon(x), \sum_p \mu(p)\right) \in \mathbb{Z}_N \times \mathbb{Z}_N$. We show that for each choice of $q \in \mathbb{Z}_N \times \mathbb{Z}_N$ the *state bundle*

$$\mathcal{B}_q := \bigcup_{\rho \in \mathcal{D}_q} (\rho, \mathcal{H}_\rho)$$

over the discrete base space $\mathcal{D}_q := \{\rho : q_\rho = q\}$ is equipped with a non-flat connection, i.e. a collection of unitary parallel transporters $U_{\rho',\rho} : \mathcal{H}_\rho \to \mathcal{H}_{\rho'}$ depending on paths Γ from ρ to ρ' in $\mathcal{D}_q$ such that $\pi_{\rho'} = \mathrm{Ad}\, U_{\rho',\rho} \circ \pi_\rho$. The holonomy of this connection is given by $\mathbb{Z}_N$-valued phases which appear as winding numbers between "electric" Wilson loops and "magnetic" vortex loops in the euclidean functional integral picture.

In an upcoming paper [10] this construction will be the starting point for an analysis of the anyonic statistics of scattering states in these models.

At this point we should also mention the previous work of J. Fröhlich and P. A. Marchetti on the construction of dyonic and anyonic states in the framework of euclidean lattice field theories ([7], [8] and [9]). In their approach the analogue of our Hilbert spaces $\mathcal{H}_\rho$ are obtained by Osterwalder-Schrader reconstruction methods which makes it difficult to discuss the representation theoretic background of the observable algebra.

In difference with their approach we work with the Hamiltonian description. In particular, we construct our states as functionals on the quasi-local observable algebra generated by the time-zero fields. As usual, in this approach the euclidean description reappears in the form of local transfer matrices T_V, whose ground state expectation values are given by functional integrals of the type (1.1). Still, we interpret and motivate our constructions mostly algebraically and consider the functional integral techniques only as a technical tool.

We now describe the plan of this paper in some more detail.

In Section 2 we introduce our basic setting. We follow the standard canonical quantization prescription of gauge theories in $A_0 = 0$ gauge to define the local algebras generated by the time-zero fields. We then define an "euclidean dynamics" in terms of local transfer matrices whose ground states give rise to finite volume euclidean functional integral expectations with actions (1.2)-(1.3). We also review the notion of *external* charges (related to a violation of Gauss' law) in order to distinguish them from the *dynamical* charges ("superselection sectors") that we are interested in this work.

Section 3 is devoted to the construction of dyonic sectors. We start with generalizing the Fredenhagen-Marcu prescription to obtain dyonic states ω_ρ on our observable algebra $\mathfrak{A}$. We then construct the associated GNS-representations $(\pi_\rho, \mathcal{H}_\rho, \Omega_\rho)$ of $\mathfrak{A}$ and implement the euclidean dynamics by a global transfer matrix on $\mathcal{H}_\rho$. We show that as representations of the "dynamic closure" $\hat{\mathfrak{A}} \supset \mathfrak{A}$ (i.e. the $*$-algebra generated by $\mathfrak{A}$ and the global transfer matrix) these representations are irreducible and pairwise equivalent provided their total charges $q_\rho \in \mathbb{Z}_N \times \mathbb{Z}_N$ coincide. (We also conjecture that they are dynamically inequivalent, if their total charges disagree). Finally we show that the infimum of the energy spectrum in the dyonic sectors is uniquely fixed by requiring charge conjugation symmetry and cluster properties of correlation functions for infinite space-like separation.

In Section 4 we construct an intertwiner algebra on $\mathbb{H}_q = \oplus_{\rho \in \mathcal{D}_q} \mathcal{H}_\rho$ by defining electric and magnetic "charge transporters" $\mathcal{E}_q(b)$, $\mathcal{M}_q(b) \in \mathcal{B}(\mathbb{H}_q)$ living on bonds b in $\mathbb{Z}^2$ and fulfilling local Weyl commutation relations. In terms of these intertwiners we obtain a unitary connection $U(\Gamma) : \mathcal{H}_\rho \to \mathcal{H}_{\rho'}$ intertwining π_ρ and $\pi_{\rho'}$ for any path $\Gamma : \rho \to \rho'$ in $\mathcal{D}_q$. The holonomy of this connection is given by $\mathbb{Z}_N$-valued phases. We conclude by applying our connection to construct a unitary implementation of the translation group in the dyonic sectors.

We remark at this point that we do not touch the question of the existence of dyonic *particles* in these models, i.e. of particles in the dyonic sectors carrying simultaneously electric and magnetic charges. To study the existence of such particles requires adaptation of the known Bethe-Salpeter kernels methods for situations involving charged particles in lattice models. This will be performed elsewhere. Clearly, if these particles exist they should be expected to show anyonic statistics among themselves.

A good part of the methods and results used here has been extracted from [1] and we will often refer to this paper when necessary. In particular, we will not repeat the proof of the convergence of the polymer expansion we are going to use since this point has been discussed in detail in [1], see however Appendix A for a short review. In fact, although polymer expansions are the main technical tool of this work (as well as of all other works on $\mathbb{Z}_N$-Higgs models cited above), our aim here is to formulate theorems and present results in a way which can be followed without any detailed knowledge on cluster expansions. Following this strategy, we abandon all statistical mechanics aspects of our proofs to Appendix B and reserve the main body of this work to exploit algebraic and quantum field theoretical concepts.

Remarks on the notation. Due to a different focus our notation will differ in many points from that of [1]. We will change our notation according to our needs of emphasis and clarity. The symbol $\square$ indicates "end of statement" and the symbol $\blacksquare$ indicates "end of proof". Products of operators run from the left to the right, i.e., $\prod_{a=1}^n A_a$ means $A_1 \cdots A_n$. For an invertible operator B, $\mathrm{Ad}\, B$ denotes the automorphism $B \cdot B^{-1}$. If $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ is an algebra acting on a Hilbert space $\mathcal{H}$ then we denote by $\mathfrak{A}'$ the *commutant* of $\mathfrak{A}$, i.e. the set of all operators of $\mathcal{B}(\mathcal{H})$ which commute with all elements of $\mathfrak{A}$. Here $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded operators acting on $\mathcal{H}$.

Acknowledgments. We would like to thank K. Fredenhagen for stimulating interest and several useful discussions.

2 The Basic Setting

We will always consider the lattice $\mathbb{Z}^d$, $d = 2,\, 3$ as a chain complex and denote by $(\mathbb{Z}^d)_p$ the elementary positively oriented p-cells in $\mathbb{Z}^d$. We also use the standard terminology "sites", "bonds" and "plaquettes" for 0-, 1- and 2-cells, respectively. By a (finite) volume $V \subset \mathbb{Z}^d$ we mean the closed chain sub-complex generated by a (finite) union of elementary d-cells in $(\mathbb{Z}^d)_d$. We denote by V_p the set of elementary oriented p-cells in V where, by definition, a p-cell is contained in $V_p \subset (\mathbb{Z}^d)_p$ if and only if it lies in the boundary of some (p+1)-cell contained in V_{p+1}. We denote by $\mathcal{C}_p(V) \equiv \mathbb{Z}V_p$ the set of p-chains in V and by $\mathcal{C}^p(V) \equiv \mathcal{C}^p(V, \mathbb{Z}_N)$ the set of $\mathbb{Z}_N$-valued cochains with support in V (i.e. group homomorphisms $\alpha : \mathcal{C}_p(V) \to \mathbb{Z}_N$). As usual we identify $\mathcal{C}^p(V)$ with the group of $\mathbb{Z}_N$-valued functions on $(\mathbb{Z}^d)_p$ with support in V_p. Hence, for $V \subset W$ we have the natural inclusion $\mathcal{C}^p(V) \subset \mathcal{C}^p(W)$. We also denote $\mathcal{C}^p := \mathcal{C}^p(\mathbb{Z}^d)$ and define $\mathcal{C}^p_{loc} \subset \mathcal{C}^p$ as the set of p-cochains with finite support. Often we will identify an elementary p-cell $c \in (\mathbb{Z}^d)_p$ with its characteristic p-cochain (i.e. taking the value $1 \in \mathbb{Z}_N$ on c and $0 \in \mathbb{Z}_N$ else).

Considered as a finite Abelian group $\mathcal{C}^p(V) \cong \mathbb{Z}_N^{|V_p|}$ is self-dual for all finite V, the pairing $\mathcal{C}^p \times \mathcal{C}^p \to U(1)$ being given by the homomorphism

$$(\alpha,\, \beta) \mapsto e^{i\langle\alpha,\beta\rangle} := \exp\left(\frac{2\pi i}{N} \sum_{c \in V_p} \alpha(c)\beta(c)\right) \tag{2.1}$$

We denote by $d : \mathcal{C}^p \to \mathcal{C}^{p+1}$ and $d^* : \mathcal{C}^p \to \mathcal{C}^{p-1}$ the exterior derivative and its adjoint, such that

$$e^{i\langle\alpha,\, d\beta\rangle} = e^{i\langle d^*\alpha,\, \beta\rangle}$$

for all $\alpha \in \mathcal{C}^p_{loc}$ and all $\beta \in \mathcal{C}^{p-1}_{loc}$.

In the main body of this paper we will be working with "time-zero" fields, i.e. cochains defined on the *spatial* lattice $\mathbb{Z}^2$. The translation to the euclidean functional integral formalism will bring us to a *space-time* lattice $\mathbb{Z}^3$. To unload the notation we will use the same symbols for both pictures as long as the meaning becomes obvious from the context. Hence, in both pictures $\varphi \in \mathcal{C}^0$ will denote the Higgs field, $\alpha \in \mathcal{C}^1$ will denote the gauge field and a gauge transformation consists of a mapping $(\varphi,\, \alpha) \mapsto (\varphi + \lambda,\, \alpha + d\lambda)$ with $\lambda \in \mathcal{C}^0_{loc}$.

2.1 The Local Algebras

As is well known, a $d+1$-dimensional lattice system described by (1.1,1.2) can typically also be described as a *quantum spin system*, using the transfer matrix formalism. By this we mean an operator algebra living on a d-dimensional spatial lattice, together with discrete "euclidean time" translations given by e^{-tH}, $t \in \mathbb{Z}$, where $T \equiv e^{-H}$ is the transfer matrix. In this formulation expectations like (1.1) represent the vacuum or ground state of the "euclidean" dynamics defined by the transfer matrix.

We have described in detail the quantum spin system of our model in [1] (see also [2]). It corresponds to the Weyl form of the usual canonical quantization prescription in temporal ($\alpha_0 = 0$) gauge. Let us recall here its main ingredients.

On the spatial lattice $\mathbb{Z}^2$ we introduce the local algebra of time-zero Higgs and gauge fields in the following way. To each $x \in (\mathbb{Z}^2)_0$ we associate the unitary $\mathbb{Z}_N$-fields $P_H(x)$ and $Q_H(x)$ and to each $b \in (\mathbb{Z}^2)_1$ we associate the unitary $\mathbb{Z}_N$-fields $P_G(b)$ and $Q_G(b)$ (the subscripts G and H stand for "gauge" and "Higgs", respectively) satisfying the relations:

$$P_H(x)^N = Q_H(x)^N = P_G(b)^N = Q_G(b)^N = \mathbb{1}, \tag{2.2}$$

and the $\mathbb{Z}_N$-Weyl algebra relations

$$P_G(\alpha)Q_G(\beta) = e^{-i\langle\alpha,\beta\rangle}Q_G(\beta)P_G(\alpha), \tag{2.3}$$

$$P_H(\gamma)Q_H(\delta) = e^{-i\langle\gamma,\delta\rangle}Q_H(\delta)P_H(\gamma), \tag{2.4}$$

where α, $\beta \in \mathcal{C}^1_{loc}$ and γ, $\delta \in \mathcal{C}^0_{loc}$ play the rôle of test functions, i.e. $P_H(\gamma) := \prod_{x\in(\mathbb{Z}^2)_0} P_H(x)^{\gamma(x)}$ etc. Operators localized at different sites and bonds commute and the G-operators commute with the H-operators.

We denote $[\delta Q_H](\alpha) := Q_H(d^*\alpha)$, $[\delta^* P_G](\beta) := P_G(d\beta)$ etc., [4] where d is the exterior derivative on cochains and d^* is its adjoint.

We will realize these operators by attaching to each lattice point x a Hilbert space $\mathcal{H}_x$ and to each lattice bond b a Hilbert space $\mathcal{H}_b$, where $\mathcal{H}_x \cong \mathcal{H}_b \cong \mathbb{C}^N$. The operators $Q_H(x)$, $P_H(x)$, $Q_G(b)$ and $P_G(b)$ are given on $\mathcal{H}_x$, and $\mathcal{H}_b$, respectively, as matrices with matrix elements:

$$P_H(x)_{a,b} = P_G(b)_{a,b} = \delta_{a,b+1(\mathrm{mod}\,N)} \quad \text{and} \tag{2.5}$$

$$Q_H(x)_{a,b} = Q_G(b)_{a,b} = \delta_{a,b}e^{\frac{2\pi i}{N}a}, \tag{2.6}$$

for $a, b \in \{0,\ldots,N-1\}$.

The operators Q_H and Q_G have to be interpreted as the $\mathbb{Z}_N$ versions of the Higgs field and gauge field, respectively: $Q_H(x) = e^{\frac{2\pi i}{N}\varphi(x)}$, $Q_G(x) = e^{\frac{2\pi i}{N}\alpha(b)}$, with φ and α taking values in $\mathbb{Z}_N$. The operators P_H and P_G are their respective canonically conjugated "exponentiated momenta", i.e. shift operators by one $\mathbb{Z}_N$-unit. Hence these operators indeed provide the Weyl form of the canonical quantization in $\alpha_0 = 0$ gauge, see also equation (2.11) below.

We denote by $\mathfrak{F}_{loc}$ the $*$-algebra generated by these operators. Denoting by $\mathfrak{F}(V)$ the C*-subalgebra generated by $Q_H(x)$, $P_H(x)$, $Q_G(b)$ and $P_G(b)$ for $x \in V_0$, $b \in V_1$, $V \subset \mathbb{Z}^2$ finite, one has $\mathfrak{F}_{loc} = \cup_{|V|<\infty}\mathfrak{F}(V)$. The algebra $\mathfrak{F}(V)$ acts on $\mathcal{H}_V := \otimes_{x\in V_0}\mathcal{H}_x \otimes_{b\in V_1} \mathcal{H}_b$. We will denote by $\mathfrak{F}$ the unique C*-algebra generated by $\mathfrak{F}_{loc}$. Without mentioning explicitly we will frequently use that by continuity states ω on $\mathfrak{F}$ or $*$-automorphisms γ of $\mathfrak{F}$ are uniquely determined by their definition on $\mathfrak{F}_{loc}$.

Let $\mathcal{S}$ denote the group of spatial lattice translations by $a \in \mathbb{Z}^2$ and rotations by multiples of $\pi/2$. $\mathcal{S}$ acts naturally on $\mathfrak{F}$ as a group of $*$-automorphisms τ_g, $g \in \mathcal{S}$, given by $\tau_g(Q_H(x)) = Q_H(gx)$ etc.

We also have the group of local (time independent) gauge transformations $\mathcal{G} \equiv \mathcal{C}^0_{loc}$ acting as $*$-automorphisms on $\mathfrak{F}$ by

$$Q_H(\alpha) \mapsto e^{-i\langle\alpha,\lambda\rangle}Q_H(\alpha)\ , \qquad Q_G(\gamma) \mapsto e^{-i\langle\gamma,d\lambda\rangle}Q_G(\gamma), \tag{2.7}$$

[4] There was a misprint in these definitions in [1].

and leaving all operators $P_H(x)$ and $P_G(b)$ invariant. These gauge transformations are implemented by the unitaries $G(\lambda) := \prod_x G(x)^{\lambda(x)}$, where [5]

$$G(x) := P_H(x) \left[\delta^* P_G\right](x). \tag{2.8}$$

Note that $G(x)^* = G(x)^{-1} = G(x)^{N-1}$. The operator $G(x)$ is the generator of a $\mathbb{Z}_N$ gauge transformation at the point x, as one can easily checks. It can be interpreted as the lattice analog of $\exp\left(-2\pi i(\text{div}\,\mathbb{E} - \rho)/N\right)$.

We denote the gauge group algebra $\mathfrak{G} \subset \mathfrak{F}$ as the Abelian C^*-sub-algebra generated by $\{G(x) \mid x \in \mathbb{Z}^2\}$ and put $\mathfrak{G}_{loc} := \mathfrak{G} \cap \mathfrak{F}_{loc}$ and $\mathfrak{G}(V) := \mathfrak{G} \cap \mathfrak{F}(V)$.

The algebra of observables $\mathfrak{A}$ is defined as the gauge invariant sub-algebra of $\mathfrak{F}$:

$$\mathfrak{A} := \{A \in \mathfrak{F} :\ G(x) A G(x)^* = A \text{ for all } x \in \mathbb{Z}^2\}, \tag{2.9}$$

i.e. the commutant of $\mathfrak{G}$ in $\mathfrak{F}$. We call $\mathfrak{A}(V) := \mathfrak{F}(V) \cap \mathfrak{A}$ and $\mathfrak{A}_{loc} := \cup_V \mathfrak{A}(V)$. Then $\mathfrak{A}_{loc}$ is the norm dense sub-algebra generated by $P_G(b)$, $P_H(x)$ and

$$Q_{GH}(b) := Q_G(b) \left[\delta Q_H\right](b)^* \tag{2.10}$$

for all $x \in (\mathbb{Z}^2)_0$ and $b \in (\mathbb{Z}^2)_1$.

We now provide a convenient "ket vector" notation for the local Hilbert spaces $\mathcal{H}_V$. Identifying $\mathcal{H}_x \cong \mathcal{H}_b \cong \mathbb{C}^N \cong \ell^2(\mathbb{Z}_N)$ we may naturally denote ON-basis elements of $\mathcal{H}_{x/b}$ by $|a\rangle,\ a \in \mathbb{Z}_N$, i.e. the characteristic functions on $\mathbb{Z}_N$. Correspondingly, ON-basis elements of $\mathcal{H}_V$ are labeled by

$$|\varphi, \alpha\rangle = \bigotimes_{x \in V_0, b \in V_1} \left(|\varphi(x)\rangle \otimes |\alpha(b)\rangle\right),$$

where (φ, α) run through all "classical configurations", i.e. $\mathbb{Z}_N$-valued 0- and 1-cochains, respectively, with support in V.

With this notation the representation of our local field algebras $\mathcal{F}(V)$ is immediately recognized as the Weyl form of the canonical quantization in $\alpha_0 = 0$ gauge, i.e.

$$\begin{array}{rcl} Q_H(\chi)\,|\varphi, \alpha\rangle &=& e^{i\langle\varphi,\chi\rangle}\,|\varphi, \alpha\rangle \\ Q_G(\beta)\,|\varphi, \alpha\rangle &=& e^{i\langle\alpha,\beta\rangle}\,|\varphi, \alpha\rangle \\ P_H(\chi)\,|\varphi, \alpha\rangle &=& |\varphi + \chi, \alpha\rangle \\ P_G(\beta)\,|\varphi, \alpha\rangle &=& |\varphi,\ \alpha + \beta\rangle \end{array} \tag{2.11}$$

where χ and β denote $\mathbb{Z}_N$-valued 0- and 1-cochains, respectively, with support in V. The action of the gauge transformations on $\mathcal{H}_V$ now takes the usual form

$$G(\lambda)\,|\varphi, \alpha\rangle = |\varphi + \lambda, \alpha + d\lambda\rangle.$$

It is well known that for a single site $x \in \mathbb{Z}^2$ the operators $Q_H(x)$ and $P_H(x)$ generate $\text{End}\,\mathcal{H}_x$ and similarly for $Q_G(b)$ and $P_G(b)$. Hence $\mathfrak{F}(V) \cong \text{End}\,(\mathcal{H}_V)$ for all V implying $\mathfrak{F}_{loc}$ and $\mathfrak{F}$ to be simple [11]. On the other hand, the observable algebra $\mathfrak{A}$ is not simple since it contains $\mathfrak{G}$ in

[5]Here we replace our notation $\mathcal{Q}(x)$ of [1] by the more common one $G(x)$ and also correct a misprint in equation. (2.9) of [1].

its center. However, since eventually we are only interested in representations π of $\mathfrak{A}$ satisfying $\pi(G(x)) = \mathbb{1}$ for all $x \in \mathbb{Z}^2$ (i.e. representations without external charges, see Section 2.4 below) we might as well consider

$$\mathfrak{B} := \mathfrak{A}/\mathfrak{J}.$$

as our "essential" observable algebra, where $\mathfrak{J} \subset \mathfrak{A}$ is the two-sided closed ideal generated by $\{G(x) - \mathbb{1}\}_{x \in \mathbb{Z}^2}$. In the obvious way we also define $\mathfrak{B}(V) = \mathfrak{A}(V)/\mathfrak{J}(V)$ and $\mathfrak{B}_{loc} = \cup_{V \subset \mathbb{Z}^2} \mathfrak{B}(V)$. Then $\mathfrak{B}$ is the C^*-closure of $\mathfrak{B}_{loc}$ and by the same argument as above $\mathfrak{B}$ is simple since we have

Lemma 2.1.1 *For all finite closed boxes $V \subset \mathbb{Z}^2$ the algebras $\mathfrak{B}(V)$ are isomorphic to full matrix algebras.*

Proof: We transform to "unitary gauge" by defining a unitary $U : \mathcal{H}_V \to \mathcal{H}_{V_0} \otimes \mathcal{H}_{V_1}$ according to

$$U\,|\varphi, \alpha\rangle := |\varphi\rangle \otimes |\alpha - d\varphi\rangle \quad .$$

This gives

$$UG(\chi)U^{-1}\,(|\varphi\rangle \otimes |\alpha\rangle) = |\varphi + \chi\rangle \otimes |\alpha\rangle$$

and therefore $U\mathfrak{G}(V)U^* = \mathfrak{M}(V) \otimes \mathbb{1}$, where $\mathfrak{M}(V)$ consists of all shift operators $|\varphi\rangle \mapsto |\varphi + \chi\rangle$ on $\mathcal{H}_{V_0} = \ell^2(\mathcal{C}^0(V)) \equiv \ell^2(\mathbb{Z}_N^{|V_0|})$. Now $\mathfrak{M}(V)$ being maximal Abelian in $\mathrm{End}\,\mathcal{H}_{V_0}$ and $\mathfrak{A}(V)$ being the commutant of $\mathfrak{G}(V)$ in $\mathfrak{F}(V) \equiv \mathrm{End}\,\mathcal{H}_V$ we conclude

$$\begin{aligned} U\mathfrak{A}(V)U^* &= (\mathfrak{M}(V)' \cap \mathrm{End}\,\mathcal{H}_{V_0}) \otimes \mathrm{End}\,\mathcal{H}_{V_1} \\ &= \mathfrak{M}(V) \otimes \mathrm{End}\,\mathcal{H}_{V_1} \end{aligned}$$

from which $\mathfrak{B}(V) \cong \mathrm{End}\,\mathcal{H}_{V_1}$ follows. ■

Note that the above transformation also gives

$$\begin{aligned} UP_G(\beta)U^*\,|\varphi\rangle \otimes |\alpha\rangle &= |\varphi\rangle \otimes |\alpha + \beta\rangle \\ UQ_{GH}(\beta)U^*\,|\varphi\rangle \otimes |\alpha\rangle &= |\varphi\rangle \otimes e^{i\langle\alpha,\beta\rangle}|\alpha\rangle \end{aligned}$$

showing that $\mathfrak{B}$ is generated by the field operators $P_G + \mathfrak{J}$ and $Q_{GH} + \mathfrak{J}$, the later being defined in (2.10).

We conclude this subsection with introducing the concept of charge conjugation. On $\mathfrak{F}_{loc}$ we define the charge conjugation i_C to be the involutive $*$-automorphism given by

$$i_C(Q_{H/G}(\chi)) = Q_{H/G}(-\chi) \tag{2.12}$$

$$i_C(P_{H/G}(\alpha)) = P_{H/G}(-\alpha) \tag{2.13}$$

Then i_C extends to $\mathfrak{F}$ by continuity. The generators of gauge transformations satisfy $i_C(G(x)) = G(x)^*$ and therefore the observable algebra is invariant under i_C:

$$i_C(\mathfrak{A}) = \mathfrak{A}.$$

Also note that the restriction of i_C to $\mathfrak{F}(V)$ is unitarily implemented on $\mathcal{H}_V$ by $|\varphi,\, \alpha\rangle \to |-\varphi,\, -\alpha\rangle$.

2.2 Local Transfer Matrices

Let us now introduce the dynamics by defining suitable finite volume transfer matrices. Infinite volume transfer matrices will be defined later. The form of the transfer matrix is justified by its finite volume ground state giving rise to the classical expectation associated with the euclidean $\mathbb{Z}_N$-Higgs model (1.1).

We will consider local transfer matrices $T_V \in \mathfrak{A}_{loc}$, for finite $V \subset \mathbb{Z}^2$, defined by:

$$T_V = e^{A_V/2} e^{B_V} e^{A_V/2}, \tag{2.14}$$

with

$$A_V := \frac{1}{\sqrt{N}} \sum_{p \in V_2} \sum_{n=0}^{N-1} \beta_g(n)\, [\delta Q_G]\,(p)^n + \frac{1}{\sqrt{N}} \sum_{b \in V_1} \sum_{n=0}^{N-1} \beta_h(n)\, Q_{GH}(b)^n, \tag{2.15}$$

and

$$B_V := \frac{1}{\sqrt{N}} \sum_{b \in V_1} \sum_{n=0}^{N-1} \gamma_g(n)\, P_G(b)^n + \frac{1}{\sqrt{N}} \sum_{x \in V_0} \sum_{n=0}^{N-1} \gamma_h(n)\, P_H(x)^n \ . \tag{2.16}$$

Here $\beta_{g/h}$ and $\gamma_{g/h}$ are even and real valued functions on $\mathbb{Z}_N$, such that T_V is in fact positive, has positive matrix elements and is invertible. Moreover, this also implies that T_V is charge conjugation invariant, $i_C(T_V) = T_V$.

In the basis $|\varphi, \alpha\rangle \in \mathcal{H}_V$ these operators have matrix elements given by

$$e^{A_V} |\varphi, \alpha\rangle = \exp - \left[\sum_{p \in V_2} S_g(d\alpha(p)) + \sum_{b \in V_1} S_h(d\varphi(b) - \alpha(b)) \right] |\varphi, \alpha\rangle \tag{2.17}$$

and

$$\langle \varphi', \alpha' |\, e^{B_V}\, |\varphi, \alpha\rangle = \prod_{b \in V_1} \rho_g(\alpha(b) - \alpha'(b)) \prod_{x \in V_0} \rho_h(\varphi(x) - \varphi'(x)) \tag{2.18}$$

where S_g and S_h are the euclidean actions (1.3) and where ρ_g and ρ_h are positive functions on $\mathbb{Z}_N$ determined by

$$\begin{aligned} \textstyle\sum_n \rho_g(n) P_G(b)^n &= \exp\left[\tfrac{1}{\sqrt{N}} \textstyle\sum_n \gamma_g(n) P_G(b)^n\right] \\ \textstyle\sum_n \rho_h(n) P_H(x)^n &= \exp\left[\tfrac{1}{\sqrt{N}} \textstyle\sum_n \gamma_h(n) P_H(x)^n\right] \ . \end{aligned}$$

With a suitable choice of the choice of the $\gamma_{g,\,h}$'s as functions of the $\beta_{g,\,h}$'s one can arrange (see [1])

$$\rho_{g,h} = e^{-S_{g,h}}$$

and hence Tr T_V^n equals the partition function appearing in the denominator of (1.1) for a volume $\Lambda \equiv V_n := V \times \{0, \ldots, n-1\} \subset \mathbb{Z}^3$ with periodic boundary conditions in "time" direction (free boundary conditions are also possible, see [2], [1] or below).

The euclidean dynamics is given by the strong limit of the non-$*$ automorphisms of $\mathfrak{F}$ generated by the local transfer matrices:

$$\alpha(A) := \lim_{V \uparrow \mathbb{Z}^2} \alpha(A)_V, \qquad A \in \mathfrak{F}, \tag{2.19}$$

where $\alpha(\cdot)_V$ is the automorphism of $\mathfrak{F}$ defined through

$$\alpha(A)_V := T_V A T_V^{-1}, \qquad A \in \mathfrak{F}. \tag{2.20}$$

For $A \in \mathfrak{F}_{loc}$ the limit in (2.19) is already reached at finite V. In [1] and [2] the notation α_i in place of α was used, due to the interpretation of $\alpha_i \equiv \alpha$ as the generator of a translation by one unit in *imaginary* (euclidean) time direction. Clearly, α commutes with the action of lattice translations and rotations and with the charge conjugation i_C.

In this work we will consider the classical expectations (1.1) in the so called "free charge phase" of the $\mathbb{Z}_N$-gauge Higgs model. Let us introduce the functions g, $h:\ \mathbb{Z}_N \mapsto \mathbb{R}_+$ defined by (see [1])

$$g(n) \ := \ \exp(-S_g(n)), \tag{2.21}$$

$$h(n) \ := \ \mathcal{F}\left[\exp(-S_h)\right](n), \tag{2.22}$$

for $n \in \mathbb{Z}_N$, where $\mathcal{F}$ is the Fourier transform of functions on $\mathbb{Z}_N$. ¿From now on we will fix the additive constants $\beta_g(0)$ and $\beta_h(0)$ through the conditions $g(0) = h(0) = 1$. In [1] the infinite volume limits of the classical expectations of local observables (1.1) have been shown to be analytic functions of the couplings $g(1), \ldots, g(N-1)$, $h(1), \ldots, h(N-1)$ whenever

$$g_c \ := \ \max\left\{|g(1)|, \ldots, |g(N-1)|\right\} \leq e^{-k_g} \ \text{ and} \tag{2.23}$$

$$h_c \ := \ \max\left\{|h(1)|, \ldots, |h(N-1)|\right\} \leq e^{-k_h}, \tag{2.24}$$

with k_g, $k_h > 0$, large enough. For this a polymer and cluster expansion has been used. The above region of analyticity (for real couplings) is contained in the "free charge region" of the phase diagram. This phase is characterized by the absence of screening and confinement. All results of our present work, specifically those concerned with the existence and the properties of the charged states are valid for g_c and h_c sufficiently small.

2.3 Ground States

In this subsection we recall the important definition of a ground state and discuss some of its basic features. We start with introducing two useful concepts.

First, the adjoint γ^* of an automorphism γ of a unital $*$-algebra $\mathfrak{C}$ is defined through $\gamma^*(A) := (\gamma(A^*))^*$, $A \in \mathfrak{C}$. We have $\gamma^{**} = \gamma$ and γ is a $*$-automorphism iff $\gamma = \gamma^*$. For the composition of automorphisms one has $(\alpha \circ \beta)^* = \alpha^* \circ \beta^*$ and consequently $\alpha^{*\,-1} = \alpha^{-1\,*}$. For an invertible element $A \in \mathfrak{C}$ one also has $(\mathrm{Ad}\ A)^* = \mathrm{Ad}\ A^{*\,-1}$. For $\gamma = \alpha$ this in particular implies $\alpha^* = \alpha^{-1}$. Finally, if ω is a γ-invariant state on $\mathfrak{C}$ then trivially it is also γ^*-invariant.

Second, we say that a state ω on a $*$-algebra $\mathfrak{C}$ has the cluster property for the automorphism γ if, for all A, $B \in \mathfrak{C}$, one has

$$\lim_{n\to\infty} \omega(A\gamma^n(B)) = \omega(A)\omega(B). \tag{2.25}$$

We now come to a central definition which has first been introduced in [2].

Definition 2.3.1 *Let γ be a (not necessarily $*$-preserving) automorphism of a unital $*$-algebra $\mathfrak{C}$. A state ω on $\mathfrak{C}$ is called a "ground state" with respect to γ and $\mathfrak{C}$ if it is γ-invariant and if*

$$0 \leq \omega(A^*\gamma(A)) \leq \omega(A^*A), \qquad \forall A \in \mathfrak{C}. \tag{2.26}$$

Actually γ-invariance follows from (2.26) (see, e.g. [1]). □

We will motivate this abstract definition below when we discuss the ground state of the finite volume transfer matrix.

The following Lemma will be very useful for proving that certain states are ground states with respect to our euclidean dynamics α (or suitable modifications of α to be introduced in Section 3.1). This Lemma was already implicitly used in [2].

Lemma 2.3.2 *Let γ be an automorphism on a $*$-algebra $\mathfrak{C}$ satisfying $\gamma^* = \gamma^{-1}$ and let ω be a γ-invariant state on $\mathfrak{C}$ which has the cluster property for γ. (Actually one just needs that, for each $A \in \mathfrak{C}$, the sequence $\omega(A^*\gamma^a(A))$, $a \in \mathbb{N}$, is bounded). Then, for all $A \in \mathfrak{C}$,*

$$|\omega(A^*\gamma(A))| \leq \omega(A^*A). \tag{2.27}$$

□

The proof is easy. See [1].

We will now exhibit translation invariant ground states of the automorphism α. First, let us explain in this context the heuristic motivation of Definition 2.3.1. If $\Omega_V \in \mathcal{H}_V$ is the Frobenius eigenvector of the finite volume transfer matrix T_V with eigenvalue $\|T_V\|_{\mathcal{H}_V}$ then, in face of the positivity of T_V, the inequalities

$$0 \leq \left(\Omega_V,\ A^*T_VAT_V^{-1}\Omega_V\right) \leq (\Omega_V,\ A^*A\Omega_V) \tag{2.28}$$

obviously hold for any $A \in \mathfrak{F}(V)$.

This motivates to consider the relation (2.26) with $\gamma = \alpha$ as a characterization of a state replacing the vector states Ω_V for infinite volumes. One should notice here that the usual characterization of a ground state of a quantum spin system as a state ω for which $\lim_{V\uparrow\mathbb{Z}^2} \omega(A^*[H_V,\, A]) \geq 0$ for all local A is inadequate for transfer matrix systems, due to the highly non-local character of the Hamilton operator $H_V \equiv -\ln T_V$ (at least in more than two space-time dimensions).

Following [2], a translation invariant ground state for α with respect to the algebra $\mathfrak{F}$ can be obtained by

$$\omega_0(B) = \lim_{V\uparrow\mathbb{Z}^2} \lim_{n\to\infty} \frac{\mathrm{Tr}_{\mathcal{H}_V}\left(T_V^n B T_V^n E_V^0\right)}{\mathrm{Tr}_{\mathcal{H}_V}\left(T_V^{2n} E_V^0\right)}, \tag{2.29}$$

for $B \in \mathfrak{F}_{loc}$, where E_V^0 is an in principle arbitrary operator with positive matrix elements. Before the limits $V \uparrow \mathbb{Z}^2$ and $n \to \infty$ are taken the expression in the right-hand side of (2.29) is identical to the classical expectation (1.1) in a volume $V \times \{-n, \ldots, n\} \subset \mathbb{Z}^3$ of a suitable classical observable B_{cl} associated with B. A convenient choice for E_V^0 giving free boundary conditions in time-direction is (see [2])

$$E_V^0 := e^{A_V/2}\, F_V^0\, e^{A_V/2} \tag{2.30}$$

where

$$F_V^0 := \sum_{(\varphi,\alpha),(\varphi',\alpha')} |\varphi,\alpha\rangle\langle\varphi',\alpha'| \tag{2.31}$$

Here the sum in (2.31) goes over all classical time-zero configurations with support in V, i.e. 0-cochains φ, $\varphi' \in \mathcal{C}^0(V)$ and 1-cochains α, $\alpha' \in \mathcal{C}^1(V)$.

The existence of the thermodynamic limit in (2.29) can be established using Griffiths' inequalities or the cluster expansions. The cluster expansions also provide a way to prove the translation and rotation invariance of the limit state ω_0. Another important fact derived from the cluster expansion is that the restriction of ω_0 to $\mathfrak{A}$ has the cluster property for the automorphism α.

One of the most useful aspects of the Definition 2.3.1 of a ground state is the possibility to define infinite volume transfer matrices. Indeed, if ω_0 is a ground state of α with respect to $\mathfrak{F}$ and $(\pi_0(\mathfrak{F}),\ \Omega_0,\ \mathcal{H}_0)$ is the GNS-triple associated with ω_0, we define, following [2], the infinite volume transfer matrix T_0 as the element of $\mathcal{B}(\mathcal{H}_0)$ given on the dense set $\pi_0(\mathfrak{F})\Omega_0$ by

$$T_0\pi_0(A)\Omega_0 := \pi_0(\alpha(A))\Omega_0, \tag{2.32}$$

$A \in \mathfrak{F}$. One checks that this is a well-defined positive operator with $0 \leq T_0 \leq 1$. If moreover ω_0 satisfies the cluster property (2.25), then $\Omega_0 \in \mathcal{H}_0$ is the unique (up to a phase) eigenvector of T_0 with eigenvalue 1. Hence we call π_0 the *vacuum representation* of $(\mathfrak{F},\ \alpha)$.

Since ω_0 is translation invariant we can also define a unitary representation of the translation group in the vacuum sector through

$$U_0(x)\pi_0(A)\Omega_0 := \pi_0(\tau_x(A))\Omega_0, \tag{2.33}$$

$A \in \mathfrak{F}$, $x \in \mathbb{Z}^2$. The momenta in the vacuum sector are therefore defined by $U_0(x) = \exp(i\mathbb{P}\cdot x)$, with $\mathrm{sp}\ \mathbb{P} \in [-\pi,\ \pi]^2$ and the vacuum Ω_0 is also the unique (up to a phase) translation invariant vector in $\mathcal{H}_0$.

Next we remark that ω_0 is also charge conjugation invariant. This can be seen from (2.29) by using $i_C(T_V) = T_V$, $i_C(E_V^0) = E_V^0$ and the fact that $i_C \restriction \mathfrak{F}(V)$ is unitarily implemented on $\mathcal{H}_V$. Hence, charge conjugation is implemented as a symmetry of the vacuum sector by the unitary operator $C_0 \in \mathcal{B}(\mathcal{H}_0)$ given on $\pi_0(\mathfrak{F})\Omega_0$ by

$$C_0\,\pi_0(A)\Omega_0 := \pi_0(i_C(A))\Omega_0 \tag{2.34}$$

When constructing charged states ω_ρ in Section 3 this will no longer hold, i.e. there we will have $\omega_\rho \circ i_C = \omega_{-\rho}$.

2.4 External Charges

Let π be a representation of the field algebra $\mathfrak{F}$ on some separable Hilbert space $\mathcal{H}_\pi$ and for $q \in \mathcal{C}^0$ let $\mathcal{H}_\pi^q \subset \mathcal{H}_\pi$ be the subspace of vectors $\psi \in \mathcal{H}_\pi$ satisfying

$$\pi(G(x))\psi = e^{\frac{2\pi i}{N}q(x)}\psi \tag{2.35}$$

for all $x \in \mathbb{Z}^2$. According to common terminology we call $\mathcal{H}_\pi^q$ the subspace with *external electric charge* q, since the operators $\pi(G(x))$ implement the gauge transformations on $\mathcal{H}_\pi$. Clearly, since $\mathfrak{A}$ commutes with $\mathfrak{G}$ we have

$$\pi(\mathfrak{A})\,\mathcal{H}_\pi^q \subset \mathcal{H}_\pi^q\,. \tag{2.36}$$

Moreover, using $Q_H(x)G(x) = e^{2\pi i/N}G(x)Q_H(x)$ for all $x \in \mathbb{Z}^2$ we have

$$\pi(Q_H(q'))\mathcal{H}_\pi^q = \mathcal{H}_\pi^{q-q'} \tag{2.37}$$

for all $q' \in \mathcal{C}^0_{loc}$. In particular, since $Q_H(q)\mathfrak{A}Q_H(q)^* = \mathfrak{A}$, the representations of $\mathfrak{A}$ on $\mathcal{H}_\pi^q$ and on $\mathcal{H}_\pi^{q-q'}$ are unitarily equivalent for all q' with finite support. [6]

Now $\mathcal{H}_\pi^q$ might be zero for general representations π and general external charge distributions q (e.g. for the vacuum representation π_0 and q with infinite support). In this work we are only interested in representations π containing a non-trivial subspace $\mathcal{H}_\pi^0 \neq 0$ of *zero* external charge (and therefore, by (2.37), also $\mathcal{H}_\pi^q \neq 0$ for all external charges with finite support). Moreover, $\mathcal{H}_\pi^0$ will always be cyclic under the action of $\pi(\mathfrak{F}_{loc})$ and therefore we will always have [7]

$$\mathcal{H}_\pi = \bigoplus_{q\in\mathcal{C}^0_{loc}} \mathcal{H}_\pi^q \tag{2.38}$$

In fact, such a decomposition is always obtained for GNS-representations $(\pi_\omega,\, \Omega_\omega,\, \mathcal{H}_\omega)$ associated with states ω on $\mathfrak{F}$, provided $\Omega_\omega \in \mathcal{H}_\omega^q$ for some $q \in \mathcal{C}^0_{loc}$ or, equivalently,

$$\omega(F\,G(x)) = e^{2\pi i\,q(x)/N}\,\omega(F) \tag{2.39}$$

for all $F \in \mathfrak{F}$ and all $x \in (\mathbb{Z}^2)_0$. In this case we call ω an *eigenstate* (with external charge $q \in \mathcal{C}^0_{loc}$) of the gauge group algebra $\mathfrak{G}$. Note that ω is an eigenstate of $\mathfrak{G}$ with external charge q if and only if $\omega \circ \mathrm{Ad}\,Q_H(q')$ is an eigenstate with external charge $q+q'$. Correspondingly, $\pi_\omega(Q_H(q')^*)\,\Omega_\omega \in \mathcal{H}_\omega^{q+q'}$ will also be a cyclic vector for $\pi_\omega(\mathfrak{F})$. Hence, without loss, we may restrict ourselves to eigenstates ω of $\mathfrak{G}$ with zero external charge. In fact, we will only be studying the restrictions of representations $\pi(\mathfrak{A})$ to $\mathcal{H}_\pi^0$ as the "physical" subspace of $\mathcal{H}_\pi$, i.e. the subspace on which "Gauss' Law", $\pi(G(x)) = \mathbb{1}$, holds.

We emphasize that this notion of external (or "background") charge is not to be confused with the concept of *(dynamically) charged states* and the associated *charged representations* of $\mathfrak{A}$. By this we mean representations of $\mathfrak{A}$ with zero external charge, which are inequivalent to the vacuum representation (at least when extended to a suitable "dynamic closure" $\hat{\mathfrak{A}} \supset \mathfrak{A}$), e.g. by the appearance of different mass spectra or the absence of a time and space translation invariant "vacuum" vector.

By analogy with the terminology of quantum field theory we call an equivalence class of such charged representations a *superselection sector.* They are the main interest of our work. Hence, from now on by charged states we will always mean dynamically charged states in this later sense.

3 The Construction of Dyonic Sectors

In [1] with the help of the cluster expansions (see Appendix A) we were able to show the existence of electrically and of magnetically charged sectors in the "free charges" phase of the $\mathbb{Z}_N$-Higgs

[6] Note, however, that this equivalence does not respect the dynamics, since $\mathrm{Ad}\,Q_H(q)$ does not commute with α.

[7] Use that $\mathfrak{F}_{loc} = \oplus_q \mathfrak{A}_{loc} Q_H(q)$ is a grading labeled by $q \in \mathcal{C}^0_{loc}$, i.e. the irreducible representations of $\mathfrak{G}$ in $\mathfrak{F}_{loc}$.

model. We also proved the existence of massive charged 1-particle states in these sectors.

This section is devoted to the construction of states and the associated sectors which are at the same time electrically and magnetically charged. Following the common use we call them *dyonic states*. Hereby we generalize and improve ideas from [6], where such states have been first constructed for the $\mathbb{Z}_2$-Higgs model. In [10] we will continue this analysis by constructing scattering states in the dyonic sectors and identifying these to constitute an "anyonic Fock space" over the above mentioned 1-particle states.

We start in Section 3.1 with reformulating the Fredenhagen-Marcu construction for charged states ω_ρ by defining them as the thermodynamic limit of ground states of *modified local transfer matrices* $T_V(\rho)$. For $\rho = (\varepsilon, \mu)$ these modified transfer matrices correspond to modified Hamiltonians, where the kinetic term in the Higgs fields is replaced by

$$\frac{1}{2}(\pi_H, \pi_H) \to \frac{1}{2}(\pi_H + \varepsilon, \pi_H + \varepsilon)$$

and the magnetic self energy is replaced by

$$\frac{1}{2}(d\alpha, d\alpha) \to \frac{1}{2}(d\alpha + \mu, d\alpha + \mu).$$

In the functional integral these states are represented by euclidean expectations in the background of infinitely long vertical Wilson and vortex lines sitting above sites $x \in (\mathbb{Z}^2)_0$ and plaquettes $p \in (\mathbb{Z}^2)_2$, respectively, of the time-zero plane $\mathbb{Z}^2$. Their $\mathbb{Z}_N$-values are given by the the values of the electric charges $\varepsilon(x)$ and the magnetic charges $\mu(p)$, respectively.

In Section 3.2 we construct the charged representations π_ρ of $\mathfrak{A}$ associated with the states ω_ρ and the global transfer matrices implementing the euclidean dynamics in π_ρ.

In Section 3.3 we prove that π_ρ and $\pi_{\rho'}$ are dynamically equivalent whenever their total charges coincide, $q_\rho = q_{\rho'}$. We also conjecture that otherwise they are dynamically inequivalent and give some criteria for a proof.

In Section 3.4 we show that the infimum of the energy spectrum in the dyonic sectors may be uniquely normalized by imposing charge conjugation symmetry and the requirement of decaying interaction energies between two charge distributions in the limit of infinite spatial separation.

We recall once more that without mentioning explicitly all results of this section are valid in the free charge phase, i.e. for g_c and h_c (defined in (2.23, 2.24)) sufficiently small.

3.1 Dyonic States

We first recall the idea behind the Fredenhagen-Marcu (FM) string operator and its use in the construction of electrically charged states [2]. Starting from the usual method to create localized electric dipole states by applying a Mandelstam string operator to the vacuum

$$\Omega_{x,y} := \phi(x)\phi(y)^* e^{i\int_x^y A_i dz^i}\,\Omega, \tag{3.1}$$

Fredenhagen and Marcu [2] proposed a modification so as to keep the energy of the dipole state $\Omega_{x,y}$ finite as $y \to \infty$. Using our lattice notation the FM-proposal reads [8]

$$\Omega^{FM}_{x,y} := \lim_{n\to\infty} c_n Q_H(x) Q_H(y)^* T_0^n Q_G(s_{xy})\Omega. \tag{3.2}$$

[8] By a convenient abuse of notation we often drop the symbol π_0 when referring to the vacuum representation.

Here $s_{x,y}$ is an arbitrary path connecting x to y in our spatial lattice, $T_0 = e^{-H}$ is the global transfer matrix in the vacuum sector and $c_n > 0$ is a normalization constant to get $\|\Omega^{FM}_{x,y}\| = 1$. Note that $\Omega^{FM}_{x,y}$ still lies in the vacuum sector and has zero external charge. In a second step one may then send one of the charges to infinity [9] to obtain dynamically charged states as expectation values on $\mathfrak{A}_{loc}$

$$\omega_x(A) := \lim_{y\to\infty} \left(\Omega^{FM}_{x,y},\ A\Omega^{FM}_{x,y}\right), \qquad A \in \mathfrak{A}_{loc}. \tag{3.3}$$

Note that as a limit of eigenstates of $\mathfrak{G}$ with zero external charge, ω_x is also an eigenstate of $\mathfrak{G}$ with zero external charge. Using duality transformations, an analogous procedure for the construction of magnetically charged states has been given in [1].

In order to be able to discuss dyonic states within a common formalism, we now pick up an observation of [1] to reformulate the above construction as follows. First we recall from [2] that in our range of couplings we have

$$\Omega^{FM}_{x,y} = Q_H(x)Q_H(y)^*\psi_{x,y}, \tag{3.4}$$

where $\psi_{x,y} \in \mathcal{H}_0^{\delta_x-\delta_y}$ is the unique (up to a phase) ground state vector of the *restricted* global transfer matrix $T_0 \restriction \mathcal{H}_0^{\delta_x-\delta_y}$, where $\mathcal{H}_0^{\delta_x-\delta_y} \subset \mathcal{H}_0$ is the subspace of an <u>external</u> electric charge-anticharge pair sitting at x and y, respectively. In fact, by looking at (3.2) we have

$$\psi_{x,y} = s - \lim_{n\to\infty} c_n T_0^n Q_G(s_{x,y})\Omega \tag{3.5}$$

and using our cluster expansion it is easy to check that the limit exists independently of the chosen string $s_{x,y}$ connecting x to y.

To avoid the use of external charges (for which we do not have a magnetic analogue) we now equivalently reformulate this by saying that $\Omega^{FM}_{x,y}$ is the unique (up to a phase) ground state vector of the *modified transfer matrix* $T_0(\delta_x - \delta_y)$ defined by

$$T_0(\delta_x - \delta_y) := Q_H(x)Q_H(y)^*T_0Q_H(y)Q_H(x)^* \restriction \mathcal{H}_0^0 \tag{3.6}$$

where $\mathcal{H}_0^0 \subset \mathcal{H}_0$ is the subspace without external charges. Since this modified transfer matrix generates a modified dynamics given by

$$\alpha' = \mathrm{Ad}\,(Q(x)Q(y)^*) \circ \alpha \circ \mathrm{Ad}\,(Q(y)Q(x)^*)$$

we conclude that the state $A \mapsto (\Omega^{FM}_{x,y},\ A\,\Omega^{FM}_{x,y})$ is a ground state of α' when restricted to the observable algebra, $A \in \mathfrak{A}$. A similar statement holds for magnetic dipole states and magnetically modified transfer matrices [1], see also below.

To treat electric and magnetic charges simultaneously we now generalize this construction as follows. Let $\mathcal{D}^E \equiv \mathcal{C}^0_{loc}$ and $\mathcal{D}^M \equiv \mathcal{C}^2_{loc}$ denote the set of $\mathbb{Z}_N$-valued 0-cochains ($\equiv$ electric charge distributions) and 2-cochains ($\equiv$ magnetic charge distributions), respectively, with finite support in our spatial lattice $\mathbb{Z}^2$, and let $\mathcal{D} = \mathcal{D}^E \times \mathcal{D}^M$. For $\rho = (\varepsilon, \mu) \in \mathcal{D}$, supp $\rho \subset V_0 \times V_2$, we generalize (3.6) and define the modified local transfer matrices in a finite volume $V \subset \mathbb{Z}^2$ as the element of $\mathfrak{A}(V)$ given by

$$T_V(\rho) := Z(\mu)^{1/2}Q_H(\varepsilon)T_VQ_H(\varepsilon)^*Z(\mu)^{1/2}, \tag{3.7}$$

[9] Actually, in [2] the two limits, (3.2) and (3.3), were performed simultaneously.

where we have used the notations

$$Q_H(\varepsilon) := \prod_{x \in (\mathbb{Z}^2)_0} Q_H(x)^{\varepsilon(x)} \tag{3.8}$$

and

$$Z(\mu) := \prod_{p \in (\mathbb{Z}^2)_2} Z^{(\mu(p))}(p). \tag{3.9}$$

Here the gauge invariant operator $Z^{(n)}(p)$, $n \in \{0, \ldots, N-1\}$, is defined by (see also [1])

$$Z^{(n)}(p) = \exp\left\{ \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \beta_g(j) \left(\exp\left(\frac{2\pi i}{N} jn \right) - 1 \right) (\delta Q_G(p))^j \right\} . \tag{3.10}$$

On our local Hilbert spaces $\mathcal{H}_V$ it acts by

$$Z^{(n)}(p) \, |\varphi, \, \alpha\rangle = \frac{e^{-S_g(d\alpha(p)+n)}}{e^{-S_g(d\alpha(p))}} \, |\varphi, \, \alpha\rangle \ . \tag{3.11}$$

Hence, this operator can be interpreted as the operator creating a vortex with magnetic charge n at the plaquette p. The definition (3.7) is also motivated by the fact that under duality transformations we roughly have $T_V(\varepsilon, 0) \leftrightarrow T_{V^*}(0, \varepsilon^*)$, see [1] for the precise statement. Also note that $i_C(T_V(\rho)) = T_V(-\rho)$ and that $T_V(\rho)$ is still gauge invariant, i.e. $T_V(\rho) \in \mathfrak{A}_{loc}$.

In a formal continuum notation, these modified transfer matrices correspond to modified Hamiltonians, where the kinetic term in the Higgs fields is replaced by

$$\frac{1}{2}(\pi_H, \, \pi_H) \to \frac{1}{2}(\pi_H + \varepsilon, \, \pi_H + \varepsilon) \tag{3.12}$$

and the magnetic self energy is replaced by

$$\frac{1}{2}(d\alpha, \, d\alpha) \to \frac{1}{2}(d\alpha + \mu, \, d\alpha + \mu). \tag{3.13}$$

Together with these modified transfer matrices we also have a modified euclidean dynamics α_ρ given by the automorphism of $\mathfrak{F}$

$$\alpha_\rho(A) := \lim_{V \uparrow \mathbb{Z}^2} T_V(\rho) A T_V(\rho)^{-1}, \qquad A \in \mathfrak{F}, \tag{3.14}$$

such that $\alpha_0 \equiv \alpha$. As in the case of α_0, for each $A \in \mathfrak{F}_{loc}$ the limit above is already reached at finite V. Moreover $\alpha_\rho(\mathfrak{A}) = \mathfrak{A}$ and

$$i_C \circ \alpha_\rho = \alpha_{-\rho} \circ i_C \ . \tag{3.15}$$

We emphasize that we introduce these modifications not as a substitute of our original "true" dynamics, but for technical reasons only. ¿From (3.7) we conclude

$$\alpha_\rho = \mathrm{Ad}\ K_\rho \circ \alpha_0 \circ \mathrm{Ad}\ K_\rho^* = \mathrm{Ad}\ L_\rho \circ \alpha_0, \tag{3.16}$$

where

$$K_\rho := Z(\mu)^{1/2} Q_H(\epsilon) \qquad \in \mathfrak{F}_{loc} \tag{3.17}$$

and

$$L_\rho := K_\rho \alpha_0(K^*_\rho) = T_V(\rho) T_V(0)^{-1} \qquad \in \mathfrak{A}_{loc}. \tag{3.18}$$

the last equality holding for V large enough. Note also that for $\rho = (\mu, \epsilon)$ and $\rho' = (\mu', \epsilon')$ such that $\mathrm{supp}\, \mu \cap \mathrm{supp}\, \mu' = \emptyset$ one has

$$K_{\rho+\rho'} = K_\rho K_{\rho'}. \tag{3.19}$$

If the distance between supp ρ and supp ρ' is large enough one also has

$$L_{\rho+\rho'} = L_\rho L_{\rho'}. \tag{3.20}$$

Next, we define the *total charge*, $q_\rho \in \mathbb{Z}_N \times \mathbb{Z}_N$, of a distribution $\rho = (\epsilon, \mu) \in \mathcal{D}$ as

$$q_\rho = (e_\epsilon,\ m_\mu) := \left(\sum_x \varepsilon(x),\ \sum_p \mu(p) \right) \tag{3.21}$$

and put

$$\mathcal{D}_q := \{\rho \in \mathcal{D} \mid \quad q_\rho = q\}. \tag{3.22}$$

Then, for all $\rho = (\varepsilon, \mu) \in \mathcal{D}_0$, (i.e. with vanishing total charge), there exist 1-cochains, s_ε and s_μ, with finite support, such that

$$d^* s_\varepsilon = -\varepsilon \qquad \text{and} \qquad d s_\mu = -\mu \tag{3.23}$$

This allows to generalize the FM-construction and define, for $\rho = (\varepsilon, \mu) \in \mathcal{D}_0$, the family of states ω_ρ on $\mathfrak{A}$ as ground states of the modified dynamics α_ρ by the following method.

As already observed in [2] and [1], ground states for the modified transfer matrices can be obtained by taking the thermodynamic limit of the state defined by the formula

$$\omega_{V,\,\rho}(A) = \lim_{n\to\infty} \omega^n_{V,\,\rho}(A), \tag{3.24}$$

where

$$\omega^n_{V,\,\rho}(A) = \frac{Tr_{\mathcal{H}_V}\left(T_V(\rho)^n A T_V(\rho)^n E^\rho_V\right)}{Tr_{\mathcal{H}_V}\left(T_V(\rho)^{2n} E^\rho_V\right)}, \qquad A \in \mathfrak{F}(V), \tag{3.25}$$

and where E^ρ_V is some suitably chosen matrix to adjust the boundary conditions. There are two possibilities we will discuss. The first one is $E^\rho_V = \mathbb{1}$, thus getting periodic boundary conditions in euclidean time direction for the classical expectations associated with $\omega^n_{V,\,\rho}(A)$. In order to understand what happens algebraically one checks that, for $\mu = 0$, the desired ground state simply becomes $\omega_0 \circ \mathrm{Ad}\, Q_H(\epsilon)$, which is a state with *external* electric charge. Hence, the resulting representation of $\mathfrak{A}$ would be equivalent to the vacuum representation. A similar statement holds for $\mu \neq 0$. Since this is not the kind of state we are interested in let us look at the second case.

There we choose E^ρ_V in such a way that we get free boundary conditions in the euclidean time direction for the classical expectations associated with $\omega^n_{V,\,\rho}(A)$, together with horizontal electric and magnetic strings "conducting" the charges among the points of the the support of ϵ and μ, respectively, and located in the highest and in the lowest time-slices of $V_n := V \times \{-n, \dots, n\}$. Hence we choose 1-cochains s_ϵ and s_μ obeying (3.23) and put

$$E^\rho_V := \left[Z(\mu)^{-1/2} M(s_\mu) Q_{GH}(s_\varepsilon)\right] E^0_V \left[Z(\mu)^{-1/2} M(s_\mu) Q_{GH}(s_\varepsilon)\right]^* . \tag{3.26}$$

Here E_V^0 has been defined in (2.30), $Q_{GH}(b)$ are the gauge invariant link operators (2.10) and

$$M(s_\mu) := e^{-A_V/2} e^{-B_V} P_G(s_\mu) e^{B_V} e^{A_V/2} \equiv e^{-A_V/2} P_G(s_\mu) e^{A_V/2} \tag{3.27}$$

Note that $M(s_\mu) \in \mathfrak{A}(V)$ for all V containing $\operatorname{supp}\mu \cup \operatorname{supp} s_\mu$ and that $M(s_\mu)$ is actually independent of V provided the distance between $\operatorname{supp} s_\mu$ and the boundary ∂V is ≥ 2. The motivation for this definition comes from its effect in the euclidean functional integral, where $Q_{GH}(s_\varepsilon)$ produces a superposition of electric Mandelstam strings connecting the charges described by ε along the support of s_ε. Similarly, the operator $Z(\mu)^{-1/2} M(s_\mu)$ creates magnetic Mandelstam strings joining the plaquettes of supp μ via the support of s_μ. In the functional integral this magnetic string will appear in the form of shifted (or frustrated) vertical plaquettes placed between the first two and the last two time-slices of the space-time volume V_n.

This can be seen by looking at the product $T_V(\rho) Z(\mu)^{-1/2} M(s_\mu)$ appearing in (3.25) due to the definition (3.26). There, the factor $Z(\mu)^{-1/2} e^{-A_V/2} e^{-B_V}$ gets canceled by a corresponding factor in $T_V(\rho)$ and the matrix elements of $P_G(s_\mu) e^{B_V}$ (coming next according to (3.27)) are given by

$$\langle \varphi',\, \alpha' |\, P_G(s_\mu) e^{B_V}\, | \varphi,\, \alpha \rangle = \exp \left[- \sum_{b \in V_1} S_g(\alpha(b) - \alpha'(b) + s_\mu(b)) - \sum_{x \in V_0} S_h(\varphi(x) - \varphi'(x)) \right] \tag{3.28}$$

After transforming the functional integral expression for (3.25) to unitary gauge the shift $s_\mu(b)$ in (3.28) appears on the vertical plaquette spanned by the horizontal bond b and the time-like bond $\langle t_{n-1},\, t_n \rangle$ (and similarly, but with opposite sign at $\langle t_{-n},\, t_{-n+1} \rangle$).

The above construction of charged states can be understood as analogous to the construction of Fredenhagen and Marcu, except that, at finite volume, the horizontal Mandelstam strings are already located at the highest (respectively lowest) time-slices.

Given the definition (3.26), the limit (3.24) is actually independent of the choice of the horizontal strings s_ε and s_μ satisfying (3.23) and we can take the thermodynamic limit to obtain

$$\omega_\rho(A) := \lim_{V \uparrow \mathbb{Z}^2} \omega_{V,\,\rho}(A). \tag{3.29}$$

Note that $G(x) E_V^0 = E_V^0$ implies $G(x) E_V^\rho = E_V^\rho$ for all $x \in V$ and therefore ω_ρ provides an eigenstate of $\mathfrak{G}$ with zero external charge. Also note that the boundary conditions (3.26 / 3.27) now imply

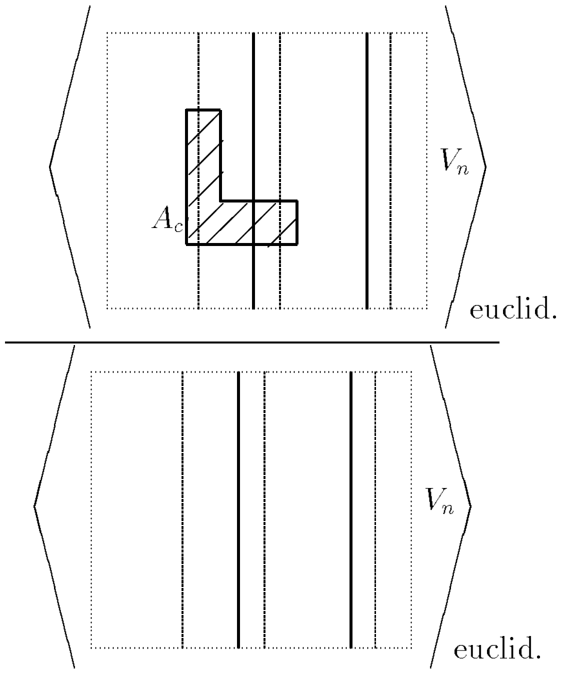
$$\omega\rho \circ i_C = \omega_{-\rho}\ . \tag{3.30}$$

Rewriting (3.29) in terms of euclidean expectation values in the unitary gauge we get an expectation of a classical function $A_{cl,\,\rho}$ associated with A in the presence of both, "electric" Wilson loops $\mathcal{L}_E(s_\varepsilon, n)$ living on bonds and "magnetic" vortex loops $\mathcal{L}_M(s_\mu, n)$ living on plaquettes, i.e.

$$\omega_\rho(A) = \langle A_{cl,\,\rho} \rangle_\rho := \lim_V \lim_n Z^{-1}_{\mathcal{L}_E, \mathcal{L}_M} \int da\; A_{cl,\,\rho}(a) e^{-S_G(da + \mathcal{L}_M) - S_H(a)} e^{\frac{2\pi i}{N}(a, \mathcal{L}_E)} \tag{3.31}$$

where a denotes the euclidean lattice $\mathbb{Z}_N$-gauge field $a := d\varphi - \alpha$ (unitary gauge) and $Z_{\mathcal{L}_E, \mathcal{L}_M}$ is the normalization such that $\omega_\rho(\mathbb{1}) = 1$. Here the vertical parts of the loops $\mathcal{L}_E$ and $\mathcal{L}_M$ run from euclidean time $t = -n$ to $t = +n$ and are spatially located at the supports of ε and μ, respectively. The horizontal parts of $\mathcal{L}_E$ and $\mathcal{L}_M$ are given by the 1-cochains $\pm s_{\varepsilon/\mu}$ (3.23), shifted to the euclidean time slice $\pm n$, respectively. To be more precise, $\mathcal{L}_E(s_\varepsilon, n) \in \mathcal{C}^1(V_n)$ is the unique 1-cochain on the euclidean space-time lattice $\mathbb{Z}^3$ satisfying $d^*\mathcal{L}_E(s_\varepsilon, n) = 0$ together with the condition that its horizontal part is nonzero only on the time slices $\pm n$, where it coincides with $\pm s_\varepsilon$. Similarly, $\mathcal{L}_M(s_\mu, n) \in \mathcal{C}^2(V_n)$ is the unique 2-cochain on $\mathbb{Z}^3$ satisfying $d\mathcal{L}_M(s_\mu, n) = 0$ plus the condition that the horizontal part of its dual 1-cochain is non-vanishing only on the time slices $\pm(n - 1/2)$, where it coincides with the dual of $\pm s_\mu$.

We will refer to such expectations by the following symbolic picture

$\omega_\rho(A) = \lim_{n,\, V}$ 

. (3.32)

The dotted box indicates the space-time volume V_n. The shaded region in the numerator indicates the support of the classical function A_{cl} associated with an observable A. The bold face vertical lines indicate the vertical part of the loops $\mathcal{L}_M$, i.e. the stacks of horizontal plaquettes whose projection onto the time-zero plane is given by μ. They are the euclidean realization of magnetic vortices located at μ. The dashed vertical lines indicate the vertical part of the loops $\mathcal{L}_E$. Their projection onto the time-zero plane is given by ε. They are the euclidean realization of electric charges located at ε. For finite n all vertical lines actually close to loops via the strings $\pm s_{\varepsilon/\mu}$ running in (or just inside of) the horizontal boundary of V_n. In the limit $n \to \infty$ the choice of these horizontal parts becomes irrelevant. In fact, applying our cluster expansion techniques [1] we have

Proposition 3.1.1 *For all $\rho \in \mathcal{D}_0$ the limit (3.29) provides a well defined state ω_ρ on $\mathfrak{F}$ which is independent of the choice of 1-chains (s_ε, s_μ) satisfying (3.23). Moreover, $\omega_\rho \restriction \mathfrak{A}$ provides a ground state with respect to the modified dynamics α_ρ fulfilling the cluster property.* □

Generalizing equation (3.2) we will show in Theorem 3.3.2 below, that for all $\rho \in \mathcal{D}_0$, i.e. with vanishing total charge, the states ω_ρ are actually induced by vectors in the vacuum sector $\mathcal{H}_0^0$.

To get charged states ω_ρ for $\rho \in \mathcal{D}_q, q \neq 0$, we now have to move a counter-charge to infinity. To this end let $\rho \cdot a$ denote the translate of ρ by $a \in \mathbb{Z}^2$. Then $\rho - \rho' \cdot a \in \mathcal{D}_0$ for all $\rho, \rho' \in \mathcal{D}_q$.

Proposition 3.1.2 *For $A \in \mathfrak{F}_{loc}$, $0 \neq q \in \mathbb{Z}_N \times \mathbb{Z}_N$ arbitrary and $\rho, \rho' \in \mathcal{D}_q$ let*

$$\omega_\rho(A) := \lim_{a \to \infty} \omega_{\rho - \rho' \cdot a}(A). \tag{3.33}$$

Then the limit exists independently of the chosen sequence $a \to \infty$ and it is independent of ρ'. Moreover $\omega_\rho \restriction \mathfrak{A}$ provides a ground state with respect to the modified dynamics α_ρ fulfilling the cluster property. □

Clearly, (3.33) implies that now (3.30) also holds for $q_\rho \neq 0$.

Next, we note that the symmetry group $\mathcal{S}$ of our spatial square lattice (i.e. consisting of translations by $a \in \mathbb{Z}^2$ and rotations by $\frac{1}{2}n\pi$) acts naturally from the right on $\mathcal{D}_q$ by $(\rho \cdot g)(x) := \rho(g \cdot x)$ and we obviously have $\omega_{V,\rho} \circ \tau_g = \omega_{g^{-1}V, \rho \cdot g}$ for all $g \in \mathcal{S}$. Hence we get the

Corollary 3.1.3 *For all $g \in \mathcal{S}$ and all $\rho \in \mathcal{D}$ we have $\omega_{\rho \cdot g} = \omega_\rho \circ \tau_g$, where τ_g denotes the natural action of $\mathcal{S}$ on $\mathfrak{F}$.* □

We now give an interpretation of these states as charged states in the following sense. For $V \subset \mathbb{Z}^2$, finite, define the charge measuring operators

$$\Phi^E(V) := \prod_{x \in V} \delta^* P_G(x) \qquad \text{and} \qquad \Phi^M(V_2) := \prod_{p \in V_2} \delta Q_G(p), \tag{3.34}$$

These operators are lattice analogues of the continuum expressions $\exp -i \int \nabla E$ and $\exp i \int dA$, respectively, i.e. they measure the total electric charge inside V and the total magnetic flux through V, respectively. As in Theorem 6.2 of [1] we have

Proposition 3.1.4 *If $V \subset \mathbb{Z}^2$ is e.g. a square centered at the origin, one has, under the conditions of Proposition 3.1.2:*

$$\lim_{V \uparrow \mathbb{Z}^2} \frac{\omega_\rho(\Phi^E(V))}{\omega_0(\Phi^E(V))} = \exp\left(\frac{2\pi i e_\epsilon}{N}\right), \tag{3.35}$$

$$\lim_{V \uparrow \mathbb{Z}^2} \frac{\omega_\rho(\Phi^M(V_2))}{\omega_0(\Phi^M(V_2))} = \exp\left(\frac{2\pi i m_\mu}{N}\right). \tag{3.36}$$

where e_ϵ and m_μ were defined in (3.21). □

We omit the proof here, since it is analogous to the equivalent one found in [1].

Finally, we check that our charges have Abelian composition rules. Since, as opposed to the DHR-theory of superselection sectors [12], the states ω_ρ are not given in terms of localized endomorphisms, we take the following statement as a substitute for this terminology.

Proposition 3.1.5 *Let $\rho, \rho' \in \mathcal{D}$ and for each $a \in \mathbb{Z}^2$ let ω_ρ and $\omega_{\rho + \rho' \cdot a}$ be given by Propositions 3.1.1 or 3.1.2, respectively. Then*

$$\lim_{a \to \infty} \omega_{\rho + \rho' \cdot a} = \omega_\rho. \square$$

Proposition 3.1.5 is actually a special case of the following more general factorization property, which will be important in the construction of scattering states in [10].

Theorem 3.1.6 *Let ρ_1, $\rho_2 \in \mathcal{D}$ and for $a \in \mathbb{Z}^2$ put $\rho(a) := \rho_1 + \rho_2 \cdot a$. Then, for all A, A', B and $B' \in \mathfrak{A}_{loc}$ and all $n \in \mathbb{N}_0$*

$$\lim_{|a|\to\infty} \omega_{\rho(a)} \left(\tau_a^{-1}(B) A \alpha_{\rho(a)}^n (A' \tau_a^{-1}(B'))\right) \quad =$$

$$\omega_{\rho_1} \left(A \alpha_{\rho_1}^n (A')\right) \omega_{\rho_2} \left(B \alpha_{\rho_2}^n (B')\right). \tag{3.37}$$

□

A sketch of the proofs of Propositions 3.1.1, 3.1.2 and Theorem 3.1.6 is given in the Appendices B.1- B.3.

3.2 Global Transfer Matrices

Given the family of dyonic states ω_ρ on $\mathfrak{A}$ we denote by $(\pi_\rho, \mathcal{H}_\rho, \Omega_\rho)$ the associated GNS representation π_ρ of $\mathfrak{A}$ on the Hilbert space $\mathcal{H}_\rho$ with cyclic vector $\Omega_\rho \in \mathcal{H}_\rho$. (From now on we will no longer consider external charges, and therefore we simplify the notation by putting $\mathcal{H}_\rho \equiv \mathcal{H}_\rho^0$). On $\mathcal{H}_\rho$ we define a *modified* global transfer matrix T_ρ by putting for $A \in \mathfrak{A}$

$$T_\rho \pi_\rho(A)\Omega_\rho = e^{-E_\rho} \pi_\rho(\alpha_\rho(A))\Omega_\rho, \tag{3.38}$$

where $E_\rho \in \mathbb{R}$ is an additive normalization constant, which will be determined later to appropriately adjust the self energies of the charge distributions ρ relative to each other.

Theorem 3.2.1 *For all $\rho \in \mathcal{D}$ equation (3.38) uniquely defines a positive bounded operator on $\mathcal{H}_\rho$,*

$$0 \le T_\rho \le e^{-E_\rho} \tag{3.39}$$

which is invertible on $\pi_\rho(\mathfrak{A})\Omega_\rho$ and implements the modified dynamics, i.e. $Ad\,T_\rho \circ \pi_\rho = \pi_\rho \circ \alpha_\rho$. Moreover, $\mathbb{C}\Omega_\rho$ is the unique eigenspace of T_ρ with maximal eigenvalue $e^{-E_\rho} \equiv \|T_\rho\|$. □

As a warning we recall that the charged states ω_ρ are ground states only with respect to the *modified euclidean dynamics* α_ρ and only when restricted to the observable algebra $\mathfrak{A} \subset \mathfrak{F}$. Hence, although π_ρ and T_ρ could easily be extended to $\mathfrak{F}$ and $\pi_\rho(\mathfrak{F})\Omega_\rho$, respectively, Ω_ρ would not necessarily be a ground state of T_ρ in this enlarged Hilbert space containing external charges. However, since we will never look at external charges we will not be bothered by such a possibility.

Proof of Theorem 3.2.1. First T_ρ is well defined on $\pi_\rho(\mathfrak{A})\Omega_\rho$, since $\pi_\rho(A)\Omega_\rho = 0$ implies $\omega_\rho(BA) = 0$ for all $B \in \mathfrak{A}$ and therefore $e^{2E_\rho}\|T_\rho\pi_\rho(A)\|^2 = \omega_\rho(\alpha_\rho(A)^*\alpha_\rho(A)) = \omega_\rho\left(\alpha_\rho^{-1}(\alpha_\rho(A)^*)A\right) = 0$ by α_ρ-invariance of ω_ρ. Since α_ρ is invertible, T_ρ is invertible on $\pi_\rho(\mathfrak{A})\Omega_\rho$. Moreover, by the ground state property we have

$$(\pi_\rho(A)\Omega_\rho,\ T_\rho\pi_\rho(A)\Omega_\rho) = e^{-E_\rho}\omega_\rho(A^*\alpha_\rho(A)) \le e^{-E_\rho}\omega_\rho(A^*A), \tag{3.40}$$

which implies $0 \leq T_\rho \leq e^{-E_\rho}$. The identity Ad $T_\rho \circ \pi_\rho = \pi_\rho \circ \alpha_\rho$ is obvious. Finally, let $P_\rho \in \mathcal{B}(\mathcal{H}_\rho)$ be the projection onto the eigenspace of T_ρ with eigenvalue e^{-E_ρ} and $\overset{\circ}{P}_\rho = P_\rho - |\Omega_\rho\rangle\langle\Omega_\rho|$. Then, for all $A,\, B \in \mathfrak{A}$,

$$\begin{aligned}\lim_n \omega_\rho(A^* \alpha_\rho^n(B)) &= \lim_n e^{nE_\rho}(\pi_\rho(A)\Omega_\rho,\, T_\rho^n \pi_\rho(B)\Omega_\rho) \\ &= (\pi_\rho(A)\Omega_\rho,\, P_\rho \pi_\rho(B)\Omega_\rho) \\ &= \omega_\rho(A^*)\omega_\rho(B) + (\pi_\rho(A)\Omega_\rho,\, \overset{\circ}{P}_\rho\, \pi_\rho(B)\Omega_\rho). \end{aligned} \tag{3.41}$$

Hence the cluster property implies $\overset{\circ}{P}_\rho = 0$. ■

Given the modified global transfer matrices T_ρ we now use relation (3.16) to define on $\mathcal{H}_\rho$ the family of transfer matrices

$$T_\rho(\rho') := \pi_\rho(L_{\rho'} L_\rho^{-1}) T_\rho, \tag{3.42}$$

where L_ρ has been given in (3.18). Hence, $T_\rho \equiv T_\rho(\rho)$ and we have

Corollary 3.2.2 *For all $\rho,\, \rho' \in \mathcal{D}$ the operators $T_\rho(\rho')$ are positive and bounded on $\mathcal{H}_\rho$ and*

$$Ad\, T_\rho(\rho') \circ \pi_\rho = \pi_\rho \circ \alpha_{\rho'}. \tag{3.43}$$

□

Proof of Corollary 3.2.2. For the purpose of this proof we consider temporarily the enlarged GNS-triple $(\pi_\rho(\mathfrak{F}),\, \Omega_\rho,\, \mathcal{H}_\rho)$, where $\mathcal{H}_\rho$ now is the closure of $\pi_\rho(\mathfrak{F})\Omega_\rho$ and contains vector states with external electric charges. Equation (3.43) follows immediately from (3.16). To see that $T_\rho(\rho')$ is positive we use $L_\rho = K_\rho \alpha_0(K_\rho^*)$, for K_ρ defined in (3.17), to conclude

$$\begin{aligned} T_\rho(\rho') &= \pi_\rho(K_{\rho'}) T_\rho(0) \pi_\rho(K_{\rho'}^*) \\ &= \pi_\rho(K_{\rho'} K_\rho^{-1}) T_\rho(\rho) \pi_\rho(K_\rho^{*\,-1} K_{\rho'}^*), \end{aligned} \tag{3.44}$$

which is positive since $T_\rho(\rho) \equiv T_\rho$ is positive. ■

In view of this result we call $T_\rho(0)$ the *unmodified* ("true") global transfer matrix, since it implements the original dynamics $\alpha \equiv \alpha_0$. We also have the following important

Corollary 3.2.3 *Let $\mathcal{A}_\rho \subset \mathcal{B}(\mathcal{H}_\rho)$ be the $*$-algebra generated by $\pi_\rho(\mathfrak{A})$ and $T_\rho(0)$. Then the commutant of $\mathcal{A}_\rho$ is trivial: $\mathcal{A}_\rho' = \mathbb{C}\,1\!\!1$.* □

Proof of Corollary 3.2.3. Since $T_\rho(\rho) \in \mathcal{A}_\rho$ we have $[B,\, T_\rho(\rho)] = 0$ for all $B \in \mathcal{A}_\rho'$ which implies $B\Omega_\rho = \lambda\Omega_\rho$ for some $\lambda \in \mathbb{C}$ by the uniqueness of the ground state vector Ω_ρ of $T_\rho(\rho)$. Hence $B\pi_\rho(A)\Omega_\rho = \pi_\rho(A)B\Omega_\rho = \lambda\pi_\rho(A)\Omega_\rho$ for all $A \in \mathfrak{A}$ and therefore $B = \lambda 1\!\!1$. ■

In view of Corollary 3.2.3 we may consider π_ρ as an irreducible representation of an extension $\hat{\mathfrak{A}} \supset \mathfrak{A}$ such that $\mathcal{A}_\rho = \pi_\rho(\hat{\mathfrak{A}})$. Algebraically $\hat{\mathfrak{A}}$ is defined to be the crossed product of $\mathfrak{A}$ with the semi-group $\mathbb{N}_0$ acting by $n \mapsto \alpha^n \in \mathrm{Aut}\,\mathfrak{A}$. Described in terms of generators and relations $\hat{\mathfrak{A}}$ is the $*$-algebra generated by $\mathfrak{A}$ and a new selfadjoint element $\mathbf{t} \equiv \mathbf{t}(0)$ with commutation relation $\mathbf{t}A = \alpha(A)\mathbf{t}$ for all $A \in \mathfrak{A}$. Note that the construction (2.32) shows that any GNS-representation π_ω of $\mathfrak{A}$ from an α-invariant state ω naturally extends to a representation of $\hat{\mathfrak{A}}$. Moreover, $T_\omega = \pi_\omega(\mathbf{t})$ is positive provided the first inequality in (2.26) holds. Using (3.18) we may also define

$$\mathbf{t}(\rho) := L_\rho \, \mathbf{t} \ \in \hat{\mathfrak{A}}. \tag{3.45}$$

Then $\mathbf{t}(\rho) = \mathbf{t}(\rho)^*$ (since $\alpha(L_\rho^*) = L_\rho$)) and $\mathbf{t}(\rho)\, A = \alpha_\rho(A)\, \mathbf{t}(\rho)$ for all $A \in \mathfrak{A}$. By (3.42) this implies that also π_ρ extends to a representation of $\hat{\mathfrak{A}}$ by defining for fixed ρ and all ρ'

$$\pi_\rho(\mathbf{t}(\rho')) := c_\rho \, T_\rho(\rho') \tag{3.46}$$

where $c_\rho > 0$ may be chosen arbitrarily.

3.3 Global Charges

In this section we show that two charged representations of $\mathfrak{A}$, π_ρ and π'_ρ, are *dynamically equivalent* if their total charges coincide, $q_{\rho_1} = q_{\rho_2}$. Here we take as an appropriate notion of equivalence the following

Definition 3.3.1 *Two representations, π and π', of $\mathfrak{A}$ are called dynamically equivalent, if they both extend to representations of $\hat{\mathfrak{A}}$ and if there exists a unitary intertwiner $U : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ such that $U\,\pi(A) = \pi'(A)\,U$ for all $A \in \mathfrak{A}$ and $U\,\pi(\mathbf{t}) = c\,\pi'(\mathbf{t})\,U$ for some constant $c > 0$.* □

The use of the constant c is to allow for the possibility of different zero-point normalizations of the energy. Clearly, by rescaling the global transfer matrices one can always achieve $c = 1$. To prove that for $q_\rho = q'_\rho$ the representations π_ρ and π'_ρ are dynamically equivalent in this sense we first sΩhow

Proposition 3.3.2 *Let $q \in \mathbb{Z}_N \times \mathbb{Z}_N$ be fixed and let $\rho, \rho' \in \mathcal{D}_q$. Then there exists a unique (up to a phase) unit vector $\Phi_{\rho,\rho'} \in \mathcal{H}_\rho$ such that $T_\rho(\rho')\,\Phi_{\rho,\rho'} = \|T_\rho(\rho')\|\,\Phi_{\rho,\rho'}$. Moreover, $\Phi_{\rho,\rho'} \in \mathcal{H}_\rho$ is cyclic for $\pi_\rho(\mathfrak{A})$ and it induces the state $\omega_{\rho'}$, i.e.*

$$(\Phi_{\rho,\rho'},\, \pi_\rho(A)\Phi_{\rho,\rho'}) = \omega_{\rho'}(A), \quad \forall A \in \mathfrak{A}. \tag{3.47}$$

□

Proof: We adapt the proof of Theorem 6.4 of [2] to our setting. Hence we pick 1-cochains $\ell_\varepsilon,\ \ell_m \in \mathcal{C}^1_{loc}$ such that $(d^*\ell_e,\, d\ell_m) = \rho - \rho' \in \mathcal{D}_0$ and define

$$\Phi^n_{\rho,\rho'} := \frac{T_\rho(\rho')^n \Omega_{\rho,\rho'}}{\|T_\rho(\rho')^n \Omega_{\rho,\rho'}\|} \tag{3.48}$$

where

$$\Omega_{\rho,\rho'} \ := \ \pi_\rho(A_{\rho,\rho'}(\ell_e,\ell_m))\,\Omega_\rho \tag{3.49}$$

$$A_{\rho,\rho'}(\ell_e,\ell_m) \ := \ Z(\mu')^{-1/2} M(\ell_m) Q_{GH}(\ell_e) Z(\mu)^{1/2} \tag{3.50}$$

and where μ, μ' denote the magnetic components in ρ, ρ'. Using the definitions (3.24-3.29) we check in Appendix B.5 that the sequence of states $(\Phi^n_{\rho,\rho'}\,,\,\pi_\rho(A)\Phi^n_{\rho,\rho'})$ converges to $\omega_{\rho'}(A)$ for all $A \in \mathfrak{A}_{loc}$. We also show, that $\Phi^n_{\rho,\rho'}$ is actually a Cauchy sequence in $\mathcal{H}_\rho$ and therefore

$$\Phi_{\rho,\rho'} := \lim_n \Phi^n_{\rho,\rho'} \ \in \mathcal{H}_\rho \tag{3.51}$$

exists. By construction it is an eigenvector of $T_\rho(\rho')$ with eigenvalue

$$\lambda_{\rho'} = \lim_n \frac{\|T_\rho(\rho')^{n+1}\Omega_{\rho,\rho'}\|}{\|T_\rho(\rho')^n\Omega_{\rho,\rho'}\|} = \lim_n (\Phi^n_{\rho,\rho'}\,,\,T_\rho(\rho')^2\,\Phi^n_{\rho,\rho'})^{1/2} \tag{3.52}$$

Let now $\mathcal{H}_{\rho,\rho'} := \overline{\pi_\rho(\mathfrak{A})\Phi_{\rho,\rho'}} \subset \mathcal{H}_\rho$. Then $T_\rho(\rho')\mathcal{H}_{\rho,\rho'} \subset \mathcal{H}_{\rho,\rho'}$ and therefore $\pi_\rho(\hat{\mathfrak{A}})\mathcal{H}_{\rho,\rho'} \subset \mathcal{H}_{\rho,\rho'}$ by (3.46). Hence $\mathcal{H}_{\rho,\rho'} = \mathcal{H}_\rho$ by Corollary 3.2.3. By the cluster property of $\omega_{\rho'}$ with respect to $\alpha_{\rho'}$ we conclude $\lambda_{\rho'} = \|T_\rho(\rho')\|$ and similarly as in (3.41) the associated eigenspace must be 1-dimensional.

■

For later purposes we emphasize that Proposition 3.3.2 in particular implies that the choice of $(\ell_e,\,\ell_m)$ in (3.49) only influences the phase of the limit vector $\Phi_{\rho,\rho'}$ in (3.51).

Using Proposition 3.3.2 we now get the equivalence of charged representations whenever their total charge coincides.

Theorem 3.3.3 *Let ρ, $\rho' \in \mathcal{D}$ and $q_\rho = q_{\rho'}$. Then π_ρ and $\pi_{\rho'}$ are dynamically equivalent and*

$$\frac{\|T_\rho(0)\|}{\|T_{\rho'}(0)\|} = \frac{\|T_\rho(\rho'')\|}{\|T_{\rho'}(\rho'')\|} \quad \forall \rho'' \in \mathcal{D} \tag{3.53}$$

Proof: Let $U : \mathcal{H}_{\rho'} \to \mathcal{H}_\rho$ be given on $\pi_{\rho'}(\mathfrak{A})\Omega_{\rho'}$ by

$$U\pi_{\rho'}(A)\Omega_{\rho'} := \pi_\rho(A)\Phi_{\rho,\rho'}\ . \tag{3.54}$$

Then U extends to a unitary intertwining $\pi_{\rho'}$ and π_ρ. Moreover, since $T_{\rho'}(\rho')$ and $T_\rho(\rho')$ implement $\alpha_{\rho'}$ on $\mathcal{H}_{\rho'}$ and $\mathcal{H}_\rho$, respectively, one immediately concludes

$$UT_{\rho'}(\rho')\|T_{\rho'}(\rho')\|^{-1} = T_\rho(\rho')\|T_\rho(\rho')\|^{-1}U \tag{3.55}$$

Using (3.42) this implies

$$UT_{\rho'}(\rho'')\|T_{\rho'}(\rho')\|^{-1} = T_\rho(\rho'')\|T_\rho(\rho')\|^{-1}U \quad \forall \rho'' \in \mathcal{D} \tag{3.56}$$

and therefore π_ρ and $\pi_{\rho'}$ are dynamically equivalent in the sense of Definition 3.3.1. Finally, (3.56) immediately implies (3.53). ■

We now recall that fixing $\|T_\rho(\rho)\| \equiv e^{-E_\rho}$ for a given $\rho \in \mathcal{D}$ amounts to fixing $\|T_\rho(\rho'')\|$ for any other $\rho'' \in \mathcal{D}$. Hence, Theorem 3.3.3 shows that the energy normalization E_ρ, $\rho \in \mathcal{D}_q$, in (3.38) may be fixed up to a constant depending only on the total charge q by requiring $\|T_\rho(\rho'')\| = \|T_{\rho'}(\rho'')\|$ for all $\rho, \rho' \in \mathcal{D}_q$ and some (and hence all) $\rho'' \in \mathcal{D}$. In particular we now take $\rho'' = 0$ and require

$$\|T_\rho(0)\| = e^{-\epsilon(q)}, \qquad \forall \rho \in \mathcal{D}_q \tag{3.57}$$

for some - as yet undetermined - function $\epsilon(q) \in \mathbb{R}$ describing the "minimal self-energy" in the sector with total charge $q \in \mathbb{Z}_N \times \mathbb{Z}_N$. In the next subsection we will appropriately fix $\epsilon(q)$ such that $2\epsilon(q) \equiv \epsilon(q) + \epsilon(-q)$ gives the minimal energy needed to create a pair of charge-anticharge configurations of total charge $\pm q$ from the vacuum and separate them apart to infinite distance.

We close this subsection by remarking that we expect conversely (at least for a generic choice of couplings in the free charge phase) that the representations π_ρ and $\pi_{\rho'}$ are dynamically inequivalent if $q_\rho \neq q_{\rho'}$. Inequivalence to the vacuum sector for $q_\rho \neq 0$ can presumably be shown by analogue methods as in [2], i.e. by proving the absence of a translation invariant vector in $\mathcal{H}_\rho$ (see Section 4.3 for the implementations of translations in the charged sectors). More generally, one could try to prove that $\|T_\rho(\rho')\|$ is not an eigenvalue of $T_\rho(\rho')$ if $q_\rho \neq q_{\rho'}$. To this end one would have to show that $q_\rho \neq q_{\rho'}$ implies

$$\mathrm{s} - \lim_n \frac{T_\rho(\rho')^n}{\|T_\rho(\rho')\|^n} \, \pi_\rho(A)\Omega_\rho = 0 \tag{3.58}$$

for all $A \in \mathfrak{A}_{loc}$. As in Proposition 5.5 of [2] a sufficient condition for this would be

$$\lim_{n\to\infty} \frac{\pi_\rho(A)\Omega_\rho,\ T_\rho(\rho')^n \pi_\rho(A)\Omega_\rho)^2}{\pi_\rho(A)\Omega_\rho,\ T_\rho(\rho')^{2n}\pi_\rho(A)\Omega_\rho)} = 0 \tag{3.59}$$

Using (3.42) this would amount to finding the asymptotics of the vertical string expectation value

$$\omega_\rho \left(A^* \prod_{k=0}^{n-1} \alpha_\rho^k (L_{\rho'} L_\rho^{-1}) \alpha_\rho^n(A) \right). \tag{3.60}$$

Similarly as in the Bricmont-Fröhlich criterion for the existence of free charges [13] equation (3.59) would follow provided (3.60) would decay like $n^{-\alpha} e^{-n\cdot const}$ for some $\alpha > 0$ (in [13] $\alpha = d/2$). Since we have not tried to prove this we only formulate the

Conjecture 3.3.4 *If $q_\rho \neq q_{\rho'}$ then $\|T_\rho(\rho')\|$ is not an eigenvalue of $T_\rho(\rho')$.*

Together with Proposition 3.3.2 and Theorem 3.3.3 the Conjecture 3.3.4 would imply that π_ρ and $\pi_{\rho'}$ are dynamically equivalent if *and only if* their total charges coincide.

3.4 The Dyonic Self Energies

In this subsection we determine the "dyonic selfenergies" $\epsilon(q)$ introduced in (3.57). In order to get charge conjugation invariant self energies it will be useful to start with implementing the charge conjugation as an intertwiner C_ρ: $\mathcal{H}_\rho \to \mathcal{H}_{-\rho}$ by putting for $A \in \mathfrak{A}$ and $\rho \in \mathcal{D}$

$$C_\rho \pi_\rho(A)\Omega_\rho := \pi_{-\rho}(i_C(A))\Omega_{-\rho}. \tag{3.61}$$

Lemma 3.4.1 *C_ρ extends to a well defined unitary $\mathcal{H}_\rho \to \mathcal{H}_{-\rho}$ such that*

i) $Ad\, C_\rho \circ \pi_\rho = \pi_{-\rho} \circ i_C$.

ii) $C_\rho^* = C_{-\rho}$.

iii) $C_\rho T_\rho(\rho')e^{E_\rho} = T_{-\rho}(-\rho')e^{E_{-\rho}}C_\rho$. □

Proof: By equation (3.30) C_ρ is well defined, unitary and obeys i). Part ii) follows from $i_C^2 = \mathrm{id}$. Finally, in the case $\rho' = \rho$ part iii) follows from equation (3.15). By (3.42) the case $\rho' \neq \rho$ follows from $i_C(L_\rho) = L_{-\rho}$. ■

By charge conjugation symmetry we would of course like to have $E_\rho = E_{-\rho}$. We now fix the self-energies $\epsilon(q)$ in (3.57) of the sectors $q \in \mathbb{Z}_N \times \mathbb{Z}_N$ by first putting the vacuum energy to zero, $\epsilon(0) = 0$, i.e.

$$\|T_\rho(0)\| = 1, \quad \forall \rho \in \mathcal{D}_0 \ . \tag{3.62}$$

This fixes $e^{-E_\rho} \equiv \|T_\rho(\rho)\| = \|\pi_\rho(L_\rho)T_\rho(0)\|$ for all $\rho \in \mathcal{D}_0$ and in particular implies $E_\rho = E_{-\rho}$, $\forall \rho \in \mathcal{D}_0$. In fact, we have

Lemma 3.4.2 *For fixed $q \in \mathbb{Z}_N \times \mathbb{Z}_N$ we have $\epsilon(q) = \epsilon(-q)$ if and only if $E_\rho = E_{-\rho}$, $\forall \rho \in \mathcal{D}_q$.* □

Proof: Putting $\rho' = 0$ in Lemma 3.4.1 iii) we see that $\|T_\rho(0)\| = \|T_{-\rho}(0)\|$, $\forall \rho \in \mathcal{D}_q$, is equivalent to $E_\rho = E_{-\rho}$, $\forall \rho \in \mathcal{D}_q$, since C_ρ is unitary. ■

We now show that under the conditions $\epsilon(0) = 0$ and $\epsilon(q) = \epsilon(-q)$ the self-energies of the charged sectors are completely determined if we adjust them such that $2\epsilon(q) \equiv \epsilon(q)+\epsilon(-q)$ becomes equal to the minimal energy needed to create a pair of charge-anticharge configurations of total charge $\pm q$ from the vacuum and separate them apart to infinity. Physically this means that for infinite separation we can consistently normalize the interaction energy between two charges to zero. Mathematically this is expressed by a factorization formula similar to Theorem 3.1.6 for matrix elements of the global transfer matrix . As it will turn out, this is further equivalent to an additivity property of the modified ground state energies, i.e. $E_{\rho+\rho'\cdot a} \to E_\rho + E_{\rho'}$ as $|a| \to \infty$.

Theorem 3.4.3 *There exists precisely one assignment of self-energies $\mathbb{Z}_N \times \mathbb{Z}_N \ni q \to \epsilon(q) \in \mathbb{R}$ such that $\|T_\rho(0)\| = e^{-\epsilon(q)}$, $\forall \rho \in \mathcal{D}_q$ and such that the following conditions hold*

i) $\epsilon(0) = 0$

ii) $\epsilon(q) = \epsilon(-q)$

iii) *The factorization formula (3.37) extends to transfer matrices, i.e.*

$$\lim_{|a|\to\infty} \left(\pi_{\rho(a)}(A\tau_{-a}(B))\Omega_{\rho(a)}\,,\ T_{\rho(a)}(0)^n \pi_{\rho(a)}(A'\tau_{-a}(B'))\Omega_{\rho(a)}\right)$$

$$= (\pi_{\rho_1}(A)\Omega_{\rho_1}\,,\ T_{\rho_1}(0)^n\pi_{\rho_1}(A')\Omega_{\rho_1})\ (\pi_{\rho_2}(B)\Omega_{\rho_2}\,,\ T_{\rho_2}(0)^n\pi_{\rho_2}(B')\Omega_{\rho_2}) \tag{3.63}$$

where we used the notation of Theorem 3.1.6. □

To prove Theorem 3.4.3 and in particular the cluster property 3.63 we have to control norm ratios of (modified) transfer matrices, for which we have to invoke our polymer expansions. First we have

Lemma 3.4.4 *Let $\rho,\,\rho' \in \mathcal{D}_q$ for some $q \in \mathbb{Z}_N \times \mathbb{Z}_N$ and choose 1-cochains $\ell_e,\,\ell_m \in \mathcal{C}^1_{loc}$ such that $(d^*\ell_e,\,d\ell_m) = \rho - \rho'$. Let $A_{\rho,\rho'} \equiv A_{\rho,\rho'}(\ell_e,\,\ell_m) \in \mathfrak{A}_{loc}$ be defined as in (3.50) and put*

$$X^{(\nu)}_{\rho,\rho'} := A^*_{\rho,\rho'} \left[\prod_{k=0}^{\nu-1} \alpha^k_\rho(L_{\rho'} L_\rho^{-1})\right] \alpha^\nu_\rho(A_{\rho,\rho'}) \,. \tag{3.64}$$

Then we have

$$\frac{\|T_\rho(\rho')\|}{\|T_\rho(\rho)\|} = \lim_{\nu\to\infty} \frac{\omega_\rho(X^{(2\nu+1)}_{\rho,\rho'})}{\omega_\rho(X^{(2\nu)}_{\rho,\rho'})}. \tag{3.65}$$

□

Proof: By Proposition 3.3.2 and equations (3.48) - (3.51) we have for all $q \in \mathbb{Z}_N \times \mathbb{Z}_N$ and all $\rho,\,\rho' \in \mathcal{D}_q$

$$\|T_\rho(\rho')\| = (\Phi_{\rho,\rho'}\,,\, T_\rho(\rho')\Phi_{\rho,\rho'}) = \lim_{\nu\to\infty} \frac{(\Omega_{\rho,\rho'},\; T_\rho(\rho')^{2\nu+1}\Omega_{\rho,\rho'})}{(\Omega_{\rho,\rho'},\; T_\rho(\rho')^{2\nu}\Omega_{\rho,\rho'})} \tag{3.66}$$

where

$$\Omega_{\rho,\rho'} := \pi_\rho(A_{\rho,\rho'}(\ell_e,\ell_m))\,\Omega_\rho$$

has been given in (3.49). The Lemma follows by using (3.42) to rewrite

$$T_\rho(\rho')^\nu = \left[\prod_{k=0}^{\nu-1} \alpha^k_\rho(L_{\rho'} L_\rho^{-1})\right] T_\rho(\rho)^\nu$$

and recalling $T_\rho(\rho)\Omega_\rho = \|T_\rho(\rho)\|\Omega_\rho$ by Theorem 3.2.1. ■

Note that the norm ratio (3.65) is obviously independent of the choice of the scale $e^{E_\rho} \equiv \|T_\rho(\rho)\|$. We also recall from Proposition 3.3.2 that the choice of $(\ell_e,\,\ell_m)$ only influences the phase of $\Phi_{\rho,\rho'}$ and hence the limit $\nu \to \infty$ in (3.65) is in fact independent of this choice. When expressing the r.h.s. of (3.65) in terms of euclidean path integrals we get a ratio of expectations of the form (3.31), where the classical function corresponding to $X^{(\nu)}_{\rho,\rho'}$ is itself a superposition of Wilson loops and vortex loops. These are built by the same recipes as before, i.e. with horizontal parts given by $\pm(\ell_e,\,\ell_m)$ in the time slices $t = \nu$ and $t = 0$, respectively, and with vertical parts joining the endpoints of these strings. Hence we get the ratio of euclidean path integral expectations of two such loop configurations with time-like extension $\nu + 1$ and ν, respectively, in the limit $\nu \to \infty$. Here these expectations have to be taken in the background determined by ρ, i.e. in the presence of the vertical parts of $\mathcal{L}_E$ and $\mathcal{L}_M$ described in (3.31), where there the limit $n \to \infty$ and $V \to \infty$ has to be taken first. [10]

Next we have to control certain factorization properties of the above norm ratios in the limit of the charge distributions being separated to infinity.

Proposition 3.4.5 *Let $\rho_1,\ \rho_2 \in \mathcal{D}$ and for $a \in \mathbb{Z}^2$ put $\rho_1(a) = \rho_1 - \rho_1 \cdot a,\ \rho_2(a) = \rho_2 - \rho_2 \cdot a$. Then*

[10] This is why in our polymer expansion only the vertical parts of $\mathcal{L}_E$ and $\mathcal{L}_M$ matter, since the horizontal parts run in the time-like boundary of V_n and since the contributions from polymers reaching from the support of a classical function A_{cl} to this boundary decay to zero for $n \to \infty$.

i) The limit

$$c_{\rho_1,\,\rho_2} := \lim_{|a|\to\infty} \frac{\|T_{\rho_1(a)}(\rho_2(a))\|}{\|T_{\rho_1(a)}(\rho_1(a))\|} > 0 \tag{3.67}$$

exists independently of the sequence $a \to \infty$ and satisfies $c_{\rho_1\cdot g,\,\rho_2\cdot g} = c_{\rho_1,\,\rho_2}$, $\forall g \in \mathcal{S}$. Moreover, for $b \in \mathbb{Z}^2$ we have the factorization property

$$\lim_{|b|\to\infty} c_{\rho_1+\rho_1'\cdot b,\,\rho_2+\rho_2'\cdot b} = c_{\rho_1,\,\rho_2} c_{\rho_1',\,\rho_2'} \tag{3.68}$$

ii) If $q_{\rho_1} = q_{\rho_2}$ then

$$c_{\rho_1,\,\rho_2} = \frac{\|T_{\rho_1}(\rho_2)\|^2}{\|T_{\rho_1}(\rho_1)\|^2}. \tag{3.69}$$

□

Proof: Part i) is proven in detail in Appendix B.4. Here we just remark that it can roughly be understood from the perimeter law of Wilson and vortex loop expectations, which guarantees that ratios of loop expectations converge for infinitely large loops if the difference of their perimeters stays finite (in the above case this difference is given by two lattice units, due to the difference by one unit in time extension, see (3.65)).

To prove ii) we pick in (3.66) the choice

$$\Omega_{\rho_1(a),\,\rho_2(a)} = \pi_{\rho_1(a)}\left(A_{\rho_1(a),\,\rho_2(a)}(\ell_e(a),\,\ell_m(a))\right)\Omega_{\rho_1(a)} \tag{3.70}$$

where we take

$$(\ell_e(a),\,\ell_m(a)) = (\ell_e - \ell_e\cdot a,\,\ell_m - \ell_m\cdot a)$$

for some fixed configuration $(\ell_e,\,\ell_m)$ satisfying $(d^*\ell_e,\,d\ell_m) = \rho_1 - \rho_2$. With this choice it is not difficult to check that for $|a|$ large enough

$$A_{\rho_1(a),\,\rho_2(a)}(\ell_e(a),\,\ell_m(a)) = A_{\rho_1,\,\rho_2}(\ell_e,\,\ell_m)\tau_{-a}(A_{\rho_1,\,\rho_2}(\ell_e,\,\ell_m)) \tag{3.71}$$

Similarly, for fixed ν and $|a|$ large enough

$$X^{(\nu)}_{\rho_1(a),\,\rho_2(a)} = X^{(\nu)}_{\rho_1,\,\rho_2}\,\tau_{-a}(X^{(\nu)}_{\rho_1,\,\rho_2}) \tag{3.72}$$

When proving part i) in Appendix B.4 we show that plugging (3.72) into (3.65) the limit $\nu \to \infty$ is uniform in a and hence we may take the limit $|a| \to \infty$ first. However, this takes us into the setting of the factorization formula (3.37) implying

$$\lim_{|a|\to\infty} \omega_{\rho(a)}(X^{(\nu)}_{\rho(a),\,\rho'(a)}) = \omega_{\rho_1}(X^{(\nu)}_{\rho_1,\,\rho_2})^2\ , \tag{3.73}$$

which proves part ii). ■

Proof of Theorem 3.4.3: The idea of the proof consists of translating the physically motivated conditions i)-iii) into conditions on the family of constants E_ρ, $\rho \in \mathcal{D}$. We have already remarked in (3.62) that i) may be obtained by suitably fixing E_ρ for all $\rho \in \mathcal{D}_0$. In Lemma 3.4.2 we have noticed that ii) is equivalent to $E_\rho = E_{-\rho}$ for all ρ. We now show that iii) holds if and only if for all $\rho_1, \rho_2 \in \mathcal{D}$

$$\lim_{|a|\to\infty} E_{\rho_1+\rho_2\cdot a} = E_{\rho_1} + E_{\rho_2} \tag{3.74}$$

To this end we use

$$T_\rho(\rho)^n = [\pi_\rho(L_\rho)T_\rho(0)]^n = \pi_\rho(Y_\rho^{(n)})T_\rho(0)^n \tag{3.75}$$

where

$$Y_\rho^{(n)} := \prod_{k=0}^{n-1} \alpha_0^k(L_\rho) \tag{3.76}$$

Furthermore, for $|a|$ large enough and $\rho(a) = \rho_1 + \rho_2 \cdot a$ we have

$$L_{\rho(a)} = L_{\rho_1}\tau_{-a}(L_{\rho_2}) = \tau_{-a}(L_{\rho_2})L_{\rho_1} \ . \tag{3.77}$$

Using that α_0 commutes with τ_{-a} we get for large enough $|a|$ and $A, A', B, B' \in \mathfrak{A}_{loc}$

$$\left(\pi_{\rho(a)}(A\tau_{-a}(B))\Omega_{\rho(a)}\,,\ T_{\rho(a)}(0)^n\pi_{\rho(a)}(A'\tau_{-a}(B'))\Omega_{\rho(a)}\right)$$

$$= \omega_{\rho(a)}\left(\tau_{-a}(B^*[Y_{\rho_2}^{(n)}]^{-1})A^*[Y_{\rho_1}^{(n)}]^{-1}\alpha^n_{\rho(a)}(A'\tau_{-a}(B'))\right) e^{-nE_{\rho(a)}} \ .$$

Here we have chosen $|a|$ large enough (depending on n) such that A^* commutes with $\tau_{-a}\left(Y_{\rho_2}^{(n)}\right)^{-1}$. On the other hand, we get for $i = 1, 2$ and all $A, A' \in \mathfrak{A}_{loc}$

$$(\pi_{\rho_i}(A)\Omega_{\rho_i}\,,\ T_{\rho_i}(0)^n\pi_{\rho_i}(A')\Omega_{\rho_i}) = \omega_{\rho_i}\left(A^*[Y_{\rho_i}^{(n)}]^{-1}\alpha^n_{\rho_i}(A')\right) e^{-nE_{\rho_i}} \ . \tag{3.78}$$

Thus, by Theorem 3.1.6 the factorization formula (3.63) is equivalent to (3.74).

Hence, there can be at most one solution ϵ of the conditions i)-iii) above. Indeed, equations (3.74) together with $E_\rho = E_{-\rho}$ imply

$$E_\rho = \frac{1}{2}\lim_{|a|\to\infty} E_{\rho-\rho a} \tag{3.79}$$

for all $\rho \in \mathcal{D}$. Since $\rho - \rho\cdot a \in \mathcal{D}_0$ and since condition i) completely fixes $E_\rho = E_{-\rho}$ for all $\rho \in \mathcal{D}_0$, equation (3.79) fixes $E_\rho = E_{-\rho}$ uniquely for all $\rho \in \mathcal{D}$.

We are left to prove that given $E_\rho = E_{-\rho}$ for all $\rho \in \mathcal{D}_0$ via condition i) the limit (3.79) indeed exists for all $\rho \in \mathcal{D}$ and satisfies (3.74). However this is an immediate consequence of Proposition 3.4.5. Indeed, putting $\rho_2 = 0$ in (3.67) and using $\|T_{\rho_1(a)}(0)\| = 1$ by condition i) we get for all $\rho \in \mathcal{D}$

$$\lim_{|a|\to\infty} e^{E_{\rho-\rho\cdot a}} = c_{\rho,0} \tag{3.80}$$

and hence (3.79) implies

$$E_\rho = \frac{1}{2}\ln c_{\rho,0} \tag{3.81}$$

Note that for $q_\rho = 0$ this is consistent with (3.69). The cluster property (3.74) now is an immediate consequence of (3.68). This concludes the proof of Theorem 3.4.3. ■

We remark that we have not worked out an analytic expression for the ratio $\frac{\|T_\rho(0)\|}{\|T_\rho(\rho)\|}$ in the case $q_\rho \neq 0$, which is why we do not have further analytic knowledge of the dyonic self-energies $\epsilon(q)$. In particular we have not tried to confirm the natural "stability conjecture" $\epsilon(q) > 0$ for all $q \neq 0$. In the purely electric (or purely magnetic) sectors the existence of 1-particle states [1] implies that $e^{-\epsilon(q_\rho)} \equiv \|T_\rho(0)\|$ lies in the spectrum of $T_\rho(0)$, i.e. $\epsilon(q_\rho)$ is precisely the 1-particle self-energy at zero momentum.

4 State Bundles and Intertwiner Connections

In this section we consider an appropriate analogue of what in the DHR-theory of super selection sectors would be called the state bundle. In our setting, by this we mean the collection of all GNS triples $(\mathcal{H}_\rho,\ \pi_\rho,\ \Omega_\rho)$ of cyclic representation of $\hat{\mathfrak{A}}$ obtained from the family of states $\omega_\rho,\ \rho \in \mathcal{D}$. Modulo Conjecture 3.3.4, these representations fall into equivalence classes labelled by their total charges q_ρ. Hence we obtain, for each value of the global charge $q \in \mathbb{Z}_N \times \mathbb{Z}_N$, a Hilbert-bundle $\mathcal{B}_q$ over the discrete base space $\mathcal{D}_q$, which is simply given as the disjoint union

$$\mathcal{B}_q := \dot{\bigcup_{\rho \in \mathcal{D}_q}} \mathcal{H}_\rho. \tag{4.1}$$

The fibers $\mathcal{H}_\rho$ of this bundle are all naturally isomorphic as $\hat{\mathfrak{A}}$-modules by Theorem 3.3.3. Since $\pi_\rho(\hat{\mathfrak{A}})$ acts irreducibly on $\mathcal{H}_\rho$ by Corollary 3.2.3, these isomorphisms are all uniquely determined up to a phase.

When trying to fix this phase ambiguity one is naturally led to the problem of constructing an *intertwiner connection* on $\mathcal{B}_q$, i.e. a family of intertwiners $U(\Gamma):\ \mathcal{H}_\rho \to \mathcal{H}_{\rho'}$ satisfying $\pi_{\rho'} = \mathrm{Ad}\, U(\Gamma) \circ \pi_\rho$ and depending on paths Γ in $\mathcal{D}_q$ from ρ to ρ'. Here, by a path in $\mathcal{D}_q$ we mean a finite sequence $(\rho_0,\ \rho_1,\ \ldots, \rho_n)$ of charge distributions $\rho_i \in \mathcal{D}_q$, such that ρ_i and ρ_{i+1} are "nearest neighbours" in $\mathcal{D}_q$ in a suitable sense to be explained below.

In order to be able to formulate a concept of locality for these intertwiners we will also consider the Hilbert direct sum

$$\mathbb{H}_q \ := \bigoplus_{\rho \in \mathcal{D}_q} \mathcal{H}_\rho \tag{4.2}$$

on which we let $A \in \hat{\mathfrak{A}}$ be represented by

$$\Pi_q(A) \ := \bigoplus_{\rho \in \mathcal{D}_q} \pi_\rho(A)\ . \tag{4.3}$$

The above mentioned connection will then be given in terms of an intertwiner $\mathbb{Z}_N$-Weyl algebra $\mathfrak{W}_q$ acting on $\mathbb{H}_q$ and commuting with $\Pi_q(\hat{\mathfrak{A}})$. Locality in this framework is formulated by the statement that $\mathfrak{W}_q$ is generated by elementary "electric string operators" $\mathcal{E}_q(b)$ and elementary "magnetic string operators" $\mathcal{M}_q(b)$ satisfying for all oriented bonds $b, b' \in (\mathbb{Z}^2)_1$

$$\mathcal{E}_q(b)^N = \mathcal{M}_q(b)^N = 1\!\!1, \tag{4.4}$$

and the local $\mathbb{Z}_N$-Weyl commutation relations

$$\begin{aligned} \mathcal{E}_q(b)\mathcal{E}_q(b') &= \mathcal{E}_q(b')\mathcal{E}_q(b) \\ \mathcal{M}_q(b)\mathcal{M}_q(b') &= \mathcal{M}_q(b')\mathcal{M}_q(b) \\ \mathcal{E}_q(b)\mathcal{M}_q(b') &= e^{\frac{2\pi i}{N}\delta_{b,\,b'}}\mathcal{M}_q(b')\mathcal{E}_q(b). \end{aligned} \tag{4.5}$$

These operators will map each fiber $\mathcal{H}_\rho$, $\rho \in \mathcal{D}_q$, isomorphically onto a "neighbouring one", i.e.

$$\mathcal{E}_q(b)\mathcal{H}_\rho \ = \ \mathcal{H}_{\rho+(d^*\delta_b,\,0)}, \tag{4.6}$$

$$\mathcal{M}_q(b)\mathcal{H}_\rho \ = \ \mathcal{H}_{\rho+(0,\,d\delta_b)}, \tag{4.7}$$

where $\delta_b \in \mathcal{C}^1_{loc}$ denotes the 1-cochain taking value one on b and zero else. Hence, one might think of $\mathcal{E}_q(b)$ ($\mathcal{M}_q(b)$) as creating an electric (magnetic) charge-anticharge pair sitting in the boundary (coboundary) of the bond b. Defining charge distributions in $\mathcal{D}_q$ to be *nearest neighbours* if they differ by either such an elementary electric or magnetic dipole, the connection in $\mathcal{B}_q$ along a path $(\rho_0,\ \rho_1,\ \ldots, \rho_n)$ in $\mathcal{D}_q$ is now given in the obvious way as an associated product of $\mathcal{E}_q(b)$'s and $M_q(b')$'s, mapping each fiber $\mathcal{H}_{\rho_i}$ isomorphically onto its successor $\mathcal{H}_{\rho_{i+1}}$ according to (4.6)-(4.7).

We remark that in a DHR-framework one would expect these intertwiners to be given on the dense subspace $\pi_\rho(\mathfrak{A})\Omega_\rho \subset \mathcal{H}_\rho$ in terms of unitary localized charge transporters $S_{el}(b)$ and $S_{mag}(b) \in \mathfrak{A}_{loc}$ by

$$\mathcal{E}_q(b)\pi_\rho(A)\Omega_\rho \;=\; \pi_{\rho+(d^*\delta_b,\,0)}(A\,S_{el}(b))\Omega_{\rho+(d^*\delta_b,\,0)} \tag{4.8}$$

$$\mathcal{M}_q(b)\pi_\rho(A)\Omega_\rho \;=\; \pi_{\rho+(0,\,d\delta_b)}(A\,S_{mag}(b))\Omega_{\rho+(0,\,d\delta_b)}\ , \tag{4.9}$$

which would be consistent and well defined if

$$\omega_{\rho+(d^*\delta_b,\,0)} \;=\; \omega_\rho \circ \mathrm{Ad}\ S_{el}(b) \tag{4.10}$$

$$\omega_{\rho+(0,\,d\delta_b)} \;=\; \omega_\rho \circ \mathrm{Ad}\ S_{mag}(b). \tag{4.11}$$

Intuitively one could think of S_{el} as a kind of "electric" Mandelstam string operator as in (3.1) and of $S_{mag}(b)$ as its dual "magnetic" analogue. Unfortunately, due to the non-local energy regularization in (3.2) (and more generally, in (3.25)), the existence of such localized charge transporters S_{el} and S_{mag} in $\mathfrak{A}_{loc}$ satisfying (4.10)-(4.11) is very questionable.

We also remark that, as opposed to the DHR-framework, in our lattice model the states ω_ρ are not given in the form $\omega_\rho = \omega_0 \circ \gamma_\rho$ for some localized automorphism γ_ρ on $\mathfrak{A}$. This is why we do not have a field algebra extension of $\mathfrak{A}$ carrying a global $\mathbb{Z}^N \times \mathbb{Z}^N$ symmetry, i.e. we are not able to define charged fields intertwining $\pi_\rho \circ \gamma_{\rho'}$ and $\pi_{\rho+\rho'}$.

It is therefore astonishing and, in our opinion, asks for further conceptual explanation that in this model one is nevertheless able to construct an intertwiner algebra with local commutation relations as given in (4.5). We will come back to this question in [10], where the existence of the local intertwiner algebra (4.4)-(4.5) will be the basic input for the construction of Haag-Ruelle scattering states in the charged sectors.

4.1 The Local Intertwiner Algebra

We now proceed to the construction of the intertwiner algebra $\mathfrak{W}_q$ obeying (4.4)-(4.5).

For $\rho,\ \rho' \in \mathcal{D}_q$, let $P_\rho(\rho') :\ \mathcal{H}_\rho \to \mathcal{H}_\rho$ be the one-dimensional projection onto the ground state of $T_\rho(\rho')$. Using the notation of (3.49)-(3.50) we then define for arbitrary 1-cochains $l \in \mathcal{C}^1_{loc}$

$$\phi^{el}_\rho(l) \;=\; \frac{P_\rho(\rho-(d^*l,\ 0))\pi_\rho(A_{\rho,\,\rho-(d^*l,\,0)}(l,\ 0))\Omega_\rho}{\left\|P_\rho(\rho-(d^*l,\ 0))\pi_\rho(A_{\rho,\,\rho-(d^*l,\,0)}(l,\ 0))\Omega_\rho\right\|} \tag{4.12}$$

$$\phi^{mag}_\rho(l) \;=\; \frac{P_\rho(\rho-(0,\ dl))\pi_\rho(A_{\rho,\,\rho-(0,\,dl)}(0,\ l))\Omega_\rho}{\left\|P_\rho(\rho-(0,\ dl))\pi_\rho(A_{\rho,\,\rho-(0,\,dl)}(0,\ l))\Omega_\rho\right\|}\ , \tag{4.13}$$

which are well defined unit vectors in $\mathcal{H}_\rho$, by Proposition 3.3.2. We may then define unitary operators $\hat{\mathcal{E}}_q(l)$, $\hat{\mathcal{M}}_q(l)$ on $\mathbb{H}_q$ commuting with $\Pi_q(\hat{\mathfrak{A}})$ by putting on the dense set $\pi_\rho(A)\Omega_\rho$, $A \in \mathfrak{A}$, $\rho \in \mathcal{D}_q$,

$$\hat{\mathcal{E}}_q(l)\pi_\rho(A)\Omega_\rho \ := \ \pi_{\rho+(d^*l,\,0)}(A)\phi^{el}_{\rho+(d^*l,\,0)}(l) \tag{4.14}$$

$$\hat{\mathcal{M}}_q(l)\pi_\rho(A)\Omega_\rho \ := \ \pi_{\rho+(0,\,dl)}(A)\phi^{mag}_{\rho+(0,\,dl)}(l), \tag{4.15}$$

which are well defined as in (3.54).

It turns out that these intertwiners do not yet obey local commutation relations, however the violation of locality can be described by a kind of "coboundary" equation.

Theorem 4.1.1 *There exists an assignment of phases $z^{el}_\rho(l)$, $z^{mag}_\rho(l) \in U(1)$ such that for all l_1 and $l_2 \in \mathcal{C}^1_{loc}$ and all $\rho \in \mathcal{D}_q$*

$$i) \qquad \Big(\hat{\mathcal{E}}_q(l_1+l_2)\Omega_\rho,\ \ \hat{\mathcal{E}}_q(l_1)\hat{\mathcal{E}}_q(l_2)\Omega_\rho\Big) = \frac{z^{el}_{\rho+(d^*l_2,\,0)}(l_1)\, z^{el}_\rho(l_2)}{z^{el}_\rho(l_1+l_2).} \tag{4.16}$$

$$ii) \qquad \Big(\hat{\mathcal{M}}_q(l_1+l_2)\Omega_\rho,\ \ \hat{\mathcal{M}}_q(l_1)\hat{\mathcal{M}}_q(l_2)\Omega_\rho\Big) = \frac{z^{mag}_{\rho+(0,\,dl_2)}(l_1)\, z^{mag}_\rho(l_2)}{z^{mag}_\rho(l_1+l_2)}. \tag{4.17}$$

$$iii) \qquad \Big(\hat{\mathcal{M}}_q(l_1)\hat{\mathcal{E}}_q(l_2)\Omega_\rho,\ \ \hat{\mathcal{E}}_q(l_2)\hat{\mathcal{M}}_q(l_1)\Omega_\rho\Big) = e^{i\langle l_1,\,l_2\rangle}\ \frac{z^{el}_{\rho+(0,\,dl_1)}(l_1)\, z^{mag}_\rho(l_1)}{z^{mag}_{\rho+(d^*l_2,\,0)}(l_1)\, z^{el}_\rho(l_2)}. \tag{4.18}$$

$$iv) \qquad \text{If } d^*l = 0 \text{ then } z^{el}_\rho(l) = 1. \tag{4.19}$$

$$\text{If } dl = 0 \text{ then } z^{mag}_\rho(l) = 1. \tag{4.20}$$

$$v) \qquad z^{el/mag}_\rho(l) = z^{el/mag}_{\rho\cdot g}(l\cdot g), \ \ \forall g \in \mathcal{S}. \tag{4.21}$$

□

The proof of this theorem and the precise definition of the phases $z^{el}_\rho(l)$ and $z^{mag}_\rho(l)$ will be given in Appendix B.6.

We now use these phases to define on $\mathbb{H}_q$ the unitaries $Z^{el}_q(l)$ and $Z^{mag}_q(l)$, $l \in \mathcal{C}^1_{loc}$, by

$$Z^{el}_q(l) \restriction \mathcal{H}_\rho \ := \ z^{el}_\rho(l)\mathbb{1}_{\mathcal{H}_\rho}, \tag{4.22}$$

$$Z^{mag}_q(l) \restriction \mathcal{H}_\rho \ := \ z^{mag}_\rho(l)\mathbb{1}_{\mathcal{H}_\rho}, \tag{4.23}$$

Then $Z^{el}_q(l)$ and $Z^{mag}_q(l)$ clearly also commute with $\Pi_q(\hat{\mathfrak{A}})$ and we may define new intertwiners

$$\mathcal{E}_q(l) \ := \ \hat{\mathcal{E}}_q(l)Z^{el}_q(l)^{-1}, \tag{4.24}$$

$$\mathcal{M}_q(l) \ := \ \hat{\mathcal{M}}_q(l)Z^{mag}_q(l)^{-1}, \tag{4.25}$$

which now obey local commutation relations.

Theorem 4.1.2 *For all $l \in \mathcal{C}^1_{loc}$ the operators $\mathcal{E}_q(l)$ and $\mathcal{M}_q(l)$ are unitaries in $\mathcal{B}(\mathbb{H}_q)$ commuting with $\Pi_q(\hat{\mathfrak{A}})$. They satisfy the Weyl algebra relations*

$$\mathcal{E}_q(l_1)\mathcal{E}_q(l_2) = \mathcal{E}_q(l_1 + l_2), \tag{4.26}$$

$$\mathcal{M}_q(l_1)\mathcal{M}_q(l_2) = \mathcal{M}_q(l_1 + l_2), \tag{4.27}$$

$$\mathcal{E}_q(l_1)\mathcal{M}_q(l_2) = e^{i\langle l_1,\, l_2\rangle}\mathcal{M}_q(l_2)\mathcal{E}_q(l_1) \tag{4.28}$$

and restricted to each subspace $\mathcal{H}_\rho \subset \mathbb{H}_q$, $\rho \in \mathcal{D}_q$, we have

$$\mathcal{E}_q(l)\mathcal{H}_\rho = \mathcal{H}_{\rho+(d^*l,\,0)}, \tag{4.29}$$

$$\mathcal{M}_q(l)\mathcal{H}_\rho = \mathcal{H}_{\rho+(0,\,dl)}. \tag{4.30}$$

□

Proof. It remains to prove the identities (4.26)-(4.28), for which it is enough to check them when applied to $\Omega_\rho \in \mathcal{H}_\rho$, $\forall \rho \in \mathcal{D}_q$. First we get

$$\mathcal{E}_q(l_1 + l_2)\Omega_\rho = z^{el}_\rho(l_1 + l_2)^{-1}\, \phi^{el}_{\rho+(d^*l_1+d^*l_2,\,0)} \tag{4.31}$$

which is a unit vector in the image of the one dimensional projection $P_{\rho+(d^*l_1+d^*l_2,\,0)}(\rho)$. On the other hand we have

$$\mathcal{E}_q(l_1)\mathcal{E}_q(l_2)\Omega_\rho = z^{el}_\rho(l_2)^{-1}\mathcal{E}_q(l_1)\phi^{el}_{\rho+(d^*l_2,\,0)}(l_2). \tag{4.32}$$

Since $\mathcal{E}_q(l_1) \restriction \mathcal{H}_{\rho'}$ intertwines $\pi_{\rho'}(\hat{\mathfrak{A}})$ and $\pi_{\rho'+(d^*l_1,\,0)}(\hat{\mathfrak{A}})$ we have

$$\mathcal{E}_q(l_1)T_{\rho'}(\rho) \restriction \mathcal{H}_{\rho'} = T_{\rho'+(d^*l_1,\,0)}(\rho)\mathcal{E}_q(l_1) \restriction \mathcal{H}_{\rho'}, \tag{4.33}$$

implying

$$\mathcal{E}_q(l_1)P_{\rho'}(\rho) \restriction \mathcal{H}_{\rho'} = P_{\rho'+(d^*l_1,\,0)}(\rho)\mathcal{E}_q(l_1) \restriction \mathcal{H}_{\rho'}. \tag{4.34}$$

Hence (4.32) implies

$$\mathcal{E}_q(l_1)\mathcal{E}_q(l_2)\Omega_\rho \in P_{\rho''}(\rho)\mathcal{H}_\rho, \tag{4.35}$$

where $\rho'' = \rho + (d^*l_1 + d^*l_2,\ 0)$. Comparing (4.31) and (4.35) we conclude that $\mathcal{E}_q(l_1)\mathcal{E}_q(l_2)\Omega_\rho$ and $\mathcal{E}_q(l_1 + l_2)\Omega_\rho$ only differ by a phase. This phase must be one since, by (4.16) and (4.24)-(4.25)

$$(\mathcal{E}_q(l_1 + l_2)\Omega_\rho,\ \ \mathcal{E}_q(l_1)\mathcal{E}_q(l_2)\Omega_\rho) = 1. \tag{4.36}$$

Equations (4.27)-(4.28) are proven similarly. ■

Putting $\mathcal{E}_q(b) \equiv \mathcal{E}_q(\delta_b)$ and $\mathcal{M}_q(b) \equiv \mathcal{M}_q(\delta_b)$, Theorem 4.1.2 provides the local intertwiner algebra $\mathfrak{W}_q$ announced in (4.4)-(4.5).

4.2 The Intertwiner Connection

We now turn to the bundle theoretic point of view, where we consider the above intertwiners as a connection on $\mathcal{B}_q$. As already explained, we call a pair $\rho,\ \rho' \in \mathcal{D}_q$ nearest neighbours if they differ by an elementary electric or magnetic dipole, i.e. if there exists a bond $b \in (\mathbb{Z}^2)_1$ such that $\rho' - \rho = (\pm d^* \delta_b,\ 0)$ or $\rho' - \rho = (0,\ \pm d\, \delta_b)$.

In the first case we put

$$\mathcal{U}_{\rho',\,\rho} = \mathcal{E}_q(b)^{\pm 1} : \quad \mathcal{H}_\rho \to \mathcal{H}_{\rho'} \tag{4.37}$$

and in the second case we put

$$\mathcal{U}_{\rho',\,\rho} = \mathcal{M}_q(b)^{\pm 1} : \quad \mathcal{H}_\rho \to \mathcal{H}_{\rho'}. \tag{4.38}$$

If $\Gamma = (\rho_0,\ \ldots,\ \rho_n)$ is a path in $\mathcal{D}_q$, i.e. a finite sequence of nearest neighbour pairs, then we put

$$\mathcal{U}(\Gamma) \ := \ \mathcal{U}_{\rho_n\, \rho_{n-1}} \cdots \mathcal{U}_{\rho_1\, \rho_0} : \quad \mathcal{H}_{\rho_0} \to \mathcal{H}_{\rho_n} \tag{4.39}$$

as the associated "parallel transport". If Γ is a closed path, i.e. $\rho_n = \rho_0$, then $\mathcal{U}(\Gamma)$ must be a phase by the ireducibility of $\pi_{\rho_0}(\hat{\mathfrak{A}})$.

In order to determine these "holonomy phases" one may use the commutation relation (4.28) which allows to restrict ourselves to purely electric or purely magnetic loops Γ (i.e. where $\mathcal{U}(\Gamma)$ only consists of a product of $\mathcal{E}_q(b)$'s or $\mathcal{M}_q(b)$'s, respectively). Using (4.26)-(4.27) such loop operators $\mathcal{U}(\Gamma)$ are always of the form $\mathcal{U}(\Gamma) = \mathcal{E}_q(l_e) \restriction \mathcal{H}_\rho$ or $\mathcal{U}(\Gamma) = \mathcal{M}_q(l_m) \restriction \mathcal{H}_\rho$, where the loop condition on Γ implies $d^* l_e = 0$ or $d l_m = 0$, respectively. Note that this is consistent with (4.29)-(4.30), i.e. these loop operators must map $\mathcal{H}_\rho$ onto itself.

To compute the holonomy phases we now use the fact that $d^* l_e = 0$ and $d l_m = 0$ implies that there exist uniquely determined cochains $s \in \mathcal{C}^2_{loc}$ and $k \in \mathcal{C}^0_{loc}$ such that

$$d^* s \ = \ l_e, \qquad dk \ = \ l_m. \tag{4.40}$$

We also recall that in these cases Theorem 4.1.1 *iv)* implies $\mathcal{E}_q(l_e) = \hat{\mathcal{E}}_q(l_e)$ and $\mathcal{M}_q(l_m) = \hat{\mathcal{M}}_q(l_m)$. Hence, being a phase when restricted to $\mathcal{H}_\rho$, it is enough to apply these operators to Ω_ρ where, by the definitions (4.22)-(4.23) and (4.24)-(4.25), they yield $\phi^{el}_\rho(l_e)$ and $\phi^{mag}_\rho(l_m)$, respectively.

Thus, we get our holonomy phases from

Proposition 4.2.3 *Let $s \in \mathcal{C}^2_{loc}$ and $k \in \mathcal{C}^0_{loc}$ such that $d^* s = l_e$ and $dk = l_m$. Then, for $\rho = (\epsilon,\ \mu)$ we have*

$$\phi^{el}_\rho(l_e) \ = \ e^{i\langle \mu,\, s\rangle}\Omega_\rho \tag{4.41}$$

$$\phi^{mag}_\rho(l_m) \ = \ e^{-i\langle \epsilon,\, k\rangle}\Omega_\rho. \tag{4.42}$$

□

By the above arguments Proposition 4.2.3 implies

$$\begin{aligned} \mathcal{E}_q(l_e) \restriction \mathcal{H}_\rho \ &= \ e^{i\langle \mu,\, s\rangle} \\ \mathcal{M}_q(l_m) \restriction \mathcal{H}_\rho \ &= \ e^{i\langle \epsilon,\, k\rangle}\ . \end{aligned} \tag{4.43}$$

If we think of s being the characteristic function of a surface encircled by an electric loop l_e, then the holonomy phase (4.41) is just the magnetic flux through this surface. Interchanging electric and magnetic (and passing to the dual lattice), equation (4.42) may be stated analogously. This state of affairs will be the origin of anyon statistics of scattering states in [10]. Proposition 4.2.3 will be proven in Appendix B.7.

4.3 The Representation of Translations

Using our local intertwiner algebra $\mathfrak{W}_q$ we are now in the position to define on each fiber $\mathcal{H}_\rho$ a unitary representation $D_\rho(a)$, $a \in \mathbb{Z}^2$, of the group of the lattice translation, such that Ad $D_\rho(a) \circ \pi_\rho = \pi_\rho \circ \tau_a$ and such that $D_\rho(a)$ commutes with $T_\rho(0)$, $\forall a \in \mathbb{Z}^2$. Moreover, for fixed q, these representations are all equivalent, i.e., $\mathcal{U}(\Gamma) D_\rho(a) = D_{\rho'}(a) \mathcal{U}(\Gamma)$ for all paths $\Gamma: \rho \to \rho'$.

First we need a lift of the natural action of translations on $\mathcal{D}_q$ to bundle automorphisms on $\mathcal{B}_q$. For the sake of generality let us formulate this by including also the lattice rotations.

Lemma 4.3.4 *For $g \in \mathcal{S}$ let $V_\rho(g): \mathcal{H}_\rho \to \mathcal{H}_{\rho\cdot g}$ be given by*

$$V_\rho(g)\pi_\rho(A)\Omega_\rho \;:=\; \pi_{\rho\cdot g}(\tau_g^{-1}(A))\Omega_{\rho\cdot g}. \tag{4.44}$$

Then $V_\rho(g)$ is a well defined unitary intertwining $\pi_\rho \circ \tau_g$ with $\pi_{\rho\cdot g}$ [11] *, i.e.*

$$V_\rho(g)\pi_\rho(\tau_g(A)) \;=\; \pi_{\rho\cdot g}(A)V_\rho(g), \tag{4.45}$$

$$V_\rho(g)T_\rho(\rho') \;=\; T_{\rho\cdot g}(\rho'\cdot g)V_\rho(g), \tag{4.46}$$

for all $A \in \mathfrak{A}$. □

Proof. $V_\rho(g)$ is well defined and unitary since $\omega_\rho \circ \tau_g = \omega_{\rho\cdot g}$. Equations (4.45)-(4.46) follow from $\alpha_\rho \circ \tau_{\rho\cdot g} = \alpha_{\rho\cdot g}$ and $E_{\rho\cdot g} = E_\rho$ (by (3.81) and $\mathcal{S}$-invariance of $c_{\rho,\rho'}$). ■

Note that the definition (4.44) implies the obvious identity

$$V_{\rho\cdot g}(h)V_\rho(g) \;=\; V_\rho(gh). \tag{4.47}$$

for all g, $h \in \mathcal{S}$, which means that the family of fiber isomorphisms $V_\rho(g)$, $\rho \in \mathcal{D}_q$, may indeed be viewed as a lift of the natural right action of $\mathcal{S}$ on $\mathcal{D}_q$ to an action by unitary bundle automorphism on $\mathcal{B}_q$. Equivalently, we now consider

$$\mathbb{V}_q(g) \;:=\; \bigoplus_{\rho\in\mathcal{D}_q} V_\rho(g) \tag{4.48}$$

as a unitary representation of $\mathcal{S}$ on $\mathbb{H}_q$, satisfying

$$\mathrm{Ad}\, \mathbb{V}_q(g) \circ \Pi_q \;=\; \Pi_q \circ \tau_{g^{-1}}. \tag{4.49}$$

Moreover, we have

[11] Note that the $*$-automorphism $\tau_g \in \mathrm{Aut}\,\mathfrak{A}$ may be extended to a $*$-automorphism on $\hat{\mathfrak{A}}$ by putting $\tau_g(\mathbf{t}) = \mathbf{t}$.

Lemma 4.3.5

$$\mathbb{V}_q(g)\mathcal{E}_q(l) = \mathcal{E}_q(l\cdot g)\mathbb{V}_q(g), \tag{4.50}$$

$$\mathbb{V}_q(g)\mathcal{M}_q(l) = \mathcal{M}_q(l\cdot g)\mathbb{V}_q(g). \tag{4.51}$$

$\square$

Proof. By Theorem 4.1.1 *v)* we have

$$\mathbb{V}_q(g)Z_q^{el/mag}(l) = Z_q^{el/mag}(l\cdot g)\mathbb{V}_q(g). \tag{4.52}$$

Hence it is enough to prove the claim with $\mathcal{E}_q$, $\mathcal{M}_q$ replaced by $\hat{\mathcal{E}}_q$, $\hat{\mathcal{M}}_q$. However this is a straight forward consequence of the definitions and the fact that by (4.46)

$$V_\rho(g)P_\rho(\rho') = P_{\rho\cdot g}(\rho'\cdot g)V_\rho(g) \tag{4.53}$$

and, therefore,

$$V_\rho(g)\phi_\rho^{el/mag}(l) = \phi_{\rho\cdot g}^{el/mag}(l\cdot g). \tag{4.54}$$

$\blacksquare$

We remark that Lemma 4.3.5 implies that the connection $\mathcal{U}(\Gamma)$ is $\mathcal{S}$-invariant, i.e.

$$V_{\rho'}(g)\mathcal{U}(\Gamma) = \mathcal{U}(\Gamma\cdot g)V_\rho(g) \tag{4.55}$$

for all paths $\Gamma:\ \rho\to\rho'$ and all $g\in\mathcal{S}$.

In order to arrive at an implementation of the translation group $\mathbb{Z}^2\subset\mathcal{S}$ mapping each fiber $\mathcal{H}_\rho$ onto itself we now have to compose $V_\rho(a)^*$, $a\in\mathbb{Z}^2$, with a parallel transporter $U_\rho(a):\mathcal{H}_\rho\to\mathcal{H}_{\rho\cdot a}$ along a suitable path $\Gamma:\rho\to\rho\cdot a$. To this end it is enough to consider the cases $a=e_i$, $i=1,\ 2$, where $e_1=(1,\ 0)$ and $e_2=(0,\ 1)$ denote the generators of $\mathbb{Z}^2$.

A minute's thought shows that for any electric charge distribution $\epsilon\in\mathcal{C}^0_{loc}$ there exists a unique 1-cochain $\ell_e(\epsilon,\ i)\in\mathcal{C}^1_{loc}$ with support only on bonds in direction i, such that the translated image $\epsilon\cdot e_i$ of ϵ by one unit in direction i satisfies

$$\epsilon\cdot e_i = \epsilon + d^*\ell_e(\epsilon,\ i). \tag{4.56}$$

Similarly, for a magnetic charge distribution $\mu\in\mathcal{C}^2_{loc}$ there exists a unique 1-cochain $\ell_m(\mu,\ i)$ with support only on the bonds perpendicular to the direction i, such that

$$\mu\cdot e_i = \mu + d\ell_m(\mu,\ i). \tag{4.57}$$

For $\rho=(\epsilon,\ \mu)\in\mathcal{D}_q$ we now define $U_\rho(e_i):\ \mathcal{H}_\rho\to\mathcal{H}_{\rho\cdot e_i}$ by putting

$$\begin{aligned} U_\rho(e_i) &:= \mathcal{E}_q(\ell_e(\epsilon,\ i))\mathcal{M}_q(\ell_m(\mu,\ i))\restriction\mathcal{H}_\rho \\ &= \mathcal{M}_q(\ell_m(\mu,\ i))\mathcal{E}_q(\ell_e(\epsilon,\ i))\restriction\mathcal{H}_\rho, \end{aligned} \tag{4.58}$$

where the second equality follows since, by construction, $\ell_m(\mu,\ i)$ and $\ell_e(\epsilon,\ i)$ always have disjoint support.

With this construction we now define the unitaries $D_\rho(e_i):\ \mathcal{H}_\rho\to\mathcal{H}_\rho$, $i=1,\ 2$, by

$$D_\rho(e_i) := V_\rho(e_i)^*\,U_\rho(e_i). \tag{4.59}$$

Theorem 4.3.6 *The unitaries $D_\rho(e_i)$, $i = 1,\ 2$, generate a representation of $\mathbb{Z}^2$ implementing the translation automorphism on $\hat{\mathfrak{A}}$, i.e.,*

$$D_\rho(e_1)D_\rho(e_2) \;=\; D_\rho(e_2)D_\rho(e_1) \tag{4.60}$$

$$D_\rho(e_i)\pi_\rho(A) \;=\; \pi_\rho(\tau_{e_i}(A))D_\rho(e_i), \tag{4.61}$$

for all $A \in \hat{\mathfrak{A}}$. □

Proof. Equation (4.61) immediately follows from (4.58) and (4.45)-(4.46). To prove (4.60) we first use Lemma 4.3.5 to conclude that

$$D_\rho(e_2)D_\rho(e_1) \;=\; V_\rho(e_2)^* V_{\rho\cdot e_2}(e_1)^* U_{\rho\cdot e_1}(e_2)U_\rho(e_1), \tag{4.62}$$

$$D_\rho(e_1)D_\rho(e_2) \;=\; V_\rho(e_1)^* V_{\rho\cdot e_1}(e_2)^* U_{\rho\cdot e_2}(e_1)U_\rho(e_2). \tag{4.63}$$

Since (4.47) implies

$$V_{\rho\cdot e_2}(e_1)V_\rho(e_2) = V_{\rho\cdot e_1}(e_2)V_\rho(e_1) = V_\rho(e_1 + e_2), \tag{4.64}$$

we are left to check

$$U_\rho(e_2)^{-1}U_{\rho\cdot e_2}(e_1)^{-1}U_{\rho\cdot e_1}(e_2)U_\rho(e_1) = 1\!\mathrm{l}_{\mathcal{H}_\rho}. \tag{4.65}$$

Using the definition (4.58) and the Weyl algebra relations (4.26)-(4.28), equation (4.65) is equivalent to

$$\mathcal{M}_q(L_m(\mu))\,\mathcal{E}_q(L_e(\epsilon)) \restriction \mathcal{H}_\rho \;=\; (u_1u_2u_3)^{-1}, \tag{4.66}$$

where

$$L_m(\mu) = \ell_m(\mu,\ 1) + \ell_m(\mu\cdot e_1,\ 2) - \ell_m(\mu\cdot e_2,\ 1) - \ell_m(\mu,\ 2), \tag{4.67}$$

$$L_e(\epsilon) = \ell_e(\epsilon,\ 1) + \ell_e(\epsilon\cdot e_1,\ 2) - \ell_e(\epsilon\cdot e_2,\ 1) - \ell_e(\epsilon,\ 2), \tag{4.68}$$

and where $u_i \in U(1)$ are the phases obtained by commuting in (4.65) all factors of $\mathcal{E}_q$'s to the right of $\mathcal{M}_q$'s, i.e.,

$$u_1(\epsilon,\ \mu) \;=\; \exp\left[-i\langle \ell_e(\epsilon,\ 2),\ \ \ell_m(\mu,\ 1) - \ell_m(\mu\cdot e_2,\ 1)\rangle\right], \tag{4.69}$$

$$u_2(\epsilon,\ \mu) \;=\; \exp\left[-i\langle \ell_e(\epsilon\cdot e_2,\ 1),\ \ \ell_m(\mu\cdot e_1,\ 2)\rangle\right], \tag{4.70}$$

$$u_3(\epsilon,\ \mu) \;=\; \exp\left[-i\langle \ell_e(\epsilon\cdot e_1,\ 2),\ \ \ell_m(\mu,\ 1)\rangle\right]. \tag{4.71}$$

To verify (4.66) we decompose ϵ and μ into a sum over "monopoles", i.e., cochains supported on a single site or a single plaquette, respectively. Using the obvious fact that ℓ_e and ℓ_m are $\mathbb{Z}_N$-module maps, i.e.,

$$\ell_e(n\epsilon_1 + \epsilon_2,\ i) \;=\; n\ell_e(\epsilon_1,\ i) + \ell_e(\epsilon_2,\ i), \tag{4.72}$$

$$\ell_m(n\mu_1 + \mu_2,\ i) \;=\; n\ell_m(\mu_1,\ i) + \ell_m(\mu_2,\ i), \tag{4.73}$$

for all $n \in \mathbb{Z}_N$, $\epsilon_{1,\,2} \in \mathcal{C}^0_{loc}$ and $\mu_{1,\,2} \in \mathcal{C}^2_{loc}$, we conclude that (4.66) holds if and only if it holds for all pairs of monopole distributions $\rho = (\epsilon,\ \mu) = (\delta_x,\ \delta_p)$, $x \in (\mathbb{Z}^2)_0$, $p \in (\mathbb{Z}^2)_2$. In fact, we have

$$u_i(\epsilon,\ \mu) = \prod_{x\in(\mathbb{Z}^2)_0} \prod_{p\in(\mathbb{Z}^2)_2} u_i(\delta_x,\ \delta_p)^{\epsilon(x)\mu(p)}$$

and similarly

$$\begin{aligned} &\left(\Omega_{(\epsilon,\,\mu)},\ \ \mathcal{M}_q(\ell_m(\mu))\mathcal{E}_q(\ell_e(\epsilon))\Omega_{(\epsilon,\,\mu)}\right) \ = \\ &\prod_{x\in(\mathbb{Z}^2)_0} \prod_{p\in(\mathbb{Z}^2)_2} \left(\Omega_{(\delta_x,\,\delta_p)},\ \ \mathcal{M}_q(\ell_m(\delta_p))\mathcal{E}_q(\ell_e(\delta_x))\Omega_{(\delta_x,\,\delta_p)}\right)^{\epsilon(x)\mu(p)}. \end{aligned} \tag{4.74}$$

Now, for $\epsilon = \delta_x$ one easily verifies

$$\ell_e(\delta_x,\ j) \ = \ -\delta_{\langle x-e_j,\,x\rangle}, \tag{4.75}$$

implying $\ell_e(\delta_x \cdot e_i,\ j) \ = \ -\delta_{\langle x-e_i-e_j,\,x-e_i\rangle}$. For $\mu = \delta_p$ and $p = \langle y,\ y+e_1,\ y+e_1+e_2,\ y+e_2\rangle$ we get

$$\ell_m(\delta_p,\ 1) \ = \ \delta_{\langle y,\,y+e_2\rangle}, \tag{4.76}$$

$$\ell_m(\delta_p,\ 2) \ = \ -\delta_{\langle y,\,y+e_1\rangle}. \tag{4.77}$$

Plugging this into (4.69)-(4.71) we get

$$u_1 \ = \ \exp\left[\frac{2\pi i}{N}\left(\delta_{x,\,y+e_2} - \delta_{x,\,y}\right)\right], \tag{4.78}$$

$$u_2 \ = \ \exp\left[-\frac{2\pi i}{N}\delta_{x,\,y+e_2}\right], \tag{4.79}$$

$$u_3 \ = \ \exp\left[-\frac{2\pi i}{N}\delta_{x,\,y+e_1+e_2}\right], \tag{4.80}$$

implying

$$(u_1u_2u_3)^{-1} \ = \ \exp\left[\frac{2\pi i}{N}\left(\delta_{x,\,y} + \delta_{x,\,y+e_1+e_2}\right)\right]. \tag{4.81}$$

Next we look at (4.67)-(4.68) to compute

$$L_m(\delta_p) \ = \ \delta_{\langle y,\,y+e_2\rangle} - \delta_{\langle y-e_1,\,y\rangle} - \delta_{\langle y-e_2,\,y\rangle} + \delta_{\langle y,\,y+e_1\rangle} \ = \ -d\delta_y \tag{4.82}$$

$$L_e(\delta_x) \ = \ -\delta_{\langle x-e_1,\,x\rangle} - \delta_{\langle x-e_1-e_2,\,x-e_1\rangle} + \delta_{\langle x-e_1-e_2,\,x-e_2\rangle} + \delta_{\langle x-e_2,\,x\rangle} \ = \ -d^*\delta_q \quad , \tag{4.83}$$

where q is the oriented plaquette $q \ = \ \langle x-e_1-e_2,\ x-e_2,\ x,\ x-e_1\rangle$. This implies

$$\mathcal{M}_q(L_m(\delta_p))\mathcal{E}_q(L_e(\delta_x))\Omega_\rho = \exp\left[\frac{2\pi i}{N}\left(\delta_{x-e_1-e_2,\,y} + \delta_{x,\,y}\right)\right]\Omega_\rho. \tag{4.84}$$

Comparing (4.81) with (4.84) we have proven (4.66) and therefore Theorem 4.3.6. ■

We remark that our definition (4.59) is consistent with the translation invariance of the vacuum Ω_0, since $\rho = 0$ implies $U_0(e_i) = \mathbb{1}$ and $V_0(e_i)\Omega_0 = \Omega_0$.

Next we show that our intertwiner connection also intertwines the representations D_ρ of the translation group $\mathbb{Z}^2$ in $\mathcal{H}_\rho$, $\forall\rho\in\mathcal{D}_q$. This is formulated most economically by putting

$$\mathbb{D}_q(e_i) \;:=\; \bigoplus_{\rho\in\mathcal{D}_q} D_\rho(e_i), \tag{4.85}$$

implying

$$\mathbb{D}_q(e_i)\Pi_q(A) \;=\; \Pi_q(\tau_{e_i}(A))\mathbb{D}_q(e_i), \tag{4.86}$$

for all $A\in\hat{\mathfrak{A}}$.

Proposition 4.3.7 *For $i=1,\ 2$ the intertwiner algebra $\mathfrak{W}_q$ commutes with $\mathbb{D}_q(e_i)$.* □

Proof. Putting $\mathbb{U}_q(e_i) := \mathbb{V}_q(e_i)\mathbb{D}_q(e_i)$ and using Lemma 4.3.5 we have to show

$$\mathbb{U}_q(e_i)\mathcal{E}_q(\ell') \;=\; \mathcal{E}_q(\ell'\cdot e_i)\mathbb{U}_q(e_i), \tag{4.87}$$

$$\mathbb{U}_q(e_i)\mathcal{M}_q(\ell') \;=\; \mathcal{M}_q(\ell'\cdot e_i)\mathbb{U}_q(e_i), \tag{4.88}$$

for all $\ell'\in\mathcal{C}^1_{loc}$. Since $\mathbb{U}_q(e_i)$ provides an intertwiner mapping $\mathcal{H}_\rho\to\mathcal{H}_{\rho\cdot e_i}$ it is again enough to check these identities on Ω_ρ, $\forall\rho\in\mathcal{D}_q$. Using (4.58), (4.59) and commuting $\mathcal{M}_q(\ell_m(\mu,\ i))$ to the left, equation (4.87) is equivalent to

$$\mathcal{E}_q(\ell_e(\epsilon+d^*\ell',\ i))\mathcal{E}_q(\ell')\Omega_{(\epsilon,\ \mu)} \;=\; e^{i\langle\ell'\cdot e_i,\ \ell_m(\mu,\ i)\rangle}\mathcal{E}_q(\ell'\cdot e_i)\mathcal{E}_q(\ell_e(\epsilon,\ i))\Omega_{(\epsilon,\ \mu)}. \tag{4.89}$$

To prove (4.89) we compute

$$\begin{aligned} L(\ell',\ i) \;&:=\; \ell_e(\epsilon+d^*\ell',\ i)+\ell'-\ell'\cdot e_i-\ell_e(\epsilon,\ i) \\ &=\; \ell_e(d^*\ell',\ i)+\ell'-\ell'\cdot e_i, \end{aligned} \tag{4.90}$$

yielding $d^*L(\ell',\ i)=0$. Let $S(\ell',\ i)\in\mathcal{C}^1_{loc}$ be such that $d^*S(\ell',\ i)=L(\ell',\ i)$. Then, by (4.43), equation (4.89) is equivalent to

$$e^{i\langle\ell'\cdot e_i,\ \ell_m(\mu,\ i)\rangle} = e^{i\langle S(\ell',\ i),\ \mu\rangle}. \tag{4.91}$$

Similarly as in the proof of Theorem 4.3.6 it is enough to check (4.91) for all $\ell'=\delta_b$, $b\in(\mathbb{Z}^2)_1$, and all $\mu=\delta_p$, $p\in(\mathbb{Z}^2)_2$. Hence, let $b=\langle x,\ x+e_j\rangle$ and $p=\langle y,\ y+e_1,\ y+e_1+e_2,\ y+e_2\rangle$. Then $d^*\delta_b=\delta_{x+e_j}-\delta_x$ and (4.75) gives

$$\begin{aligned} L(\delta_{\langle x,\,x+e_j\rangle},\ i) \;&=\; \ell_e(\delta_{x+e_j},\ i)-\ell_e(\delta_x,\ i)+\delta_{\langle x,\,x+e_j\rangle}-\delta_{\langle x-e_i,\,x+e_j-e_i\rangle} \\ &=\; \delta_{\langle x,\,x+e_j\rangle}-\delta_{\langle x+e_j-e_i,\,x+e_j\rangle}+\delta_{\langle x-e_i,\,x\rangle}-\delta_{\langle x-e_i,\,x+e_j-e_i\rangle}. \end{aligned} \tag{4.92}$$

Thus, if $j=i$ then $L(\delta_{\langle x,\,x+e_j\rangle},\ i)=0$ implying (4.91), since in this case $\ell_m(\mu,\ i)$ is perpendicular to the direction j. We are left to check the case $j\neq i$, which gives $L(\delta_{\langle x,\,x+e_j\rangle},\ i)=(-1)^j d^*\delta_{q-e_i}$, where $q=\langle x,\ x+e_1,\ x+e_1+e_2,\ x+e_2\rangle$. Thus we get for $i\neq j$

$$e^{i\langle S(\delta_{\langle x,\,x+e_j\rangle}),\ \delta_p\rangle} = \exp(\frac{2\pi i}{N}(-1)^j\delta_{x-e_i,\,y})\ . \tag{4.93}$$

On the other hand, equations (4.76)-(4.77) give for $i\neq j$

$$\ell_m(\delta_p,\ i) \;=\; (-1)^j\delta_{\langle y,\,y+e_j\rangle} \tag{4.94}$$

and hence (4.93)-(4.94) imply (4.91) in the case $i\neq j$. Thus we have proven (4.87). Equation (4.88) is proven by similar methods. ■

4.4 Conclusions

In this work we have investigated the dyonic sector structure of $2+1$-dimensional lattice $\mathbb{Z}_N$-Higgs models described by the Euclidean action (1.2) in a range of couplings (2.23)-(2.24) corresponding to the "free charge phase" of the Euclidean statistical mechanics model (1.1).

We have worked in the Hamiltonian picture by formulating the model in terms of its observable algebra $\mathcal{A}$ generated by the time-zero fields. A Euclidean (modified) dynamics $\alpha_\rho = \lim_V \mathrm{Ad}\, T_V(\rho)$ has been defined on $\mathcal{A}$ in terms of local (modified) transfer matrices $T_V(\rho)$, where $\rho = (\varepsilon, \mu)$ is a superposition of electric and magnetic $\mathbb{Z}_N$-charge distributions with finite support on the spatial lattice $\mathbb{Z}^2$.

Dyonic states ω_ρ have been constructed as ground states of the modified dynamics α_ρ on $\mathcal{A}$. The associated charged representations $(\pi_\rho, \mathcal{H}_\rho, \Omega_\rho)$ of $\mathcal{A}$ extend to irreducible representations of the "dynamic closure" $\hat{\mathcal{A}} \supset \mathcal{A}$, where $\hat{\mathcal{A}} = \langle \mathcal{A}, \mathbf{t} \rangle$ is the abstract $*$-algebra generated by $\mathcal{A}$ and a global positive transfer matrix $\mathbf{t}$ implementing the "true" (i.e. unmodified) dynamics α_0. π_ρ and $\pi_{\rho'}$ are equivalent as representations of $\hat{\mathcal{A}}$, provided their total charges coincide, $q_\rho = q_{\rho'}$. We have conjectured that the total charges $q_\rho \in \mathbb{Z}_N \times \mathbb{Z}_N$ indeed label the sectors of the model, i.e. $\pi_\rho \not\simeq \pi_{\rho'}$ if $q_\rho \neq q_{\rho'}$. The ground state energy $\epsilon(q)$ of each sector has been shown to be uniquely fixed by the conditions $\epsilon(0) = 0$ and $\epsilon(q) = \epsilon(-q)$ and the requirement of decaying of interaction energies for infinite spatial separation.

In Section 4 we have analyzed structural algebraic aspects of the state bundle $\mathcal{B}_q = \bigcup_{\rho \in \mathcal{D}_q} (\rho, \mathcal{H}_\rho)$, $\mathcal{D}_q := \{\rho \mid q_\rho = q\}$, by constructing on $\mathbb{H}_q = \oplus_{\rho \in \mathcal{D}_q} \mathcal{H}_\rho$ a local intertwiner algebra $\mathfrak{W}_q$ commuting with $\oplus_{\rho \in \mathcal{D}_q} \pi_\rho(\hat{\mathcal{A}})$. The generators of $\mathfrak{W}_q$ are given by electric and magnetic "charge transporters", $\mathcal{E}_q(b)$ and $\mathcal{M}_q(b)$, localized on bonds b in $\mathbb{Z}^2$ and fulfilling local Weyl commutation relations (Theorem 4.1.2). In terms of these charge transporters we have obtained a unitary connection $U(\Gamma) : \mathcal{H}_\rho \to \mathcal{H}_{\rho'}$ intertwining π_ρ and $\pi_{\rho'}$ for any path $\Gamma : \rho \to \rho'$ in $\mathcal{D}_q$. The holonomy of this connection is given by $\mathbb{Z}_N$-valued phases related to the electric and magnetic charges enclosed by Γ (see (4.43)).

Finally, the connection Γ has been used to construct on each $\mathcal{H}_\rho$ a unitary representation of the group of spatial lattice translations acting covariantly on $\pi_\rho(\hat{\mathcal{A}})$ and being intertwined by $U(\Gamma)$. We remark that the existence of such representations in the charged sectors of our model is by no means an obvious feature.

In [10] the holonomy phases of our connection Γ will be the main ingredient for establishing the anyonic nature of multiparticle scattering states of electrically and magnetically charged particles whose existence has been shown in [1].

Appendix

A A Brief Sketch of the Polymer and Cluster Expansions

A.1 Expansions for the Vacuum Sector

In this Appendix we present the basics of the polymer and cluster expansion developed in [1]. We intend to present here only the most relevant facts to make the main ideas in the proofs of this Appendix understandable. For more details see [1].

Definition A.1.1 *For a 1-cochains E with $d^*E = 0$ and a 2-cochain D with $dD = 0$, both with finite support, we define the "winding number of E around D" as*

$$[D:\, E] := \exp\left(\frac{2\pi i}{N}\langle u^D,\, E\rangle\right). \tag{A.1}$$

One easily sees that this definition does not depend of the choice of the particular configuration u^D. □

Let us now prepare the definition of our polymers and their activities. Define the sets

$$\mathcal{P} = \left\{P \in (\mathbb{Z}^3)_2^+ :\ P \text{ is finite, co-connected and } P = \operatorname{supp} D, \text{ for some } D \in (\mathbb{Z}^3)^2,\ dD = 0,\ D \neq 0\right\},$$

$$\mathcal{B} = \left\{M \in (\mathbb{Z}^3)_1^+ :\ M \text{ is finite, connected and } M = \operatorname{supp} E, \text{ for some } E \in (\mathbb{Z}^3)^1,\ d^*E = 0,\ E \neq 0\right\},$$

where $(\mathbb{Z}^3)_1^+$ (respectively $(\mathbb{Z}^3)_2^+$) refers to the set of positively oriented bonds (plaquettes) of $\mathbb{Z}^3$ and the sets

$$\begin{aligned}
\mathcal{P}_{total} &= \left\{P \in (\mathbb{Z}^3)_2^+ \text{ finite, so that } P = \operatorname{supp} D, \text{ for some } D \in (\mathbb{Z}^3)^2,\ dD = 0\right\}, \\
\mathcal{B}_{total} &= \left\{M \in (\mathbb{Z}^3)_1^+ \text{ finite, so that } M = \operatorname{supp} E, \text{ for some } E \in (\mathbb{Z}^3)^1,\ d^*E = 0\right\}.
\end{aligned}$$

Note that the sets $\mathcal{P}_{total}$ and $\mathcal{B}_{total}$ contain the empty set and that the non-empty elements of $\mathcal{P}_{total}$ and of $\mathcal{B}_{total}$ are build up by unions of co-disjoint elements of $\mathcal{P}$, respectively, by unions of disjoint elements of $\mathcal{B}$. One has naturally $\mathcal{P} \subset \mathcal{P}_{total}$ and $\mathcal{B} \subset \mathcal{B}_{total}$.

Each non-empty set $P \in \mathcal{P}_{total}$ and $M \in \mathcal{B}_{total}$ can uniquely be decomposed into disjoint unions $P = P_1 + \cdots + P_{A_P}$, $M = M_1 + \cdots + M_{B_M}$ (the symbol "+" indicates here disjoint union) where $P_i \in \mathcal{P}$ and $M_j \in \mathcal{B}$. Then, if $D \in (\mathbb{Z}^3)^2$ is such that $\operatorname{supp} D = P$, there is a unique decomposition $D = D_1 + \cdots + D_{A_P}$ with $D_i \in (\mathbb{Z}^3)^2$, $\operatorname{supp} D_i = P_i$. Moreover if $E \in (\mathbb{Z}^3)^1$ is such that $\operatorname{supp} E = M$ then there is a unique decomposition $E = E_1 + \cdots + E_{B_M}$ with $E_i \in (\mathbb{Z}^3)^1$, $\operatorname{supp} E_i = M_i$. One can also decompose $u = u^{D_1} + \cdots + u^{D_{A_P}}$ with $u^{D_i} \in (\mathbb{Z}^3)^1$, $du^{D_i} = D_i$.

For $P \in \mathcal{P}_{total}$ and $M \in \mathcal{B}_{total}$ we define the sets

$$\begin{aligned}
\mathcal{D}(P) &:= \{D \in (\mathbb{Z}^3)^2 \text{ so that } \operatorname{supp} D = P \text{ and } dD = 0\}, \\
\mathcal{E}(M) &:= \{E \in (\mathbb{Z}^3)^1 \text{ so that } \operatorname{supp} E = M \text{ and } d^*E = 0\}.
\end{aligned}$$

We consider now pairs $(P,\ D)$ with $P \in \mathcal{P}_{total}$ and $D \in \mathcal{D}(P)$ and pairs $(M,\ E)$ with $M \in \mathcal{B}_{total}$ and $E \in \mathcal{D}(M)$ and define $w((P,\ D),\ (M,\ E)) = w((M,\ E),\ (P,\ D)) \in \{0, \ldots, N-1\}$ as the "$\mathbb{Z}_N$-winding number" of $(M,\ E)$ around $(P,\ D)$:

$$w((P,\ D),\ (M,\ E)) = w((M,\ E),\ (P,\ D)) := [D:\ E]. \tag{A.2}$$

The pairs with $P \in \mathcal{P}$ and $M \in \mathcal{B}$ will be the building blocks of our polymers.

With the help of w we can establish a connectivity relation between pairs $(P,\ D)$ with $P \in \mathcal{P}$, $D \in \mathcal{D}(P)$ and pairs $(M,\ E)$ with $M \in \mathcal{B}$, $E \in \mathcal{E}(M)$: we say that $(P,\ D)$ and $(M,\ E)$ are "w-connected" if $w((P,\ D),\ (M,\ E)) \neq 1$ and "w-disconnected" otherwise. We arrive then at the following

Definition A.1.2 *A polymer γ is formed by two pairs*

$$\{(P^\gamma,\ D^\gamma),\ (M^\gamma,\ E^\gamma)\},$$

with $P^\gamma \in \mathcal{P}_{total}(V)$, $M^\gamma \in \mathcal{B}_{total}(V)$ and $D^\gamma \in \mathcal{D}(P^\gamma)$, $E^\gamma \in \mathcal{E}(M^\gamma)$, so that the set

$$\{(P_1^\gamma,\ D_1^\gamma), \ldots, (P_{A_\gamma}^\gamma,\ D_{A_\gamma}^\gamma), (M_1^\gamma,\ E_1^\gamma), \ldots, (M_{B_\gamma}^\gamma,\ E_{B_\gamma}^\gamma)\} \tag{A.3}$$

formed by the decompositions $P^\gamma = P_1^\gamma + \cdots + P_{A_\gamma}^\gamma$, $M^\gamma = M_1^\gamma + \cdots + M_{B_\gamma}^\gamma$ with $P_i^\gamma \in \mathcal{P}(V)$, $M_j^\gamma \in \mathcal{B}(V)$ and $D^\gamma = D_1^\gamma + \cdots + D_{A_\gamma}^\gamma$, $E^\gamma = E_1^\gamma + \cdots + E_{B_\gamma}^\gamma$ with $D_i^\gamma \in \mathcal{D}(P_i^\gamma)$, $E_j^\gamma \in \mathcal{E}(M_j^\gamma)$ is a w-connected set. □

Below, when we write $(M,\ E) \in \gamma$ and $(P,\ D) \in \gamma$ we are intrinsically be assuming that $M \in \mathcal{B}$ with $E \in \mathcal{E}(M)$ and that $P \in \mathcal{P}$ with $D \in \mathcal{D}(P)$.

For a polymer $\gamma = ((P^\gamma,\ D^\gamma),\ (M^\gamma,\ E^\gamma))$ we call the pair $\gamma_g := (P^\gamma,\ M^\gamma)$ the geometrical part of γ and the pair $\gamma_c := (D^\gamma,\ E^\gamma)$ is the "coloring" of γ. Of course the coloring determines uniquely the geometric part. Each pair $(D,\ E)$, $D \in \mathcal{D}(P)$, $E \in \mathcal{E}(M)$ with $P \in \mathcal{P}$, $M \in \mathcal{B}$ is a color for $(P,\ M)$. Another important definition is the "size" of a polymer. We define the size of γ by $|\gamma| = |\gamma_g| := |P^\gamma| + |M^\gamma|$, where $|P^\gamma|$ (respectively $|M^\gamma|$) is the number of plaquettes (respectively bonds) making up P^γ (respectively M^γ).

The activity $\mu(\gamma) \in \mathbb{C}$ of a polymer γ is defined to be

$$\mu(\gamma) := [D^\gamma:\ E^\gamma] \left\{ \prod_{i=1}^{A_\gamma} \left[\prod_{p \in P_i^\gamma} g(D_i^\gamma(p)) \right] \right\} \left\{ \prod_{j=1}^{B_\gamma} \left[\prod_{b \in M_j^\gamma} h(E_j^\gamma(b)) \right] \right\}, \tag{A.4}$$

with $\mu(\emptyset) = 1$.

For a polymer model we need the notions of "compatibility" and "incompatibility" between pairs of polymers. This is defined in the following way. Two polymers γ and γ' are said to be incompatible, $\gamma \not\sim \gamma'$, if at least one of the following conditions hold:

i) There exist $M_a^\gamma \in \gamma_g$ and $M_b^{\gamma'} \in \gamma_g'$, so that M_a^γ and $M_b^{\gamma'}$ are connected (i.e., there exists at least one lattice point x so that $x \in \partial b$ and $x \in \partial b'$ for some bonds $b \in M_a^\gamma$ and $b' \in M_b^{\gamma'}$);

ii) There exists $P_a^\gamma \in \gamma_g$ and $P_b^{\gamma'} \in \gamma_g'$, so that P_a^γ and $P_b^{\gamma'}$ are co-connected (i.e., there exists at least one cube c in the lattice so that $p \in \partial c$ and $p' \in \partial c$ for some plaquettes $p \in P_a^\gamma$ and $p' \in P_b^{\gamma'}$);

iii) There exists $(M_a^\gamma,\, E_a^\gamma) \in \gamma$ and $(P_b^{\gamma'},\, D_b^{\gamma'}) \in \gamma'$, so that $(M_a^\gamma,\, E_a^\gamma)$ and $(P_b^{\gamma'},\, D_b^{\gamma'})$ are w-connected. Or the same with γ and γ' interchanged.

They are said to be compatible, $\gamma \sim \gamma'$, otherwise.

We will denote by $\mathcal{G}(V)$ the set of all polymers in $V \subset \mathbb{Z}^3$ and by $\mathcal{G}_{com}(V)$ the set of all finite sets of compatible polymers.

We want to express the vacuum expectation of classical observables in terms of our polymer expansion. We consider the following

Definition A.1.3 *Let α be a 1-cochain and β 2-cochain, both with finite support. Define the classical observable*

$$B(\alpha,\, \beta) := \exp\left[-\frac{2\pi i}{N}\langle\alpha,\, u\rangle\right] \prod_p \frac{g((du+\beta)(p))}{g(du(p))} \quad . \tag{A.5}$$

Any classical observable can be written as a linear combination of such functions. □

For a finite volume $V \subset \mathbb{Z}^3$ (say, a cube) we have the following[12]

$$\langle B(\alpha,\, \beta)\rangle_V = \frac{1}{Z_V^1} \sum_{\substack{D \in V^2 \\ d(D-\beta)=0}} \sum_{\substack{E \in V^1 \\ d^*(E-\alpha)=0}} [D-\beta \,:\, E-\alpha] \left[\prod_{p \in \mathrm{supp} D} g(D(p))\right] \left[\prod_{b \in \mathrm{supp} E} h(E(b))\right] . \tag{A.6}$$

Here the normalization factor Z_V^1 is given by

$$Z_V^1 = \sum_{\Gamma \in \mathcal{G}_{com}(V)} \mu^\Gamma , \tag{A.7}$$

in multi-index notation, i.e., $\mu^\Gamma := \prod_{\gamma \in \Gamma} \mu(\gamma)$. We will often identify the elements of $\mathcal{G}_{com}$ with their characteristic functions.

The cochains D appearing in the sums in (A.6) can uniquely be decomposed in such a way that $D = D_0 + D_1$ with $d(D_0 - \beta) = 0$ and $dD_1 = 0$ and so that supp D_0 is co-connected and co-disconnected from supp D_1. If $d\beta = 0$ we choose $D_0 = 0$. Analogously, the cochains E appearing in the sums in (A.6) can be decomposed uniquely in such a way that $E = E_0 + E_1$ with $d^*(E_0 - \alpha) = 0$ and $d^* E_1 = 0$ and so that supp E_0 is connected and disconnected from supp E_1. If $d^*\alpha = 0$ we choose $E_0 = 0$.

We denote by $\mathcal{C}_1(\alpha)$ the set of the supports of all such E_0's, for a given α and by $\mathcal{C}_2(\beta)$ the set of the supports of all such D_0's, for a given β. For $d^*\alpha = 0$ we have $\mathcal{C}_1(\alpha) = \emptyset$ and for $d\beta = 0$ we

[12] For simplicity we will neglect some boundary terms, which can be controlled with more work.

have $\mathcal{C}_2(\beta) = \emptyset$. We define the sets of pairs

$$\mathrm{Conn}_1(\alpha,\, V) := \Big\{(M,\ E),\ \text{so that } M \in \mathcal{C}_1(\alpha) \text{ and } E \in V^1,\ \text{with } \operatorname{supp} E = M \text{ and } d^*E = d^*\alpha\Big\}, \tag{A.8}$$

$$\mathrm{Conn}_2(\beta,\, V) := \Big\{(P,\ D),\ \text{so that } P \in \mathcal{C}_2(\beta) \text{ and } D \in V^2,\ \text{with } \operatorname{supp} D = P \text{ and } dD = d\beta\Big\}. \tag{A.9}$$

We then write

$$\begin{aligned}\langle B(\alpha,\, \beta)\rangle_V \;=\; & \sum_{\substack{(M,\,E)\in \mathrm{Conn}_1(\alpha,\,V)\\ (P,\,D)\in \mathrm{Conn}_2(\beta,\,V)}} [D-\beta :\ E-\alpha] \\ & \times \left[\prod_{p\in P} g(D(p))\right]\left[\prod_{b\in M} h(E(b))\right] \frac{\displaystyle\sum_{\Gamma\in\mathcal{G}_{com}} a^{\Gamma}_{(M,E),\alpha}\, b^{\Gamma}_{(P,D),\beta}\, \mu^{\Gamma}}{\displaystyle\sum_{\Gamma\in\mathcal{G}_{com}} \mu^{\Gamma}},\end{aligned} \tag{A.10}$$

for

$$a_{(M,E),\alpha}(\gamma) := \begin{cases} 0, & \text{if } M^{\gamma} \text{ is connected with } M, \\ [D^{\gamma} :\ E-\alpha]\,, & \text{otherwise} \end{cases}, \tag{A.11}$$

and

$$b_{(P,D),\beta}(\gamma) := \begin{cases} 0, & \text{if } P^{\gamma} \text{ is co-connected with } P \\ [D-\beta :\ E^{\gamma}]\,, & \text{otherwise} \end{cases}. \tag{A.12}$$

It is for many purposes useful to write the normalization factor Z^1_V in the form

$$Z^1_V = \exp\left\{\sum_{\Gamma\in\mathcal{G}_{clus}(V)} c_{\Gamma}\mu^{\Gamma}\right\}. \tag{A.13}$$

Let us explain the symbols used above. Our notation is close to that of [2]. $\mathcal{G}_{clus}(V)$ is the set of all finite clusters of polymers in V, i.e., an element $\Gamma \in \mathcal{G}_{clus}$ is a finite set of (not necessarily distinct) polymers building a connected "incompatibility graph". An incompatibility graph is a graph which has polymers as vertices and where two vertices are connected by a line if the corresponding polymers are incompatible. We will often identify an elements $\Gamma \in \mathcal{G}_{clus}$ with a function $\Gamma\colon\ \mathcal{G} \to \mathbb{N}$, where $\Gamma(\gamma)$ is the multiplicity of γ in $\Gamma \in \mathcal{G}_{clus}$. The coefficients c_{Γ} are the "Ursell functions" and are of purely combinatorial nature. They are defined (see [2] and [14]) by

$$c_{\Gamma} := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\mathcal{N}_n(\Gamma), \tag{A.14}$$

where $\mathcal{N}_n(\Gamma)$ is the number of ways of writing Γ in the form $\Gamma = \Gamma_1 + \cdots + \Gamma_n$ where $0 \neq \Gamma_i \in \mathcal{G}_{com}$, $i = 1, \ldots, n$.

Relation (A.13) makes sense provided the sum over clusters is convergent. As discussed in [2] and [1] a sufficient condition for this is $\|\mu\| \leq \|\mu\|_c$, where $\|\mu\| := \sup_{\gamma\in\mathcal{G}} |\mu(\gamma)|^{1/|\gamma|}$, and $\|\mu\|_c$ is a constant defined in [2]. By (A.4),

$$|\mu(\gamma)| \leq [\max\{g(1), \ldots, g(N-1), h(1), \ldots, h(N-1)\}]^{|\gamma|}, \tag{A.15}$$

what justifies the conditions (2.23)-(2.24).

Calling $\mathrm{Conn}_1(\alpha) := \mathrm{Conn}_1(\alpha,\, \mathbb{Z}^3)$, $\mathrm{Conn}_2(\beta) := \mathrm{Conn}_2(\beta,\, \mathbb{Z}^3)$ and $\mathcal{G}_{clus} := \mathcal{G}_{clus}(\mathbb{Z}^3)$, we can also write the thermodynamic limit of $\langle B(\alpha,\, \beta)\rangle_V$ as

$$\langle B(\alpha,\, \beta)\rangle = \sum_{\substack{(M,\,E)\in\mathrm{Conn}_1(\alpha)\\(P,\,D)\in\mathrm{Conn}_2(\beta)}} [D-\beta :\, E-\alpha]\left[\prod_{p\in P} g(D(p))\right]\left[\prod_{b\in M} h(E(b))\right]$$

$$\times \exp\left(\sum_{\Gamma\in\mathcal{G}_{clus}} c_\Gamma \left(a^\Gamma_{(M,E),\,\alpha}\, b^\Gamma_{(P,\,D),\,\beta} - 1\right)\mu^\Gamma\right). \tag{A.16}$$

The presence of the phases $[D-\beta :\, E-\alpha]$ is an important feature of this last expression and is responsible for the emergence of the anyonic statistics. Note that for α and β such that $d\beta = 0$ and $d^*\alpha = 0$ the expectation $\langle B(\alpha,\, \beta)\rangle$ is proportional to $[\beta :\, \alpha]$, i.e. to the winding number of β around α.

Let Γ be a cluster of polymers. We say that a polymer γ is incompatible with Γ, i.e.., $\gamma \not\sim \Gamma$, if there is at least one $\gamma' \in \Gamma$ with $\gamma \not\sim \gamma'$. For two clusters Γ, Γ' we have $\Gamma \not\sim \Gamma'$ if there is at least one $\gamma \in \Gamma$ with $\gamma \not\sim \Gamma'$.

For the polymer system discussed in this work we have the following result:

Theorem A.1.4 *There is a convex, differentiable, monotonically decreasing function F_0: $(a_0,\, \infty) \to \mathbb{R}_+$, for some $a_0 \geq 0$, with $\lim_{a\to\infty} F_0(a) = 0$ such that, for all sets of polymers Γ, and for all $a > a_0$,*

$$\sum_{\gamma\not\sim\Gamma} e^{-a|\gamma|} \leq F_0(a)\, \|\Gamma\|, \tag{A.17}$$

where $\|\Gamma\| = \sum \Gamma(\gamma')|\gamma'|$, $\Gamma(\gamma')$ being the multiplicity of γ' in Γ.

Once inequality (A.17) has been established, it has been proven in [2], Appendix A.1, that the two following results hold:

$$\sum_{\substack{\Gamma\in\mathcal{G}_{clus}\\ \Gamma\not\sim\Gamma_0}} |c_\Gamma|\, |\mu^\Gamma| \leq F_1(-\ln\|\mu\|)\, \|\Gamma_0\|, \tag{A.18}$$

$$\sum_{\substack{\Gamma\in\mathcal{G}_{clus}\\ \Gamma\not\sim\Gamma_0\\ \|\Gamma\|\geq n}} |c_\Gamma|\, |\mu^\Gamma| \leq \left(\frac{\|\mu\|}{\|\mu_c\|}\right)^n \|\Gamma_0\| F_0(a_c), \tag{A.19}$$

where a_c and $\|\mu_c\| > 0$ are constants defined in [2], F_1: $(a_c + F_0(a_c),\, \infty) \to \mathbb{R}_+$ is the solution of $F_1(a + F_0(a)) = F_0(a)$ and $\|\mu\| := \sup_\gamma |\mu(\gamma)|^{1/|\gamma|}$. □

For a proof we refer the reader to [1] and [2]. The inequalities (A.18) and (A.19) are of central importance in the theory of cluster expansions and are often used for proving theorems. For instance, (A.19) tells us that the sums like $\sum_\Gamma |c_\Gamma||\mu^\Gamma|$ involving only clusters with size larger than a certain n (and which are incompatible with some Γ_0 fixed) decay exponentially with n.

A.2 Expansions for the Dyonic Sectors

Let us now present the corresponding expansions for the states ω_ρ for $\rho \in \mathcal{D}_0$.

To each $B \in \mathfrak{F}_0$ we can associate a classical observable $B_{cl,\,\rho} = B_{cl,\,\rho}(d\varphi - A)$ (see (3.31)). A possible but non-unique choice is (see [1] and [2])

$$B_{cl,\,\rho} = \frac{Tr_{\mathcal{H}_{\underline{V}}}\left(F_{(\varphi(0),A(0)),(\varphi(1),A(1))} B\, T_{\underline{V}}(\rho)\right)}{Tr_{\mathcal{H}_{\underline{V}}}\left(F_{(\varphi(0),A(0)),(\varphi(1),A(1))}\, T_{\underline{V}}(\rho)\right)} \tag{A.20}$$

where

$$F_{(\varphi(0),A(0)),(\varphi(1),A(1))} = \sum_{(\varphi(0),A(0)),(\varphi(1),A(1))} |\varphi(0),\, A(0)\rangle\langle\varphi(1),\, A(1)| \tag{A.21}$$

and where $\varphi(k)$, $A(k)$ refers to the variables in the k-th euclidean time plane.

Since any such classical observable can be written as a finite linear combination of the functions $B(\alpha,\,\beta)$ previously introduced (with coefficients eventually depending on ρ) we concentrate on expectations of such functions.

Proceeding as in the previous sections we can express $\langle B(\alpha,\,\beta)\rangle_\rho$ defined in (3.31) as

$$\begin{aligned}\langle B(\alpha,\,\beta)\rangle_\rho \quad &= \sum_{\substack{(M,\,E)\in\mathrm{Conn}_1(\alpha)\\(P,\,D)\in\mathrm{Conn}_2(\beta)}} [D-\beta:\,E-\alpha]\,[D-\beta:\,-\tilde{\epsilon}]\,[-\tilde{\mu}:\,E-\alpha]\left[\prod_{p\in P} g(D(p))\right]\\ &\times\left[\prod_{b\in M} h(E(b))\right]\exp\left(\sum_{\Gamma\in\mathcal{G}_{clus}} c_\Gamma\left(a^\Gamma_{(M,E),\,\alpha}\, b^\Gamma_{(P,D),\beta}-1\right) a_\epsilon^\Gamma b_\mu^\Gamma \mu^\Gamma\right),\end{aligned} \tag{A.22}$$

where $\tilde{\mu}$ is the 2-cochain on $\mathbb{Z}^3$ defined by $\tilde{\mu}((x,\,n)) := \mu(x)$ for all $x \in (\mathbb{Z}^2)_2$ and $n \in \mathbb{Z}$ and $\tilde{\epsilon}$ in the 1-cochain on $\mathbb{Z}^3$ defined by $\tilde{\epsilon}((y,\,n+1/2)) := \epsilon(y)$, for all $y \in \mathbb{Z}^2$ and $n \in \mathbb{Z}$. Here $(y,\,n+1/2)$ indicates the vertical bond in $(\mathbb{Z}^3)_1^+$ whose projection onto $\mathbb{Z}^2$ is y and is located between the euclidean time-planes n and $n+1$. Beyond this we defined

$$a_\epsilon(\gamma) \;:=\; [D^\gamma:\,-\tilde{\epsilon}], \tag{A.23}$$

$$b_\mu(\gamma) \;:=\; [-\tilde{\mu}:\,E^\gamma]. \tag{A.24}$$

The cochains $\tilde{\mu}$ and $\tilde{\epsilon}$ do not have finite support but, since the polymers are finite, the right hand side of the last two expressions can be defined using some limit procedure, for instance, by closing $\tilde{\mu}$ and $\tilde{\epsilon}$ at infinity by adding, before the thermodynamic limit is taken, the cochains s_μ and s_ϵ to them. Since the polymers are finite, the limit does not depend on the particular s_μ and s_ϵ chosen.. The same can be said about the winding numbers $[D-\beta:\,-\tilde{\epsilon}]$ and $[-\tilde{\mu}:\,E-\alpha]$ in (A.22).

Concerning the sum over clusters in (A.22), the following estimate can be established

Proposition A.2.1 *For $(M,\,E) \in Conn_1(\alpha)$ and $(P,\,D) \in Conn_2(\beta)$ one has*

$$\left|\sum_{\Gamma\in\mathcal{G}_{clus}} c_\Gamma\left(a^\Gamma_{(M,E),\,\alpha}\, b^\Gamma_{(P,D),\beta}-1\right) a_\epsilon^\Gamma b_\mu^\Gamma \mu^\Gamma\right| \le c_0\left((|M|+|sup\ \alpha|)+(|P|+|sup\ \beta|)\right), \tag{A.25}$$

where c_0 is a positive constant. □

Proof. One has

$$
\begin{aligned}
\left| \sum_{\Gamma \in \mathcal{G}_{clus}} c_\Gamma \left(a^\Gamma_{(M,E),\alpha} \, b^\Gamma_{(P,D),\beta} - 1 \right) a^\Gamma_\epsilon b^\Gamma_\mu \mu^\Gamma \right| &\leq \sum_{\Gamma \in \mathcal{G}_{clus}} |c_\Gamma| \left| a^\Gamma_{(M,E),\alpha} \, b^\Gamma_{(P,D),\beta} - 1 \right| \, |\mu^\Gamma| \\
&\leq \sum_{\substack{\Gamma \in \mathcal{G}_{clus} \\ \Gamma \not\sim \gamma_1}} |c_\Gamma| \, |\mu^\Gamma| + \sum_{\substack{\Gamma \in \mathcal{G}_{clus} \\ \Gamma \not\sim \gamma_2}} |c_\Gamma| \, |\mu^\Gamma| \\
&\leq F_1(-\ln \|\mu\|)(\|\gamma_1\| + \|\gamma_2\|), \qquad \text{(A.26)}
\end{aligned}
$$

by (A.18), where $\gamma_1 = (\sup\, (E+\alpha),\ E+\alpha)$ and $\gamma_2 = (\sup\, (D+\beta),\ D+\beta)$. Clearly, $\|\gamma_1\| \leq |M| + |\sup\, \alpha|$ and $\|\gamma_2\| \leq |P| + |\sup\, \beta|$. ■

Proposition A.2.1 has a simple corollary

Proposition A.2.2 *If g_c and h_c are small enough and if $\min\{\alpha_l,\ \beta_l\} \geq n$, where*

$$\alpha_l \ := \ \inf\{|M|, \quad (M,\ E) \in Conn_1(\alpha)\}, \qquad \text{(A.27)}$$

$$\beta_l \ := \ \inf\{|P|, \quad (P,\ D) \in Conn_2(\beta)\}, \qquad \text{(A.28)}$$

then

$$|\langle B(\alpha,\ \beta)\rangle_\rho| \leq c_a e^{-c_b\, n}, \qquad \text{(A.29)}$$

for positive constants c_a and c_b. □

Proof. Using the representation (A.22) of $\langle B(\alpha,\ \beta)\rangle_\rho$ in terms of cluster expansions and the estimate (A.25), one gets

$$|\langle B(\alpha,\ \beta)\rangle_\rho| \leq \sum_{\substack{(M,\ E) \in \mathrm{Conn}_1(\alpha) \\ (P,\ D) \in \mathrm{Conn}_2(\beta)}} g_c^{|P|} \, h_c^{|M|} \, \exp\left\{ c_0 \left((|M| + |\sup\ \alpha|) + (|P| + |\sup\ \beta|) \right) \right\}, \qquad \text{(A.30)}$$

By standard arguments one has

$$\sum_{(M,E) \in \mathrm{Conn}_1(\alpha)} (h_c \, e^{c_0})^{|M|} \leq const.\, e^{-c_a\, n/2}, \qquad \text{(A.31)}$$

for some positive c_a, provided h_c is small enough and, analogously,

$$\sum_{(P,D) \in \mathrm{Conn}_2(\beta)} (g_c \, e^{c_0})^{|P|} \leq const.\, e^{-c_a\, n/2}, \qquad \text{(A.32)}$$

provided g_c is small enough. This proves the proposition. ■

An important particular case of (A.22) occurs when $d\beta = d^*\alpha = 0$. In this case we get simply

$$\langle B(\alpha,\ \beta)\rangle_\rho = [\beta : \alpha] \, [\beta : \tilde{\epsilon}] \, [\tilde{\mu} : \alpha] \exp\left(\sum_{\Gamma \in \mathcal{G}_{clus}} c_\Gamma \left(a^\Gamma_{(\emptyset,0),\alpha} \, b^\Gamma_{(\emptyset,0),\beta} - 1 \right) a^\Gamma_\epsilon b^\Gamma_\mu \mu^\Gamma \right). \qquad \text{(A.33)}$$

Notice the presence of the $\mathbb{Z}_N$-factors $[\beta : \alpha] \, [\beta : \tilde{\epsilon}] \, [\tilde{\mu} : \alpha]$ related to winding numbers involving α, β and the background charges ρ.

B The Remaining Proofs

B.1 Proof of Proposition 3.1.1

Let E_G be the projection $\mathfrak{F}_{loc} \to \mathfrak{A}_{loc}$. Since $\omega_{V,\,\rho}$ is gauge invariant, it is enough to prove the existence of $\lim_{V \uparrow \mathbb{Z}^2} \omega_{V,\,\rho} \restriction \mathfrak{A}_{loc}$. Expectations like $\omega_{V,\,\rho}(A)$ for $A \in \mathfrak{A}_{loc}$ can be written as finite linear combinations of the previously introduced classical expectations $\langle B(\alpha,\ \beta)\rangle_{V,\,\rho}$, whose thermodynamic limit was described in subsection A.2.

To show that $\omega_\rho \restriction \mathfrak{A}_{loc}$ is a ground state with respect to α_ρ we first notice that, by the representation of $\omega_{V,\,\rho}(A)$, $A \in \mathfrak{A}_{loc}$ in terms of cluster expansions we can write, in analogy to (3.25),

$$\omega_{V,\rho}(A) \;=\; \lim_{n\to\infty} \frac{Tr_{\mathcal{H}_V}\left(T_V(\rho)^n A T_V(\rho)^{n-1} E_V^\rho\right)}{Tr_{\mathcal{H}_V}\left(T_V(\rho)^{2n-1} E_V^\rho\right)}, \qquad A \in \mathfrak{F}(V), \tag{B.1}$$

and, hence, for V large enough, one has

$$\omega_{V,\rho}(A^*\alpha_\rho(A)) \;=\; \lim_{n\to\infty} \frac{Tr_{\mathcal{H}_V}\left(T_V(\rho)^n A^* T_V(\rho) T_V(\rho)^n E_V^\rho\right)}{Tr_{\mathcal{H}_V}\left(T_V(\rho)^{2n-1} E_V^\rho\right)}, \qquad A \in \mathfrak{F}(V). \tag{B.2}$$

Now, by (3.26) and (2.30)-(2.31), E_V^ρ is a positive operator and so, the numerator in (B.2) is clearly positive. This proves that $\omega_\rho(A^*\alpha_\rho(A)) \geq 0$.

To show that $\omega_\rho(A^*\alpha_\rho(A)) \leq \omega_\rho(A^*A)$ we can make use of Lemma 2.3.2 and show that ω_ρ fulfills the cluster property with respect to α_ρ. We can represent $\omega_\rho(A^*\alpha_\rho^n(A))$ in terms of classical expectations of the classical functions associated to the operator $A^*\alpha_\rho^n(A)$. These classical expectations can be written as a finite linear combination of expectations like $\langle B(\alpha_n,\ \beta_n)\rangle_\rho$, where the local cochains α_n and β_n can be written, for n large enough, as sums $\alpha_n = \alpha(0) + \alpha(n)$ and $\beta_n = \beta(0) + \beta(n)$, where the local cochain $\alpha(n)$ (respect. $\beta(n)$) is the complex conjugate of the translate of $\alpha(0)$ (respect., of $\beta(0)$) by n units in euclidean time direction.

Recalling now the representation (A.22) of $\langle B(\alpha,\ \beta)\rangle_\rho$ in terms of cluster expansions we notice that, by Proposition A.2.2, the contributions of sets $(M,\ E) \in \mathrm{Conn}_1(\alpha)$ connecting the support of $\alpha(0)$ to the support of $\alpha(n)$ decay exponentially with n, the same happening with the contribution of the sets $(P,\ D) \in \mathrm{Conn}_2(\beta)$ connecting the support of $\beta(0)$ to the support of $\beta(n)$. The only surviving terms, after taking the limit $n \to \infty$ correspond to sets $(M,\ E) \in \mathrm{Conn}_1(\alpha)$ and sets $(P,\ D) \in \mathrm{Conn}_2(\beta)$ connecting the supports of $\alpha(0)$, $\alpha(n)$, $\beta(0)$ and $\beta(n)$ with themselves. The contributions of these last terms converges to the product $\langle B(\alpha(0),\ \beta(0))\rangle_\rho \langle B(\alpha(n),\ \beta(n))\rangle_\rho$. This implies that $\omega_\rho(A^*\alpha_\rho^n(A)) \to \omega_\rho(A^*)\omega_\rho(A)$, $n \to \infty$, thus proving the ground state property. The general cluster property $\omega_\rho(A\alpha_\rho^n(B)) \to \omega_\rho(A)\omega_\rho(B)$, $A,\ B \in \mathfrak{A}_{loc}$, follows from the same arguments.

■

B.2 Proof of Propositions 3.1.2 and 3.1.5

In order to prove Proposition 3.1.2 we have to study $\lim_{a\to\infty} \langle B(\alpha,\ \beta)\rangle_{\rho-\rho' a}$. We recall the representation (A.22) of $\langle B(\alpha,\ \beta)\rangle_{\rho-\rho' a}$ and notice that, since

$$\left| c_\Gamma \left(a^\Gamma_{(M,E),\,\alpha}\, b^\Gamma_{(P,D),\,\beta} - 1 \right) a^\Gamma_{\epsilon-\epsilon' a} b^\Gamma_{\mu-\mu' a} \mu^\Gamma \right| \leq \left| c_\Gamma \left(a^\Gamma_{(M,E),\,\alpha}\, b^\Gamma_{(P,D),\,\beta} - 1 \right) \mu^\Gamma \right| \tag{B.3}$$

which is summable, we can write

$$\lim_{a\to\infty}\sum_{\Gamma\in\mathcal{G}_{clus}} c_\Gamma\left(a^\Gamma_{(M,E),\alpha}\, b^\Gamma_{(P,D),\beta}-1\right) a^\Gamma_{\epsilon-\epsilon' a} b^\Gamma_{\mu-\mu' a}\mu^\Gamma \quad =$$

$$\sum_{\Gamma\in\mathcal{G}_{clus}} c_\Gamma\left(a^\Gamma_{(M,E),\alpha}\, b^\Gamma_{(P,D),\beta}-1\right)(\lim_{a\to\infty} a^\Gamma_{\epsilon-\epsilon' a} b^\Gamma_{\mu-\mu' a})\mu^\Gamma. \tag{B.4}$$

But, clearly, $\lim_{a\to\infty} a^\Gamma_{\epsilon-\epsilon' a} b^\Gamma_{\mu-\mu' a} = a^\Gamma_\epsilon b^\Gamma_\mu$ for every cluster Γ, since the polymers are finite. The limit does not depend on the particular way as $a\to\infty$. This shows that the representation (A.22) holds also for $\rho\in\mathcal{D}_q$, $q\neq 0$, and can be used to describe $\omega_\rho(A)$, $A\in\mathfrak{A}_{loc}$ with $\rho\in\mathcal{D}_q$, $q\neq 0$. The cluster property, and consequently the ground state property for $A\in\mathfrak{A}_{loc}$, can be proven in the same way as in the previous case. The proof of Proposition 3.1.5 is analogous to the proof of Proposition 3.1.2 and does not need to be repeated but in the next subsection we present the proof of the more general Theorem 3.1.6. ■

B.3 Proof of Theorem 3.1.6

Let us consider $\omega_{\rho(a)}\left(\tau_a^{-1}(B)A\alpha^n_{\rho(a)}(A'\tau_a^{-1}(B'))\right)$ for A, B, A' and $B'\in\mathfrak{A}_{loc}$. According to (3.16) one has, for $|a|$ large enough, $\alpha^n_{\rho(a)}(A')=\alpha^n_{\rho_1}(A')$ and $\alpha^n_{\rho(a)}(\tau_a^{-1}(B'))=\alpha^n_{\rho_2 a}(\tau_a^{-1}(B'))=\tau_a^{-1}(\alpha^n_{\rho_2}(B'))$. Hence, for $|a|$ large enough,

$$\omega_{\rho(a)}\left(\tau_a^{-1}(B)A\alpha^n_{\rho(a)}(A'\tau_a^{-1}(B'))\right)=\omega_{\rho(a)}\left(A\alpha^n_{\rho_1}(A')\tau_a^{-1}(B\alpha^n_{\rho_2}(B'))\right). \tag{B.5}$$

The representation of the last expectation in terms of classical expectations is given by finite sums of classical expectations like $\langle B(\alpha(a),\ \beta(a))\rangle_{\rho(a)}$, where $\alpha(a)=\alpha_1+\alpha_2 a$ and $\beta(a)=\beta_1+\beta_2 a$, for local cochains $\alpha_{1,2}$ and $\beta_{1,2}$, where the cochains α_1 and β_1 are related to the operators $A\alpha^n_{\rho_1}(A')$ and where the cochains α_2 and β_2 are related to the operators $B\alpha^n_{\rho_2}(B')$.

Let us now consider the representation of $\langle B(\alpha(a),\ \beta(a))\rangle_{\rho(a)}$ in terms of cluster expansions. It is given by

$$\langle B(\alpha(a),\ \beta(a))\rangle_{\rho(a)} =$$

$$\sum_{\substack{(M,\,E)\in \mathrm{Conn}_1(\alpha(a))\\ (P,\,D)\in\mathrm{Conn}_2(\beta(a))}} [D-\beta(a):\ E-\alpha(a)]\left[D-\beta(a):\ -\tilde{\epsilon(a)}\right][-\tilde{\mu}(a):\ E-\alpha(a)]\left[\prod_{p\in P} g(D(p))\right]$$

$$\times\left[\prod_{b\in M} h(E(b))\right]\exp\left(\sum_{\Gamma\in\mathcal{G}_{clus}} c_\Gamma\left(a^\Gamma_{(M,E),\alpha(a)}\, b^\Gamma_{(P,D),\beta(a)}-1\right) a_\epsilon(a)^\Gamma b_\mu(a)^\Gamma\mu^\Gamma\right), \tag{B.6}$$

Let us assume $|a|$ so large as to include the set sup $\alpha_1\cup$ sup $\beta_1\cup$ sup $\alpha_2\cup$ sup β_2 in the ball of radius $|a|/8$ centered at the origin and let us consider two infinite cylinders C_1 and $C_2=C_1+a$ with radius $|a|/4$, parallel to the euclidean time axis and extending from $+\infty$ to $-\infty$. The cylinder C_1 contains the set sup $\alpha_1\cup$sup β_1 and C_2 contains the set (sup $\alpha_2\cup$sup β_2) $+\ a$. By construction, the sets $\mathrm{Conn}_1(\alpha(a))$ and $\mathrm{Conn}_2(\beta(a))$ will contain some elements which are entirely contained in

$C_1 \cup C_2$ and some which are not. These last ones must have a size larger than $|a|/4$ and, therefore, by arguments analogous to those used in the proof of Proposition A.2.2, their contribution to (B.6) decay exponentially with $|a|$. So, up to exponentially falling error, we can restrict the sums over $\mathrm{Conn}_1(\alpha(a))$ and $\mathrm{Conn}_2(\beta(a))$ to elements contained only in $C_1 \cup C_2$. The next question is, what happens to the sums over clusters, provided the elements M and P are now contained in $C_1 \cup C_2$?

Let us denote $M_a := M \cap C_a$ and $P_a := M \cap C_a$, for $a = 1,\ 2$, with $M = M_1 \cup M_2$ and $P = P_1 \cup P_2$, as disjoint unions and let $E = E_1 + E_2$ and $D = D_1 + D_2$ with $\sup E_a = M_a$ and $\sup D_a = P_a$ for $a = 1,\ 2$.

We claim that the difference

$$\sum_{\Gamma \in \mathcal{G}_{clus}} c_\Gamma \left(a^\Gamma_{(M,E),\alpha(a)}\, b^\Gamma_{(P,D),\beta(a)} - 1 \right) a_\epsilon(a)^\Gamma b_\mu(a)^\Gamma \mu^\Gamma \ -$$

$$\left[\sum_{\Gamma \in \mathcal{G}_{clus}} c_\Gamma \left(a^\Gamma_{(M_1,E_1),\alpha_1}\, b^\Gamma_{(P_1,D_1),\beta_1} - 1 \right) a^\Gamma_{\epsilon_1} b^\Gamma_{\mu_1} \mu^\Gamma + \sum_{\Gamma \in \mathcal{G}_{clus}} c_\Gamma \left(a^\Gamma_{(M_2,E_2),\alpha_2 a}\, b^\Gamma_{(P_2,D_2),\beta_2 a} - 1 \right) a^\Gamma_{\epsilon_2 a} b^\Gamma_{\mu_2 a} \mu^\Gamma \right] \tag{B.7}$$

decays exponentially to zero with $|a|$. For, notice that the difference above is given by sums over clusters connecting C_1 to C_2, having thus a size larger than $|a|/2$. Therefore, by (A.19), their contribution decay exponentially with $|a|$.

Using now the exact factorization

$$[D - \beta(a) :\ E - \alpha(a)] \left[D - \beta(a) :\ -\tilde{\epsilon(a)} \right] [-\tilde{\mu}(a) :\ E - \alpha(a)] \left[\prod_{p \in P} g(D(p)) \right] \left[\prod_{b \in M} h(E(b)) \right] =$$

$$[D_1 - \beta_1 :\ E_1 - \alpha_1]\, [D_1 - \beta_1 :\ -\tilde{\epsilon}_1]\, [-\tilde{\mu}_1 :\ E_1 - \alpha_1] \left[\prod_{p \in P_1} g(D(p)) \right] \left[\prod_{b \in M_1} h(E(b)) \right] \times$$

$$[D_2 - \beta_2 a :\ E_2 - \alpha_2 a]\, [D_2 - \beta_2 a :\ -\tilde{\epsilon}_2 a]\, [-\tilde{\mu}_2 a :\ E_2 - \alpha_2 a] \left[\prod_{p \in P_2} g(D(p)) \right] \left[\prod_{b \in M_2} h(E(b)) \right] \tag{B.8}$$

valid in $C_1 \cup C_2$, we get using the translation invariance of the cluster expansions and taking $|a| \to \infty$,

$$\lim_{|a| \to \infty} \langle B(\alpha(a),\ \beta(a)) \rangle_{\rho(a)} = \langle B(\alpha_1,\ \beta_1) \rangle_{\rho_1} \langle B(\alpha_2,\ \beta_2) \rangle_{\rho_2}. \tag{B.9}$$

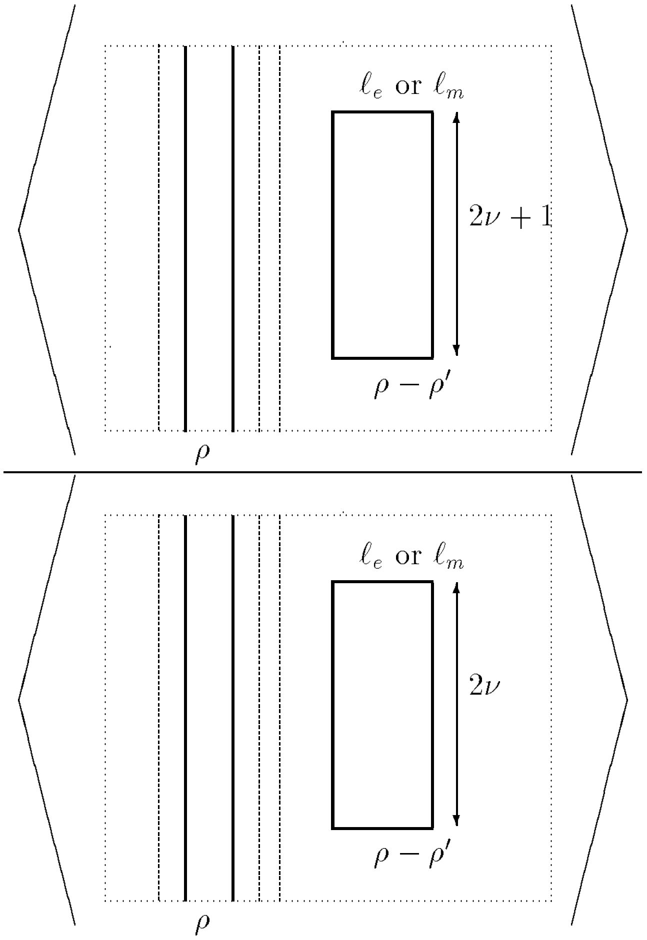
With this, the proof of Theorem 3.1.6 is complete. ■

B.4 Proof of Proposition 3.4.5

Here we establish part i of Proposition 3.4.5.

Let $\rho = (\epsilon,\ \mu)$ and $\rho' = (\epsilon',\ \mu')$ and define $\rho_0 = (\epsilon_0,\ \mu_0) = (\epsilon - \epsilon',\ \mu - \mu')$. We need first an expression in terms of the cluster expansions for the ratio $\omega_\rho(X^{(2\nu+1)}_{\rho,\,\rho'})/\omega_\rho(X^{(2\nu)}_{\rho,\,\rho'})$. Using a pictorial notation, this ratio can be written in terms of classical expectations as

$$\frac{\omega_\rho(X^{(2\nu+1)}_{\rho,\,\rho'})}{\omega_\rho(X^{(2\nu)}_{\rho,\,\rho'})} = \qquad . \tag{B.10}$$



The infinite vertical lines indicate the background charges ρ and the finite loops are constructed over the charge distribution $\rho-\rho'$. Their horizontal lines represent the strings ℓ_e and/or ℓ_m used in the definition (3.50) and their vertical lines have length $2\nu+1$ in the numerator and 2ν in the denominator, respectively Notice that ρ and $\rho-\rho'$ may have a non-empty overlap, a circumstance not shown in the figure for reasons of clarity.

The next step is to find an expansion for the last expression in terms of our cluster expansions. The result is

$$\frac{\omega_\rho(X^{(2\nu+1)}_{\rho,\,\rho'})}{\omega_\rho(X^{(2\nu)}_{\rho,\,\rho'})} = \exp\left\{\sum_\Gamma\; c_\Gamma\;\left(a^\Gamma_{\epsilon_0,\,0,\,2\nu+1} b^\Gamma_{\mu_0,\,0,\,2\nu+1} - a^\Gamma_{\epsilon_0,\,0,\,2\nu} b^\Gamma_{\mu_0,\,0,\,2\nu}\right)\; a^\Gamma_\epsilon\; b^\Gamma_\mu\; \mu^\Gamma\right\}, \tag{B.11}$$

where, in an almost self-explanatory notation, $a_{\epsilon_0,\,\alpha,\,\beta}(\gamma)$, $\alpha<\beta$, represents the winding number of the magnetic part of the polymer γ with the electric loop built by the horizontal strings ℓ_e located at euclidean times α and $\beta\in\mathbb{Z}$ and by the vertical electric lines located over the support of ϵ_0 with length $\beta-\alpha$. The quantity $b_{\mu_0,\,\alpha,\,\beta}$ is defined analogously.

The right hand side of (B.11) can be written as

$$\exp\left\{\sum_\Gamma\; c_\Gamma\;\left(a^\Gamma_{\epsilon_0,\,0,\,1} b^\Gamma_{\mu_0,\,0,\,1} - 1\right)\; a^\Gamma_{\epsilon_0,\,-2\nu,\,0} b^\Gamma_{\mu_0,\,-2\nu,\,0} a^\Gamma_\epsilon\; b^\Gamma_\mu\; \mu^\Gamma\right\}, \tag{B.12}$$

where, above, we used the factorization properties

$$a_{\epsilon_0,\,\alpha,\,\beta}(\gamma)a_{\epsilon_0,\,\beta,\,\delta}(\gamma) \;=\; a_{\epsilon_0,\,\alpha,\,\delta}(\gamma) \tag{B.13}$$

$$b_{\mu_0,\,\alpha,\,\beta}(\gamma)b_{\mu_0,\,\beta,\,\delta}(\gamma) \;=\; b_{\mu_0,\,\alpha,\,\delta}(\gamma), \tag{B.14}$$

for $\alpha < \beta < \delta \in \mathbb{Z}$ and we used translation invariance. Taking the limit $\nu \to \infty$ of expression (B.12) is easy and gives

$$\frac{\|T_\rho(\rho')\|}{\|T_\rho(\rho)\|} = \exp\left\{\sum_\Gamma c_\Gamma \left(a^\Gamma_{\epsilon_0,\,0,\,1}b^\Gamma_{\mu_0,\,0,\,1} - 1\right) a^\Gamma_{\epsilon_0,\,-\infty,\,0}b^\Gamma_{\mu_0,\,-\infty,\,0}a^\Gamma_\epsilon\, b^\Gamma_\mu\, \mu^\Gamma\right\}, \tag{B.15}$$

where $a_{\epsilon_0,\,-\infty,\,0}(\gamma) = \lim\limits_{j\to\infty} a_{\epsilon_0,\,-j,\,0}(\gamma)$ etc., which is a well defined limit for each γ, since the polymers are finite (for each γ, the limit is reached at finite j).

Next, we are interested in studying the limit $\lim\limits_{|a|\to\infty} \dfrac{\|T_{\rho_1(a)}(\rho_2(a))\|}{\|T_{\rho_2(a)}(\rho_2(a))\|}$ using its representation in terms of cluster expansions.

The main technical problem we have to confront with is the fact that, if $\rho_1 - \rho_2$ have a non-zero total charge, the strings ℓ_e and ℓ_m have to connect elements of the support of $\rho_1 - \rho_2$ with elements of the support of $(\rho_1 - \rho_2)\cdot a$ and have, hence, a length which increases with $|a|$. The crucial observation is, however, that the left hand side of (B.15) does not depend on the strings ℓ_e and ℓ_m, although this independence cannot apparently be seen from the representation in terms of cluster expansions.

Let us consider two cylinders $C_1(r)$ and $C_2(r) = C_1(r) + a$, such that $C_1(r)$ is centered on the euclidean time axis, extending from $-\infty$ to ∞ and has a radius r. Denote by r_0 the largest distance from the set sup $(\rho_1) \cup$ sup (ρ_2) to the origin of the lattice and consider $|a|$ large enough so that sup $(\rho_1) \cup$ sup (ρ_2) is contained in $C_1(|a|/8)$ (by taking, say, $|a| > 16r_0$).

We first observe that the sum over clusters contained in $\mathbb{Z}^3 \setminus (C_1(|a|/8) \cup C_2(|a|/8))$ does not contribute to (B.12). This can be seen at best in (B.11) by noticing that: 1) clusters contained in $\mathbb{Z}^3\setminus(C_1(|a|/8)\cup C_2(|a|/8))$ crossing the $t = 0$ euclidean plane and having a side smaller than 2ν have a zero contribution (for them, one has $a^\Gamma_{\epsilon_0,\,0,\,2\nu+1}b^\Gamma_{\mu_0,\,0,\,2\nu+1} = a^\Gamma_{\epsilon_0,\,0,\,2\nu}b^\Gamma_{\mu_0,\,0,\,2\nu}$); 2) clusters contained in $\mathbb{Z}^3\setminus(C_1(|a|/8)\cup C_2(|a|/8))$ crossing the $t = 2\nu$ euclidean plane, having a side smaller than 2ν and having a non-zero contribution cancel with their translates by one unit in euclidean time direction; 3) the only surviving clusters in $\mathbb{Z}^3 \setminus (C_1(|a|/8) \cup C_2(|a|/8))$ must cross the planes $t = 0$ and $t = 2\nu$, and therefore, their side is larger than 2ν and their contribution decays exponentially when the limit $\nu \to \infty$ is taken.

It remains to consider two classes of clusters: **a)** those entirely contained in $C_1(|a|/4)\cup C_2(|a|/4)$ and having a non-empty intersection with $C_1(d_0) \cup C_2(d_0)$ for a fixed d_0 with $r_0 < d_0 < a/8$ and **b)** those having a non-empty intersection with both $C_1(d_0) \cup C_2(d_0)$ and $\mathbb{Z}^3 \setminus (C_1(|a|/4) \cup C_2(|a|/4))$.

The contribution to the clusters belonging to the class **b** decays exponentially with $|a|$. For, note that the clusters which give a non-zero contribution to (B.12) must either cross the $t = 0$ plane or the $t = 1$ plane (or eventually both). The clusters of this sort having a non-empty intersection with both $C_1(d_0) \cup C_2(d_0)$ and $\mathbb{Z}^3 \setminus (C_1(|a|/4) \cup C_2(|a|/4))$ must have a size larger that $|a|/8$ and , hence, by (A.19), their contribution decays exponentially with $|a|$.

It remains now to consider the clusters belonging to the class **a** above. They are entirely contained inside one of the cylinders $C_1(|a|/4)$ or $C_2(|a|/4)$. Since we have freedom to choose the strings ℓ_e and ℓ_m at our will, we choose them depending on a such that, inside of $C_1(|a|/4) \setminus C_1(d_0)$ and $C_2(|a|/4) \setminus C_2(d_0)$ they run parallel to a fixed direction, say to the positive x-axis of $\mathbb{Z}^2$. Now, taking the limit $|a| \to \infty$ is straight forward and gives $c_{\rho_1,\,\rho_2}$ independent on the way the sequence a goes to infinity. The result is that $c_{\rho_1,\,\rho_2} = d_{\rho_1,\,\rho_2} d_{-\rho_1,\,-\rho_2}$, where

$$d_{\rho_1,\,\rho_2} \;:=\; \exp\left\{ \sum_{\substack{\Gamma \\ \Gamma\cap C_1(d_0)\neq\emptyset}} c_\Gamma \;\left(a^\Gamma_{\epsilon_{12},\,0,\,1;\,\infty} b^\Gamma_{\mu_{12},\,0,\,1;\,\infty} - 1 \right)\; a^\Gamma_{\epsilon_{12},\,-\infty,\,0;\,\infty} b^\Gamma_{\mu_{12},\,-\infty,\,0;\,\infty} a^\Gamma_{\epsilon_1}\, b^\Gamma_{\mu_1}\, \mu^\Gamma \right\}, \tag{B.16}$$

for any sufficiently large d_0, where $\rho_1 - \rho_2 = (\epsilon_{12},\ \mu_{12})$ and where

$$a_{\epsilon_{12},\,\alpha,\,\beta;\,\infty}(\gamma) = \lim_{|a|\to\infty} a_{(\epsilon_1-\epsilon_2)-(\epsilon_1-\epsilon_2)\cdot a,\,\alpha,\,\beta}(\gamma), \tag{B.17}$$

etc., where $\alpha < \beta \in \mathbb{Z}$ and in $a_{(\epsilon_1-\epsilon_2)-(\epsilon_1-\epsilon_2)\cdot a,\,\alpha,\,\beta}(\gamma)$ the strings ℓ_e depend on a in the way described above, i.e., such that, inside of $C_1(|a|/4) \setminus C_1(d_0)$ and $C_2(|a|/4) \setminus C_2(d_0)$ they point parallel to the positive x-axis of $\mathbb{Z}^2$. Note that, for each γ, the limit above is reached at finite values of $|a|$. The condition $\Gamma \cap C_1(d_0) \neq \emptyset$ means that the geometrical part of the cluster Γ must have a non-empty intersection with the cylinder $C_1(d_0)$. Note also that the convergence of the sum over clusters in (B.16) can be shown using the fact that the contributing clusters have a non-empty intersection with $C_1(d_0)$ and with the $t = 0$ and/or $t = 1$ euclidean time slices together with the exponential decay provided by (A.19). We can say, for instance, that the sum over clusters in (B.16) can be bounded by

$$const.\ \sum_{t=-\infty}^{\infty} \sum_{\substack{\Gamma \\ \Gamma\cap(C_1(d_0)\cap T_t)\neq\emptyset \\ \|\Gamma\|\geq t}} |c_\Gamma|\ |\mu^\Gamma| \leq const.\ \sum_{t=-\infty}^{\infty} e^{-c_a|t|} < \infty \tag{B.18}$$

with some positive constant c_a where T_t is the euclidean time-plane at euclidean time t.

Using the representation above in terms of cluster expansions one can also easily show that $d_{-\rho_1,\,-\rho_2} = \overline{d_{\rho_1,\,\rho_2}}$.

The next problem is to prove the factorization property (3.68). The arguments used are analogous to those leading to (B.16). We can namely prove that $\lim_{b\to\infty} d_{\rho_1+\rho_1'\cdot b,\,\rho_2+\rho_2'\cdot b} = d_{\rho_1,\,\rho_2} d_{\rho_1',\,\rho_2'}$. This can be obtained using the representation (B.16) with d_0 depending on b such that $C_1(d_0)$ contains $\sup\,\rho_1 \cup \sup\,\rho_2 \cup \sup\,(\rho_1' \cdot b) \cup \sup\,(\rho_2' \cdot b)$. We consider again cylinders $D_1(|b|/4) = C_1(|b|/4)$ and $D_2(|b|/4) = D_1(|b|/4) + b$, both contained in $C_1(d_0(b))$, with $D_1(|b|/4)$ containing the set $\sup\,\rho_1 \cup \sup\,\rho_2$ and $D_2(|b|/4)$ containing the set $\sup\,(\rho_1' \cdot b) \cup \sup\,(\rho_2' \cdot b)$ for some $|b|$ large enough. Repeating the previous arguments, we can neglect contributions from clusters contained outside of $\mathbb{Z}^3 \setminus (D_1(|b|/4) \cup D_2(|b|/4))$ and take the pieces of the strings ℓ_e and ℓ_m which join the supports of ρ_1 and ρ_2 with the supports of $\rho_1' \cdot b$ and $\rho_2' \cdot b$ so that they again run parallel to the x-axis at sufficiently large distances. The desired relation will follow again from the usual clustering properties of the cluster expansions established above. ∎

B.5 Completing the Proof of Proposition 3.3.2

We will here complete some missing points in the proof of Proposition 3.3.2. The ideas are actually contained in [2] and therefore we will concentrate only on the more relevant details.

To show that the sequence of unit vectors $\Phi^n_{\rho,\,\rho'}$, $n \in \mathbb{N}$ is a Cauchy sequence it is enough to show that, for any $\epsilon > 0$, one has $|(\Phi^n_{\rho,\,\rho'},\ \Phi^m_{\rho,\,\rho'}) - 1| < \epsilon$ provided n is large enough, with $m > n$. The scalar product $(\Phi^n_{\rho,\,\rho'},\ \Phi^m_{\rho,\,\rho'})$ can be expressed as the exponential of a sum over clusters and, hence, it is enough to show that this sum is small enough provided n is large enough, with $m > n$.

This sum can be written as

$$\sum_{\Gamma} c_\Gamma \left(a^\Gamma_{l_1} b^\Gamma_{l_1} - \frac{1}{2} a^\Gamma_{l_1,\,l_2} b^\Gamma_{l_1,\,l_2} - \frac{1}{2} \right) \ a^\Gamma_{m,\,m}\, b^\Gamma_{m,\,m}\ a^\Gamma_\rho\, b^\Gamma_\rho\ \mu^\Gamma. \tag{B.19}$$

Above $a^\Gamma_{l_1}$, $b^\Gamma_{l_1}$, $a^\Gamma_{l_1,\,l_2}$ and $b^\Gamma_{l_1,\,l_2}$ are the electric and magnetic winding numbers on the loops l_1 and $l_1 \cup l_2$ schematically represented in Figure 1 (where $l_2 = \theta l_1$, θ meaning reflection on the $t = 0$ euclidean time plane). Also above $a^\Gamma_{a,\,b}$ and $b^\Gamma_{a,\,b}$ (with $0 \le a < b$) are the electric and magnetic winding numbers around the infinite vertical lines schematically represented in Figure 2.

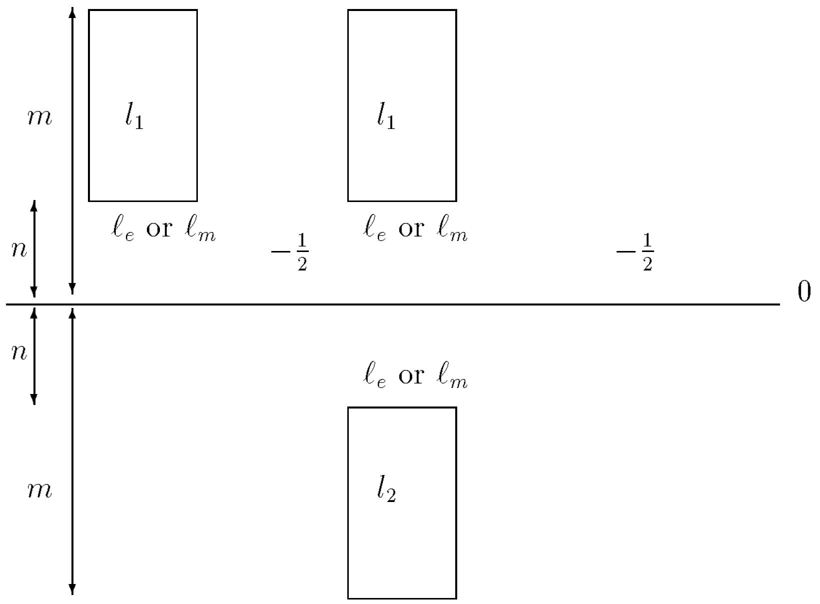


Figure 1: Schematic representation of the expression $a^\Gamma_{l_1} b^\Gamma_{l_1} - \frac{1}{2} a^\Gamma_{l_1,\,l_2} b^\Gamma_{l_1,\,l_2} - \frac{1}{2}$ and the loops l_1 (above) and l_2 (below).

By a straight forward inspection we can verify that a cluster Γ with a size smaller than n with a non-trivial winding number with, say, the loop l_1 are canceled in the sum (B.19) by the contribution of the reflected cluster $\theta\Gamma$. The contribution of the clusters entirely contained between the time-slices n and m and the contribution of the clusters entirely contained between the time-slices $-n$ and $-m$ also cancel mutually. The only surviving clusters must have non-trivial winding numbers with both l_1 and l_2 simultaneously and must cross both planes at time n and $-n$. Therefore, they must have a size which increases with n. By estimate (2.31) their contributions disappear when $n \to \infty$ uniformly m, completing thus the proof.

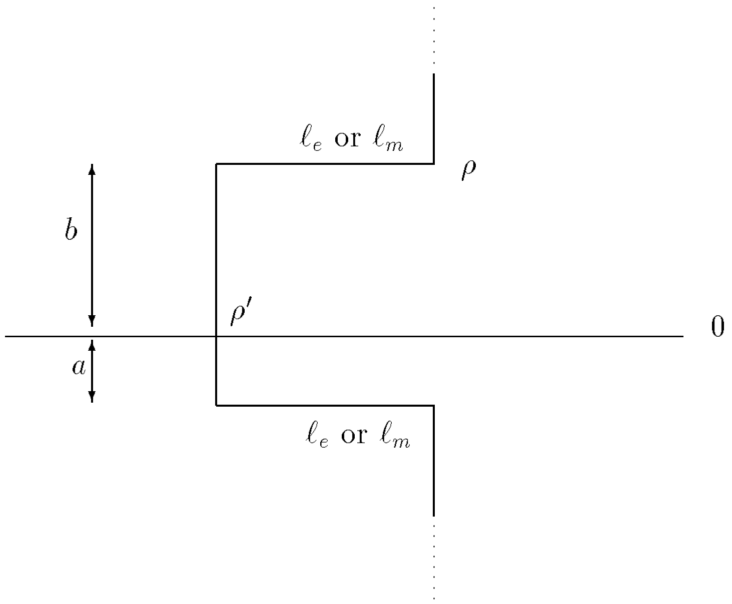


Figure 2: Schematic representation of the partial replacement of the infinite vertical line representing the background charge distribution ρ by ρ'. The connections are performed at euclidean time planes b and $-a$, $0 \leq a < b$, through the strings ℓ_e and/or ℓ_m.

B.6 Proof of Theorem 4.1.1

Let us start proving *i)*. We will first consider the case where $d^* l_1 \neq 0$ and $d^* l_2 \neq 0$. Without loss, we will take l_1 and l_2 as having support on single lattice links. According to the definitions we have

$$\begin{aligned} \hat{\mathcal{E}}_q(l_1 + l_2)\Omega_\rho &= \phi^{el}_{\rho_{1,2}}(l_1 + l_2) \\ &= \lim_{n\to\infty} \frac{\pi_{\rho_{1,2}}\left(\alpha^n_\rho\left(A_{\rho_{1,2},\,\rho}((l_1+l_2),\ 0)\right)\right)\Omega_{\rho_{1,2}}}{N_1(n)}, \end{aligned} \tag{B.20}$$

with

$$N_1(n) := \left\|\pi_{\rho_{1,2}}\left(\alpha^n_\rho\left(A_{\rho_{1,2},\,\rho}((l_1+l_2),\ 0)\right)\right)\Omega_{\rho_{1,2}}\right\| \tag{B.21}$$

and

$$\begin{aligned} \hat{\mathcal{E}}_q(l_1)\hat{\mathcal{E}}_q(l_2)\Omega_\rho &= \lim_{p\to\infty} \hat{\mathcal{E}}_q(l_1)\frac{\pi_{\rho_2}\left(\alpha^p_\rho\left(A_{\rho_2,\,\rho}(l_2,\ 0)\right)\right)\Omega_{\rho_2}}{\|\pi_{\rho_2}\left(\alpha^p_\rho\left(A_{\rho_2,\,\rho}(l_2,\ 0)\right)\right)\Omega_{\rho_2}\|} \\ &= \lim_{p\to\infty} \frac{\pi_{\rho_{1,2}}\left(\alpha^p_\rho\left(A_{\rho_2,\,\rho}(l_2,\ 0)\right)\right)\phi^{el}_{\rho_{1,2}}(l_1)}{\|\pi_{\rho_2}\left(\alpha^p_\rho\left(A_{\rho_2,\,\rho}(l_2,\ 0)\right)\right)\Omega_{\rho_2}\|}, \end{aligned} \tag{B.22}$$

where $\rho_i = \rho + (d^* l_i,\ 0)$, $i = 1,\ 2$ and $\rho_{1,2} = \rho + (d^*(l_1 + l_2),\ 0)$,

The vector in the right hand side can be written as

$$\lim_{p\to\infty}\lim_{q\to\infty} \frac{\pi_{\rho_{1,2}}\left(\alpha^p_\rho\left(A_{\rho_2,\,\rho}(l_2,\ 0)\right)\alpha^q_{\rho_2}\left(A_{\rho_{1,2},\,\rho_2}(l_1,\ 0)\right)\right)\Omega_{\rho_{1,2}}}{N_2(p)N_3(q)}, \tag{B.23}$$

where $N_1(p)$ and $N_2(q)$ are the normalization factors

$$N_2(p) \ := \ \left\| \pi_{\rho_2} \left(\alpha^p_\rho \left(A_{\rho_2,\, \rho}(l_2,\ 0) \right) \right) \Omega_{\rho_2} \right\| , \tag{B.24}$$

and

$$N_3(q) \ := \ \left\| \pi_{\rho_{1,\,2}} \left(\alpha^q_{\rho_2} \left(A_{\rho_{1,\,2},\, \rho_2}(l_1,\ 0) \right) \right) \Omega_{\rho_{1,\,2}} \right\| . \tag{B.25}$$

The scalar product in (4.16) can now be written as

$$\lim_{n\to\infty} \lim_{p\to\infty} \lim_{q\to\infty} \frac{\left(\Omega_{\rho_{1,\,2}}, \quad \pi_{\rho_{1,\,2}}(A)\, \Omega_{\rho_{1,\,2}} \right)}{N_1(n) N_2(p) N_3(q)} , \tag{B.26}$$

with

$$A \ := \ \alpha^n_\rho \left(A_{\rho_{1,\,2},\, \rho}((l_1 + l_2),\ 0) \right)^* \alpha^p_\rho \left(A_{\rho_2,\, \rho}(l_2,\ 0) \right) \alpha^q_{\rho_2} \left(A_{\rho_{1,\,2},\, \rho_2}(l_1,\ 0) \right) . \tag{B.27}$$

After expressing the expectation values above in terms of classical expectations (which involve only closed loops) and these in terms of cluster expansions, we arrive at the following expression:

$$\lim_{n\to\infty} \lim_{p\to\infty} \lim_{q\to\infty} \exp \left(\sum_{\Gamma} c_\Gamma\, \mu^\Gamma \ \left[a^\Gamma_1 - \frac{1}{2} \left(a^\Gamma_2 + a^\Gamma_3 + a^\Gamma_4 - a^\Gamma_5 \right) \right] a^\Gamma_{\rho_{1,\,2}}\, b^\Gamma_{\rho_{1,\,2}} \right) , \tag{B.28}$$

where $a_i(\gamma)$ represent winding numbers of γ with respect to the loops successively presented in Figure 3. The quantities $a_{\rho_{1,\,2}}(\gamma)$ and $b_{\rho_{1,\,2}}(\gamma)$ are electric and magnetic winding numbers with respect to the background charge $\rho_{1,\,2}$.

We have to perform a detailed analysis of the sum over clusters appearing in (B.28). For the sake of brevity we will sketch the main arguments.

Define the time planes $H_a := \{(\underline{x},\ x_0) \in \mathbb{Z}^3$ with $x_0 = a\}$ and denote by $\mathcal{G}_B$ the set of all clusters Γ not crossing any of the planes $H_{\pm n}$, $H_{\pm p}$ and $H_{\pm q}$. It is easy to verify that for a cluster $\Gamma \in \mathcal{G}_B$ one either has $a^\Gamma_1 - \frac{1}{2}\left(a^\Gamma_2 + a^\Gamma_3 + a^\Gamma_4 - a^\Gamma_5 \right) = 0$ or it happens that its contribution cancels that of other cluster in $\mathcal{G}_B$ obtained by translating Γ in time direction. This is, for instance, what happens for clusters located between H_p and H_q and translated clusters located between H_{-q} and H_{-p}.

On the other hand, the size of clusters which cross at least two of the planes $H_{\pm n}$, $H_{\pm p}$ or $H_{\pm q}$ is at least $\min\{2n,\ p-n,\ q-p\}$ (assuming $q > p > n$). After the limits $q \to \infty$, $p \to \infty$ and $n \to \infty$ are taken the total contribution of such clusters is zero, what can be shown using the exponential decay given in (A.19) and noticing that the size of the loops of Figure 3 grows only linearly in n, p or q. The remaining terms belong to clusters crossing one and only one of the planes $H_{\pm n}$, $H_{\pm p}$ or $H_{\pm q}$. Using the translation invariance of the sum over clusters, we may express these remaining terms (after the limits are taken) in the following form:

$$\begin{aligned} &\frac{1}{2} \sum_{\Gamma \not\sim H_0} c_\Gamma\, \mu^\Gamma \ \left(f^\Gamma_{l_1} - (f^t_{l_1})^\Gamma \right) a^\Gamma_{\rho_{1,\,2}}\, b^\Gamma_{\rho_{1,\,2}} + \frac{1}{2} \sum_{\Gamma \not\sim H_0} c_\Gamma\, \mu^\Gamma \ \left(f^\Gamma_{l_2} - (f^t_{l_2})^\Gamma \right) a^\Gamma_{(d^* l_1,\, 0)} a^\Gamma_{\rho_{1,\,2}}\, b^\Gamma_{\rho_{1,\,2}} + \\ &\frac{1}{2} \sum_{\Gamma \not\sim H_0} c_\Gamma\, \mu^\Gamma \ \left((f^t_{l_1})^\Gamma (f^t_{l_2})^\Gamma - f^\Gamma_{l_1} f^\Gamma_{l_2} \right) a^\Gamma_{\rho_{1,\,2}}\, b^\Gamma_{\rho_{1,\,2}} , \end{aligned} \tag{B.29}$$

where, with some abuse of notation, $\Gamma \not\sim H_0$ indicates that the geometric part of at least one polymer composing Γ crosses the plane H_0. Above, $f_l(\gamma)$ (respectively, $f^t_l(\gamma)$) represents the winding number

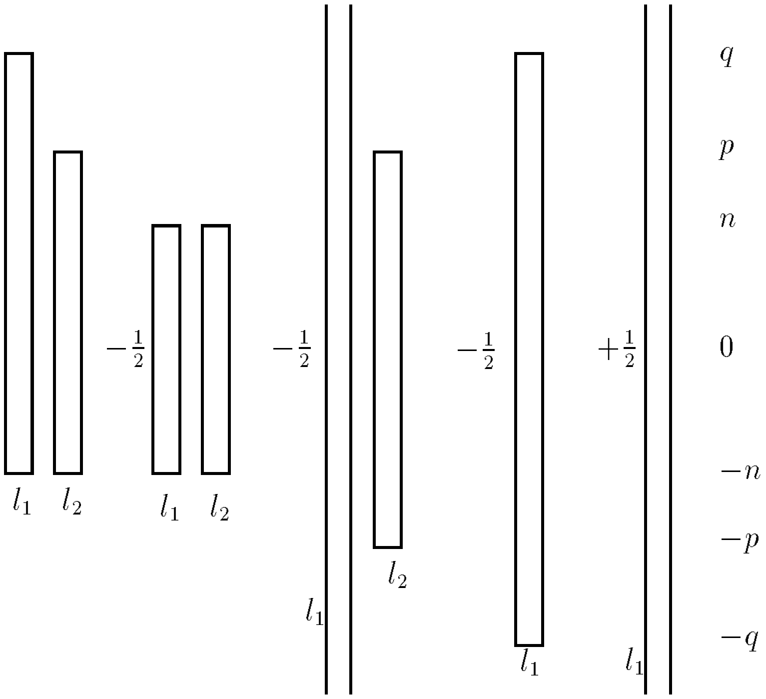


Figure 3: Pictorial representation of the expression $a_1^\Gamma - \frac{1}{2}\left(a_2^\Gamma + a_3^\Gamma + a_4^\Gamma - a_5^\Gamma\right)$ appearing in (B.28). The a_i's are winding numbers with respect to the sets of loops presented in the picture (counted from the left to the right and separated by the associated factor $\pm 1/2$). The vertical lines are parallel to the euclidean time-axis. The open loops cross $d^* l_1$ and extend to the euclidean time infinity. At the right we indicate the different time planes.

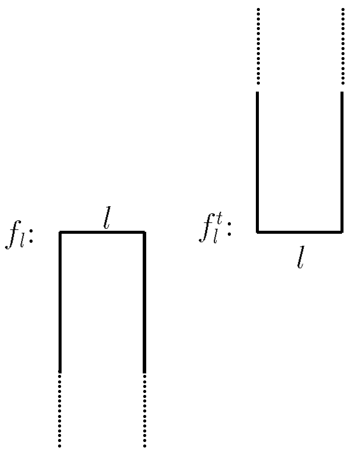


Figure 4: The semi-infinite loops for which $f_l(\gamma)$ and $f_l^t(\gamma)$ are defined. The horizontal lines represent the link l located at H_0. The vertical lines are parallel to the euclidean time axis and extend to the negative (left) or positive (right) euclidean time infinity.

of the polymer γ with respect to the semi-infinite loops formed by l and by vertical lines starting at d^*l and extending to the negative (positive) euclidean time infinity. See Figure 4. Note that the second sum in (B.29) can be simplified, since $a^{\Gamma}_{(d^*l_1,\,0)} a^{\Gamma}_{\rho_{1,\,2}}\, b^{\Gamma}_{\rho_{1,\,2}} = a^{\Gamma}_{\rho_2}\, b^{\Gamma}_{\rho_2}$.

It is easy to show that each of the sums over clusters in (B.29) is absolutely convergent. Analogously, sums like

$$\sum_{\Gamma \not\sim H_0} c_\Gamma\, \mu^\Gamma\, \left(f^{\Gamma}_{l_1} - 1\right) a^{\Gamma}_{\rho_{1,\,2}}\, b^{\Gamma}_{\rho_{1,\,2}} \tag{B.30}$$

are also absolutely convergent. It can be seen by reflecting polymers on the plane H_0 that the last expression is the complex conjugate of $\sum_{\Gamma \not\sim H_0} c_\Gamma\, \mu^\Gamma\, \left((f^t_{l_1})^{\Gamma} - 1\right) a^{\Gamma}_{\rho_{1,\,2}}\, b^{\Gamma}_{\rho_{1,\,2}}$. This means that each of the sums over clusters in (B.29) is purely imaginary. Defining

$$z^{el}_\rho(l_2) \;:=\; \exp\left\{ \frac{1}{2} \sum_{\Gamma \not\sim H_0} c_\Gamma\, \mu^\Gamma\, \left(f^{\Gamma}_{l_2} - (f^t_{l_2})^{\Gamma}\right) a^{\Gamma}_{\rho_2}\, b^{\Gamma}_{\rho_2} \right\}, \tag{B.31}$$

which is a pure phase, we conclude from (B.29) the proof of part *i)* of Theorem 4.1.1. Part *ii)* can be proven analogously, and we do not need to show the details.

The proof of part *iii)* is also analogous but with an important difference. Since in this case l_1 is a magnetic link and l_2 an electric one, the closed loops formed by l_1 and by l_2, appearing in the left Figure 3, can have a nontrivial winding number, which can contribute to the classical expectations in the numerator of (B.26) with an additional $\mathbb{Z}_N$ phase factor, as the phase factor $[\beta :\ \alpha]$ emerging from (A.33). This phase equals $e^{i\langle l_1,\, l_2\rangle}$.

In order to prove *iv)*, consider that the support that l_2 is, say, an elementary plaquette at H_0. Following the same steps of the proof of *i)* we would arrive at relations like (B.29) and (B.31) where both $f_{l_2}(\gamma)$ and $f^t_{l_2}(\gamma)$ represent the winding number of γ around this plaquette. Therefore, for any polymer γ, $f_{l_2}(\gamma) = f^t_{l_2}(\gamma)$ and hence $z^{el}_\rho(l_2) = 1$. The proof of *v)* is analogous. ∎

B.7 Proof of Proposition 4.2.3

We will prove only (4.41) since (4.42) is analogous. Relation (4.41) means $\|\phi^{el}_\rho(l_e) - e^{i\langle\mu,\, s\rangle}\Omega_\rho\| = 0$ and to prove one has only to show that $\left(\Omega_\rho,\ \phi^{el}_\rho(l_e)\right) = e^{i\langle\mu,\, s\rangle}$. According to the definitions

$$\left(\Omega_\rho,\ \phi^{el}_\rho(l_e)\right) \;=\; \frac{\left(\Omega_\rho,\ \ P_\rho(\rho - (d^*l_e,\ 0))\pi_\rho(A_{\rho,\,\rho-(d^*l_e,\,0)}(l_e,\ 0))\Omega_\rho\right)}{\left\| P_\rho(\rho - (d^*l_e,\ 0))\pi_\rho\left(A_{\rho,\,\rho-(d^*l_e,\,0)}(l_e,\ 0)\right)\Omega_\rho \right\|}. \tag{B.32}$$

Under the hypothesis $d^*l_e = 0$ and, hence, we can write the right hand side of (B.32) as

$$\lim_{n\to\infty} \frac{\left(\Omega_\rho,\ \ T_\rho(\rho)^n \pi_\rho\left(A_{\rho,\,\rho}(l_e,\ 0)\right)\Omega_\rho\right)}{\left\|T_\rho(\rho)^n \pi_\rho\left(A_{\rho,\,\rho}(l_e,\ 0)\right)\Omega_\rho\right\|} \;=\; \lim_{n\to\infty} \frac{\left(\Omega_\rho,\ \ \pi_\rho\left(\alpha^n_\rho(A_{\rho,\,\rho}(l_e,\ 0))\right)\Omega_\rho\right)}{\left\|\pi_\rho\left(\alpha^n_\rho(A_{\rho,\,\rho}(l_e,\ 0))\right)\Omega_\rho\right\|}. \tag{B.33}$$

We now expand the right hand side of (B.33) in terms of our cluster expansions and treat it with the same methods used in the proof of Theorem 4.1.1 above. We get

$$\left(\Omega_\rho,\ \phi^{el}_\rho(l_e)\right) \;=\; z^{el}_\rho(l_e)\, e^{i\langle\mu,\, s\rangle}, \tag{B.34}$$

where the $\mathbb{Z}_N$ phase factor $e^{i\langle\mu,\, s\rangle}$ emerges in this expression as the factor $[\tilde{\mu} : \alpha]$ emerges from (A.33): it represents the winding number of d^*s in the background charge ρ. Actually $e^{i\langle\mu,\, s\rangle} = [\tilde{\mu} : s]$. Since $d^* l_e = 0$, one has $z_\rho^{el}(l_e) = 1$ and the proposition is proven. ■

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