EDGE RESONANCE IN AN ELASTIC SEMI-STRIP

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Abstract. We study the elasticity operator in a semi-strip subject to free boundary conditions. In the case of zero Poisson ratio we prove the existence of a positive eigenvalue embedded in the essential spectrum. Physically, the eigenvalue corresponds to a "trapped mode", that is, a harmonic oscillation of the semi-strip localized near the edge. This effect, known in mechanics as the "edge resonance", has been extensively studied numerically and experimentally. Our result provides a mathematical justification.

1. Introduction

In this paper we study a particular self-adjoint operator A with spectrum $\sigma(A) = [0, +\infty)$ and prove the existence of a positive eigenvalue λ_e. Our proof is based on the use of certain symmetries of A which allow us to identify suitable invariant subspaces. The crucial step is to choose an invariant subspace in such a way that the restriction of A to this subspace has essential spectrum separated from zero. Then the existence of an eigenvalue below this essential spectrum is established by a standard variational argument.

The approach outlined above is typical for recent rigorous results on trapped modes [3], [5], [6], [7]. However, these papers dealt only with the Laplacian acting on scalar functions, whereas we shall consider an operator acting on vector-functions. This leads to a major difficulty because in the case of a system of equations spatial symmetries are insufficient for separating the essential spectrum from zero. Therefore we will have to make use of an additional internal (hidden) symmetry of our operator.

Our operator A describes the harmonic vibrations of an elastic semi-strip subject to free boundary conditions. The existence of an eigenvalue for this operator was conjectured a long time ago on the basis of extensive numerical and experimental data, as well as heuristic mechanical arguments; see, for example, [4], [8], [12], [13], [17], [20], [21], [22]. We establish the existence of an eigenvalue under the assumption that Poisson's ratio is zero. To our knowledge this is the first rigorous proof in the subject of the edge resonance.

When Poisson's ratio is nonzero the internal symmetry is destroyed and our argument fails. One may assume that in this case the eigenvalue turns into a complex resonance lying close to the real axis.

Let us stress that the phenomenon described in this paper does not occur in the scalar situation: the Neumann Laplacian in a semi-strip does not have eigenvalues.

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2. Main result

2.1. Notation. By $G := \{x = (x_1, x_2) \in \mathbb{R}^2 :\ x_1 > 0,\ x_2 \in J\}$, $J := (-\pi/2, \pi/2)$, we denote a (infinite) semi-strip of width π. Its boundary consists of three parts: $\Gamma_e = \{x :\ x_1 = 0,\ x_2 \in \overline{J}\}$ and $\Gamma_\pm = \{x :\ x_1 \geq 0,\ x_2 = \pm\pi/2\}$. By

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : G \to \mathbb{C}^2$$

(or by $\mathbf{v}$) we denote vector-functions which have the physical meaning of the displacement of the elastic medium. We set for brevity $\partial_{x_k} = \partial/\partial x_k$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$, where $\alpha = (\alpha_1, \alpha_2)$ is a multiindex and $|\alpha| = \alpha_1 + \alpha_2$. We denote

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2, \quad \operatorname{grad} = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix}, \quad \operatorname{div} = (\partial_{x_1}, \partial_{x_2}).$$

We will be working in the Hilbert space $H = L_2(G, \mathbb{C}^2)$ with inner product $(\mathbf{u}, \mathbf{v})_H = \int_G (u_1\overline{v_1} + u_2\overline{v_2})dx$. By $W_2^l(G, \mathbb{C}^2)$ we denote the Sobolev space with inner product

$$(\mathbf{u}, \mathbf{v})_{W_2^l} \ = \ \sum_{k=1}^{2} \sum_{|\alpha| \leq l} \iint_G (\partial_x^\alpha u_k)(\partial_x^\alpha \overline{v_k}) \ dx \ .$$

The constants $\rho > 0$, $E > 0$ and $\nu \geq 0$ are, respectively, the volume density, Young's modulus and Poisson's ratio of our elastic material. The constants $c_l \geq \sqrt{2}c_t > 0$ are the velocities of longitudinal and transverse waves. By ω we denote the (circular) vibration frequency, and by $\lambda = \omega^2$ the spectral parameter. By $\sigma(\,\cdot\,)$ we denote the spectrum of an operator, and by $\sigma_{ess}(\,\cdot\,)$ the essential spectrum.

2.2. Statement of the problem. Consider the quadratic form

(2.1)
$$a[\mathbf{u}, \mathbf{u}] = \int_G \left(c_l^2\, |\partial_{x_1} u_1 + \partial_{x_2} u_2|^2 - 4c_t^2 \operatorname{Re} \partial_{x_1} u_1 \partial_{x_2} \overline{u_2} + c_t^2\, |\partial_{x_2} u_1 + \partial_{x_1} u_2|^2\right) dx,$$

on the domain $d[a] = W_2^1(G, \mathbb{C}^2)$.

The form (2.1) appears as a result of separation of variables ($u_3 \equiv 0$, $\partial_{x_3} \equiv 0$, $-\infty < x_3 < +\infty$) in three-dimensional elasticity. In this case

$$c_l = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}}, \qquad c_t = \sqrt{\frac{E}{2\rho(1+\nu)}},$$

and $\rho a[\mathbf{u},\mathbf{u}]/2$ is the potential energy of elastic deformation per unit length in x_3; see [16], Sections 4 and 22.

The form (2.1) also appears in plate theory. In this case

$$c_l = \sqrt{\frac{E}{\rho(1-\nu^2)}}, \qquad c_t = \sqrt{\frac{E}{2\rho(1+\nu)}},$$

and $h\rho a[\mathbf{u},\mathbf{u}]/2$ is the potential energy of tangential elastic deformation, h being the thickness of the plate; see [11].

Note that for zero Poisson ratio we have $c_l = \sqrt{2}c_t$ in both physical models.

For some sufficiently large positive constant C we have

$$C^{-1}\|\mathbf{u}\|^2_{W_2^1} \le a[\mathbf{u},\mathbf{u}] + \|\mathbf{u}\|^2_H \le C\|\mathbf{u}\|^2_{W_2^1} \tag{2.2}$$

uniformly over all $\mathbf{u} \in d[a]$. Here the right inequality is obvious, whereas the left one is nontrivial and known as Korn's inequality. Korn's inequality holds [10] for any bounded domain satisfying the cone condition. As our semi-strip can be represented as a union of squares of side π, the constant C in (2.2) is the same as for such a square.

By (2.2) the form a is closed and semibounded from below, therefore it induces the (unique) self-adjoint operator $A : D(A) \to H$ with some domain $D(A) \subset d[a]$.

Alternatively, the operator A can be defined as follows. Denote by $D(\tilde{A})$ the set of all vector-functions $\mathbf{u} \in W_2^2(G, \mathbb{C}^2)$, satisfying the boundary conditions

$$c_l^2\partial_{x_1}u_1 + (c_l^2 - 2c_t^2)\partial_{x_2}u_2 = 0, \quad \partial_{x_2}u_1 + \partial_{x_1}u_2 = 0 \qquad \text{for} \quad x \in \Gamma_e\ , \tag{2.3}$$

$$c_l^2\partial_{x_2}u_2 + (c_l^2 - 2c_t^2)\partial_{x_1}u_1 = 0, \qquad \partial_{x_2}u_1 + \partial_{x_1}u_2 = 0 \qquad \text{for} \quad x \in \Gamma_\pm\ . \tag{2.4}$$

Let $\tilde{A} : D(\tilde{A}) \to H$ be the operator defined by

$$\tilde{A} = -c_t^2\Delta - (c_l^2 - c_t^2)\,\mathrm{grad}\,\mathrm{div}\,. \tag{2.5}$$

It is easy to see that $(\tilde{A}\mathbf{u},\mathbf{u})_H = a[\mathbf{u},\mathbf{u}]$, $\forall \mathbf{u} \in D(\tilde{A})$, and that $D(\tilde{A})$ is dense in $d[a]$ with respect to the norm $\sqrt{a[\mathbf{u},\mathbf{u}]} + \|\mathbf{u}\|_H$. Therefore A is the Friedrichs extension of $\tilde{A}$.

We have $\sigma(A) = [0, +\infty)$. Indeed, the operator A is nonnegative and for any positive λ one can easily construct a sequence $\mathbf{u}_n \in D(\tilde{A})$ such that $\|\mathbf{u}_n\|_H = 1$ and $\|A\mathbf{u}_n - \lambda\mathbf{u}_n\|_H \to 0$ as $n \to \infty$.

Naturally, $\sigma(A) = \sigma_{ess}(A)$. The fact that the spectrum of A is not discrete is, of course, a consequence of the unboundedness of G.

Further on we will be dealing with the special case $c_l = \sqrt{2}c_t$ which, as pointed out earlier, corresponds to zero Poisson ratio. In order to simplify notation we shall assume that $c_t = 1$, which can always be achieved by a rescaling of the spectral parameter.

In the case $c_l = \sqrt{2}$, $c_t = 1$ formulae (2.1), (2.3), (2.4), (2.5) turn into

$$a[\mathbf{u},\mathbf{u}] = \int_G \left(2\,|\partial_{x_1}u_1|^2 + 2\,|\partial_{x_2}u_2|^2 + |\partial_{x_2}u_1 + \partial_{x_1}u_2|^2\right)dx, \tag{2.6}$$

$$\partial_{x_1}u_1 = 0, \quad \partial_{x_2}u_1 + \partial_{x_1}u_2 = 0 \quad \text{for} \quad (x_1,x_2) \in \Gamma_e\ , \tag{2.7}$$

$$\partial_{x_2}u_2 = 0, \quad \partial_{x_2}u_1 + \partial_{x_1}u_2 = 0 \quad \text{for} \quad (x_1,x_2) \in \Gamma_\pm, \tag{2.8}$$

$$\tilde{A} = -\Delta - \operatorname{grad}\operatorname{div} = -\begin{pmatrix} 2\partial_{x_1x_1} + \partial_{x_2x_2} & \partial_{x_1x_2} \\ \partial_{x_1x_2} & \partial_{x_1x_1} + 2\partial_{x_2x_2} \end{pmatrix} \tag{2.9}$$

respectively. An examination of formulae (2.6), (2.7), (2.8), (2.9) shows that in the case $c_l = \sqrt{2}$, $c_t = 1$ the spectral problem still remains nontrivial and is not reduced to that for the Laplacian.

2.3. Statement of the result. Set

$$\lambda_+ := 2 + \left(\frac{16}{3\pi}\right)^2 - \sqrt{4 - 4\left(\frac{16}{3\pi}\right)^2 + 2\left(\frac{16}{3\pi}\right)^4} \approx 1.86805. \tag{2.10}$$

Our main result is

Theorem 2.1. *If $c_l = \sqrt{2}$ and $c_t = 1$ then the operator A has an eigenvalue $\lambda_e \in (0, \lambda_+]$.*

Numerical results suggest that $\lambda_e \approx 1.60$, see [12] and [13], p. 267.

The remainder of the paper is devoted to the proof of Theorem 2.1.

3. Spatial and internal symmetries.

3.1. Orthogonal decompositions. We recall the notion of an orthogonal decomposition of an unbounded self-adjoint operator. For a detailed exposition see [2], chapter 5, sections 5 and 6. Let H be a separable Hilbert space and let H_1 be a (closed) subspace of H. Let H_2 be the orthogonal complement of H_1. Denote by P_j the respective orthogonal projections onto H_j, $j = 1, 2$. Let $A : D(A) \to H$ be a self-adjoint operator in H. We shall call H_1, H_2 a *pair of invariant subspaces* of the operator A if

$$P_j D(A) \subset D(A), \qquad j = 1, 2, \tag{3.1}$$

$$P_j A\mathbf{u} = AP_j\mathbf{u} \qquad \text{for all} \qquad u \in D(A), \qquad j = 1, 2. \tag{3.2}$$

In this case we have $AP_jD(A) \subset H_j$, so the operator A breaks up into the orthogonal sum $A = A_1 \oplus A_2$ where

$$A_j : D(A_j) \to H_j, \quad D(A_j) = P_jD(A), \quad A_j = A|_{D(A_j)}, \qquad j = 1, 2,$$

and $D(A) = D(A_1) \oplus D(A_2)$. The operators A_j turn out to be self-adjoint (in H_j), and the spectral analysis of the operator A is reduced to that of the operators A_j. In particular, an eigenvalue of A_j is an eigenvalue of A.

Suppose now that the operator A is associated with a closed nonnegative sesquilinear form a with domain $d[a]$, and that $H = H_1 \oplus H_2$,

$$P_j d[a] \subset d[a], \qquad j = 1, 2, \tag{3.3}$$

$$a[P_1\mathbf{u}, P_2\mathbf{v}] = 0 \qquad \text{for all} \qquad \mathbf{u}, \mathbf{v} \in d[a]. \tag{3.4}$$

It is easy to see that in this case H_1, H_2 is a pair of invariant subspaces of the operator A, and that A_j is associated with the restriction a_j of a to $d[a_j] = P_j d[a]$. Clearly, the forms a_j are closed and nonnegative in H_j.

3.2. Symmetric and antisymmetric vector-functions. Let A be the operator corresponding to (2.1) with general c_l, c_t and let P_j, $j = 1, 2$, be the orthogonal projections

$$P_j \begin{pmatrix} u_1(x_1,x_2) \\ u_2(x_1,x_2) \end{pmatrix} := \frac{1}{2} \begin{pmatrix} u_1(x_1,x_2) + (-1)^{j-1} u_1(x_1,-x_2) \\ u_2(x_1,x_2) - (-1)^{j-1} u_2(x_1,-x_2) \end{pmatrix} \tag{3.5}$$

onto the subspaces

$$H_j := \left\{ \mathbf{u} \in H : u_1(x_1,x_2) = (-1)^{j-1} u_1(x_2,-x_2),\ u_2(x_1,x_2) = (-1)^j u_2(x_1,-x_2) \right\}$$

respectively. Vector-functions $\mathbf{u} \in H_1$ are called *symmetric*, whereas vector functions $\mathbf{u} \in H_2$ are called *antisymmetric*. Obviously $H = H_1 \oplus H_2$, and (3.3), (3.4) hold. Thus, H_1, H_2 is a pair of invariant subspaces for A, and $A = A_1 \oplus A_2$. The operators A_j correspond to the differential expression $\tilde{A}$ acting on the symmetric (antisymmetric) vector-functions, respectively. Naturally, we shall denote $a_j := a|_{d[a_j]}$, $d[a_j] := P_j d[a]$, $j = 1, 2$.

3.3. Internal symmetries. Suppose now that $c_l = \sqrt{2}$ and $c_t = 1$. In this case the operator $A_1 : D(A_1) \to H_1$ obtains an additional "internal" symmetry. Set

$$P_3 \begin{pmatrix} u_1(x_1,x_2) \\ u_2(x_1,x_2) \end{pmatrix} := \frac{1}{\pi} \begin{pmatrix} \int_J u_1(x_1,x_2)dx_2 \\ 0 \end{pmatrix}, \quad \mathbf{u} \in H_1. \tag{3.6}$$

Then P_3 is an orthogonal projection from H_1 onto the subspace

$$H_3 := \{ \mathbf{u} \in H_1 :\ u_1(x_1,x_2) = u_1(x_1),\ u_2 \equiv 0 \}.$$

The orthogonal complement of H_3 in H_1 is

$$H_4 := \left\{ \mathbf{u} \in H_1 : \int_J u_1(x_1,x_2)dx_2 = 0 \text{ in } L_2(\mathbb{R}_+, \mathbb{C}) \right\}.$$

Let P_4 be the orthogonal projection from H_1 onto H_4. Elementary calculations show that

$$P_k d[a_1] \subset d[a_1], \qquad k = 1, 2,$$

$$a_1[P_3\mathbf{u}, P_4\mathbf{v}] = 0 \qquad \text{for all} \qquad \mathbf{u}, \mathbf{v} \in d[a_1],$$

and thus $A_1 = A_3 \oplus A_4$ with $A_k := A_1|_{D(A_k)}$, $D(A_k) := P_k D(A_1)$ for $k = 3, 4$. These operators correspond to the quadratic forms $a_k := a_1|_{d[a_k]}$ on the domains $d[a_k] := P_k d[a_1]$. Hence the operator A decomposes as $A = (A_3 \oplus A_4) \oplus A_2$. Note that this argument fails for general coefficients c_l, c_t.

4. Two auxiliary statements and the proof of Theorem 2.1.

Theorem 2.1 is an immediate consequence of the following two lemmas, the proofs of which are given in subsequent sections (recall that λ_+ is defined by formula (2.10)).

Lemma 4.1. *We have* $\min \sigma(A_4) \le \lambda_+$.

Lemma 4.2. *We have* $\lambda_+ < \min \sigma_{ess}(A_4)$.

Proof of Theorem 2.1. Set $\lambda_e := \min \sigma(A_4)$. Lemmas 4.1, 4.2 imply that $\lambda_e \in \sigma(A_4)$ and $\lambda_e \notin \sigma_{ess}(A_4)$. Thus, λ_e is an eigenvalue of the operator A_4, and, consequently, an eigenvalue of the original operator A. It remains to show that $0 < \lambda_e \le \lambda_+$. The right inequality is obvious, so let us prove that $\lambda_e > 0$. Suppose $\lambda_e = 0$. Then the corresponding eigenfunction $\mathbf{u}$ satisfies

$$a[\mathbf{u}, \mathbf{u}] = 0. \tag{4.1}$$

Formulae (4.1), (2.6) imply $\partial_{x_1} u_1 \equiv 0$, that is, u_1 is a function of the variable x_2 only. As $u_1 \in L_2(G)$ and the strip G is infinite in the x_1-direction we conclude that $u_1 \equiv 0$. Turning once again to (4.1), (2.6) we get now $\partial_{x_1} u_2 \equiv 0$, which by the same argument implies $u_2 \equiv 0$. Thus, $\mathbf{u} \equiv \mathbf{0}$, which contradicts the fact that $\mathbf{u}$ is an eigenfunction. $\square$

5. Proof of Lemma 4.1.

By the variational principle we have

$$\min \sigma(A_4) = \inf_{0 \not\equiv \mathbf{u} \in P_4 P_1 W_2^1(G, \mathbb{C}^2)} \frac{\int_G \left\{ 2\,|\partial_{x_1} u_1|^2 + 2\,|\partial_{x_2} u_2|^2 + |\partial_{x_2} u_1 + \partial_{x_1} u_2|^2 \right\} dx}{\int_G (|u_1|^2 + |u_2|^2) dx}.$$

Let us choose a test vector-function of the form

$$\mathbf{u} = \begin{pmatrix} f(x_1) \cos 2x_2 \\ g(x_1) \sin x_2 \end{pmatrix} \tag{5.1}$$

with $f,g\in W_2^1(\mathbb{R}_+,\mathbb{R})$, $f^2+g^2\not\equiv 0$. Then $\mathbf{u}\in P_4P_1W_2^1(G,\mathbb{R}^2)$, and hence

$$\min\sigma(A_4)\le\frac{\int_0^\infty\left\{2(\partial_{x_1}f)^2+(\partial_{x_1}g)^2-\frac{32}{3\pi}f\partial_{x_1}g+4f^2+2g^2\right\}dx_1}{\int_0^\infty(f^2+g^2)dx_1}. \tag{5.2}$$

In order to estimate the variational quotient in the right-hand side we consider the corresponding spectral problem

$$\begin{pmatrix}-2\partial_{x_1}^2+4 & -\frac{16}{3\pi}\partial_{x_1}\\ \frac{16}{3\pi}\partial_{x_1} & -\partial_{x_1}^2+2\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}=\lambda\begin{pmatrix}f\\ g\end{pmatrix},\qquad x_1>0, \tag{5.3}$$

$$(\partial_{x_1}f)|_{x_1=0}=\left(\partial_{x_1}g-\frac{16}{3\pi}f\right)\Big|_{x_1=0}=0. \tag{5.4}$$

When $\lambda=\lambda_+$ (see (2.10)) the system (5.3), (5.4) has the non-trivial solution

$$f(x_1)=\operatorname{Im}\left(\overline{\zeta}e^{\zeta x_1}\right),\quad g(x_1)=\frac{3\pi}{16}|\zeta|^2\operatorname{Im}\left(((4-\lambda_+)\zeta^{-2}-2)e^{\zeta x_1}\right), \tag{5.5}$$

where ζ is the root of the biquadratic equation

$$2\zeta^4+\left(\left(\frac{16}{3\pi}\right)^2+3\lambda_+-8\right)\zeta^2+(4-\lambda_+)(2-\lambda_+)=0$$

in the second quadrant [1]. Substituting (5.5) into (5.2) we conclude that $\min\sigma(A_4)\le\lambda_+$. $\square$

6. Proof of Lemma 4.2.

6.1. Reduction to the problem in an infinite strip. Denote $\breve G=\mathbb{R}\times J$. The domain $\breve G$ is an infinite strip (as opposed to the semi-strip G) with boundary $\partial\breve G=\breve\Gamma_+\cup\breve\Gamma_-$, where $\breve\Gamma_\pm=\mathbb{R}\times\{\pm\frac{\pi}{2}\}$. Let $\breve H=L_2(\breve G,\mathbb{C}^2)$, $\breve H_1$ be the subspace of symmetric vector-functions in $\breve H$ (cf. subsection 3.2), and let $\breve H_4$ be the subspace of $\breve H_1$ singled out by the condition $\int_J u_1(x_1,x_2)\,dx_2=0$ in $L_2(\mathbb{R},\mathbb{C})$ (cf. subsection 3.3). Denote by $D(\breve A_4)$ the set of all vector-functions $\mathbf{u}\in\breve H_4$ satisfying the boundary conditions $\partial_{x_2}u_2=0$ and $\partial_{x_2}u_1+\partial_{x_1}u_2=0$ for $(x_1,x_2)\in\breve\Gamma_\pm$, and let $\breve A_4:D(\breve A_4)\to\breve H_4$ be the self-adjoint operator defined by the differential expression (2.9). We have

Lemma 6.1. *The essential spectra of the operators A_4 and $\breve A_4$ coincide.*

Proof of Lemma 6.1. Let us take two copies of our semi-strip and form a strip by pointing the semi-strips in opposite directions and bringing the edges together. We have now a strip cut in half with free boundary conditions on all sides, including the cut. Clearly, the spectra of the corresponding operator and its orthogonal components are (up to multiplicity) the same as the spectra of our original operator A

[1] $\zeta\approx -0.3560149552+0.4982995888i$

and its orthogonal components. Now it remains only to remove the cut, which does not change the essential spectra of the operator and its orthogonal components by [1], [9]. □

6.2. Separation of variables in an infinite strip. With Lemma 6.1 the proof of Lemma 4.2 is reduced to the proof of

$$\lambda_+ < \min \sigma_{ess}(\breve{A}_4). \tag{6.1}$$

The advantage of dealing with the operator $\breve{A}_4$ is that it admits separation of variables, so its spectral analysis is quite straightforward.

Making a formal Fourier transform in the x_1-direction $\mathbf{u}(x) = \hat{\mathbf{u}}(x_2)e^{ix_1\xi}$ (ξ is a real parameter) we obtain the operator $\hat{A}_4(\xi) : D(\hat{A}_4(\xi)) \to \hat{H}_4$ defined by

$$\hat{A}_4(\xi) = \begin{pmatrix} 2\xi^2 - \partial_{x_2}^2 & -i\xi\partial_{x_2} \\ -i\xi\partial_{x_2} & \xi^2 - 2\partial_{x_2}^2 \end{pmatrix}, \tag{6.2}$$

where $\hat{H}_4$ is the subspace of $L_2(J, \mathbb{C}^2)$ singled out by the conditions

$$\hat{u}_1(x_2) = \hat{u}_1(-x_2), \quad \hat{u}_2(x_2) = -\hat{u}_2(-x_2), \tag{6.3}$$

$$\int_J \hat{u}_1(x_2)\, dx_2 = 0, \tag{6.4}$$

and $D(\hat{A}_4(\xi))$ is the the subspace of $W_2^2(J, \mathbb{C}^2)$ singled out by (6.3), (6.4) and

$$(\partial_{x_2}\hat{u}_2)|_{x_2=\pm\pi/2} = (\partial_{x_2}\hat{u}_1 + i\xi\hat{u}_2)|_{x_2=\pm\pi/2} = 0. \tag{6.5}$$

Clearly, the operator $\hat{A}_4(\xi)$ is self-adjoint and is associated with the closed quadratic form

$$\hat{a}_4(\xi)[\hat{\mathbf{u}}, \hat{\mathbf{u}}] = \int_J \left(2\xi^2 |\hat{u}_1|^2 + 2|\partial_{x_2}\hat{u}_2|^2 + |\partial_{x_2}\hat{u}_1 + i\xi\hat{u}_2|^2\right) dx_2 \tag{6.6}$$

the domain $d[\hat{a}_4(\xi)]$ of which is the the subspace of $W_2^1(J, \mathbb{C}^2)$ singled out by (6.3), (6.4).

The spectral analysis of the operator $\hat{A}_4(\xi)$ reduces to the following eigenvalue problem on the interval $(0, \pi/2)$:

$$\begin{pmatrix} 2\xi^2 - \partial_{x_2}^2 & -i\xi\partial_{x_2} \\ -i\xi\partial_{x_2} & \xi^2 - 2\partial_{x_2}^2 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} = \lambda \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}, \qquad 0 < x_2 < \pi/2, \tag{6.7}$$

$$(\partial_{x_2}\hat{u}_1)|_{x_2=0} = \hat{u}_2|_{x_2=0} = 0, \tag{6.8}$$

$$(\partial_{x_2}\hat{u}_2)|_{x_2=\pi/2} = (\partial_{x_2}\hat{u}_1 + i\xi\hat{u}_2)|_{x_2=\pi/2} = 0. \tag{6.9}$$

The eigenfunctions of $\hat{A}_4(\xi)$ are obtained from the eigenfunctions of (6.7)–(6.9) by symmetric continuation (see (6.3)) to the full interval J. One only has to remember to exclude (in view of (6.4)) the parasitic eigenvalue $\lambda = 2\xi^2$ and the corresponding eigenspace $\hat{u}_1 \equiv const$, $\hat{u}_2 \equiv 0$.

For a given $\xi \in \mathbb{R}$ the problem (6.7)–(6.9) is an eigenvalue problem for a self-adjoint ordinary differential operator with constant coefficients, and its analysis is performed by elementary techniques. Below we list the main facts resulting from this analysis.

The operator $\hat{A}_4(\xi)$ has a discrete positive spectrum $0 < \lambda_1(\xi) \le \lambda_2(\xi) \le \dots$ accumulating to $+\infty$. Each eigenvalue $\lambda_k(\xi)$ depends continuously on ξ, and there exists a positive constant C such that $\lambda_1(\xi) \ge C(1+|\xi|)^2$ uniformly over all $\xi \in \mathbb{R}$. This estimate for eigenvalues leads to an estimate for the resolvent. Let λ be a fixed complex number such that $\lambda \notin \left[\min_{\xi\in\mathbb{R}} \lambda_1(\xi), +\infty\right)$, and let I be the identity operator. Then

$$\text{(6.10)} \quad \left\|(\hat{A}_4(\xi) - \lambda I)^{-1}\right\|_{\hat{H}_4 \to W_2^l(J,\mathbb{C}^2)} \le C_\lambda (1+|\xi|)^{l-2}, \qquad \forall \xi \in \mathbb{R}, \quad l = 0, 1, 2,$$

with some positive constant C_λ.

The relation between the operator $\breve{A}_4$ and the family of operators $\hat{A}_4(\xi)$ is established by the following

Lemma 6.2. *We have* $\sigma(\breve{A}_4) = \left[\min_{\xi\in\mathbb{R}} \lambda_1(\xi), +\infty\right)$.

In other words, Lemma 6.2 states that the spectrum of the operator $\breve{A}_4$ coincides with the superposition of the spectra of the operators $\hat{A}_4(\xi)$ when ξ runs through $\mathbb{R}$.

Proof of Lemma 6.2. Take an arbitrary $\lambda \notin \left[\min_{\xi\in\mathbb{R}} \lambda_1(\xi), +\infty\right)$. Then $(\breve{A}_4 - \lambda I)^{-1}$ can be constructed explicitly as a Fourier integral of $(\hat{A}_4(\xi) - \lambda I)^{-1}$. The estimates (6.10) guarantee that the resolvent $(\breve{A}_4 - \lambda I)^{-1}$ constructed in such a way maps $\breve{H}_4$ to $D(\breve{A}_4)$. This proves that $\sigma(\breve{A}_4) \subset \left[\min_{\xi\in\mathbb{R}} \lambda_1(\xi), +\infty\right)$.

Take now an arbitrary $\lambda \in \left[\min_{\xi\in\mathbb{R}} \lambda_1(\xi), +\infty\right)$. Then one can easily construct a sequence $\mathbf{u}_n \in D(\breve{A}_4)$ such that $\|\mathbf{u}_n\|_{\breve{H}_4} = 1$ and $\|\breve{A}_4\mathbf{u}_n - \lambda\mathbf{u}_n\|_{\breve{H}_4} \to 0$ as $n \to \infty$. This proves that $\left[\min_{\xi\in\mathbb{R}} \lambda_1(\xi), +\infty\right) \subset \sigma(\breve{A}_4)$. $\square$

An alternative way of proving Lemma 6.2 is by the use of the notion of a Neumann integral of Hilbert spaces, see [2] Chapter 7 and [19] Section XIII.16.

6.3. Analysis of the dispersion equation. With Lemma 6.2 the proof of Lemma 4.2 is reduced to the proof of

$$\text{(6.11)} \qquad \lambda_+ < \min_{\xi\in\mathbb{R}} \lambda_1(\xi).$$

In other words, we have reduced the proof of Lemma 4.2 to the proof of an inequality for the eigenvalue problem (6.7)–(6.9).

The problem (6.7)–(6.9) is an eigenvalue problem for a self-adjoint ordinary differential operator with constant coefficients, so its characteristic determinant is expressed explicitly through elementary functions. The resulting transcendent equation for the eigenvalues $\lambda_k(\xi)$ has been studied in great detail. The analysis of (6.7)–(6.9) was initiated by Rayleigh [18] and H. Lamb [14], [15], and followed by numerous works of other authors. The graphs of the functions $\lambda_k(\xi)$ are known as the *Rayleigh-Lamb dispersion curves*, and their basic properties are described in textbooks on elasticity.

The number $\min_{\xi\in\mathbb{R}}\lambda_1(\xi)$ is called the *threshold of the second symmetric Rayleigh-Lamb mode*. It is known [13], p. 125 that the function $\lambda_1(\xi)$ has the highly unusual property of being non-convex and that $\min_{\xi\in\mathbb{R}}\lambda_1(\xi)\ne\lambda_1(0)$. Consequently, there is no simple analytic expression for this threshold. A numerical evaluation [13], p.266 gives

$$\min_{\xi\in\mathbb{R}}\lambda_1(\xi)\approx 1.374^2\approx 1.888. \tag{6.12}$$

Comparing (2.10) and (6.12) we arrive at (6.11). Note that the margin between the left- and right-hand sides of (6.11) is only 1%, which shows how fortunate we were with our choice (5.1), (5.5) of the test vector-function. This completes the proof of Lemma 4.2, and with it the proof of Theorem 2.1. $\square$

7. Appendix

The proof of (6.11) given above relies on the approximate numerical estimates (2.10), (6.12), and may seem to be insufficiently rigorous. Therefore in this appendix we outline a more rigorous proof of (6.11).

We have $\lambda_+<1.87$; this fact can be derived from the analytic expression (2.10) using known rational appoximations for the number π. We also know that $\lambda_1(\xi)$ is a continuous even function. Therefore in order to prove (6.11) it is sufficient to prove

$$1.87\le\inf_{\xi>0}\lambda_1(\xi). \tag{7.1}$$

Lemma 7.1. *For any $\xi\in\mathbb{R}$ we have $\lambda_1(\xi)\ge\min\{2\xi^2,2\}$.*

Proof of Lemma 7.1. By the variational principle

$$\lambda_1(\xi)\ge\inf_{0\not\equiv\mathbf{\hat u}\in C^\infty,\ \hat u_2(0)=0}\frac{\int_0^{\pi/2}\left(2\xi^2\,|\hat u_1|^2+2\,|\partial_{x_2}\hat u_2|^2+|\partial_{x_2}\hat u_1+i\xi\hat u_2|^2\right)dx_2}{\int_0^{\pi/2}\left(|\hat u_1|^2+|\hat u_2|^2\right)dx_2}$$

(cf. (6.6)); here we have an inequality instead of an equality because we have disregarded the condition (6.4). Dropping the term $|\partial_{x_2}\hat u_1+i\xi\hat u_2|^2$ in the above inequality we get

$$\lambda_1(\xi) \geq \min\left\{ \inf_{0\not\equiv u_1\in C^\infty} \frac{\int_0^{\pi/2}\left(2\xi^2\left|\hat{u}_1\right|^2\right)dx_2}{\int_0^{\pi/2}\left|\hat{u}_1\right|^2 dx_2}, \inf_{0\not\equiv \hat{u}_2\in C^\infty,\ \hat{u}_2(0)=0} \frac{\int_0^{\pi/2}\left(2\left|\partial_{x_2}\hat{u}_2\right|^2\right)dx_2}{\int_0^{\pi/2}\left|\hat{u}_2\right|^2 dx_2} \right\}$$

which yields the required result. □

With Lemma 7.1 the proof of (7.1) is reduced to the proof of

$$1.87 \leq \inf_{0<\xi<\sqrt{0.935}} \lambda_1(\xi). \tag{7.2}$$

Further on we assume $\xi \neq 0$.

The two linearly independent solutions of (6.7) satisfying the boundary conditions (6.8) are

$$\begin{pmatrix} \xi\cos\alpha x_2 \\ i\alpha\sin\alpha x_2 \end{pmatrix}, \quad \begin{pmatrix} i\cos\beta x_2 \\ \xi\beta^{-1}\sin\beta x_2 \end{pmatrix}, \tag{7.3}$$

where

$$\alpha = \alpha(\xi,\lambda) = \sqrt{\frac{\lambda}{2}-\xi^2}, \quad \beta = \beta(\xi,\lambda) = \sqrt{\lambda-\xi^2}, \tag{7.4}$$

and $\beta^{-1}\sin\beta x_2 := x_2$ when $\beta = 0$. The solutions (7.3) are entire functions in all the variables, that is, they are analytic with respect to $(\xi,\lambda,x_2) \in \mathbb{C}^3$. The choice of signs of the square roots (7.4) does not matter because the power series expansions of (7.3) contain only even powers of α and β.

Substituting the solutions (7.3) into the boundary conditions (6.9) we obtain the characteristic determinant

$$D = D(\xi,\lambda) = \alpha^4\beta^{-1}\cos\frac{\pi\alpha}{2}\sin\frac{\pi\beta}{2} + \alpha\xi^2\cos\frac{\pi\beta}{2}\sin\frac{\pi\alpha}{2} \tag{7.5}$$

of the eigenvalue problem (6.7)–(6.9); see also [13], Chapter 4, Sections 2,3. Thus, λ is an eigenvalue of (6.7)–(6.9) if and only if $D(\xi,\lambda) = 0$. Moreover, it is easy to see that the spectral multiplicity of the eigenvalue coincides with the order of the zero of the function D with respect to λ.

As the spectra of the problem (6.7)–(6.9) and of the operator $\hat{A}_4(\xi)$ differ by one eigenvalue $\lambda = 2\xi^2$, the function

$$D_1 = D_1(\xi,\lambda) = \alpha^{-2}D \tag{7.6}$$

is the characteristic determinant of the operator $\hat{A}_4(\xi)$. Therefore $\lambda_1(\xi)$ is the smallest λ-root of the equation $D_1(\xi,\lambda) = 0$.

As $\lambda_1(0) = 2 > 1.87$, in order to prove (7.2) it is sufficient to prove

$$D_1(\xi, 1.87) \neq 0, \qquad \forall \xi \in (0,\sqrt{0.935}) \tag{7.7}$$

(this would mean that the graph $\lambda = \lambda_1(\xi)$, $0 < \xi < \sqrt{0.935}$, does not intersect the horizontal line $\lambda = 1.87$). Dealing with (7.7) will be much easier than with (7.2) because we will no longer have to work with an implicitly defined function.

Let us denote for brevity $z = \xi^2$, $F(z) = D_1(\sqrt{z}, 1.87)$. Then (7.7) is rewritten as

$$F(z) \neq 0, \qquad \forall z \in (0, 0.935), \tag{7.8}$$

with

$$F(z) = \frac{0.935 - z}{\sqrt{1.87 - z}} \cos \frac{\pi\sqrt{0.935 - z}}{2} \sin \frac{\pi\sqrt{1.87 - z}}{2} + \frac{z}{\sqrt{0.935 - z}} \cos \frac{\pi\sqrt{1.87 - z}}{2} \sin \frac{\pi\sqrt{0.935 - z}}{2} \tag{7.9}$$

(see (7.4)–(7.6)). Thus, we have to establish (7.8) for the (entire) function F given by the explicit formula (7.9). The graph of F is plotted on Figure 1.

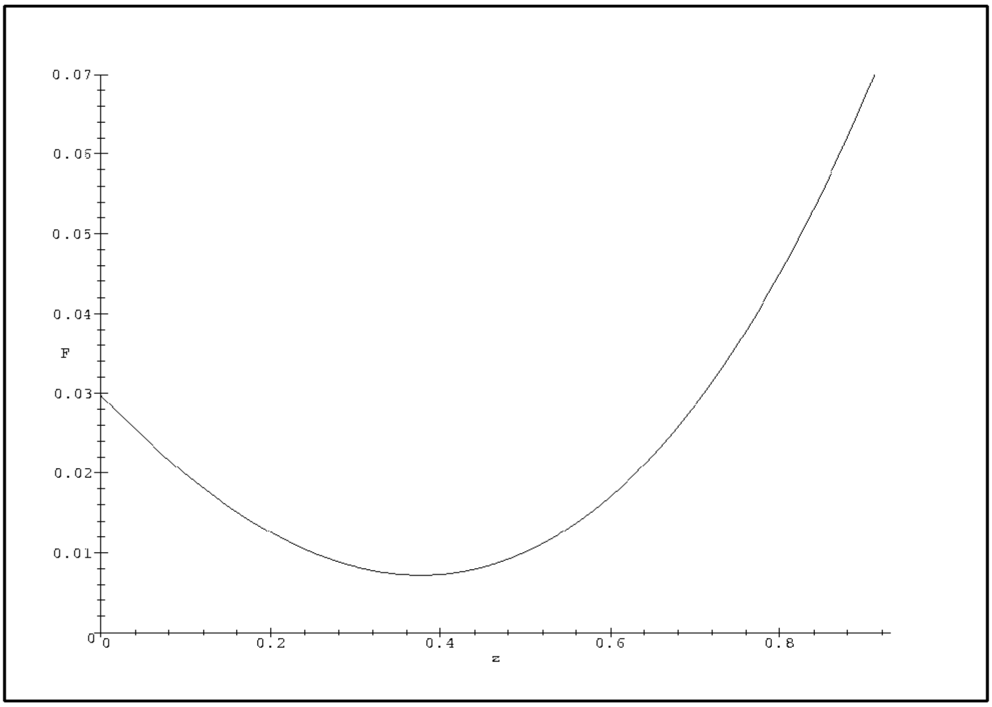

Figure 1

This graph does not really constitute a mathematical proof of (7.8), but looking at it one can suggest ways of providing a rigorous proof. The most straightforward (though rather time-consuming) way is as follows. Examination of (7.9) gives the rough estimate $|F'(z)| \leq 100$, $\forall z \in (0, 0.935)$, so in order to prove (7.8) it is sufficient to show that

$$F(10^{-5}k) \geq 10^{-3}, \qquad \forall k = 1, 2, \ldots, 93499. \tag{7.10}$$

Using the power series expansions for cos and sin one can prove each of the 93499 inequalities (7.10) by performing a finite number of arithmetic operations with rational numbers.

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