ON A COUPLED KURAMOTO-SIVASHINSKY AND GINZBURG-LANDAU TYPE MODEL FOR THE MARANGONI CONVECTION

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Abstract. The surface tension driven Marangoni convection is an interesting pattern formation system. The "primitive" governing equations are too complicated to be investigated analytically. In this paper, the authors consider a simplified model for this system. This simplified model is in the form of coupled Kuramoto-Sivashinsky and Ginzburg-Landau type partial differential equations. The authors prove the existence and uniqueness of global solutions of this simplified mathematical model, under the condition that the Marangoni number $Ma > Ma_c + \frac{k}{2^5}$, where Ma_c is the critical Marangoni number at which the trivial stationary state becomes linearly unstable, and k is a positive constant related to other system parameters. The authors use the contraction mapping principle in the proof as the usual semigroup method does not apply directly to this system. This work sets the foundation for further study of this model.

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§1. Introduction

Like the thermally driven Benard convection, the surface tension driven Marangoni convection is an interesting pattern formation nonlinear system. A typical setup for the Marangoni convection with mass transfer from liquid phase to gas phase is a liquid layer resting on a rigid surface, with a free deformable upper interface contacting an infinite layer of gas. However, the governing equations for the Marangoni convection, i.e. Navier-Stokes equations and mass (surfactant) transfer equation, are not amenable to analysis at the present time, except for numerical simulation. Very few works have been done on nonlinear Marangoni convection.

A simplified model, aimed to capture important nonlinear feature yet more amenable to analysis, has been derived recently by Golovin et al. [1994]:

$$A_t = A + A_{xx} - |A|^2 A + Ah, \tag{1-1}$$

$$h_t = -h_{xx} - h_{xxxx} + \alpha |A|^2_{xx}, \tag{1-2}$$

where $A(x,t)$ is the complex amplitude for the Marangoni convective mode, $h(x,t)$ is the interface deformation (real function). The positive constant $\alpha = \frac{k}{(Ma - Ma_c)^{\frac{1}{5}}}$, where Ma is the Marangoni number measuring the gradient (derivative) of surface tension with respect to surfactant concentration, Ma_c is the critical Marangoni number at which the trivial stationary state becomes linearly unstable, and $k > 0$ is a constant related to other system parameters. This simplified model presumably models the interaction (for $Ma > Ma_c$) of the two most important unstable modes in the Marangoni convection: the short-scale convective mode ($A(x,t)$), driven by surface tension gradient alone, and the long-scale surface deformation mode ($h(x,t)$), driven by gravity and capillary forces.

The equation (1-1) without the interaction term Ah is the well-known Ginzburg-Landau equation (cf. Hocking et al [1972], Doering et al [1988], and references therein), while the equation (1-2) without the term $|A|^2_{xx}$ is the "linearized" version of the so-called Kuramoto-Sivanshinsky equation(cf. Nicolaenko et al [1985] and references therein). Both the Ginzburg-

Landau equation and Kuramoto-Sivanshinsky equation are interesting prototypical nonlinear partial differential equations, and have been extensively investigated as model examples of infinite dimensional dynamical systems.

Kazhdan et al [1995] has done numerical simulation, using a "de-aliased" pseudo-spectral method, of this coupled system of Kuramoto-Sivanshinsky and Ginzburg-Landau type equations (hereafter, $KS-GL$ system), under periodic boundary conditions

$$A(x,t), \quad h(x,t) \quad are \quad l-periodic, \tag{1-3}$$

and initial conditions

$$A(x,0)=A_0(x), \quad h(x,0)=h_0(x). \tag{1-4}$$

In this paper, we consider more analytic aspect of the model, i.e., we discuss the existence and uniqueness of global solutions of this coupled system. This will set the foundation for further study of this simplified model of the Marangoni convection. In Section 2, we will prove the local existence and uniqueness by the contraction mapping principle. In Section 3, we will derive necessary energy estimates to complete the existence proof of global solutions.

§2. Local Existence and Uniqueness

We denote $L^2_{per}=L^2_{per}[0,l]$ and $H^k_{per}=H^k_{per}[0,l], k=1,2,3,\cdots$ the usual Sobolev spaces of $l-$periodic functions with norms $||\cdot||_2$ and $||\cdot||_{H^k}$, respectively, and with the usual L^2-inner product.

We first comment that the usual semigroup method for evolutionary differential equations does not directly apply for the coupled system of Kuramoto-Sivanshinsky and Ginzburg-Landau type equations. In fact we can rewrite the equations (1-1) and (1-2) in the following form

$$\frac{du}{dt}+\mathcal{L}u=N(u), \tag{1-5}$$

where

$$u=\begin{pmatrix} A \\ h \end{pmatrix}, \tag{1-6}$$

$$\mathcal{L} = \begin{pmatrix} -\partial_{xx} & 0 \\ 0 & \partial_{xxxx} \end{pmatrix}, \tag{1-7}$$

$$N(u) = \begin{pmatrix} A - |A|^2 A + Ah \\ -h_{xx} + \alpha |A|^2_{xx} \end{pmatrix}, \tag{1-8}$$

We consider this problem in the Sobolev space

$$X = L^2_{per} \bigotimes L^2_{per}, \tag{1-9}$$

The domain of definition of the linear differential operator $\mathcal{L}$ is

$$D(\mathcal{L}) = H^2_{per} \bigotimes H^4_{per}. \tag{1-10}$$

It is well-known that both linear operators $-\partial_{xx}$ and ∂_{xxxx} are sectorial operators in L^2_{per}, i.e. ∂_{xx} and $-\partial_{xxxx}$ each generates an analytic semigroup of linear bounded operators; see Henry [1981, p.18] for definition of a sectorial operator. It is easy to see that the "diagonal" linear operator $\mathcal{L}$ above is also sectorial in $X = L^2_{per} \bigotimes L^2_{per}$. Hence we can define the fractional powers $\mathcal{L}^\gamma$ and $X^\gamma = D(\mathcal{L}^\gamma)$ for $\gamma \in [0, 1)$. However, we can not find a $\gamma \in [0, 1)$ such that the nonlinear operator defined above

$$N : X^\gamma \to X = L^2_{per} \bigotimes L^2_{per} \tag{1-11}$$

is locally Lipschitzian, and N maps bounded sets in X^γ into bounded sets in $L^2_{per} \bigotimes L^2_{per}$. Therefore, we can not use Theorem 3.3.3 in Henry [1981] to conclude the local existence.

In the rest of this section, We use contraction mapping principle to prove the local existence and uniqueness of (1-1),(1-2) with periodic boundary conditions. In the rest of this paper, $C, c, c', c'', \hat{c}, \tilde{c}$ denote various positive constants, perhaps depdending on T (of the finite time interval $[0, T]$). We will frequently use the following three inequalities.

$Young's\ inequality$ (Renardy and Rogers [1993, p.180]):

$$AB \le \frac{1}{2}A^2 + \frac{1}{2}B^2,$$

or

$$AB \leq \epsilon A^2 + \frac{1}{\epsilon} B^2,$$

where A, B are non-negative real numbers and $\epsilon > 0$.

Cauchy – Schwarz inequality (Renardy and Rogers [1993, p.183]):

$$\int |fg| dx \leq ||f|| \cdot ||g||,$$

for $f, g \in L^2(0,1)$.

Gronwall inequality (Temam [1988, p.89]): If an integrable function $y(t)$ satisfies that

$$\frac{dy}{dt} \leq Ay + B, \quad t \geq 0,$$

for some constants A, B with $A \neq 0$, then

$$y(t) \leq [y(0) + \frac{B}{A}] e^{At} - \frac{B}{A}, \quad t > 0.$$

Let

$$\sum_T = \{(A,h) | A \in C([0,T]; H^2_{per}) \cap L^2(0,T; H^3_{per}), A_t \in C(0,T; L^2_{per}) \cap L^2(0,T; H^1_{per});$$

$$h \in C([0,T]; H^2_{per} \cup L^2(0,T; H^4_{per}), h_t \in L^2(0,T; L^2_{per})\}.$$

For $(\tilde{A}; \tilde{h}) \in \sum_T$, we consider the following auxiliary problem for A and h respectively,

$$A_t - A_{xx} = \tilde{A} - |\tilde{A}|^2 \tilde{A} + \tilde{A}\tilde{h} \doteq f, \tag{2-1}$$

$$A(x,0) = A_0(x) \in H^2_{per}, \tag{2-2}$$

and

$$h_t + h_{xxxx} = -\tilde{h}_{xx} + \alpha |\tilde{A}|^2_{xx} \doteq g, \tag{2-3}$$

$$h(x,t) = h_0(x) \in H^2_{per}. \tag{2-4}$$

Then, we denote by F the mapping from $(\tilde{A}, \tilde{h})$ to (A, h). By the well-known results for linear parabolic equations (Lions and Magenes [1972]), we have

Lemma 1. *For any $(\tilde{A}, \tilde{h}) \in \sum_T$, (2-1)-(2-4) admit uniquely a pair of solution $(A, h) \in \sum_T$. Moreover, the following energy estimates hold for $0 \le t \le T$.*

$$\|A\|^2 \le e^t(\|A_0\|^2 + \int_0^t \|f\|^2 d\tau), \tag{2-5}$$

$$\|A_x\|^2 \le \|A_{0x}\|^2 + \int_0^t \|f\|^2 d\tau, \tag{2-6}$$

$$\|A_{xx}\|^2 \le e^t(\|A_{0xx}\|^2 + \int_0^t \|f_x\|^2 d\tau), \tag{2-7}$$

$$\|h\|^2 \le e^t\|h_0\|^2 \le e^t(\|h_0\|^2 + \int_0^t \|g\|^2 d\tau), \tag{2-8}$$

$$\|h_x\|^2 \le e^t(\|h_{ox}\|^2 + \int_o^t \|g\|^2 d\tau), \tag{2-9}$$

$$\|h_{xx}\|^2 \le \|h_{0xx}\|^2 + \int \|g\|^2 d\tau. \tag{2-10}$$

The straight application of energy method and the Gronwall inequality yields inequalities (2-5)-(2-10). We omit the details here.

Let

$$\sum_T(C_1, C_2)\{(A,h) \in \sum_T |\|A\|_{H^2} \le \|A_{0x}\|^2 + C_1, \|A_t\| \le \|A_{0x}\|^2 + C_1; \|h\|_{H^2} \le \|h_{0xx}\|^2 + C_2\},$$

we have

Lemma 2. *There exist constants C_1, C_2 and a positive constant t^* depending only on $\|A_0\|_{H^2}, \|h\|_{H^2}$ such that the mapping F maps $\sum_{t^*}(C_1, C_2)$ into itself. Moreover, the mapping F is a contraction mapping from $\sum_{t^*}(C_1, C_2)$ into itself.*

Proof: For $(\tilde{A}, \tilde{h}) \in \sum_{t*}(C_1, C_2)$, the estimates of (2-5)-(2-10) hold, and

$$\|A\|_{H^2}^2 \le e^t(\|A_0\|^2 + \|A_{0xx}\|^2 + \int_0^t (\|f\|^2 + \|f_x\|^2) d\tau) + \|A_{0x}\|^2 + \int_0^t \|f\|^2 d\tau, \tag{2-11}$$

$$\|h\|_{H^2}^2 \le e^t(\|h_0\|^2 + \|h_{0x}\|^2 + 2\int_0^t \|g\|^2 d\tau) + \|h_{0xx}\|^2 + \int_0^t \|g\|^2 d\tau, \qquad \text{(2-12.)}$$

Using the Sobolev imbedding theorem(see, e.g. Renardy and Rogers [1993, p.215])

$$H^k_{per} \hookrightarrow L^\infty_{per}$$

and the fact that H^k is a Banach algebra for $k > \frac{1}{2}$, i.e. $uv \in H^k$ and $||uv||_{H^k} \leq ||u||_{H^k}||v||_{H^k}$ whenever $u, v \in H^k$, we can verify that F defined above satisfied

$$||f||^2 \leq ||\tilde{A}||^2 + ||\tilde{A}||^4_{H^1}||\tilde{A}||^2 + ||\tilde{A}||^2_{H^1}||h||^2 \leq C_3,$$

$$||f_x||^2 \leq ||\tilde{A}_x||^2 + ||\tilde{A}||^4_{H^2}||\tilde{A}||^2_{H^1} + ||\tilde{A}||^2_{H^2}||h||^2_{H^2} \leq C_4,$$

$$||g||^2 \leq ||h_{xx}||^2 + ||A||^2_{H^2} \leq C_5.$$

Now, we choose t_1 such that for $t \leq t_1$

$$e^t(||A_0||^2 + ||A_{0xx}||^2 + (C_3 + C_4)t) + C_3 t \leq C_1,$$

$$e^t(||h_0||^2 + ||h_{0x}||^2 + 2C_5 t) + tC_5 \leq C_2,$$

so,

$$(A, h) \in \sum_{t_1}(C_1, C_2).$$

Let $(A_1, h_1), (A_2, h_2)$ be two solutions corresponding to $(\tilde{A}_1, \tilde{h}_1)$ and $(\tilde{A}_2, \tilde{h}_2)$, respectively. Let

$$\tilde{A} = \tilde{A}_1 - \tilde{A}_2, \;\; \tilde{h} = \tilde{h}_1 - \tilde{h}_2; \;\; A = A_1 - A_2, \;\; h = h_1 - h_2.$$

Then A and h satisfy that

$$A_t - A_{xx} = \tilde{A} - |\tilde{A}_1|^2\tilde{A} + \tilde{A}_1\bar{\tilde{A}}\tilde{A}_2 + |\tilde{A}_2||^2\tilde{A} + \tilde{A}\tilde{h}_1 + \tilde{A}_2 h, \tag{2-13}$$

$$h_t + h_{xxxx} = -\tilde{h}_{xx} + \alpha(\tilde{A}_1\bar{\tilde{A}}_{xx} + \alpha(\tilde{A}\tilde{A}_2)_{xx}. \tag{2-14}$$

Applying the energy method as before, we obtain

$$||A||^2_{H^2} \leq C_6\int_0^t(||\tilde{A}||^2_{H^2} + ||h||^2_{H^2})d\tau,$$

$$\|h\|_{H^2}^2 \leq C_7 \int_0^t (\|\tilde{h}\|^2 + \|A\|_{H^2}^2) d\tau.$$

It turns out that

$$Sup_{0\leq\tau\leq t}(\|A\|_{H^2}^2 + \|h\|_{H^2}^2) \leq C_8 t Sup_{0\leq\tau\leq t}(\|\tilde{A}\|_{H^2}^2 + \|\tilde{h}\|_{H^2}^2.$$

We choose t_2 small enough so that for $t \leq t_2$

$$C_8 t < \frac{1}{2}.$$

If we choose $t^* = min(t_1, t_2)$, then Lemma 2 is proved. By the contraction mapping principle (Renardy and Rogers [1993, p.336]), we obtain the following theorem

Local Existence Theorem. *For initial data $A(x,0) \in H^2_{per}$ and $h(x,0) \in H^2_{per}$, there exists a unique (local) solution of the coupled initial-boundary value problem (1-1)-(1-2)-(1-3)-(1-4) for $0 < t < t^*$, where $t^* > 0$ depends only on initial data. Moreover,*

$$A(t) \in C([0,t^*]; H^2_{per}) \cap C([0,t^*); L^2_{per}) \tag{2-15}$$

$$h(t) \in C([0,t^*]; H^2_{per}) \cap L^2([0,t^*]; H^4_{per}). \tag{2-16}$$

§3. Global Existence Result

In this section, we show that, under the condition $0 < \alpha < 2$, the above local solution can be extended to any time $t > 0$ and hence we obtain global existence and uniqueness. To achieve this we need to show that $||A||_{H^2}$ and $||h||_{H^2}$ are bounded (a priori) on *any* finite time interval $[0,T]$ under the condition that $0 < \alpha < 2$.

Recall that

$$A_t = A + A_{xx} - |A|^2 A + Ah, \tag{3-1}$$

$$h_t = -h_{xx} - h_{xxxx} + \alpha |A|^2_{xx}. \tag{3-2}$$

First we differentiate $|A|^2$ and integrate from 0 to l to obtain:

$$\frac{d}{dt}\|A\|_2^2 = \int_0^l (A_t\bar{A} + \bar{A}_t A)dx \tag{3-3}$$

$$= 2Re\int_0^l \bar{A}(A + A_{xx} - |A|^2A + Ah)dx$$

$$= 2\|A\|_2^2 - 2\|A_x\|_2^2 - 2\|A\|_4^4 + 2\int_0^l |A|^2hdx.$$

Similarly, doing the same thing to $|h|^2$ we have

$$\frac{d}{dt}\|h\|_2^2 = \int_0^l 2hh_tdx = 2\int_0^l (-hh_{xx} - hh_{xxxx} + \alpha|A|_{xx}^2h)dx \tag{3-4}$$

$$= 2\|h_x\|_2^2 - 2\|h_{xx}\|_2^2 + 2\alpha\int_0^l |A|^2h_{xx}dx.$$

Adding (3-3) and (3-4) and writing $\alpha = 2-\epsilon$ with $\epsilon > 0$ we get:

$$\frac{d}{dt}(\|A\|_2^2 + \|h\|_2^2) = 2\|A\|_2^2 - 2\|A_x\|_2^2 - 2\|A\|_4^4 \tag{3-5}$$

$$+2\int_0^l |A|^2hdx + 2\|h_x\|_2^2 - 2\|h_{xx}\|_2^2 + (4-2\epsilon)\int_0^l |A|^2h_{xx}dx$$

$$\leq 2\|A\|_2^2 - 2\|A_x\|_2^2 - 2\|A\|_4^4 + \epsilon\|A\|_4^4 + \frac{1}{\epsilon}\|h\|_2^2$$

$$+2\|h_x\|_2^2 - 2\|h_{xx}\|_2^2 + (2-\epsilon)(\|A\|_4^4 + \|h_{xx}\|_2^2)$$

$$= 2\|A\|_2^2 - 2\|A_x\|_2^2 + \frac{1}{\epsilon}\|h\|_2^2 + 2\|h_x\|_2^2 - \epsilon\|h_{xx}\|_2^2.$$

By the Young's inequality, we have

$$2\|h_x\|_2^2 = -2\int_0^l hh_{xx}dx \leq \epsilon\|h_{xx}\|_2^2 + \frac{1}{\epsilon}\|h\|_2^2. \tag{3-6}$$

Thus, the estimate (3-5) becomes:

$$\frac{d}{dt}(\|A\|_2^2 + \|h\|_2^2) \leq 2\|A\|_2^2 - 2\|A_x\|_2^2 + \frac{1}{\epsilon}\|h\|_2^2 + \epsilon\|h_{xx}\|_2^2 + \frac{1}{\epsilon}\|h\|_2^2 - \epsilon\|h_{xx}\|_2^2$$

$$= 2\|A\|_2^2 - 2\|A_x\|_2^2 + 2\frac{1}{\epsilon}\|h\|_2^2$$

$$\leq c_0(\|A\|_2^2 + \|h\|_2^2). \tag{3-7}$$

We integrate (3-7) from 0 to t and apply the Gronwall inequality to get

$$\|A\|_2^2 + \|h\|_2^2 \leq (\|A_0\|_2^2 + \|h_0\|_2^2)\exp(c_0 t). \tag{3-8}$$

Therefore, the L^2 norm of the solution is bounded on any finite interval $[0, T]$.

Next we differentiate $\|A_x\|_2^2$ with respect to the t variable:

$$\frac{d}{dt}\|A_x\|_2^2 = 2Re\int_0^l \bar{A}_x A_{xt}dx = -2Re\int_0^l \bar{A}_{xx}A_t dx$$

$$= -2Re\int_0^l \bar{A}_{xx}(A + A_{xx} - |A|^2A + Ah)dx$$

$$= -2Re\int_0^l (\bar{A}_{xx}A + |A_{xx}|^2 - |A|^2A\bar{A}_{xx} + Ah\bar{A}_{xx})dx$$

$$= 2\int_0^l |A_x|^2dx - 2\|A_{xx}\|_2^2 + 2Re\int_0^l |A|^2A\bar{A}_{xx}dx + 2Re\int_0^l Ah\bar{A}_{xx}dx$$

$$\leq 2\|A_x\|_2^2 - 2\|A_{xx}\|_2^2 + 2\int_0^l |A|^3|A_{xx}|dx + 2\int_0^l |Ah||A_{xx}|dx$$

$$\leq 2\|A_x\|_2^2 - 2\|A_{xx}\|_2^2 + 2\|A\|_6^6 + \frac{1}{2}\|A_{xx}\|_2^2 + 2\int_0^l |Ah|^2dx + \frac{1}{2}\|A_{xx}\|_2^2$$

$$= 2\|A_x\|_2^2 - \|A_{xx}\|_2^2 + 2\|A\|_6^6 + \|A\|_4^4 + \|h\|_4^4. \tag{3-9}$$

By the Gagliardo-Nirenberg inequalities (cf. Nirenberg [1959] or Henry [1981]) and the fact (proved above) that $\|A\|_2^2 + \|h\|_2^2$ is bounded on any finite interval $[0, T]$, we can estimate the last three terms in (3-9) above on any finite interval $[0, T]$:

$$\|A\|_6^6 \leq \lambda_1\|A_x\|_2^2\|A\|_2^4 \leq c_1\|A_x\|_2^2, \tag{3-10}$$

$$\|A\|_4^4 + \|h\|_4^4 \leq \lambda_2(\|A_x\|_2\|A\|_2^3 + \|h_x\|_2\|h\|_2^3) \leq c_2(\|A_x\|_2 + \|h_x\|_2), \tag{3-11}$$

where λ_1 and λ_2 are positive constants in the Gagliardo-Nirenberg inequalities, and the coefficient c_1, c_2 depends on the Gagliardo-Nirenberg constant and T. Inserting (3-10) and (3-11) in (3-9) we obtain

$$\frac{d}{dt}\|A_x\|_2^2 \le 2\|A_x\|_2^2 - \|A_{xx}\|_2^2 + c_1\|A_x\|_2^2 + c_2(\|A_x\|_2 + \|h_x\|_2) \tag{3-12}$$
$$\le c_3 + c_4\|A_x\|_2^2 + c_5\|h_x\|_2^2 - \|A_{xx}\|_2^2.$$

Now we differentiate $\|h_x\|_2^2$ with respect to the t variable and integrate by parts to obtain

$$\frac{d}{dt}\|h_x\|_2^2 = 2\int_0^l h_{xt}h_x dx = -2\int_0^l h_{xx}h_t dx \tag{3-13}$$
$$= -2\int_0^l h_{xx}(-h_{xx} - h_{xxxx} + \alpha|A|^2_{xx})dx$$
$$= 2\|h_{xx}\|_2^2 - 2\|h_{xxx}\|_2^2 + 2\alpha\int_0^l h_{xxx}|A|^2_x dx$$
$$\le 2\|h_{xx}\|_2^2 - 2\|h_{xxx}\|_2^2 + \|h_{xxx}\|_2^2 + \alpha^2\int_0^l |AA_x|^2 dx$$
$$= 2\|h_{xx}\|_2^2 - \|h_{xxx}\|_2^2 + \alpha^2\|A_x\|_\infty^2\|A\|^2.$$

Putting the following Gagliardo-Nirenberg estimates on $\|A_x\|_\infty$ and $\|h_{xx}\|_2^2$

$$\|A_x\|_\infty^2 \le \lambda_3\|A_{xx}\|_2^s\|A\|_2^r, \tag{3-14}$$

$$\|h_{xx}\|_2^2 \le \lambda_4\|h_{xxx}\|_2^{s'}\|h\|_2^{r'} \tag{3-15}$$

where $0 < s, s' < 2, 0 < r, r' < 2, s + r = 2, s' + r' = 2$ and where λ_3 and λ_4 are positive constants in the Gagliardo-Nirenberg inequalities, into (3-13) and noting that $\|A\|_2$ is bounded on any finite interval $[0, T]$, we have the following estimate

$$\frac{d}{dt}\|h_x\|_2^2 \le 2\|h_{xx}\|_2^2 - \|h_{xxx}\|_2^2 + \alpha^2\|A_x\|_\infty^2\|A\|_2^2$$
$$\le 2\lambda_4\|h_{xxx}\|_2^{s'}\|h\|_2^{r'} - \|h_{xxx}\|_2^2 + \alpha^2\lambda_3\|A_{xx}\|_2^s\|A\|_2^{2+r}$$

$$\leq \|h_{xxx}\|_2^2 + c_6 - \|h_{xxx}\|_2^2 + c_5\|A_{xx}\|_2^s + c_7 \leq \hat{c} + \|A_{xx}\|_2^2 \tag{3-16}$$

The coefficients c_5, c_6, c_7 in the above inequality depend on Gagliardo-Nirenberg constant and T. The estimates in the last few lines above are based on the fact that $c_5\|A_{xx}\|_2^s \leq \|A_{xx}\|_2^2 + \tilde{c}$ and $2\lambda_4\|h_{xxx}\|_2^{s'}\|h\|_2^{r'} \leq \|h_{xxx}\|_2^2 + c_6$ (from the Cauchy-Schwarz inequality).

Adding (3-12) and (3-16) we have the following

$$\frac{d}{dt}(\|A_x\|_2^2 + \|h_x\|_2^2) \leq c_3 + c_4\|A_x\|_2^2 + c_5\|h_x\|_2^2 - \|A_{xx}\|_2^2 + \hat{c} + \|A_{xx}\|_2^2 \tag{3-17}$$

$$\leq c_8 + c_9(\|A_x\|_2^2 + \|h_x\|_2^2).$$

Now we integrate (3-17) from 0 to t and apply the Gronwall inequality to obtain

$$\|A_x\|_2^2 + \|h_x\|_2^2 \leq (\|\frac{\partial}{\partial x}A_0\|_2^2 + \|\frac{\partial}{\partial x}h_0\|_2^2 + \frac{c_8}{c_9}T)\exp(c_9 t). \tag{3-18}$$

By combining (3-8) and (3-18) we conclude that $\|A\|_{H^1}$ and $\|h\|_{H^1}$ are both bounded on any finite interval $[0, T]$.

To complete the proof that $\|A\|_{H^2}$ and $\|h\|_{H^2}$ are both bounded on any finite interval $[0, T]$, which will imply the existence of global solutions, we now only need to prove that $\|A_{xx}\|_2^2$ and $\|h_{xx}\|_2^2$ are bounded on $[0, T]$.

To this end, we differentiate $\|A_{xx}\|_2^2$ with respect to t

$$\frac{d}{dt}\|A_{xx}\|_2^2 = 2Re\int_0^l \bar{A}_{xx}A_{xxt}dx = 2Re\int_0^l \bar{A}_{xxxx}A_t dx$$

$$= 2\|A_{xx}\|_2^2 - 2\|A_{xxx}\|_2^2 - 2\int_0^l |A|^2 A\bar{A}_{xxxx}dx + 2\int_0^l Ah\bar{A}_{xxxx}dx$$

$$\leq 2\|A_{xx}\|_2^2 - 2\|A_{xxx}\|_2^2 + 3\|A\|_{L^\infty}^6 \int_0^L |A_x||A_{xxx}|dx$$

$$+2(\|A\|_{L^\infty} + \|h\|_{L^\infty})\int_0^l (|A_x| + |h_x|)|A_{xxx}|dx$$

$$\leq \|A_{xx}\|^2 + c_{10}(\|A\|_{H^1}, \|h\|_{H^1}).$$

By the Gronwall inequality, we conculde that $\|A_{xx}\|_2^2$ is bounded on $[0, T]$.

We now only need to prove that $\|h_{xx}\|_2^2$ is bounded on $[0,T]$. So we differentiate $\|h_{xx}\|_2^2$ with respect to t

$$\frac{d}{dt}\|h_{xx}\|^2 = \int_0^l 2h_{xxxx}h_t dx$$

$$= 2\int_0^l h_{xxxx}(-h_{xx} - h_{xxxx} + \alpha|A|^2_{xx})dx$$

$$\leq 2||h_{xxx}||_2^2 - 2||h_{xxxx}||_2^2 + (||h_{xxxx}||_2^2 + c_{11}||\partial_{xx}|A|^2||_2^2)$$

$$\leq 2||h_{xxx}||_2^2 - ||h_{xxxx}||_2^2 + c_{11}\int_0^l (|AA_{xx}| + 2|A_x|^2)^2 dx$$

$$\leq (||h_{xxxx}||_2^2 + 4||h_{xx}||_2^2) - ||h_{xxxx}||_2^2 + c_{11}(||A||_\infty^2||A_{xx}||_2^2 + 2||A_x||_4^4)$$

$$\leq 4||h_{xx}||_2^2 + c_0(||A||_\infty^2||A_{xx}||_2^2 + 2||A_x||_\infty^2||A_x||_2^2), \tag{3-19}$$

where we have used the fact that $2||h_{xxx}||_2^2 \leq ||h_{xxxx}||_2^2 + 4||h_{xx}||_2^2$ due to integration by parts and Cauchy-Schwarz inequality.

Moreover, from (3-9) we have

$$\|A_x\|_2^2 + \int_0^t ||A_{xx}||d\tau \leq \int_0^t (2\|A_x\|_2^2 + 2\|A\|_6^6 + \|A\|_4^4 + \|h\|_4^4)d\tau. \tag{3-20}$$

Since A, h are bounded in H^1 norm on any finite interval $[0,T]$, $2\|A_x\|_2^2, \|A\|_6^6, \|A\|_4^4$ and $\|h\|_4^4$ are all bounded. Thus (3-20) implies that there exists $c_{12} > 0$, depending on T, such that for all $0 \leq t \leq T$

$$\int_0^t ||A_{xx}||_2^2 d\tau < c_{12}. \tag{3-21}$$

We apply the Gagliardo-Nirenberg estimates (3-14) in (3-19) to get

$$\frac{d}{dt}\|h_{xx}\|^2 \leq 4||h_{xx}||_2^2 + c_{11}(||A||_\infty^2||A_{xx}||_2^2 + 2||A_x||_\infty^2||A_x||_2^2)$$

$$\leq 4||h_{xx}||_2^2 + c_{11}(||A||_\infty^2||A_{xx}||_2^2 + \lambda_3||A_{xx}||_2^s||A||_2^r||A_x||_2^2). \tag{3-22}$$

Note that on $[0,T]$, $||A_{xx}||_2^s \leq ||A_{xx}||_2^2 + c_{13}$. Further, $||A||_\infty^2, ||A||_2^r$ and $||A_x||_2^2$ are all bounded on $[0,T]$. Thus (3-22) becomes

$$\frac{d}{dt}||h_{xx}||_2^2 \leq 4||h_{xx}||_2^2 + c'||A_{xx}||_2^2 + c''. \tag{3-23}$$

Integrating (3-23) in t and applying the time-averaged estimate (3-21) we finally get

$$||h_{xx}||_2^2 \leq ||h_0||_{H^2}^2 + \int_0^t 4||h_{xx}||_2^2 d\tau + c' \int_0^t ||A_{xx}||_2^2 d\tau + c''t$$

$$\leq ||h_0||_{H^2}^2 + \int_0^t 4||h_{xx}||_2^2 d\tau + c'c_{12} + c''T$$

$$\leq c_{14} + 4\int_0^t 4||h_{xx}||_2^2 d\tau. \tag{3-24}$$

By the Gronwall inequality, we conclude that $||h_{xx}||_2^2$ is bounded on $[0,T]$.

By now we have proved that both $||A||_{H^2}$ and $||h||_{H^2}$ are bounded (a priori) on any finite time interval $[0,T]$. Since T is arbitrary, both H^2 norms of A and h are bounded (a priori) for any $t > 0$ and hence can not blow up at any finite time. Combining with the local existence result of the last section, we obtain the global existence and uniqueness result.

We summarize the above result in the following theorem:

Global Existence Theorem. *Assume that the "Marangoni coefficient" $0 < \alpha < 2$ and $A(x,0), h(x,0) \in H_{per}^2[0,l]$, then the unique solution to the KS-GL system (1-1)-(1-2) with periodic boundary conditions (1-3) and initial condition (1-4) exists for any time $t > 0$.*

We recall that constant $\alpha = \frac{k}{(Ma - Ma_c)^{\frac{1}{5}}}$, where Ma is the Marangoni number measuring the gradient (derivative) of surface tension with respect to surfactant concentration, Ma_c is the critical Marangoni number at which the trivial stationary state becomes linearly unstable, and $k > 0$ is a constant related to other system parameters. So the global existence condition $0 < \alpha < 2$ means that $Ma > Ma_c + \frac{k}{2^5}$.

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