THE WAVE EQUATION WITH ONE POINT INTERACTION AND THE (LINEARIZED) CLASSICAL ELECTRODYNAMICS OF A POINT PARTICLE

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Abstract. We study the point limit of the linearized Maxwell–Lorentz equations describing the interaction, in the dipole approximation, of an extended charged particle with the electromagnetic field. We find that this problem perfectly fits into the framework of singular perturbations of the Laplacian; indeed we prove that the solutions of the Maxwell–Lorentz equations converge – after an infinite mass renormalization which is necessary in order to obtain a non trivial limit dynamics – to the solutions of the abstract wave equation defined by the self–adjoint operator describing the Laplacian with a singular perturbation at one point. The elements in the corresponding form domain have a natural decomposition into a regular part and a singular one, the singular subspace being three–dimensional. We obtain that this three–dimensional subspace is nothing but the velocity particle space, the particle dynamics being therefore completely determined – in an explicit way – by the behaviour of the singular component of the field. Moreover we show that the vector coefficient giving the singular part of the field evolves according to the Abraham–Lorentz–Dirac equation.

1. Introduction

One of the most important, and difficult, problems in classical and quantum field theory is the passage to the so called "local limit", i.e. the limit in which the interaction takes place at one point.

The most meaningful theory in which this problem shows up is electrodynamics of point particles, both in its classical and its quantum version ([D],[Be],[CN]). The goal of this paper is to show that it is possible to construct a complete and mathematically consistent theory of the dynamical system constituted by the electromagnetic field interacting with a charged point particle, at least in the linear (also called dipole) approximation.

Such a program goes back to the early studies of Kramers ([Kr]) in renormalization theory, the most comprehensive report on which is perhaps the Ph.D. thesis of Kramers' pupil N.G. VanKampen ([VK]).

But these efforts did not reach the goal, and, up to now, a system of equations already renormalized and describing the dynamics of the coupled system was lacking. The model here studied is the point limit of a regularization of the Maxwell–Lorentz system, also called, in this particular case, the Pauli–Fierz model ([PF],[A],[BG],[Bl]): it consists of an extended charged particle, described by a spherically symmetric form factor ρ, interacting with the electromagnetic field and in linear approximation. In Coulomb gauge this corresponds, for the vector potential A and for the particle position q, to the equations

$$\begin{cases} \dfrac{1}{c^2}\ddot{A} =\Delta A + \dfrac{4\pi e}{c} M\dot{q}\rho \\ m_0\ddot{q} = -\dfrac{e}{c}\displaystyle\int_{\mathbb{R}^3} \rho(x)\dot{A}(x)\, dx \\ A(0) =A_0, \quad \dot{A}(0) = \dot{A}_0, \quad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0 \ ; \end{cases} \tag{1.1}$$

here M is the projection onto the divergenceless part, e is the electric charge, c is the light velocity, and m_0 is the bare, or mechanical, mass, a parameter to be suitably redefined to get the equations in the point limit, i.e. when ρ weakly converges to δ_0, the Dirac mass at zero. Notice that in the Coulomb gauge the scalar potential ϕ plays no role, in that it is purely static, being the solution of $-\Delta\phi = 4\pi e\rho$, and for a spherically symmetric form factor ρ it gives no contribution to the Lorentz force on the particle (see e.g. [H]).

As it stands the system is meaningless in the point limit, due to the fact that the solution for the field is not sufficiently regular to be evaluated at the origin, as would be required in the right hand side of the particle equation in (1.1), so that the Lorentz force on the particle appears to be ill defined.

The traditional way of dealing with this problem consists of decoupling the system, firstly eliminating the degrees of freedom of the field by integrating the field equation in (1.1) and then substituting into the particle equation. This procedure leads, through heuristic manipulations, to the classical Abraham–Lorentz–Dirac equation for the particle, namely

$$-m\tau_0 \dddot{q}(t) + m\ddot{q}(t) = F_{ext}(t, q(t)) \ , \tag{1.2}$$

([D],[L]), where $\tau_0 = \frac{2e^2}{3mc^3}$, while m is the renormalized (or phenomenological) mass. A first rigorous analysis of the behaviour of the particle in the point limit appears in [BN] and more thoroughly in [B].

But the dynamics of the field is usually not even dealt with, the only notable exception being, as far as we know, [K].

The procedure here adopted is more functional analytic minded, but equally simple in spirit. We too decouple the system, in the reverse way however, obtaining an equation for the field variable alone as follows. Integrate the particle equation in (1.1) and substitute into the field equation, neglecting for the moment the contribution of the initial data $\dot{q}_0, A_0$ – a contribution that, as a matter of fact, will turn out to disappear in the limit (see lemma 2.6 for this non completely trivial fact). The result is

$$\frac{1}{c^2}\ddot{A} = \Delta A - \frac{4\pi e^2}{m_0 c^2} M \int_{\mathbb{R}^3} \rho(x) A(x) dx \ \rho \ .$$

Now the right hand side is a finite rank perturbation of the Laplacian, which, as we will show, turns out to converge in the point limit, in norm resolvent sense, to a well defined self–adjoint operator $-H_m$, provided mass is renormalized according to the classical prescription adopted also in the quantum context ([Be])

$$m_0 = m - m_{em} = m - \frac{8\pi e^2}{3c^2} E(\rho) \ ,$$

where $E(\rho)$ is the electrostatic energy of the form factor, and m is the phenomenological mass which appears here as an arbitrary parameter not contained in the original system (1.1) (see thm. 2.1). Instead, if such a renormalization is not performed, the limit dynamics turns out to be trivial in a sense to be explained in thm. 2.11.

The limit equation for the field is then

$$\frac{1}{c^2}\ddot{A} = -H_m A \ , \tag{1.3}$$

an abstract wave equation where H_m is an operator that is self–adjoint, bounded from below, containing the renormalized mass only.

Operators of this kind, when m is replaced by a parameter to be interpreted as the scattering length, are well known in ordinary quantum mechanics, and a wide mathematical literature exists on them ([AGHH]). In the quantum mechanics context they are related to the so called “Fermi pseudopotentials”, which describe potentials with a singular support – one point in our situation (see remark 2.4). Such an operator H_m represents a rigorous version of the formal writing “$-\Delta + M \cdot \delta_0$”, in which δ_0 would act as a multiplication operator.

A main characteristics of the operator H_m is the structure of its form domain, the elements of which being the sum of a regular part and of a singular one with a fixed singularity, of the Coulomb type, at the origin (see thm. 2.3). The remarkable fact is that the solutions of the Maxwell–Lorentz system (1.1) converge to the ones of (1.3) if and only if the coefficient of this singular component is proportional to the velocity $\dot{q}$ of the particle through the relation

$$\dot{q} = \frac{3c}{2e} \lim_{r\downarrow 0} r \frac{1}{4\pi r^2} \int_{S_r} A(x) d\mu_r(x) \tag{1.4}$$

(see corollary 2.9 and (2.5)). So to determine the flow of the complete system it suffice to calculate it for the field alone, i.e. for the equation (1.3). This is easily done (see thm. 3.2), and the particle evolution is then readily and explicitly calculated by (1.4) (see (3.3) and (3.4)). Moreover it turns out that the vector coefficient determining the singular part of the solution of (1.3), i.e. $\dot{q}(t)$, evolves according to Abraham–Lorentz–Dirac equation (1.2), where $F_{ext}(t, q(t))$ is now the (linearized) Lorentz force due to the free evolution; more precisely it satisfies its integrated version

$$-m\tau_0 \ddot{q}(t) + m\dot{q}(t) = -\frac{e}{c} A_f(t, 0) \ ,$$

where A_f denotes the solution of the free wave equation with the given initial data (see thm. 3.3). With such a $q(t)$ the solution of (1.3) satisfies then the distributional equation

$$\frac{1}{c^2} \ddot{A} = \Delta A + \frac{4\pi e}{c} M \dot{q} \delta_0$$

(see remark 3.5). This establishes the link with the traditional description.

We conclude with two remarks. The first one deals with the well–known problem of runaway solutions ([D],[CN]). The operator H_m has a negative eigenvalue (see (2.6)), so that equation (1.3), and hence the particle motion too, admits runaways. We do not discuss here this relevant physical problem because we are concerned only with the mathematical description of the system. In any case, on the purely mathematical side, the runaways can be readily eliminated by projection onto the absolutely continuous subspace of H_m.

The second remark concerns the possibility of extending the procedure here pursued to the non linear complete Maxwell–Lorentz system. Its point limit resists, up to now, every effort of a mathematical study. We conjecture the possible form of the non linear limit equations in remark 3.1.

2. The point limit of the Maxwell–Lorentz equations.

Let us start with some notations. We denote by $L^2_*(\mathbb{R}^3)$ the Hilbert space of square integrable, divergence–free, vector fields on $\mathbb{R}^3$. M will be the projection from $L^2_3(\mathbb{R}^3)$, the Hilbert space of square integrable vector fields on $\mathbb{R}^3$, onto $L^2_*(\mathbb{R}^3)$ and we will use the same symbol $\langle\cdot,\cdot\rangle$ ($\|\cdot\|_2$ is the corresponding Hilbert norm) to indicate the scalar products in $L^2(\mathbb{R}^3)$, $L^2_3(\mathbb{R}^3)$, $L^2_*(\mathbb{R}^3)$ and also to indicate the obvious pairing between an element of $L^2_3(\mathbb{R}^3)$ and one of $L^2(\mathbb{R}^3)$ (the result being a vector in $\mathbb{R}^3$). By the same abuse of notation, given two functions f and g in $L^2(\mathbb{R}^3)$, by $f \otimes g$ we will indicate the operator in $L^2_3(\mathbb{R}^3)$ defined by $f \otimes g(A) := f\langle g, A\rangle$. $H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$, indicates the usual scale of Sobolev–Hilbert spaces, and the meaning of $H^s_3(\mathbb{R}^3)$ and $H^s_*(\mathbb{R}^3)$ should now be clear. Finally, given a measurable function ρ we define its energy $E(\rho)$ as

$$E(\rho) := \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(x)\rho(y)}{|x-y|}dx\ dy\ .$$

On $L^2_*(\mathbb{R}^3)\times\mathbb{R}^3$ (the correct domain of definition will be specified later) let us consider the system of equations ($r>0$)

$$\begin{cases} \frac{1}{c^2}\ddot{A}_r = \Delta A_r + \frac{4\pi e}{c}M\dot{q}_r\rho_r \\ m_r\ddot{q}_r = -\frac{e}{c}\langle\rho_r,\dot{A}_r\rangle \\ A_r(0) = A^r_0, \quad \dot{A}_r(0)=\dot{A}_0, \quad q_r(0)=q_0, \quad \dot{q}_r(0)=\dot{q}_0\ , \end{cases} \tag{2.1}$$

where

$$m_r := m - \frac{8\pi e^2}{3c^2}E(\rho_r)\ , \tag{2.2}$$

and where $\rho_r(x) := r^{-3}\rho(r^{-1}x)$, $\rho\in L^2(\mathbb{R}^3)$ a spherically symmetric probability density with bounded support. Therefore

$$m_r = m - \frac{1}{r}\frac{8\pi e^2}{3c^2}E(\rho)\ ,$$

(observe that $E(\rho)$ is finite by Sobolev lemma since $\rho\in\cap_{1\le p\le 2}L^p(\mathbb{R}^3)$) and ρ_r weakly converges, as $r\downarrow 0$, to δ_0, the Dirac mass at zero. Integrating the particle equation in (2.1) one obtains

$$\dot{q}_r = -\frac{e}{m_r c}\langle\rho_r,A_r\rangle + \dot{q}_0 + \frac{e}{m_r c}\langle\rho_r,A^r_0\rangle\ .$$

Inserting this expression into the field equation one gets

$$\frac{1}{c^2}\ddot{A}_r = -H_rA_r + \frac{4\pi e}{c}Mv_r\rho_r\ ,$$

where

$$H_r := -\Delta + \frac{4\pi e^2}{m_r c^2}M\cdot\rho_r\otimes\rho_r\ ,$$

and

$$v_r := \dot{q}_0 + \frac{e}{m_r c}\langle\rho_r,A^r_0\rangle\ .$$

Since $M\cdot\rho_r\otimes\rho_r$ is a bounded symmetric operator, H_r is a self–adjoint operator on $L^2_*(\mathbb{R}^3)$ with operator domain $H^2_*(\mathbb{R}^3)$ and form domain $H^1_*(\mathbb{R}^3)$.

Theorem 2.1. *As $r\downarrow 0$, i.e as ρ_r weakly converges to δ_0, the self–adjoint operator H_r defined above converges in norm resolvent sense in $L^2_*(\mathbb{R}^3)$ to a self–adjoint operator H_m, where H_m has the resolvent*

$$(H_m+\lambda)^{-1} = (-\Delta+\lambda)^{-1} + \Gamma_m(\lambda)^{-1}M\cdot G_\lambda\otimes G_\lambda\ ,$$

and where

$$-\lambda \in \rho(H_m), \quad \lambda > 0, \qquad \Gamma_m(\lambda) = -\frac{mc^2}{4\pi e^2} + \frac{\sqrt{\lambda}}{6\pi}, \qquad G_\lambda(x) := \frac{1}{4\pi}\frac{e^{-\sqrt{\lambda}|x|}}{|x|} \ .$$

Proof. The proof proceeds as in [AGHH, §II.1.1]. Since

$$\langle f, Mvg\rangle = \frac{2}{3}v\langle f, g\rangle$$

if $v \in \mathbb{R}^3$, $f, g \in L^2(\mathbb{R}^3)$ spherically symmetric, a straightforward computation gives

$$(H_r + z^2)^{-1} = (-\Delta + z^2)^{-1} + \Gamma_r(z^2)^{-1} M \cdot (-\Delta + z^2)^{-1}\rho_r \otimes (-\Delta + z^2)^{-1}\rho_r \ ,$$

where Im $z \neq 0$, Re $z > 0$, and

$$\begin{aligned} \Gamma_r(z^2) &= -\frac{m_r c^2}{4\pi e^2} - \frac{2}{3}\langle(-\Delta + z^2)^{-1}\rho_r, \rho_r\rangle \\ &= -\frac{mc^2}{4\pi e^2} + \frac{2}{3}\frac{1}{r}\left(E(\rho) - \langle(-\Delta + r^2 z^2)^{-1}\rho, \rho\rangle\right) \ . \end{aligned} \tag{2.3}$$

Obviously

$$\lim_{r\downarrow 0} \Gamma_r(z^2) = \Gamma_m(z^2), \quad \lim_{r\downarrow 0} \|(-\Delta + z^2)^{-1}\rho_r - G_{z^2}\|_2 = 0 \ ,$$

and so

$$\lim_{r\downarrow 0} \|(H_r + z^2)^{-1} - (H_m + z^2)^{-1}\|_{HS} = 0$$

(here HS stands for Hilbert–Schmidt). In order to prove that $(H_m + z^2)^{-1}$ is the resolvent of a closed operator one needs to prove that it is injective. This is done as in [AGHH, page 112]. □

Remark 2.2. As clearly indicated by (2.3), H_r has a non trivial, i.e. different from $-\Delta$, limit if and only if m_r diverges according to the classical prescription (2.2).

We summarize the properties of the operator H_m in the following

Theorem 2.3. *The vectors A in the operator domain $D(H_m)$ of H_m are of the type*

$$A = A_\lambda + \Gamma_m(\lambda)^{-1} M A_\lambda(0) G_\lambda, \quad A_\lambda \in H^2_*(\mathbb{R}^3), \quad -\lambda \in \rho(H_m), \quad \lambda > 0 \ .$$

The above decomposition in a regular part A_λ, and a corresponding singular one, is unique, and with $A \in D(H_m)$ of this form one has

$$(H_m + \lambda)A = (-\Delta + \lambda)A_\lambda \ . \tag{2.4}$$

Let F_m be the quadratic form corresponding to H_m. Then the vectors A in the form domain $D(F_m)$ are of the type

$$A = A_\lambda + \frac{4\pi e}{c} M Q_A G_\lambda, \qquad A_\lambda \in H^1_*(\mathbb{R}^3), \quad Q_A \in \mathbb{R}^3, \quad \lambda > 0 \ .$$

Given $A \in D(F_m)$, Q_A can be explicitly computed by the formula

$$Q_A = \frac{3c}{2e} \lim_{r\downarrow 0} r \ \frac{1}{4\pi r^2} \int_{S_r} A(x) d\mu_r(x) \ , \tag{2.5}$$

where S_r denotes the sphere of radius r and μ_r is the corresponding surface measure. The above decomposition is unique, and with $A \in D(F_m)$ of this form one has

$$\begin{aligned} F^\lambda_m(A, A) :=& F_m(A, A) + \lambda \|A\|^2_2 \\ =& \|(-\Delta + \lambda)^{\frac{1}{2}} A_\lambda\|^2_2 + \frac{2}{3}\left(\frac{4\pi e}{c}\right)^2 \Gamma_m(\lambda) |Q_A|^2 \ . \end{aligned}$$

Moreover

$$\sigma_{ess}(H_m) = \sigma_{ac}(H_m) = [0, +\infty), \quad \sigma_{sc}(H_m) = \emptyset \ ,$$

and

$$\sigma_p(H_m) = \left\{ -\left(\frac{3mc^2}{2e^2}\right)^2 \right\} \equiv \{-\lambda_0\} \ , \tag{2.6}$$

where $-\lambda_0$ has a threefold degeneration and, given an orthonormal basis $\{e_j\}^3_1$,

$$X^0_j = 2\sqrt{2\pi m} \ \frac{c}{e} M e_j G_{\lambda_0} \ ,$$

are the corresponding normalized eigenvectors.

Proof. Everything follows proceeding as in [AGHH, §I.1.1, §II.1.1] as regards the operator, and as in [T, §2] as regards the form, the only difference being in a different definition of Q_A. Concerning this point the assertion follows from the observation that $Q_{MQ_AG_\lambda} = (\frac{4\pi e}{c})^{-1} Q_A$ and that the $H^{-1}(\mathbb{R}^3)$–norm of the measure $(4\pi r^2)^{-1}\mu_r$ is of the same order as $r^{-1/2}$, and so $Q_{A_\lambda} = 0$. □

Remark 2.4. By (2.4) we have that for any $A \in H^2_*(\mathbb{R}^3)$ such that $A(0) = 0$, $H_m A = -\Delta A$. Therefore H_m is a singular perturbation of $-\Delta$, the perturbation taking place only at $x = 0$.

Remark 2.5. By thm. 2.3, for any $A \in D(F_m)$, $\lambda > \lambda_0$, there exists an unique $X \in L^2_*(\mathbb{R}^3)$ such that

$$A = (H_m + \lambda)^{-\frac{1}{2}} X = (-\Delta + \lambda)^{-\frac{1}{2}} X + \frac{4\pi e}{c} M L_m(X) G_\lambda \ ,$$

where $L_m \ : \ L^2_*(\mathbb{R}^3) \to \mathbb{R}^3$ is the bounded linear operator defined by

$$L_m(X) := Q_{(H_m+\lambda)^{-1/2}X} \ .$$

Moreover for any $A \in H^1_*(\mathbb{R}^3)$, $\lambda > \lambda_0$, r sufficiently large so that $-\lambda \in \rho(H_r)$, there exists an unique $X \in L^2_*(\mathbb{R}^3)$ such that

$$A = (H_r + \lambda)^{-\frac{1}{2}} X = (-\Delta + \lambda)^{-\frac{1}{2}} X + \frac{4\pi e}{c} M L_r(X)(-\Delta + \lambda)^{-1} \rho_r \ ,$$

where, since $(H_r + \lambda)^{-\frac{1}{2}} X$ converges to $(H_m + \lambda)^{-\frac{1}{2}} X$, $L_r \ : \ L^2_*(\mathbb{R}^3) \to \mathbb{R}^3$ is a bounded linear operator such that

$$\lim_{r \downarrow 0} \ \|L_r - L_m\|_{L^2_*, \mathbb{R}^3} = 0 \ .$$

Let us now give some general results on second order linear differential equations in Hilbert spaces which we will need below (for the proofs of such results see [F, Chaps. II and III],[Ki],[S1,2]). Let H be a bounded from below (this is a necessary condition) self–adjoint operator on $L^2_*(\mathbb{R}^3)$, and let F the corresponding quadratic form. Then H generates a cosine operator function $C \ : \ \mathbb{R} \ \to \ \mathcal{L}(L^2_*(\mathbb{R}^3); L^2_*(\mathbb{R}^3))$, i.e. C is a strongly continuous function such that

$$C(0) = 1, \qquad C(s+t)C(s-t) = 2C(s)C(t) \ ,$$

$$D(H) = \{ \ A \in L^2_*(\mathbb{R}^3) \ : \ \lim_{t \to 0} \frac{2}{t^2}(C(t)A - A) \text{ exists } \} \ ,$$

$$\forall A \in D(H) \qquad HA = \lim_{t \to 0} \frac{2}{t^2}(C(t)A - A) \ ,$$

$$C(t)D(H) \subseteq D(H) \ .$$

Moreover $\forall z \in \mathbb{C}$ such that $-z^2 \in \rho(H)$ and Re $z > |\inf \sigma(H) \wedge 0|^{\frac{1}{2}}$

$$z(H + z^2)^{-1} A = \int_0^\infty e^{-zt} C(t) A \ dt \ ,$$

and so, by $(H+z^2)^{-1}A = z^{-2}A - z^{-2}(H+z^2)HA$, and by inverse Laplace transform, $\forall t \geq 0$, $\forall A \in D(H)$,

$$C(t)A = A - \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{zt} \frac{1}{z}(H + z^2)^{-1} HA \ dz \ , \qquad x > |\inf \sigma(H) \wedge 0|^{\frac{1}{2}} \ . \quad (2.7)$$

One defines then the sine operator function $S \ : \ \mathbb{R} \ \to \ \mathcal{L}(L^2_*(\mathbb{R}^3); L^2_*(\mathbb{R}^3))$ by

$$S(t)A := \int_0^t C(s)A \ ds \ .$$

Given $A_0, \dot{A}_0, X \in L^2_*(\mathbb{R}^3)$, let

$$A(t) := C(t)A_0 + S(t)\dot{A}_0 + \int_0^t S(s)X ds \ .$$

Then $A \in C(\mathbb{R}; L^2_*(\mathbb{R}^3))$,

$$A_0 \in D(F), \dot{A}_0 \in L^2_*(\mathbb{R}^3) \quad \Rightarrow \quad A \in C(\mathbb{R}; D(F)) \cap C^1(\mathbb{R}; L^2_*(\mathbb{R}^3)) \ ,$$

$$A_0 \in D(H), \dot{A}_0 \in D(F) \quad \Rightarrow \quad A \in C(\mathbb{R}; D(H)) \cap C^1(\mathbb{R}; D(F)) \cap C^2(\mathbb{R}; L^2_*(\mathbb{R}^3)) \ ,$$

and $A(t)$ solves the inhomogeneous Cauchy problem

$$\begin{cases} \ddot{A}(t) = -HA(t) + X \\ A(0) = A_0, \quad \dot{A}(0) = \dot{A}_0 \ . \end{cases}$$

Moreover there is a well defined dynamics on the "phase–space" in the following sense: on $D(F) \times L^2_*(\mathbb{R}^3)$ define the Hilbert norm

$$|||(A, \dot{A})|||^2 := \|((H + \lambda)^{\frac{1}{2}} \times 1)(A, \dot{A})\|^2_{L^2_* \times L^2_*} \equiv F(A, A) + \lambda \|A\|_2^2 + \|\dot{A}\|_2^2 \ ,$$

with λ s.t. $H + \lambda$ is strictly positive. Then $U(t)$ defined by

$$U(t) := \begin{bmatrix} C(t) & S(t) \\ \dot{C}(t) & C(t) \end{bmatrix} \quad + \quad \begin{bmatrix} \int_0^t S(s) X ds \\ S(t) X \end{bmatrix}$$

is a strongly continuous (w.r.t. $||| \cdot |||$) one–parameter group on $D(F) \times L^2_*(\mathbb{R}^3)$ with generator

$$\begin{bmatrix} 0 & 1 \\ -H & 0 \end{bmatrix} \quad + \quad \begin{bmatrix} 0 \\ X \end{bmatrix} \ ,$$

and $U(t) D(H) \times D(F) \subseteq D(H) \times D(F)$.

The lemma below is the key ingredient in the proof of the next theorem and hence of the successive corollary. It essentially says that convergence of the solutions of the Cauchy problem (2.1) occurs only and only if some compatibility conditions on the initial data are satisfied and the inhomogeneous term $M v_r \rho_r$ disappears in the limit.

Lemma 2.6. *Given $X, X_0 \in L^2_*(\mathbb{R}^3)$, $\dot{q}_0 \in \mathbb{R}^3$, $\lambda > \lambda_0$, let*

$$A_0^r := (H_r + \lambda)^{-\frac{1}{2}} X_0, \quad v_r := \dot{q}_0 + \frac{e}{m_r c} \langle \rho_r, A_0^r \rangle \ .$$

Then

$$\lim_{r \downarrow 0} \|(H_r + \lambda)^{-\frac{1}{2}} M v_r \rho_r - (H_m + \lambda)^{-\frac{1}{2}} X\|_2 = 0$$

if and only if

$$X = 0, \quad \lim_{r \downarrow 0} L_r(X_0) = \dot{q}_0, \quad \lim_{r \downarrow 0} \langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X_0 \rangle = -\frac{mc}{e} \dot{q}_0 \ . \tag{2.8}$$

Remark 2.7. The third condition in (2.8) requires, obviously, some regularity at the origin for X_0. Such a limit exists if $X_0 \in H_*^{\frac{1}{2}+\epsilon}(U_0)$, U_0 a nbh. of the origin. Moreover note that if we define $A_0^r := (H_r + \lambda)^{-1} Y_0$, $Y_0 \in L^2_*(\mathbb{R}^3)$, then one can prove (by proceeding as in the proof below) that lemma 2.6 holds true under the condition, beside $X = 0$,

$$(-\Delta+\lambda)^{-1}Y_0(0) = \frac{4\pi e}{c}\Gamma_m(\lambda)\dot{q}_0 \ ,$$

which coincides with the relation, holding for an element in $D(H_m)$, between Q_A and $A_\lambda(0)$ that can be obtained by thm. 2.3.

Proof of lemma 2.6. By remark 2.5 we can write

$$A_0^r = (-\Delta+\lambda)^{-\frac{1}{2}}X_0 + \frac{4\pi e}{c}ML_r(X_0)(-\Delta+\lambda)^{-1}\rho_r \ .$$

So

$$v_r = \dot{q}_0 + \frac{e}{m_r c}\langle\rho_r, (-\Delta+\lambda)^{-\frac{1}{2}}X_0\rangle + \frac{8\pi e^2}{3m_r c^2}L_r(X_0)\langle(-\Delta+\lambda)^{-1}\rho_r, \rho_r\rangle \ ,$$

and, by the expression for $(H_r+\lambda)^{-1}$ we have

$$\begin{aligned}\|(H_r+\lambda)^{-\frac{1}{2}}Mv_r\rho_r\|_2^2 =& \frac{2}{3}|v_r|^2\langle(-\Delta+\lambda)^{-1}\rho_r, \rho_r\rangle \\ &+\frac{4}{9}\Gamma_r(\lambda)^{-1}(|v_r|\langle(-\Delta+\lambda)^{-1}\rho_r, \rho_r\rangle)^2 \ .\end{aligned}$$

Since

$$\lim_{r\downarrow 0}\ \Gamma_r(\lambda) = \Gamma_m(\lambda) = \frac{1}{6\pi}(\sqrt{\lambda}-\sqrt{\lambda_0}) > 0 \ ,$$

we have the sum of two positive terms, and so $|v_r|\langle(-\Delta+\lambda)^{-1}\rho_r, \rho_r\rangle$ needs to converge. However $\langle(-\Delta+\lambda)^{-1}\rho_r, \rho_r\rangle$ diverges, and so $|v_r|$ needs to converge to zero. Since $(-\Delta+\lambda)^{-\frac{1}{2}}X_0 \in H_*^1(\mathbb{R}^3) \subset L_*^6(\mathbb{R}^3)$, and $\|\rho_r\|_{6/5} = r^{-1/2}\|\rho\|_{6/5}$, by Hölder inequality we obtain

$$\lim_{r\downarrow 0}\ \frac{\langle\rho_r, (-\Delta+\lambda)^{-\frac{1}{2}}X_0\rangle}{m_r} = 0 \ ,$$

and so we need

$$\lim_{r\downarrow 0}\ \dot{q}_0 + \frac{8\pi e^2}{3c^2}L_r(X_0)\frac{\langle(-\Delta+\lambda)^{-1}\rho_r, \rho_r\rangle}{m_r} = 0 \ .$$

Since

$$\lim_{r\downarrow 0}\ \frac{\langle(-\Delta+\lambda)^{-1}\rho_r, \rho_r\rangle}{m_r} = -\frac{3c^2}{8\pi e^2} \ ,$$

we obtain

$$\lim_{r\downarrow 0}\ L_r(X_0) = L_m(X_0) = \dot{q}_0 \ .$$

Therefore

$$v_r\langle(-\Delta+\lambda)^{-1}\rho_r, \rho_r\rangle = \langle(-\Delta+\lambda)^{-1}\rho_r, \rho_r\rangle\times$$

$$\times\left(\dot q_0\left(1+\frac{8\pi e^2}{3c^2}\frac{\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}\right)+\frac{e}{c}\frac{\langle\rho_r,(-\Delta+\lambda)^{-\frac12}X_0\rangle}{m_r}\right)\ .$$

As

$$\lim_{r\downarrow 0}\ \langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle\left(1+\frac{8\pi e^2}{3c^2}\frac{\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle}{m_r}\right)=-\frac{3c^2m}{8\pi e^2}\ ,$$

we have

$$\lim_{r\downarrow 0}\ v_r\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle=-\frac{3c}{8\pi e}\left(\lim_{r\downarrow 0}\ \langle\rho_r,(-\Delta+\lambda)^{-\frac12}X_0\rangle+\frac{mc}{e}\dot q_0\right)\ .$$

Since H_r converges in norm resolvent sense to H_m,

$$\lim_{r\downarrow 0}\ \|(H_r+\lambda)^{-\frac12}Mv_r\rho_r-(H_m+\lambda)^{-\frac12}X\|_2=0$$

if and only if

$$\lim_{r\downarrow 0}\ \|(H_r+\lambda)^{-\frac12}(Mv_r\rho_r-X)\|_2=0\ .$$

Moreover

$$\begin{aligned}\|(H_r+\lambda)^{-\frac12}(Mv_r\rho_r-X)\|_2^2&=\|(-\Delta+\lambda)^{-\frac12}(Mv_r\rho_r-X)\|_2^2\\&\quad+\Gamma_r(\lambda)^{-1}\langle(-\Delta+\lambda)^{-1}\rho_r,Mv_r\rho_r-X\rangle^2\ ,\end{aligned}$$

so we have the sum of two positive terms, and therefore we need

$$\lim_{r\downarrow 0}\ \|(-\Delta+\lambda)^{-\frac12}(Mv_r\rho_r-X)\|_2=0\ .$$

Since $v_r\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle$ converges, and v_r converges to zero, we have

$$\lim_{r\downarrow 0}\ \|(-\Delta+\lambda)^{-\frac12}Mv_r\rho_r\|_2=0\ .$$

Therefore

$$\|(-\Delta+\lambda)^{-\frac12}X\|_2=0,$$

and so $X=0$. This implies that $v_r\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle$ needs to converge to zero, and so

$$\lim_{r\downarrow 0}\ \langle\rho_r,(-\Delta+\lambda)^{-\frac12}X_0\rangle=-\frac{mc}{e}\dot q_0\ .$$

□

Theorem 2.8. *Given $X,X_0\in L^2_*(\mathbb{R}^3)$, $\dot q_0\in\mathbb{R}^3$, $\lambda>\lambda_0$, let*

$$A_0^r:=(H_r+\lambda)^{-\frac12}X_0,\quad v_r:=\dot q_0+\frac{e}{m_rc}\langle\rho_r,A_0^r\rangle\ ,$$

let $U_r(t)$ be the strongly continuous group on $H^1_*(\mathbb{R}^3)\times L^2_*(\mathbb{R}^3)$ with generator

$$\begin{bmatrix}0&1\\-c^2H_r&0\end{bmatrix}\quad+\quad 4\pi ec\begin{bmatrix}0\\Mv_r\rho_r\end{bmatrix}\ ,$$

and let $U_m(t)$ be the strongly continuous group on $D(F_m)\times L^2_*(\mathbb{R}^3)$ with generator

$$\begin{bmatrix} 0 & 1 \\ -c^2H_m & 0 \end{bmatrix} \quad + \quad \begin{bmatrix} 0 \\ X \end{bmatrix} .$$

Then

$$((H_r+\lambda)^{\frac{1}{2}}\times 1)\cdot U_r(t)\cdot((H_r+\lambda)^{-\frac{1}{2}}\times 1)$$

converges, strongly in $L^2_*(\mathbb{R}^3)\times L^2_*(\mathbb{R}^3)$, uniformly in t over compact intervals, to

$$((H_m+\lambda)^{\frac{1}{2}}\times 1)\cdot U_m(t)\cdot((H_m+\lambda)^{-\frac{1}{2}}\times 1)$$

if and only if

$$X=0,\quad \lim_{r\downarrow 0}\, L_r(X_0)=\dot q_0,\quad \lim_{r\downarrow 0}\,\langle\rho_r,(-\Delta+\lambda)^{-\frac{1}{2}}X_0\rangle=-\frac{mc}{e}\dot q_0\ . \tag{2.8}$$

Proof. Let us at first prove that the theorem holds true for $\tilde U_r(t)$ and $\tilde U_m(t)$, the affine, strongly continuous, groups with generators

$$\begin{bmatrix} 0 & 1 \\ -c^2(H_r+\lambda) & 0 \end{bmatrix} \quad + \quad 4\pi ec\begin{bmatrix} 0 \\ Mv_r\rho_r \end{bmatrix} ,$$

$$\begin{bmatrix} 0 & 1 \\ -c^2(H_m+\lambda) & 0 \end{bmatrix} \quad + \quad \begin{bmatrix} 0 \\ X \end{bmatrix} .$$

By functional calculus we have

$$\tilde U_r(t)=\hat U_r(t)\quad + \quad 4\pi e\begin{bmatrix} \frac{1}{c}(H_r+\lambda)^{-1}(1-\cos ct(H_r+\lambda)^{\frac{1}{2}})Mv_r\rho_r \\ (H_r+\lambda)^{-\frac{1}{2}}\sin ct(H_r+\lambda)^{\frac{1}{2}}Mv_r\rho_r \end{bmatrix} ,$$

$$\hat U_r(t):=\begin{bmatrix} \cos ct(H_r+\lambda)^{\frac{1}{2}} & \frac{1}{c}(H_r+\lambda)^{-\frac{1}{2}}\sin ct(H_r+\lambda)^{\frac{1}{2}} \\ -c(H_r+\lambda)^{\frac{1}{2}}\sin ct(H_r+\lambda)^{\frac{1}{2}} & \cos ct(H_r+\lambda)^{\frac{1}{2}} \end{bmatrix} ,$$

$$\tilde U_m(t)=\hat U_m(t)\quad + \quad \begin{bmatrix} \frac{1}{c^2}(H_m+\lambda)^{-1}(1-\cos ct(H_m+\lambda)^{\frac{1}{2}})X \\ \frac{1}{c}(H_m+\lambda)^{-\frac{1}{2}}\sin ct(H_m+\lambda)^{\frac{1}{2}}X \end{bmatrix} ,$$

$$\hat U_m(t):=\begin{bmatrix} \cos ct(H_m+\lambda)^{\frac{1}{2}} & \frac{1}{c}(H_m+\lambda)^{-\frac{1}{2}}\sin ct(H_m+\lambda)^{\frac{1}{2}} \\ -c(H_m+\lambda)^{\frac{1}{2}}\sin ct(H_m+\lambda)^{\frac{1}{2}} & \cos ct(H_m+\lambda)^{\frac{1}{2}} \end{bmatrix} .$$

By norm resolvent convergence of H_r to H_m we have that

$$((H_r+\lambda)^{\frac{1}{2}}\times 1)\cdot \hat U_r(t)\cdot((H_r+\lambda)^{-\frac{1}{2}}\times 1)$$

converges, strongly in $L^2_*(\mathbb{R}^3)\times L^2_*(\mathbb{R}^3)$, uniformly in t over compact intervals, to

$$((H_m+\lambda)^{\frac{1}{2}}\times 1)\cdot \hat U_m(t)\cdot((H_m+\lambda)^{-\frac{1}{2}}\times 1) ,$$

and so we only need to study the behaviour of $(H_r+\lambda)^{-\frac{1}{2}}Mv_r\rho_r$ as $r\downarrow 0$. The thesis then follows from the previous lemma. Let us now go back to $U_r(t)$ and $U_m(t)$. Putting

$$\Lambda := \begin{bmatrix} 0 & 0 \\ c^2\lambda & 0 \end{bmatrix} ,$$

define the non autonomous, affine, vector fields

$$\Lambda_r(t) := \hat{U}_r(t)^{-1}\cdot\Lambda\cdot\tilde{U}_r(t) ,$$
$$\Lambda_m(t) := \hat{U}_m(t)^{-1}\cdot\Lambda\cdot\tilde{U}_m(t) ,$$

and let $P_r(t,s)$ and $P_m(t,s)$ be the corresponding evolution operators. Since

$$((H_r+\lambda)^{\frac{1}{2}}\times 1)\cdot\tilde{U}_r(t)\cdot((H_r+\lambda)^{-\frac{1}{2}}\times 1)$$

converges, strongly in $L^2_*(\mathbb{R}^3)\times L^2_*(\mathbb{R}^3)$, to

$$((H_m+\lambda)^{\frac{1}{2}}\times 1)\cdot\tilde{U}_m(t)\cdot((H_m+\lambda)^{-\frac{1}{2}}\times 1)$$

if and only if (2.8) holds, we have that

$$((H_r+\lambda)^{\frac{1}{2}}\times 1)\cdot\Lambda_r(t)\cdot((H_r+\lambda)^{-\frac{1}{2}}\times 1)$$

converges, strongly in $L^2_*(\mathbb{R}^3)\times L^2_*(\mathbb{R}^3)$, to

$$((H_m+\lambda)^{\frac{1}{2}}\times 1)\cdot\Lambda_m(t)\cdot((H_m+\lambda)^{-\frac{1}{2}}\times 1)$$

if and only if (2.8) holds. Therefore

$$((H_r+\lambda)^{\frac{1}{2}}\times 1)\cdot P_r(t,s)\cdot((H_r+\lambda)^{-\frac{1}{2}}\times 1)$$

converges, strongly in $L^2_*(\mathbb{R}^3)\times L^2_*(\mathbb{R}^3)$, to

$$((H_m+\lambda)^{\frac{1}{2}}\times 1)\cdot P_m(t,s)\cdot((H_m+\lambda)^{-\frac{1}{2}}\times 1)$$

if and only if (2.8) holds. Since

$$U_r(t)=\tilde{U}_r(t)\cdot P_r(t,0),\qquad U_m(t)=\tilde{U}_m(t)\cdot P_m(t,0) ,$$

the proof is done. □

We can now state our main result about the convergence of the solutions of the Maxwell–Lorentz system:

Corollary 2.9. *Let $(A_r,q_r)\in C(\mathbb{R};H^1_*(\mathbb{R}^3)\times\mathbb{R}^3)\cap C^1(\mathbb{R};L^2_*(\mathbb{R}^3)\times\mathbb{R}^3)$ be the mild solution of the Cauchy problem*

$$\begin{cases} \dfrac{1}{c^2}\ddot{A}_r =\Delta A_r+\dfrac{4\pi e}{c}M\dot{q}_r\rho_r \\ m_r\ddot{q}_r =-\dfrac{e}{c}\langle\rho_r,\dot{A}_r\rangle \\ A_r(0) =(-\Delta+\lambda)^{-\frac{1}{2}}X_0+\dfrac{4\pi e}{c}ML_r(X_0)(-\Delta+\lambda)^{-1}\rho_r \\ \dot{A}_r(0)=\dot{A}_0,\quad q_r(0)=q_0,\quad \dot{q}_r(0)=\dot{q}_0 , \end{cases} \tag{2.9}$$

with $\lambda>\lambda_0$, $\dot A_0\in L^2_*(\mathbb{R}^3)$, and with $X_0\in L^2_*(\mathbb{R}^3)$ such that

$$\lim_{r\downarrow 0}\ L_r(X_0)=\dot q_0,\quad \lim_{r\downarrow 0}\ \langle\rho_r,(-\Delta+\lambda)^{-\frac12}X_0\rangle=-\frac{mc}{e}\dot q_0\ . \tag{2.10}$$

Then $\forall T>0$

$$\begin{aligned}&\lim_{r\downarrow 0}\ \sup_{|t|\le T}\|(H_r+\lambda)^{\frac12}A_r(t)-(H_m+\lambda)^{\frac12}A(t)\|_2=0,\\ &\lim_{r\downarrow 0}\ \sup_{|t|\le T}\|\dot A_r(t)-\dot A(t)\|_2=0,\\ &\lim_{r\downarrow 0}\ \sup_{|t|\le T}|\dot q_r(t)-Q_{A(t)}|=0\ ,\end{aligned} \tag{2.11}$$

where $A\in C(\mathbb{R};D(F_m))\cap C^1(\mathbb{R};L^2_*(\mathbb{R}^3))$ is the mild solution of the Cauchy problem

$$\begin{cases}\dfrac{1}{c^2}\ddot A=-H_mA\\ A(0)=(-\Delta+\lambda)^{-\frac12}X_0+\dfrac{4\pi e}{c}M\dot q_0G_\lambda\\ \dot A(0)=\dot A_0\ .\end{cases} \tag{2.12}$$

Moreover if X_0 does not satisfy the relations (2.10) *then there is no convergence of the A_r's.*

Remark 2.10. If in the corollary 2.9 we are concerned with strict solutions of (2.9), i.e. if $A_r(0)=(H_r+\lambda)^{-1}Y_0$, $\dot A_0=(H_r+\lambda)^{-\frac12}\dot Y_0$, $Y_0,\dot Y_0\in L^2_*(\mathbb{R}^3)$, then, by remark 2.7, the (2.11)'s hold under the condition

$$(-\Delta+\lambda)^{-1}Y_0(0)=\frac{4\pi e}{c}\Gamma_m(\lambda)\dot q_0\ ,$$

and $A(t)$ is now the strict solution of (2.12) with initial data

$$A(0)=(-\Delta+\lambda)^{-1}Y_0+\frac{4\pi e}{c}M\dot q_0G_\lambda,\quad \dot A(0)=(H_m+\lambda)^{-\frac12}\dot Y_0\ .$$

Proof of corollary 2.9. The assertions about the convergence of the A_r's immediately follows from the previous theorem equating the initial data $\dot q_0, X_0$ to the elements defining the vector v_r. Let us now prove the convergence of the q_r's.
Let $C_r(t)$, $S_r(t)$ and $C_m(t)$, $S_m(t)$ be the cosine and the sine operator functions of c^2H_r and c^2H_m respectively. Then

$$\begin{aligned}A_r(t)&=C_r(t)(H_r+\lambda)^{-\frac12}X_0+S_r(t)\dot A_0+4\pi ec\int_0^t S_r(s)Mv_r\rho_r ds\\ &\equiv(H_r+\lambda)^{-\frac12}X_r(t)\ ,\end{aligned}$$

where

$$X_r(t):=C_r(t)X_0+(H_r+\lambda)^{\frac12}\left(S_r(t)\dot A_0+4\pi ec\int_0^t S_r(s)Mv_r\rho_r ds\right)\ ,$$

and

$$A(t) = C_m(t)(H_m + \lambda)^{-\frac{1}{2}} X_0 + S_m(t)\dot{A}_0 \equiv (H_m + \lambda)^{-\frac{1}{2}} X(t) \ ,$$

where

$$X(t) := C_m(t) X_0 + (H_m + \lambda)^{\frac{1}{2}} S_m(t) \dot{A}_0 \ .$$

By the previous theorem

$$\lim_{r\downarrow 0} \sup_{|t|\le T} \|X_r(t) - X(t)\|_2 = 0 \ .$$

Since

$$\dot{q}_r(t) = -\frac{e}{m_r c}\langle \rho_r, A_r(t)\rangle + v_r$$

$$= -\frac{e}{c}\frac{\langle \rho_r, (-\Delta+\lambda)^{-\frac{1}{2}} X_r(t)\rangle}{m_r} - \frac{8\pi e^2}{3c^2} L_r(X_r(t)) \frac{\langle (-\Delta+\lambda)^{-1}\rho_r, \rho_r\rangle}{m_r} + v_r \ ,$$

the thesis follows from

$$\lim_{r\downarrow 0} \ v_r = 0,$$

$$\lim_{r\downarrow 0} \sup_{|t|\le T} \left| \frac{\langle \rho_r, (-\Delta+\lambda)^{-\frac{1}{2}} X_r(t)\rangle}{m_r} \right| = 0 \ ,$$

$$\lim_{r\downarrow 0} \sup_{|t|\le T} |L_r(X_r(t)) - L_m(X(t))| = 0 \ ,$$

$$\lim_{r\downarrow 0} \frac{\langle (-\Delta+\lambda)^{-1}\rho_r, \rho_r\rangle}{m_r} = -\frac{3c^2}{8\pi e^2} \ .$$

□

In the next theorem we study the case in which no mass renormalization is done, i.e. m_r =const. By abuse of notation we will denote such a constant by the symbol m. In this situation we prove that the limit dynamics is trivial (As regards the particle dynamics, and for zero initial data for the field, the same result already appeared in [BN]).

Theorem 2.11. *Let* $(A_r, q_r) \in C(\mathbb{R}; H^2_*(\mathbb{R}^3) \times \mathbb{R}^3) \cap C^2(\mathbb{R}; L^2_*(\mathbb{R}^3) \times \mathbb{R}^3)$ *be the strict solution of the Cauchy problem*

$$\begin{cases} \dfrac{1}{c^2}\ddot{A}_r = \Delta A_r + \dfrac{4\pi e}{c} M \dot{q}_r \rho_r \\ m\ddot{q}_r = -\dfrac{e}{c}\langle \rho_r, \dot{A}_r\rangle \\ A_r(0) = A_0 \in H^2_*(\mathbb{R}^3), \quad \dot{A}_r(0) = \dot{A}_0 \in H^1_*(\mathbb{R}^3), \quad q_r(0) = q_0, \quad \dot{q}_r(0) = \dot{q}_0 \ . \end{cases}$$

Then $\forall T > 0$

$$\begin{aligned} &\lim_{r\downarrow 0} \sup_{|t|\le T} \|A_r(t) - A_f(t)\|_2 = 0 \ , \\ &\lim_{r\downarrow 0} \sup_{|t|\le T} |q_r(t) - q_0| = 0 \ , \end{aligned} \qquad (2.13)$$

where A_f is the solution of the free wave equation with the same initial data $A_0, \dot{A}_0$.

Proof. By remark 2.2 H_r converges in norm resolvent sense to $-\Delta$. Therefore, proceeding as in thm. 2.8, the first formula in (2.13) follows from

$$\begin{aligned}&\lim_{r\downarrow 0}\ \|(H_r+\lambda)^{-1}Mv_r\rho_r\|_2\\ =&\lim_{r\downarrow 0}\ \left(1+\frac{2}{3}\Gamma_r(\lambda)^{-1}\langle(-\Delta+\lambda)^{-1}\rho_r,\rho_r\rangle\right)\|(-\Delta+\lambda)^{-1}Mv_r\rho_r\|_2=0\ .\end{aligned}$$

Since, if $t\ge 0$,

$$\begin{aligned}q_r(t)=&q_0+v_rt-\frac{e}{mc}\int_0^t\langle\rho_r,A_r(s)\rangle ds\\ =&q_0+v_rt-\frac{e}{mc}\left(\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{1}{c^2}e^{zt}\langle\rho_r,(H_r+(z/c)^2)^{-1}A_0\rangle\ dz\right.\\ &+\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{1}{c^2}e^{zt}\frac{1}{z}\langle\rho_r,(H_r+(z/c)^2)^{-1}\dot{A}_0\rangle\ dz\\ &\left.+\,4\pi ec\ \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{1}{c^2}e^{zt}\frac{1}{z^2}\langle\rho_r,(H_r+(z/c)^2)^{-1}Mv_r\rho_r\rangle\ dz\right)\ ,\end{aligned}$$

by

$$\begin{aligned}&\lim_{r\downarrow 0}\ \langle\rho_r,(H_r+z^2)^{-1}X\rangle\\ =&\lim_{r\downarrow 0}\ \left(1+\frac{2}{3}\Gamma_r(z^2)^{-1}\langle(-\Delta+z^2)^{-1}\rho_r,\rho_r\rangle\right)\langle\rho_r,(-\Delta+z^2)^{-1}X\rangle=0\ ,\end{aligned}$$

$X\in L^2_*(\mathbb{R}^3)$, $-z^2\in\rho(H_m)$, by

$$\lim_{r\downarrow 0}\ v_r=\dot{q}_0+\lim_{r\downarrow 0}\ \frac{e}{mc}\langle\rho_r,A_0\rangle=\dot{q}_0+\frac{e}{mc}A_0(0)\ ,$$

by

$$\lim_{r\downarrow 0}\ \langle\rho_r,(H_r+z^2)^{-1}Mv_r\rho_r\rangle=\frac{mc^2}{4\pi e^2}\left(\dot{q}_0+\frac{e}{mc}A_0(0)\right)\ ,$$

and by Lebesgue dominated convergence theorem, we obtain

$$\lim_{r\downarrow 0}\ \sup_{|t|\le T}|q_r(t)-q_0|=0\ ,$$

and the proof is done. □

3. The limit dynamics.

The results in corollary 2.9 induce us to declare that the system

$$\begin{cases} \frac{1}{c^2}\ddot{A} = -H_m A \\ \dot{q} = Q_A \\ A(0) = A_0 \in D(F_m),\ Q_{A_0} = \dot{q}_0,\ \dot{A}(0) = \dot{A}_0 \in L^2_*(\mathbb{R}^3),\ q(0) = q_0\ . \end{cases} \tag{3.1}$$

describes the classical electrodynamics of a point charged particle in the dipole approximation.

Remark 3.1. The equations (3.1) strongly suggest what could be the point limit of the non linear complete Maxwell–Lorentz system

$$\begin{cases} \frac{1}{c^2}\ddot{A} = \Delta A + \frac{4\pi e}{c} M\dot{q}\rho_q \\ \frac{d}{dt} m_0\gamma_{\dot{q}}\dot{q} = -\frac{e}{c}\int_{\mathbb{R}^3} \rho_q(x)\dot{A}(x)\ dx + \frac{e}{c}\dot{q}\wedge\int_{\mathbb{R}^3}\rho_q(x)\mathrm{curl}\ A(x)\ dx \\ A(0) = A_0,\quad \dot{A}(0) = \dot{A}_0,\quad q(0) = q_0,\quad \dot{q}(0) = \dot{q}_0\ , \end{cases}$$

where $\gamma_{\dot{q}} := (1 - |\dot{q}|^2/c^2)^{-1/2}$, and $\rho_q(x) := \rho(x-q)$. In such a general case we conjecture the limit equations

$$\begin{cases} \frac{1}{c^2}\ddot{A}(t) = -H_{m,q(t)}A(t) \\ \dot{q}(t) = \Phi(Q_{A(t)}) \\ A(0) = A_0 \in D(F_m(0)),\ Q_{A_0} = \dot{q}_0,\ \dot{A}(0) = \dot{A}_0 \in L^2_*(\mathbb{R}^3),\ q(0) = q_0\ , \end{cases} \tag{3.2}$$

for some vector field Φ. Here $H_{m,q(t)}$ is the time–dependent self–adjoint operator with resolvent given by

$$(H_{m,q(t)} + \lambda)^{-1} = (-\Delta+\lambda)^{-1} + \Gamma_m(\lambda)^{-1} M\cdot G_{\lambda,q(t)}\otimes G_{\lambda,q(t)}\ ,$$

where

$$G_{\lambda,y}(x) := \frac{1}{4\pi}\frac{e^{-\sqrt{\lambda}|x-y|}}{|x-y|}\ .$$

Let us remark that the non linear system (3.2) poses a non trivial mathematical problem. Indeed, since neither the form domain $D(F_m(t))$ of $H_{m,q(t)}$ is time–independent, nor the resolvent $(H_{m,q(t)}+\lambda)^{-1}$ is a regular function w.r.t. t, we can not use known general results in order to prove that (3.2) has a solution. The results in [DFT] will be surely useful for this point.

The next theorem gives us the solution of (3.1).
Theorem 3.2. *Let $C_m(t)$, $S_m(t)$ the cosine and the sine operator functions of c^2H_m and let $C_0(t)$, $S_0(t)$ the cosine and the sine operator functions of $-c^2\Delta$. Then, $\forall t\geq 0$,*

$$C_m(t) = C_0(t) + M\cdot K_1(t) + c\sqrt{\lambda_0}M\cdot K_2(t)\ ,\qquad S_m(t) = S_0(t) + M\cdot K_2(t)\ ,$$

where

$$K_1(t)A(x) = \frac{3}{8\pi}\frac{\theta(ct-|x|)}{|x|}\frac{1}{ct-|x|}\int_{S_{ct-|x|}} A(y)d\mu_{ct-|x|}(y) \ ,$$

and $K_2(t)$ is the Hilbert–Schmidt operator with kernel given by

$$K_2(t;x,y) = \frac{3}{8\pi c}\frac{1}{|x|\,|y|}\theta(ct-|x|-|y|)\exp\sqrt{\lambda_0}(ct-|x|-|y|) \ .$$

Here θ denotes the Heaviside function. Moreover $\dot{C}_m(t)$, which is defined on $D(F_m)$, is given by

$$\dot{C}_m(t) = \dot{C}_0(t) + c\sqrt{\lambda_0}M\cdot K_1(t) + c^2\lambda_0 M\cdot K_2(t) + cM\cdot K_1(t)\cdot\nabla_r + M\cdot K_3(t) \ ,$$

where ∇_r denotes the radial derivative and

$$K_3(t)A(x) = -\frac{c}{ct-|x|}K_1(t)A(x) \ .$$

Proof. By the explicit expression for the resolvent of H_m we have $C_m(t) = C_0(t) + M\cdot K(t)$ where $K(t)$ has a Laplace transform given by the integral operator with kernel

$$\begin{aligned}\tilde{K}(z;x,y) =&\frac{z}{c^2}\Gamma_m(z^2/c^2)G_{z^2/c^2}(x)G_{z^2/c^2}(y)\\ =&\frac{3}{8\pi c}\ \frac{2ze^2}{-3mc^3+2e^2z}\ \frac{1}{|x|\,|y|}\exp-\frac{z}{c}(|x|+|y|)\\ =&\frac{3}{8\pi}\left(\frac{1}{c}+\frac{3mc^2}{2e^2}\ \frac{2e^2}{-3mc^3+2e^2z}\right)\ \frac{1}{|x|\,|y|}\exp-\frac{z}{c}(|x|+|y|) \ .\end{aligned}$$

Since (here we use the same trick as in [ABD, §3.1])

$$\frac{2e^2}{-3mc^3+2e^2z}\exp-\frac{z}{c}(|x|+|y|) =$$

$$= \frac{\exp-\frac{z}{c}(|x|+|y|)}{z} + \frac{3mc^2}{2e^2}\int_0^\infty \exp\frac{3mc^2}{2e^2}s\ \frac{\exp-\frac{z}{c}(|x|+|y|+s)}{z}ds \ ,$$

since δ_{t-s} and $\theta(t-s)$ are the inverse Laplace transforms of e^{-zs} and $(1/z)e^{-zs}$ respectively, and since

$$\theta(ct-|x|-|y|) + \frac{3mc^2}{2e^2}\int_0^\infty \exp\frac{3mc^2}{2e^2}s\ \theta(ct-|x|-|y|-s)ds =$$

$$= \theta(ct-|x|-|y|)\exp\frac{3mc^2}{2e^2}(ct-|x|-|y|) \ ,$$

the statements about $C_m(t)$ and $\dot{C}_m(t)$ then readily follows. By the definition of $S_m(t)$ the proof is then concluded by straightforward calculations. □

Suppose now that $A_0 \in D(F_m)$, $\dot{A}_0 \in L^2_*(\mathbb{R}^3)$ are such that

$$Q_{A_0} = \dot{q}_0, \quad A_\lambda(r,\theta,\phi) \equiv A_0(r,\theta,\phi) - \frac{4\pi e}{c} M\dot{q}_0 G_\lambda(r) = A^1_\lambda(r)A^2_\lambda(\theta,\phi) \ ,$$

and

$$\dot{A}_0(r,\theta,\phi) = \dot{A}^1_0(r)\dot{A}^2_0(\theta,\phi)$$

(here we are using spherical coordinates). Then, if $\langle\cdot\rangle$ denotes the spherical mean,

$$K_1(t)A_0(x) = \left(\frac{3}{2}(ct-|x|)\langle A^2_\lambda\rangle A^1_\lambda(ct-|x|) + \frac{e}{c}\dot{q}_0 \ e^{-\sqrt{\lambda}(ct-|x|)}\right)\frac{\theta(ct-|x|)}{|x|} \ ,$$

$$\begin{aligned} K_2(t)A_0(x) = &\left(\frac{3}{2c}\langle A^2_\lambda\rangle e^{\sqrt{\lambda_0}(ct-|x|)}\int_0^{ct-|x|} A^1_\lambda(r)e^{-\sqrt{\lambda_0}r}rdr\right. \\ &\left. - \frac{e}{c^2}\dot{q}_0\frac{1}{\sqrt{\lambda_0}+\sqrt{\lambda}}\left(e^{-\sqrt{\lambda}(ct-|x|)} - e^{\sqrt{\lambda_0}(ct-|x|)}\right)\right)\frac{\theta(ct-|x|)}{|x|} \ , \\ K_2(t)\dot{A}_0(x) = &\left(\frac{3}{2c}\langle \dot{A}^2_0\rangle e^{\sqrt{\lambda_0}(ct-|x|)}\int_0^{ct-|x|} \dot{A}^1_0(r)e^{-\sqrt{\lambda_0}r}rdr\right)\frac{\theta(ct-|x|)}{|x|} \ . \end{aligned}$$

Since $C_0(t)A_\lambda + S_0(t)\dot{A}_0 \in H^1_*(\mathbb{R}^3)$, since

$$C_0(t)M\dot{q}_0G_\lambda = M\dot{q}_0\Phi_\lambda(t) \ ,$$

where

$$\Phi_\lambda(t,x) = \frac{\lambda}{4\pi}\int_{B_{ct}(x)}\frac{G_\lambda(y)}{|x-y|}dy - \frac{1}{4\pi}\frac{\theta(ct-|x|)}{|x|} + G_\lambda(x) \ ,$$

$B_r(x) := \{\ y \ : \ |x-y| < r\ \}$, we have that $C_0(t)A_0 + S_0(t)\dot{A}_0$ gives no contribution to $Q_{A(t)}$, and so

$$\begin{aligned} Q_{A(t)} = &\frac{3c^2}{2e}\ t\ \langle A^2_\lambda\rangle A^1_\lambda(ct) + \dot{q}_0 e^{-\sqrt{\lambda}ct} \\ &+ \frac{3c\sqrt{\lambda_0}}{2e}\langle A^2_\lambda\rangle e^{\sqrt{\lambda_0}ct}\int_0^{ct} A^1_\lambda(r)e^{-\sqrt{\lambda_0}r}rdr - \dot{q}_0\frac{\sqrt{\lambda_0}}{\sqrt{\lambda_0}+\sqrt{\lambda}}\left(e^{-\sqrt{\lambda}ct} - e^{\sqrt{\lambda_0}ct}\right) \\ &+ \frac{3}{2e}\langle \dot{A}^2_0\rangle e^{\sqrt{\lambda_0}ct}\int_0^{ct} \dot{A}^1_0(r)e^{-\sqrt{\lambda_0}r}rdr \ . \end{aligned} \tag{3.3}$$

Let us now consider strict solutions, i.e. we suppose that $A_0 \in D(H_m)$, $\dot{A}_0 \in D(F_m)$. Then

$$A_\lambda(0) = \langle A^2_\lambda\rangle A^1_\lambda(0) = \frac{4\pi e}{c}\Gamma_m(\lambda)\dot{q}_0 = \frac{3c}{2e}(\sqrt{\lambda} - \sqrt{\lambda_0})\dot{q}_0 \ ,$$

$$\dot{A}_\lambda(r,\theta,\phi) \equiv \dot{A}_0(r,\theta,\phi) - \frac{4\pi e}{c}MQ_{\dot{A}_0}G_\lambda(r) = \dot{A}^1_\lambda(r)\dot{A}^2_\lambda(\theta,\phi) \ ,$$

and

$$
\begin{aligned}
Q_{A(t)} = & \frac{3c^2}{2e}\, t\, \langle A^2_\lambda \rangle A^1_\lambda(ct) + \dot q_0 e^{-\sqrt{\lambda}ct} \\
& + \frac{3c\sqrt{\lambda_0}}{2e} \langle A^2_\lambda \rangle e^{\sqrt{\lambda_0}ct} \int_0^{ct} A^1_\lambda(r) e^{-\sqrt{\lambda_0}r} r dr - \dot q_0 \frac{\sqrt{\lambda_0}}{\sqrt{\lambda_0}+\sqrt{\lambda}} \left(e^{-\sqrt{\lambda}ct} - e^{\sqrt{\lambda_0}ct} \right) \\
& + \frac{3}{2e} \langle \dot A^2_\lambda \rangle e^{\sqrt{\lambda_0}ct} \int_0^{ct} \dot A^1_\lambda(r) e^{-\sqrt{\lambda_0}r} r dr - \frac{Q_{\dot A_0}}{c} \frac{1}{\sqrt{\lambda_0}+\sqrt{\lambda}} \left(e^{-\sqrt{\lambda}ct} - e^{\sqrt{\lambda_0}ct} \right) .
\end{aligned} \tag{3.4}
$$

Moreover $A \in C^1(\mathbb{R}; D(F_m))$, $Q_A \in C^1(\mathbb{R};\mathbb{R}^3)$, and

$$
\begin{aligned}
Q_{\dot A(t)} = \frac{d}{dt} Q_{A(t)} = & c\sqrt{\lambda_0} Q_{A(t)} - c\sqrt{\lambda}\dot q_0 e^{-\sqrt{\lambda}ct} + \frac{3c^2}{2e}\langle A^2_\lambda \rangle A^1_\lambda(ct) \\
& + \frac{3c^3}{2e}\, t\, \langle A^2_\lambda \rangle \frac{d}{dr} A^1_\lambda(ct) + Q_{\dot A_0} e^{-\sqrt{\lambda}ct} + \frac{3c^2}{2e} \langle \dot A^2_\lambda \rangle \dot A^1_\lambda(ct) \ .
\end{aligned} \tag{3.5}
$$

These results, and the classical Kirchhoff formula for the solution of the free wave equation, give us the following

Theorem 3.3. *Let $A \in C(\mathbb{R}; D(F_m)) \cap C^1(\mathbb{R}; L^2_*(\mathbb{R}^3))$ be the mild solution of the Cauchy problem*

$$
\begin{cases}
\dfrac{1}{c^2}\ddot A = -H_m A \\
A(0) = A_0 \in D(F_m),\ Q_{A_0} = \dot q_0, \quad \dot A(0) = \dot A_0 \in L^2_*(\mathbb{R}^3) \ .
\end{cases}
$$

Then

$$
A(t) = A_f(t) + \frac{4\pi e}{c} M A_\delta(t),
$$

where $A_f(t)$ is the solution of the free wave equation with initial data $A_0, \dot A_0$, and

$$
A_\delta(t,x) = \frac{1}{4\pi} \frac{\theta(ct-|x|)}{|x|} Q_{A(t-|x|/c)} \ .
$$

Moreover if $A_0 \in D(H_m)$ and $\dot A_0 \in D(F_m)$, i.e. if we are dealing with strict solutions, then $Q_A \in C^1(\mathbb{R};\mathbb{R}^3)$, and it solves the Cauchy problem

$$
\begin{cases}
\dot Q_{A(t)} = c\sqrt{\lambda_0} Q_{A(t)} + \dfrac{3c^2}{2e} A_f(t,0) \\
Q_{A(0)} = \dot q_0 \ .
\end{cases}
$$

Remark 3.4. Let us note that in the above Cauchy problem the function $t \mapsto A_f(t,0)$ is well defined for any $t>0$, and, by (3.4) and the Kirchhoff formula, one has

$$
\lim_{t\downarrow 0} A_f(t,0) = \frac{2e}{3c^2}\left(Q_{\dot A_0} - c\sqrt{\lambda_0}\dot q_0 \right) \ .
$$

Remark 3.5. One can rephrase the previous theorem saying that the strict solution of the Cauchy problem

$$\begin{cases} \frac{1}{c^2}\ddot{A} = -H_m A \\ A(0) = A_0 \in D(H_m),\ Q_{A_0} = \dot{q}_0, \quad \dot{A}(0) = \dot{A}_0 \in D(F_m) \end{cases}$$

coincides with the first component of (A, Q), the solution of

$$\begin{cases} \frac{1}{c^2}\ddot{A} = \Delta A + \frac{4\pi e}{c} M Q \delta_0 \\ \dot{Q}(t) = c\sqrt{\lambda_0} Q(t) + \frac{3c^2}{2e} A_f(t,0) \\ A(0) = A_0 \in D(H_m),\ Q_{A_0} = \dot{q}_0, \quad \dot{A}(0) = \dot{A}_0 \in D(F_m), \quad Q(0) = \dot{q}_0\ . \end{cases}$$

Moreover if $A_\lambda \in H^3_*(\mathbb{R}^3)$, and $\dot{A}_\lambda \in H^2_*(\mathbb{R}^3)$, then, by (3.5), $Q_A \in C^2(\mathbb{R}; \mathbb{R}^3)$ and one obtains, defining

$$q(t) := q_0 + \int_0^t Q_{A(s)} ds\ ,$$

the Abraham–Lorentz–Dirac equation

$$\begin{cases} -m\tau_0 \dddot{q}(t) + m\ddot{q}(t) = -\frac{e}{c}\dot{A}_f(t,0) \\ q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0, \quad \ddot{q}(0) = Q_{\dot{A}_0}\ , \end{cases}$$

where

$$\tau_0 := \frac{2e^2}{3mc^3}\ .$$

Observe that $-\frac{e}{c}\dot{A}_f(t,0)$ is nothing but the (linearized) Lorentz force due to the free field evaluated at $x = 0$. Moreover let us point out that the relation $\ddot{q}(0) = Q_{\dot{A}_0}$, as the correct initial acceleration for the Abraham–Lorentz–Dirac equation, already appeared in [B].

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