Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations

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Abstract. We construct an invariant manifold of periodic orbits for a class of non-linear Schrödinger equations. Using standard ideas of the theory of center manifolds, we rederive the results of Soffer and Weinstein ([SW1], [SW2]) on the large time asymptotics of small solutions (scattering theory).

1. Introduction

In this paper we illustrate the applicability of invariant manifold techniques to a class of partial differential equations. While invariant manifold theorems have found a host of uses in the study of the time evolution of dissipative pde's, this is the first case we know of in which they have been applied to dispersive equations. (Though some intriguing non-rigorous computations using invariant manifold theory in dispersive equations are presented in [R]). The problem we consider here is the behavior of solutions with small initial data for the equation

$$i\,\frac{\partial\phi}{\partial t} = (-\Delta + V)\phi + \lambda|\phi|^{m-1}\phi, \qquad (x,t) \in \mathbf{R}^n \times \mathbf{R}, \tag{1.1}$$

where $\lambda \in \mathbf{R}$ and the potential $V(x)$ is chosen so that the spectrum of the linear part $H_0 \equiv -\Delta + V$ consists of a simple eigenvalue $E_0 < 0$ and absolutely continuous spectrum filling the positive real half-line.

As is known from the work of Soffer and Weinstein ([SW1] and [SW2]), as time goes to infinity solutions with small initial data will approach a periodic orbit of definite period and phase. Their method of proof is to make the Ansatz that the solution can be written as a sum of a periodic piece with time dependent amplitude and phase and a dispersive piece. They then derive modulation equations for these quantities and prove that these equations have solutions. As we prove below, this Ansatz can be avoided by the use of the invariant manifold theory. This theory allows one to split the phase space into two pieces – one spanned by the eigenfunction of the linear problem with eigenvalue E_0, and one corresponding to the continuous spectrum of H_0, and work entirely in terms of this splitting. This should be contrasted with the modulation equation approach which involves a time dependent splitting of the phase space – essentially determining at each time, t, a part of the solution which is periodic, and a part which is dispersive. Because the way in which we split the solution is time independent, the estimates necessary to establish convergence of solutions with small initial data to one of the periodic solution of the non-linear problem are considerably simplified. We simply demonstrate the existence of a "center-manifold" for the non-linear problem, (which consists of periodic orbits bifurcating from the eigenfunction with eigenvalue E_0 of H_0) and then show that all orbits near that invariant manifold approach it as $t \to \infty$. This latter step is quite standard in classical treatments of center manifold theory [C] though its implementation here is slightly complicated by the dispersive nature of the problem. However, we also note that while we believe the present approach has advantages in terms of simplicity and intuitiveness, we have so far only recovered the results of Soffer and Weinstein, not extended them.

To explain our results more precisely, we make the following assumptions.

(H1) $n \geq 3$.

(H2) $\max\left(2, 1 + \frac{4}{n}\right) < m < \frac{n+2}{n-2}$.

(H3) The potential $V(x)$ satisfies:

(i) There exists constants $\sigma > n$ and $n/2 < \eta \leq 3$ such that the multiplication operator $(1 + |x|^2)^{\sigma/2}V(x)$ is bounded on the Sobolev space $H^\eta(\mathbf{R}^n)$.

(ii) $\widehat{V} \in L^1(\mathbf{R}^n)$.

(iii) 0 is neither an eigenvalue nor a resonance of H_0.

(iv) H_0 acting on $L^2(\mathbf{R}^n)$ has exactly one negative eigenvalue $E_0 < 0$ with normalized eigenfunction ψ_0.

Remark 1. Hypotheses H1-H2 impose an upper bound to the space dimension, namely $n \leq 5$. Note also that for integer m, the only choices are $m = 3, 4$ and $n = 3$.

Remark 2. Conditions (i)–(iii) of Hypothesis H3 guarantee the applicability of the decay estimates of Journé, Soffer, and Sogge [JSS] to the Schrödinger group $\mathrm{e}^{-iH_0 t}$.

We now proceed as follows. For the linear Schrödinger equation

$$i\frac{\partial \phi}{\partial t} = H_0\phi,$$

there is a manifold of periodic orbits,

$$E^p \equiv \left\{\rho \mathrm{e}^{i\theta}\psi_0(x)\colon\ \rho \geq 0 \quad \text{and} \quad 0 \leq \theta \leq 2\pi\right\},$$

and all solutions $\phi(x, t)$ with initial conditions $\phi(x, 0) \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ approach one of the periodic orbits $\rho \mathrm{e}^{-i(E_0 t - \theta)}\psi_0(x) \in E^p$, due to the dispersion of the part of the solution in the continuous spectral subspace. We will show in the next section that the nonlinear equation (1.1) has an invariant manifold W^p (the center manifold) which is close to E^p. On W^p all orbits are periodic of the form $\mathrm{e}^{-i(Et-\theta)}\psi_E(x)$, where ψ_E is a non-linear bound state: A positive solution of

$$(H_0 + \lambda|\psi_E(x)|^{m-1})\psi_E = E\psi_E.$$

We will then demonstrate in the two succeeding sections that all small solutions of (1.1) approach one of the periodic orbits in W^p as time goes to infinity, thereby recovering Theorem 2.4 of [SW2].

Let P_c and P_p be the projections in $L^2(\mathbf{R}^n)$ onto the continuous and pure point spectral subspaces of H_0. If we apply the projections P_c and P_p to (1.1) and use the fact that Ran(P_p) is one dimensional, we can rewrite (1.1) as a system:

$$\begin{aligned} i\,\dot{u}_p &= E_0 u_p + \lambda f_p(u_p, u_c), \\ i\,\dot{u}_c &= H_0 u_c + \lambda f_c(u_p, u_c), \end{aligned} \tag{1.2}$$

where $u_p \in \mathbf{C}$, $u_c \in$ Ran(P_c), and

$$\begin{aligned} f_p(u_p, u_c) &\equiv \langle \psi_0, |u_p\psi_0 + u_c|^{m-1}(u_p\psi_0 + u_c)\rangle, \\ f_c(u_p, u_c) &\equiv P_c|u_p\psi_0 + u_c|^{m-1}(u_p\psi_0 + u_c). \end{aligned} \tag{1.3}$$

If we think of E^p as the "central subspace" in the ordinary center manifold theorem, we would be led to look for a function $h\colon E^p \to E^c \equiv$ Ran(P_c), whose graph is invariant under the flow generated by (1.2). In the next section we will prove that such a function exists and that the "center manifold" defined by its graph is filled with periodic orbits. We then show that all small solutions of (1.2) approach this center-manifold and, in an extension of this argument, that they approach a particular periodic orbit on the manifold of given period and phase.

Notation. We will work primarily with the spaces $L^p(\mathbf{R}^n)$ and the weighted spaces $L^2_\sigma(\mathbf{R}^n)$ in this paper. We define their norms as follows:

$$\begin{aligned} |f|_p &\equiv \left(\int |f(x)|^p dx\right)^{1/p}, \\ \|f\|_\sigma &\equiv \left(\int (1+|x|^2)^\sigma |f(x)|^2 dx\right)^{1/2}. \end{aligned}$$

The quantity $(1+|x|^2)^{\sigma/2}$ will arise frequently and we denote it as $\langle x\rangle^\sigma$. Finally, on a few occasions we will need the ordinary Sobolev spaces $H^s(\mathbf{R}^n)$, whose norms we will denote by $\|f\|_{H^s}$. If X and Y are two of the above spaces, we define the norm on $X \cap Y$ by

$$\|f\|_{X\cap Y} \equiv \max(\|f\|_X, \|f\|_Y).$$

Our main result is:

Theorem 1.1. *Suppose that Hypotheses H1–H3 are satisfied. If $\|\phi_0\|_{H_1\cap L^2_\sigma}$ is sufficiently small, there exists smooth functions $E(t)$, $\theta(t)$ such that the limits*

$$\begin{aligned} E_\pm &\equiv \lim_{t\to\pm\infty} E(t), \\ \theta_\pm &\equiv \lim_{t\to\pm\infty} \theta(t), \end{aligned} \tag{1.4}$$

exist and

$$\lim_{t\to\pm\infty}\left|\phi(t)-\mathrm{e}^{-i\left(\int_0^t E(s)ds-\theta(t)\right)}\psi_{E(t)}\right|_{m+1}=0, \tag{1.5}$$

where $\phi(t)$ is the solution of (1.1) with initial condition ϕ_0.

Remark. One can also give estimates on the rate of convergence in (1.4) and (1.5). See the remark at the end of Sections 3 and Equ. (4.4), (4.5).

2. Motions on the invariant manifold

In the present example, the invariant manifold is very simple, and almost explicitly constructible. It consists of a family of periodic orbits of the form $\mathrm{e}^{iEt}\psi_E(x)$, where ψ_E is a family of nonlinear eigenfunctions "close" to ψ_0. If one substitutes $u(x,t)=\mathrm{e}^{iEt}\psi_E(x)$ into (1.1), one sees that ψ_E must satisfy the following non-linear eigenvalue problem

$$(-\Delta+V(x)+\lambda|\psi_E(x)|^{m-1})\psi_E(x)=E\psi_E(x). \tag{2.1}$$

The existence and properties of solutions of (2.1) was discussed extensively in [SW1]. They proved:

Theorem 2.1. *For E close to E_0, there exists a positive solution $\psi_E(x)$ of Equ. (2.1) such that:*

(a) $\psi_E\in H^{\eta+2}(\mathbf{R}^n)$.

(b) The function $E\mapsto\|\psi_E\|_{H^2}$ is smooth for $E\neq E_0$ and

$$\lim_{E\to E_0}\|\psi_E\|_{H^2}=0.$$

(c) For any $\sigma\in\mathbf{R}$, there exists a finite constant C_σ such that

$$\|\langle x\rangle^\sigma\psi_E\|_{H^2}\le C_\sigma\|\psi_E\|_{H^2}.$$

Define $W_\mu^p\equiv\{\mathrm{e}^{i\theta}\psi_E(x)\colon |E-E_0|<\mu \quad\text{and}\quad 0\le\theta\le 2\pi\}$. Then W_μ^p is a two (real) dimensional invariant manifold for (1.1). It will be the "local center manifold" in our interpretation.

Many properties of W_μ^p can be read off immediately from Theorem 2.1, however, we will be particularly concerned with its form when we choose special coordinates. Keeping in mind our goal of viewing this problem from the perspective of the classical invariant manifold theorem, we will write W^p_μ as the graph of a function from the linear subspace spanned by ψ_0, into its complement Ran(P_c). Given a point $e^{i\theta}\psi_E(x) \in W^p_\mu$, we write it as

$$e^{i\theta}\psi_E = u_p\psi_0 + h(u_p), \tag{2.2}$$

where $u_p \in \mathbf{C}$ and $h(u_p) \in \mathrm{Ran}(P_c)$.

If we substitute $u_p\psi_0 + h(u_p)$ into (2.1) we obtain the pair of equations:

$$\begin{aligned} h(u_p) &= -\lambda(H_0 - E)^{-1} f_c(u_p, h(u_p)), \\ E_0 - E &= -\lambda u_p^{-1} f_p(u_p, h(u_p)), \end{aligned} \tag{2.3}$$

where the functions f_c and f_p are given by Equ. (1.3). The first thing we note about these equations is that for any $\theta \in \mathbf{R}$, $u_p \in \mathbf{C}$ and $u_c \in \mathrm{Ran}(P_c)$, we have $f_c(e^{i\theta}u_p, e^{i\theta}u_c) = e^{i\theta} f_c(u_p, u_c)$. A similar identity holds for the function f_p. Therefore,

$$h(u_p) = \frac{u_p}{|u_p|} h(|u_p|),$$

and it suffices to consider h as a real function of a real variable r.

If $I_1 = (E_0 - \delta, E_0 + \delta)$ and $I_2 = (-\delta, \delta)$, one can show that

$$\mathcal{F}(E, r, h) \equiv h + \lambda(H_0 - E)^{-1} f_c(r, h),$$

is a C^1 function from $I_1 \times I_2 \times L^2_\sigma \cap H^\eta$ to $L^2_\sigma \cap H^\eta$, provided δ is sufficiently small (see the Appendix for a sketch of the proof). Furthermore, $\mathcal{F}(E, 0, 0) = 0$ and $D_h\mathcal{F}(E, 0, 0) = I$, so by the implicit function theorem there exists a smooth function $\tilde{h}(E, r)$, from $I_1 \times I_2$ to $L^2_\sigma \cap H^\eta$ such that $\tilde{h}(E, 0) = 0$ and $\mathcal{F}(E, r, \tilde{h}(E, r)) = 0$. If one substitutes $\tilde{h}(E, r)$ into the second equation in (2.3) and applies the implicit function theorem a second time (see the Appendix for details), one obtains a C^1 function $E(r)$ such that $E(0) = E_0$ and

$$E_0 - E(r) = -\lambda r^{-1} f_p(r, \tilde{h}(E(r), r)). \tag{2.4}$$

If we now define

$$h(u_p) \equiv \frac{u_p}{|u_p|}\, \tilde{h}(E(|u_p|), |u_p|), \tag{2.5}$$

then $h(u_p)$ and $E(|u_p|)$ satisfy the system (2.3). Moreover one can check that $h(u_p)$ is C^1, thus we have

Proposition 2.2. *For μ sufficiently small, there exists $\delta > 0$ and a C^1 function*

$$h\colon \{u_p \in \mathbf{C}\colon |u_p| < \delta\} \to L^2_\sigma(\mathbf{R}^n) \cap H^\eta(\mathbf{R}^n),$$

such that the local center manifold is given by $W^p_\mu = \{\psi = u_p\psi_0 + h(u_p)\colon |u_p| < \delta\}$.

3. Approach to the invariant manifold

In the present section, we demonstrate that solutions starting near the center manifold will approach it. Recall that in Section 1 we wrote the solution $\phi(t)$ of the Schrödinger equation (1.1) as $\phi(t) = u_p(t)\psi_0 + u_c(t)$, where $u_c \in \mathrm{Ran}(P_c)$, while in the previous section we showed that the center manifold can be written as $u_p\psi_0 + h(u_p)$ in a neighborhood of the origin. Define

$$\begin{aligned} \varepsilon &\equiv \|\phi(0)\|_{L^2_\sigma \cap H^1}, \\ z(t) &\equiv h(u_p(t)) - u_c(t). \end{aligned} \tag{3.1}$$

We will prove that any solution of (1.1) for which ε is sufficiently small, $z(t) \to 0$ as $t \to \infty$ and hence the solution approaches the invariant manifold. An elementary computation shows that $z(t) \in \mathrm{Ran}(P_c)$ satisfies

$$i\dot{z}(t) = H_0 z + \lambda N(u_p(t), z(t)), \tag{3.2}$$

where

$$\begin{aligned} N(u_p, z) = &\Big(f_c(u_p, h(u_p)) - f_c(u_p, h(u_p) - z)\Big) \\ &- Dh(u_p)\Big(f_p(u_p, h(u_p)) - f_p(u_p, h(u_p) - z)\Big). \end{aligned} \tag{3.3}$$

We will bound solutions of (3.2) in $L^2_{-\sigma} \cap L^{1+m}$. To this end, we start by estimating the non-linearity N in L^p for all $p \geq 1$.

Recalling the definitions (1.3), we write

$$\begin{aligned} f_c(u_p, u_c) &= P_c G(u_p\psi_0 + u_c), \\ f_p(u_p, u_c) &= \langle \psi_0, G(u_p\psi_0 + u_c) \rangle, \end{aligned} \tag{3.4}$$

where $G(z) \equiv |z|^{m-1}z$. Since $\psi_0 \in L^1 \cap L^\infty$, the projections P_p and $P_c = I - P_p$ are bounded operators on any L^p. An elementary calculation shows that, for $m > 2$ (Hypothesis H2) one has

$$|G(a+b) - G(a)| \leq m\, 2^{m-2} \left(|a|^{m-1} + |b|^{m-1}\right) |b|, \tag{3.5}$$

for arbitrary $a, b \in \mathbf{C}$. Using this information we find that the first contribution to N in Equ. (3.3) satisfies the estimate

$$\begin{aligned} &|f_c(u_p, h(u_p)) - f_c(u_p, h(u_p) - z)|_p \\ &\qquad \leq C \left(|\, |u_p\psi_0 + h(u_p)|^{m-1} |z|\, |_p + |\, |z|^m |_p \right), \end{aligned} \tag{3.6}$$

for some constant C.

Notation. Here and in the sequel, C denotes a generic positive constant whose value may change from one equation to the other.

To estimate the second contribution to N remark that, by Proposition 2.2, the derivative $Dh(u_p)\colon \mathbf{C} \to L^2_\sigma \cap H^\eta$ is bounded as long as $|u_p| < \delta$. By Hypothesis H3 one has the following continuous inclusions: $L^2_\sigma \subset L^1 \cap L^2$ since $\sigma > n/2$, and $H^\eta \subset L^2 \cap L^\infty$ since $\eta > n/2$. Therefore, $L^2_\sigma \cap H^\eta \subset L^1 \cap L^\infty$, and we have

$$\begin{aligned} &|Dh(u_p)(f_p(u_p, h(u_p)) - f_p(u_p, h(u_p) - z))|_p \\ &\qquad \le C\left(\left| |u_p\psi_0 + h(u_p)|^{m-1}|z| \right|_p + \left| |z|^m \right|_p \right). \end{aligned} \tag{3.7}$$

Combining Definition (3.3) with the estimates (3.6) (3.7), we obtain

$$|N(u_p, z)|_p \le C\left(\left| |u_p\psi_0 + h(u_p)|^{m-1}|z| \right|_p + \left| |z|^m \right|_p \right), \tag{3.8}$$

for $p \ge 1$.

Rewriting Equ. (3.2) as an integral equation gives

$$z(t) = \mathrm{e}^{-iH_0t}z(0) + \lambda \int_0^t \mathrm{e}^{-iH_0(t-s)}N(u_p(s), z(s))ds. \tag{3.9}$$

We start by estimating the first term on the right hand side of (3.9). For $|t| \le 1$, the continuous inclusion $H^1 \subset L^{1+m}$ (Hypotheses H1-H2), and the fact that V is bounded on H^1 (Hypothesis H3) easily lead to

$$|\mathrm{e}^{-iH_0t}\psi|_{1+m} \le C\,\|\psi\|_{H^1}. \tag{3.10}$$

The bound

$$\|\mathrm{e}^{-iH_0t}\psi\|_{-\sigma} \le C\,|\psi|_2, \tag{3.11}$$

is immediate. For $t \ne 0$, we use the estimates in [JSS]:

$$\begin{aligned} |\mathrm{e}^{-iH_0t}\psi|_{1+m} &\le C|t|^{-\mu}|\psi|_{1+m^{-1}}, \\ \|\mathrm{e}^{-iH_0t}\psi\|_{-\sigma} &\le C|t|^{-n/2}\|\psi\|_\sigma, \end{aligned} \tag{3.12}$$

which hold for $\psi \in \mathrm{Ran}(P_c)$. Here we have set

$$\frac{1}{2} < \mu \equiv \frac{n}{2}\,\frac{m-1}{m+1} < 1, \tag{3.13}$$

the inequalities following from Hypotheses H1-H2. Combining (3.10) and (3.11) with (3.12), and using the continuous inclusion $L^2_\sigma \subset L^{1+m^{-1}}$ gives the desired estimates

$$\begin{aligned} |\mathrm{e}^{-iH_0t}\psi|_{1+m} &\le C\langle t\rangle^{-\mu}\,\|\psi\|_{L^2_\sigma\cap H^1}, \\ \|\mathrm{e}^{-iH_0t}\psi\|_{-\sigma} &\le C\langle t\rangle^{-n/2}\,\|\psi\|_{L^2_\sigma\cap H^1}, \end{aligned} \tag{3.14}$$

valid for $\psi \in \mathrm{Ran}(P_c)$.

Next we estimate the integral in Equ. (3.9), which we denote by $J(t)$. We start with the L^{1+m} norm. From the estimates (3.8) and (3.12) one has

$$|J(t)|_{1+m} \leq C \int_0^t |t-s|^{-\mu} \left(\left| |u_p(s)\psi_0 + h(u_p(s))|^{m-1} |z(s)| \right|_{1+m^{-1}} + \left| |z(s)|^m \right|_{1+m^{-1}} \right) . ds \tag{3.15}$$

To simplify this expression note that

$$\left| |z|^m \right|_{1+m^{-1}} = |z|^m_{1+m},$$

while

$$\left| |u_p\psi_0 + h(u_p)|^{m-1} |z| \right|_{1+m^{-1}} \leq |\langle x\rangle^\sigma \, |u_p\psi_0 + h(u_p)|^{m-1}|_{\frac{2m}{m+2}} \, \|z\|_{-\sigma}.$$

If we now recall that $u_p(t)\psi_0 + h(u_p(t))$ lies in W^p_μ, then we see from Equ. (2.2) that

$$u_p(t)\psi_0 + h(u_p(t)) = \mathrm{e}^{i\theta(t)} \psi_{e(t)},$$

where $\theta(t) = \mathrm{Arg}(u_p(t))$ and $e(t) = E(|u_p(t)|)$, with $E(r)$ given by Equ. (2.4). Since $|\phi(t)|_2 = |\phi(0)|_2 \leq \varepsilon$, we also see that $|u_p(t)| = |\langle\psi_0, \phi(t)\rangle| \leq \varepsilon$. Thus if $\varepsilon < \delta$ we have, by Theorem 2.1,

$$\begin{aligned} |\langle x\rangle^\sigma |u_p(s)\psi_0 + h(u_p(s))|^{m-1}|_{\frac{2m}{m+2}} &= |\langle x\rangle^{\sigma/m-1} \psi_{e(t)}|^{m-1}_{\frac{2m(m-1)}{m+2}} \\ &\leq C \, \|\langle x\rangle^{\sigma/m-1} \psi_{e(t)}\|^{m-1}_{H^2} \\ &\leq C \, \|\psi_{e(t)}\|^{m-1}_{H^2}, \\ &\leq C \, \nu(\varepsilon)^{m-1}, \end{aligned} \tag{3.16}$$

where we have introduced the function

$$\nu(\varepsilon) \equiv \sup_{|r|\leq\varepsilon} \|\psi_{E(r)}\|_{H^2}. \tag{3.17}$$

Thus we have obtained the bound

$$|J(t)|_{1+m} \leq C \int_0^t |t-s|^{-\mu} \left(\nu(\varepsilon)^{m-1} \|z(s)\|_{-\sigma} + |z(s)|^m_{1+m} \right) ds. \tag{3.18}$$

Following Soffer and Weinstein we now define

$$\begin{aligned} M_1(T) &= \sup_{|t|\le T} \langle t\rangle^{\zeta} \|z(t)\|_{-\sigma}, \\ M_2(T) &= \sup_{|t|\le T} \langle t\rangle^{\mu} |z(t)|_{1+m}, \\ M_3(T) &= \sup_{|t|\le T} |z(t)|_2, \end{aligned}$$

where $T > 0$ is arbitrary and $\zeta > 1$ will be specified below.

Multiplying both side of Equ. (3.18) by $\langle t\rangle^{\mu}$ we immediately get, for $|t| \le T$,

$$\langle t\rangle^{\mu} |J(t)|_{1+m} \le C \int_0^t \langle t\rangle^{\mu} |t-s|^{-\mu} \left(\nu(\varepsilon)^{m-1} M_1(T)\langle s\rangle^{-\zeta} + M_2(T)^m \langle s\rangle^{-m\mu}\right) ds.$$

To estimate this integral we use the simple fact that, for $0 < a < 1$ and $b > 1$, one has

$$\sup_{t\in\mathbf{R}} \int_0^t |t-s|^{-a} \langle t\rangle^{a} \langle s\rangle^{-b}\, ds < \infty.$$

To check that this formula is indeed applicable, recall that $\mu < 1$ (Equ. (3.13)) and $\zeta > 1$ by the above definition. Moreover, since $m > 2$ by Hypothesis H2, it follows from Equ. (3.13) that $m\mu > 1$. Thus we have shown that

$$\langle t\rangle^{\mu} |J(t)|_{1+m} \le C \left(\nu(\varepsilon)^{m-1} M_1(T) + M_2(T)^m\right), \tag{3.19}$$

for $|t| \le T$.

We now turn to the estimate of $J(t)$ in $L^2_{-\sigma}$. We consider $t > 0$, and split the integral in two pieces

$$\begin{aligned} J_1(t) &\equiv \int_0^{\max(0,t-1)} \mathrm{e}^{-iH_0(t-s)} N(u_p(s), z(s))\, ds, \\ J_2(t) &\equiv \int_{\max(0,t-1)}^{t} \mathrm{e}^{-iH_0(t-s)} N(u_p(s), z(s))\, ds. \end{aligned} \tag{3.20}$$

We bound $J_2(t)$ by estimating its integrand in the following way:

$$\|\mathrm{e}^{-iH_0(t-s)} N(u_p(s), z(s))\|_{-\sigma} \le |\langle x\rangle^{-\sigma}|_{2\frac{m+1}{m-1}}\ |\mathrm{e}^{-iH_0(t-s)} N(u_p(s), z(s))|_{1+m}.$$

The first factor on the right hand side is finite provided $\sigma > \mu$, which is true by Hypothesis H3 and Equ. (3.13). Thus J_2 can be handled in exactly the same way as (3.15), to get

$$\|J_2(t)\|_{-\sigma} \le C \int_{\max(0,t-1)}^{t} |t-s|^{-\mu} \left(\nu(\varepsilon)^{m-1} \|z(s)\|_{-\sigma} + |z(s)|^m_{1+m}\right) ds. \tag{3.21}$$

We now multiply both side of this inequality by $\langle t\rangle^\zeta$, and use the fact that, for $0 < a < 1$ and $0 < b \le c$,

$$\sup_{t\ge 0}\int_{\max(0,t-1)}^t |t-s|^{-a}\,\langle t\rangle^b \langle s\rangle^{-c}\,ds < \infty.$$

Under the condition

$$\zeta \le m\mu, \tag{3.22}$$

we can apply this formula to Equ. (3.21) to obtain the estimate

$$\langle t\rangle^\zeta \|J_2(t)\|_{-\sigma} \le C\left(\nu(\varepsilon)^{m-1}M_1(T) + M_2(T)^m\right), \tag{3.23}$$

for $0 \le t \le T$.

To bound $J_1(t)$ we start with the following estimate from [SW2]

$$\|\mathrm{e}^{-iH_0 t}\psi\|_{-\sigma} \le C|t|^{-n(1/p-1/2)}|\psi|_p,$$

valid for $\psi \in \mathrm{Ran}(P_c)$, $1 \le p \le 2$ and $\sigma > n/2$. Together with Estimate (3.8), this leads to

$$\begin{aligned}\|J_1(t)\|_{-\sigma} \le C\int_0^{\max(0,t-1)} &|t-s|^{-n(1/p-1/2)}\\ &\left(\big|\,|u_p(s)\psi_0 + h(u_p(s))|^{m-1}|z(s)|\,\big|_p + |z(s)|_{mp}^m\right)\,ds.\end{aligned} \tag{3.24}$$

The integrand in the last formula can be further estimated with the help of

$$\big|\,|u_p\psi_0 + h(u_p)|^{m-1}|z|\,\big|_p \le |\langle x\rangle^\sigma |u_p\psi_0 + h(u_p)|^{m-1}|_{2p/(2-p)}\,\|z\|_{-\sigma}.$$

Using again the properties of the periodic solutions $u_p(t)\psi_0 + h(u_p(t))$ described in Theorem 2.1, we obtain

$$\big|\,|u_p\psi_0 + h(u_p)|^{m-1}|z|\,\big|_p \le \nu(\varepsilon)^{m-1}\,\|z\|_{-\sigma}.$$

Inserting this estimate in Equ. (3.24) gives

$$\|J_1(t)\|_{-\sigma} \le C\int_0^{\max(0,t-1)} |t-s|^{-n(1/p-1/2)}\,\left(\nu(\varepsilon)^{m-1}\|z(s)\|_{-\sigma} + |z(s)|_{mp}^m\right)\,ds. \tag{3.25}$$

Following [SW2], we note that $n(1/p-1/2) \downarrow 1$ as $p \uparrow 2n/(n+2)$. Therefore, we set

$$n(1/p-1/2) \equiv 1+\kappa,$$

and use the generic name $O(\kappa)$ to denote continuous functions such that $O(\kappa) \downarrow 0$ as $\kappa \downarrow 0$. By the previous discussion, we have $p(\kappa) = 2n/(n+2) - O(\kappa)$. We now interpolate $L^{mp(\kappa)}$ between L^2 and L^{1+m}. This leads to

$$|z|_{mp(\kappa)}^m \le |z|_2^{\beta(\kappa)}\,|z|_{1+m}^{\gamma(\kappa)}, \tag{3.26}$$

where

$$\beta(\kappa) = \frac{n-2}{n(m-1)}\left(\frac{n+2}{n-2} - m\right) + O(\kappa),$$
$$\gamma(\kappa) = \frac{m+1}{m-1}\left(m - 1 - \frac{2}{n}\right) - O(\kappa).$$

Inserting Inequality (3.26) into our estimate (3.25), we get

$$\|J_1(t)\|_{-\sigma} \leq C \int_0^{\max(0,t-1)} |t-s|^{-(1+\kappa)} \left(\nu(\varepsilon)^{m-1}\|z(s)\|_{-\sigma} + |z(s)|_2^{\beta(\kappa)}\, |z(s)|_{1+m}^{\gamma(\kappa)}\right) ds. \tag{3.27}$$

We now use the fact that

$$\sup_{t\geq 0} \int_0^{\max(0,t-1)} |t-s|^{-a}\, \langle t\rangle^b \langle s\rangle^{-c}\, ds < \infty,$$

provided that $a > 1$ and $a, c \geq b$. Under the conditions

$$\begin{aligned} \zeta &\leq \gamma(\kappa)\mu, \\ \zeta &\leq 1 + \kappa, \end{aligned} \tag{3.28}$$

we can apply this to Equ. (3.27) to obtain

$$\langle t\rangle^\zeta \|J_1(t)\|_{-\sigma} \leq C\left(\nu(\varepsilon)^{m-1} M_1(T) + M_2(T)^{\gamma(\kappa)} M_3(T)^{\beta(\kappa)}\right), \tag{3.29}$$

for $0 \leq t \leq T$. We shall now determine κ and ζ in such a way that conditions (3.22) and (3.28) are satisfied. First we note that, by Hypotheses H1-H2, $\gamma(0)\mu > 1$. Thus we can find $\kappa_0 > 0$ such that $\gamma(\kappa_0)\mu > 1$. As already mentioned, we also have $m\mu > 1$. Therefore, setting

$$1 < \zeta \equiv \min(m\mu, \gamma(\kappa_0)\mu, 1+\kappa_0), \tag{3.30}$$

validates our estimate (3.29). Collecting Estimates (3.23), (3.29) we finally get

$$\langle t\rangle^\zeta \|J(t)\|_{-\sigma} \leq C\left(\nu(\varepsilon)^{m-1} M_1(T) + M_2(T)^m + M_2(T)^{\gamma(\kappa_0)} M_3(T)^{\beta(\kappa_0)}\right), \tag{3.31}$$

for $0 \leq t \leq T$. By a straightforward modification of our argument, the same bound holds for $-T \leq t \leq 0$.

We can now insert the linear bounds (3.14) and our uniform bounds on $J(t)$, Equ. (3.19), (3.31), in the integral equation (3.9). Taking the supremum over $|t| \leq T$ in the resulting inequalities gives

$$\begin{aligned} M_1(T) &\leq C\Big(\|z(0)\|_{L^2_\sigma \cap H^1} + |\lambda|\nu(\varepsilon)^{m-1} M_1(T) + |\lambda| M_2(T)^m\Big), \\ M_2(T) &\leq C\Big(\|z(0)\|_{L^2_\sigma \cap H^1} + |\lambda|\nu(\varepsilon)^{m-1} M_1(T) + |\lambda| M_2(T)^m \\ &\qquad + |\lambda| M_2(T)^{\gamma(\kappa_0)} M_3(T)^{\beta(\kappa_0)}\Big). \end{aligned} \tag{3.32}$$

To close this set of inequalities we need an estimate on $|z(t)|_2$. From Equ. (3.2), we immediately get

$$\frac{d}{dt}\frac{1}{2}|z(t)|_2^2 = \lambda \operatorname{Im} \left\langle z(t), N(u_p(t), z(t))\right\rangle . \tag{3.33}$$

Going back to the definition of N, Equ. (3.3), and using Proposition 2.2, we can derive the following inequality

$$\left|\left\langle z(t), N(u_p(t), z(t))\right\rangle\right| \le C\left(|z(t)|_{1+m}^2 + |z(t)|_{1+m}^{1+m}\right).$$

Inserting this in Equ. (3.33), we obtain

$$\frac{d}{dt}|z(t)|_2^2 \le C|\lambda|\left(M_2(T)^2 + M_2(T)^{m+1}\right)\langle t\rangle^{-2\mu}.$$

Since $2\mu > 1$ by Equ. (3.13), we can integrate this inequality to get

$$M_3(T)^2 \le C\left(|z(0)|_2^2 + |\lambda| M_2(T)^2 + |\lambda| M_2(T)^{m+1}\right).$$

Reinserting this inequality into the estimates (3.32) now gives a closed set of inequalities for $M_1(T)$ and $M_2(T)$. By Definitions (3.1), (3.17) and Theorem 2.1 we have $\nu(\varepsilon) \to 0$ and $\|z(0)\|_{L^2_\sigma \cap H^1} \to 0$ as $\varepsilon \to 0$. We conclude that for ε sufficiently small, there exists C such that $\max(M_1(T), M_2(T)) \le C$ for all T. Thus we have proven:

Theorem 3.1. *Assume that Hypotheses H1-H3 hold and let $\phi(t)$ be a solution of Equ. (1.1) with initial condition $\phi(0) \in L^2_\sigma \cap H^1$. Then, if $\|\phi(0)\|_{L^2_\sigma \cap H^1}$ is sufficiently small, $z(t)$ satisfies the estimates*

$$\|z(t)\|_{-\sigma} \le C_\sigma \langle t\rangle^{-\zeta},$$
$$|z(t)|_{m+1} \le C_{m+1}\langle t\rangle^{-n\left(\frac{1}{2}-\frac{1}{m+1}\right)},$$

for some $\zeta > 1$.

Remark. It is a simple exercise to optimize the choice of κ_0 in Equ. (3.30). The best estimate of the exponent ζ we can get by our method is $\zeta = (m-1)n/4$.

4. Convergence to a periodic orbit

In this section we will establish that not only does every solution of (1.1) approach the center manifold as we demonstrated in the previous section, but in fact, it approaches a particular orbit on that manifold. From the previous section we know that if we write the solution of Equ. (1.1) as $\phi(t) = u_p(t)\psi_0 + u_c(t)$, with $u_p(t) \in \mathbf{C}$, then

$$\phi(t) = u_p(t)\psi_0 + h(u_p(t)) - z(t),$$

with $z(t) = h(u_p(t)) - u_c(t) \to 0$ at a rate given by Theorem 3.1. By Equ. (1.2) the "center" part of the solution satisfies the equation

$$i\dot{u}_p = E_0 u_p + \lambda f_p(u_p, h(u_p)) + \lambda Q(u_p, z), \tag{4.1}$$

where $Q(u_p, z) \equiv f_p(u_p, h(u_p) - z) - f_p(u_p, h(u_p))$. Going to polar coordinates $u_p = r\mathrm{e}^{i\varphi}$, and using Equ. (2.3) we obtain

$$\phi(t) = \mathrm{e}^{i\varphi(t)}\psi_{E(r(t))} - z(t), \tag{4.2}$$

together with the following set of equations

$$\begin{aligned} \dot{r} &= \mathrm{Im}\,\left(\mathrm{e}^{-i\varphi}Q(r\mathrm{e}^{i\varphi}, z)\right), \\ \dot{\varphi} &= -E(r) - \mathrm{Re}\,\left(\mathrm{e}^{-i\varphi}r^{-1}Q(r\mathrm{e}^{i\varphi}, z)\right). \end{aligned} \tag{4.3}$$

By a simple variation of Equ. (3.7) and Theorem 3.1 we have the estimate

$$|Q(u_p(t), z(t))| \le C\langle t\rangle^{-\zeta}.$$

Integrating the first equation in (4.3) shows that $r(t)$ satisfies

$$r(t) - r_\pm = O(t^{\zeta-1}), \tag{4.4}$$

as $t \to \pm\infty$. In a similar way we obtain, from the second equation in (4.3)

$$\varphi(t) = -\int_0^t E(r(s))\,ds + \theta(t),$$

where

$$\theta(t) - \theta_\pm = O(t^{\zeta-1}), \tag{4.5}$$

as $t \to \pm\infty$. Combining (4.2), (4.4) and (4.5) we obtain Equ. (1.5) in Theorem 1.1.

Appendix

In this appendix we discuss the smoothness of the map $\mathcal{F}$ and a few other technical details required in the construction of the local center manifold W^p_μ. As already remarked in Section 2, it suffices to consider $\mathcal{F}$ as a real valued function of real arguments. Consequently, in what follows, all function spaces will be real vector spaces of real valued functions on $\mathbf{R}^n$. We write $\mathcal{F}(E, r, h) = h - \lambda F(E, r\psi_0 + h)$, where

$$F(E, \phi) \equiv (H_0 - E)^{-1} P_c G(\phi),$$

and $G(x) \equiv |x|^{m-1}x$ is real function of a real variable.

We first claim that $\phi \mapsto G(\phi)$ is of class $C^1(H^\eta, H^1)$. Since

$$G(\phi + \psi) - G(\phi) - G'(\phi)\psi = \int_0^1 \big(G'(\phi + t\psi) - G'(\phi)\big)\, \psi \, dt, \tag{5.1}$$

with $G'(x) = m|x|^{m-1}$, a simple application of Lebesgue's dominated convergence theorem reduces the claim to the following set of estimates:

$$\begin{aligned}
\big|\big(G'(\phi_1) - G'(\phi_2)\big)\, \psi\big|_2 &\leq C\|\phi_1 - \phi_2\|_{H^\eta} \left(\|\phi_1\|_{H^\eta} + \|\phi_2\|_{H^\eta}\right)^{m-2} \|\psi\|_{H^\eta}, \\
\big|\big(G'(\phi_1) - G'(\phi_2)\big)\, \nabla\psi\big|_2 &\leq C\|\phi_1 - \phi_2\|_{H^\eta} \left(\|\phi_1\|_{H^\eta} + \|\phi_2\|_{H^\eta}\right)^{m-2} \|\psi\|_{H^\eta}, \\
\big|\big[\nabla, G'(\phi_1) - G'(\phi_2)\big]\, \psi\big|_2 &\leq C\|\phi_1 - \phi_2\|_{H^\eta} \left(\|\phi_1\|_{H^\eta} + \|\phi_2\|_{H^\eta}\right)^{m-2} \|\psi\|_{H^\eta}.
\end{aligned}$$

All these inequalities follow from explicit calculations and the use of the continuous inclusions $H^\eta \subset H^1$ and $H^\eta \subset L^\infty$, which hold under our current Hypotheses.

Since $\eta \leq 3$, the usual elliptic regularity argument applied to $(H_0 - E)^{-1} P_c G(\phi)$ allows us to conclude that $F \in C^1(I_1 \times H^\eta, H^\eta)$.

Using again Equ. (5.1) and repeating the above argument with the simple estimate

$$\big\|\big(G'(\phi_1) - G'(\phi_2)\big)\, \psi\big\|_\sigma \leq C\|\phi_1 - \phi_2\|_{H^\eta} \left(\|\phi_1\|_{H^\eta} + \|\phi_2\|_{H^\eta}\right)^{m-2} \|\psi\|_\sigma,$$

we see that $\phi \mapsto G(\phi)$ is of class $C^1(H^\eta \cap L^2_\sigma, L^2_\sigma)$. Finally a standard Combes-Thomas argument ([CT]) shows that

$$\langle x\rangle^\sigma (H_0 - E)^{-1} P_c \langle x\rangle^{-\sigma}$$

is an analytic function of $E \in I_1$ with values in the bounded operators on L^2, provided δ is small enough. This clearly implies that $F \in C^1(I_1 \times L^2_\sigma \cap H^\eta, L^2_\sigma \cap H^\eta)$ which proves the smoothness of $\mathcal{F}$ asserted in Section 2.

We now justify the use of the implicit function theorem to obtain the function $E(r)$ from Equ. (2.4). We must check that

$$r^{-1} f_p(r, \tilde{h}(E,r)) = \langle \psi_0, r^{-1} G(r\psi_0 + \tilde{h}(E,r)) \rangle,$$

is of class C^1. Clearly the only problem is at $r = 0$. Since $\partial_r \mathcal{F}(E,0,0) = 0$, one has $\partial_r \tilde{h}(E,0) = 0$ and hence $\tilde{h}(E,r) = o(r)$. This implies that

$$\left| r^{-1} G(r\psi_0 + \tilde{h}(E,r)) \right| = O(|r|^{m-1}) = o(|r|),$$

and hence $r^{-1} G(r\psi_0 + \tilde{h}(E,r))$ is differentiable at $r = 0$, and its derivative vanishes there. A simple calculation shows that the derivative is indeed continuous at $r = 0$.

A similar argument shows that the function $h \colon \mathbf{C} \to \mathbf{C}$ defined in Equ. (2.5) satisfies

$$\left| h(u_p) \right| = \left| \tilde{h}(E(|u_p|), |u_p|) \right| = o(|u_p|),$$

as $u_p \to 0$. It is again easy to conclude that h is C^1.

References

[C] Carr, J.: *Applications of Centre Manifold Theory*, New York, Heidelberg, Berlin, Springer (1981).

[CT] Combes, J.-M. and L. Thomas: Asymptotic behavior of eigenfunctions for multiparticle Schrödinger operators. Commun. Math. Phys. **34**, 251-270 (1973).

[JSS] Journé, J.L., A. Soffer and C. Sogge: Decay estimates for Schrödinger operators. Comm. Pure Appl. Math **44**, 573–604 (1991).

[R] Roberts, A.J.: A sub-centre manifold description of the evolution and interaction of nonlinear dispersive waves.. Preprint (1991).

[SW1] Soffer, A., and M. Weinstein: Multichannel nonlinear scattering for non-integrable equations. Comm. Math. Phys. **133**, 119–146 (1990).

[SW2] Soffer, A., and M. Weinstein: Multichannel nonlinear scattering, II. The case of anisotropic potentials and data. J. Diff. Eqns. **98**, 376–390 (1992).