Decay to Equilibrium in Random Spin Systems on a Lattice II

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Abstract

We study the continuous spin systems on a $d \geq 3$ dimensional lattice with random ferromagnetic interactions of finite range. We show that, if the temperature is sufficiently high and the probability of interaction to be large is small enough, the almost sure decay to equilibrium has a subexponential upper bound.

1 Introduction

In [12] we have studied the stochastic dynamics of random spin systems on a lattice. In particular we have shown there, improving the results of [31], that for discrete as well as continues spin systems if the random couplings are allowed to take on sufficiently large values, the generator of stochastic dynamics has no spectral gap with probability one even at high temperatures. As a consequence one concludes that the decay to equilibrium for local observable cannot be exponentially fast. Moreover in that paper, under mild natural assumptions, we have shown that on two dimensional lattice one has an almost sure stretched exponential upper bound on the decay to equilibrium in the uniform norm for continuous as well as discrete spin systems with random finite range interactions. By this we have improved the previous results of [31], where a similar estimate has been obtained for one dimensional systems, and the results of an excellent paper [10], where a general but weaker upper bound has been proven for discrete spin systems with finite range random interactions in any dimensions. In this paper we consider the continuous spin random ferromagnet with finite range of interaction on $d \geq 3$ lattice. We prove that at high temperatures one still has faster decay to equilibrium than the one obtained in [10]. Our method is based on application of Logarithmic Sobolev inequalities.

The organisation of our paper is as follows. In Section 2, which has a preliminary character, we introduce the notation and definitions. In Section 3 we formulate the technical assumptions about our spin systems and present our results: the almost sure upper bound on the decay to equilibrium and the basic ingredient an almost sure estimate on the growth of the log Sobolev coefficient. The prove of the upper bound of the decay to equilibrium is based on this basic ingredient and a clever argument of [19] (which extends the one of Holley's) using the attractivity of the interaction. The rest of the paper is devoted to the proof of our main technical result.

2 Preliminaries

Let $\mathbb{Z}^d$ be a $d \geq 3$ dimensional integer lattice with a metric $d(i,j) \equiv \sum_{\alpha=1}^{d} |i^\alpha - j^\alpha|$, $j, i \in \mathbb{Z}^d$. By $\mathcal{F}$ we denote the family of all finite subsets of the lattice. Let $\mathcal{F}_0 \equiv \{\Lambda_n \in \mathcal{F}\}_{n \in \mathbb{N}}$ be a countable exhaustion of the lattice, i.e. an increasing sequence of finite sets invading all the lattice. We will assume that $\mathcal{F}_0$ is a Fisher sequence, [22]. A cardinality (the volume) of a finite set $X \in \mathcal{F}$ will be denoted by $|X|$. Given a positive number R we define an R-boundary of a set $\Lambda \subset \mathbb{Z}^d$ as follows

$$\partial_R \Lambda \equiv \{j \in \Lambda^c : d(j, \Lambda) \leq R\}$$

where $\Lambda^c \equiv \mathbb{Z}^d \setminus \Lambda$ and $d(j, \Lambda)$ denotes the distance of j from the set Λ. Moreover, $\bar{\Lambda}^R$ will denote the R-closure of a set Λ, i.e. $\bar{\Lambda}^R = \Lambda \cup \partial_R \Lambda$.

We will restrict ourselves to continuous spins, i.e. we will consider spins taking values in a smooth, compact, connected (finite dimensional) Riemannian manifold M. In this context, we will call configuration space of the infinite spin system on the lattice $\mathbb{Z}^d$ the product space $\Omega \equiv M^{\mathbb{Z}^d}$ with a Borel σ-algebra Σ given by the product topology. Given a finite set $X \in \mathcal{F}$ we define a projection

$$\Omega \ni \omega \longmapsto \omega_X \in M^X$$

in particular if $X \equiv \{j\}$ is a single point set, such projection map is called *a spin at site* $j \in \mathbb{Z}^d$ and is denoted by σ_j. Given a set $X \in \mathcal{F}$, we define a measurable mapping

$$\Omega \times \Omega \ni (\omega, \tilde{\omega}) \longmapsto \omega \bullet_X \tilde{\omega} \in \Omega$$

by setting

$$(\omega \bullet_X \tilde{\omega})_j = \begin{cases} \omega_j \text{ if } j \in X \\ \tilde{\omega}_j \text{ otherwise} \end{cases}$$

In the case when $X = \{j\}$ is a one point set, we will simplify this notation by writing $\omega \bullet_j \tilde{\omega} \equiv \omega \bullet_{\{j\}} \tilde{\omega}$. We introduce also a measurable mapping

$$M^X \times M^{X^c} \ni (y_X, z_{X^c}) \longmapsto y_X \circ_X z_{X^c} \in \Omega$$

by setting

$$(y_X \circ_X z_{X^c})_j = \begin{cases} y_j \text{ if } j \in X \\ z_j \text{ otherwise} \end{cases}$$

For $\Lambda \subset \mathbb{Z}^d$ we define Σ_Λ as a smallest σ-algebra for which all the spins σ_j, $j \in \Lambda$ are measurable. A set of continuous and bounded measurable functions on (Ω, Σ) will be denoted by $\mathcal{C} \equiv \mathcal{C}(\Omega)$ and $\mathbb{B} \equiv \mathbb{B}(\Omega)$, respectively. By $||\cdot||_u$ we denote the supremum norm on $\mathbb{B}$. A measurable or continuous real function f on the configuration space is called *a local function* if there is a finite set $Y \in \mathcal{F}$ such that f is Σ_Y measurable. This means that there is a measurable (or continuous) function φ_Y on M^Y such that $f(\omega) = \varphi_Y(\omega_Y)$. A set of all local continuous and local measurable functions will be denoted by $\mathcal{C}_0 \equiv \mathcal{C}_0(\Omega)$ and $\mathbb{B}_0 \equiv \mathbb{B}_0(\Omega)$, respectively. For a given local function f, a smallest set $Z \in \mathcal{F}$ for which f is Σ_Z measurable will be denoted by $\Lambda(f)$ and called *a set of localization of the function* f. Frequently it will be convenient to use the following notation $d(f, Y) \equiv d(\Lambda(f), Y)$, where on the right hand side we have the distance between two subsets $\Lambda(f)$ and Y of the lattice $\mathbb{Z}^d$.

By ν_M we will denote a probability measure on $(M, \mathcal{B}_M)$ given the normalized Riemannian volume corresponding to a given Riemannian metric. A product measure $\mu_0 \equiv \nu_M^{\otimes \mathbf{Z}^d}$ on (Ω, Σ) is called *a free measure.* We will use a notation μf to denote an expectation value of measurable function f and a probability measure μ given on the same measurable space (whenever it is well defined). A covariance of two functions f and g with the measure μ will be denoted by

$$\mu(f, g) \equiv \mu fg - \mu f \mu g$$

In the configuration space Ω we have a natural representation $(T_j)_{j \in \mathbf{Z}^d}$ of the group of translations of the lattice given by the following measurable maps

$$(T_j \omega)_k \equiv \omega_{k-j}$$

We can use it to induce a representation on the space of measurable functions by setting

$$(T_j f)(\omega) \equiv f(T_j \omega)$$

Given a measurable function f on (Ω, Σ), for every configuration $\omega \in \Omega$ and a finite set $X \in \mathcal{F}$, we define a function

$$M^X \ni y \longmapsto f_X(y|\omega) \equiv f(y \circ_X \omega_{X^c})$$

In particular case of one point set $X = \{j\}$ we use a simplified notation $f_j(y|\omega) \equiv f(y \circ_j \omega) \equiv f_{\{j\}}(y|\omega)$. In continuous case, if for every $j \in \mathbb{Z}^d$ and ω, the function $f_j(\cdot|\omega)$ is differentiable, we say that f is differentiable. For such function we introduce its gradient $\nabla_\Lambda f \equiv (\nabla_j f)_{j \in \Lambda}$, with respect to coordinates in a set $\Lambda \subseteq \mathbb{Z}^d$, where

$$\nabla_j f(\omega) \equiv \partial_M f_j(x|\omega)_{|x=\omega_j}$$

with ∂_M denoting the gradient operator on the Riemannian manifold M. Using the scalar product $(\cdot|\cdot)_j$ associated to the Riemannian manifold M (at site i), we introduce the square of gradient $\nabla_\Lambda f$ by the following formula

$$|\nabla_\Lambda f|^2(\omega) \equiv \sum_{j \in \Lambda} (\nabla_j f(\omega)|\nabla_j f(\omega))_j$$

If $\Lambda = \mathbb{Z}^d$, for simplicity we will omit the corresponding index in the notation of the gradient. The space of all differentiable functions for which the following seminorm is finite

$$|||f||| \equiv \sum_{j \in \mathbf{Z}^d} || \, |\nabla_j f| \, ||_u$$

will be denoted by $\mathcal{C}^1$. Inductively, we say that a function f is n-times differentiable if for every finite set $X \in \mathcal{F}$, and $\omega \in \Omega$ the corresponding function $f_X(\cdot|\omega)$ on M^X is n-times differentiable. Frequently we will use also the following abbreviated notation

$$||\nabla_j f|| \equiv || \, |\nabla_j f| \, ||_u$$

and

$$||\nabla_i \nabla_j f|| \equiv || \, |\nabla_i \nabla_j f| \, ||_u$$

Let $(\mathbb{J}, \mathcal{B}_{\boldsymbol{J}}, \mathbb{E})$ be a probability space. A potential is a family $\Phi \equiv (\Phi_X)_{X \in \mathcal{F}}$ consisting of measurable functions $\Phi_X : \mathbb{J} \times \Omega \longrightarrow \mathbb{R}$, such that:

- for every $X \in \mathcal{F}$, the function $\Phi_X(J, \cdot)$ is continuous and Σ_X measurable (i.e. depends only on the coordinates $(\omega_j, \, j \in X)$), and

- for every $i \in \mathbb{Z}^d$ we have

$$\sum_{\substack{X \in \mathcal{F} \\ X \ni i}} ||\Phi_X(J, \cdot)||_u < \infty$$

- The family of random variables $\{\Phi_X(\cdot, \omega), X \in \mathcal{F}\}$ on the probability space $(\mathbb{J}, \mathcal{B}_{\boldsymbol{J}}, \mathbb{E})$ are mutually independent. Moreover, the random variables $\Phi_{X+j}(\cdot, T_j\omega)$ and $\Phi_X(\cdot, \omega)$ are identically distributed.

In this paper we restrict ourselves to potentials of finite range, i.e. we assume that there is a positive integer R such that $\Phi_X \equiv 0$ whenever $diam(X) \geq R$. Additionally we assume that the potential is twice differentiable, in the sense that for every $X \in \mathcal{F}$ the corresponding function $\Phi_X(J, \cdot)$ is twice differentiable, for every $J \in \mathbb{J}$.

Given a potential Φ, we define an interaction energy U_Λ in a finite set Λ by

$$U_\Lambda(J, \omega) \equiv \sum_{\substack{X \in \mathcal{F} \\ X \cap \Lambda \neq \emptyset}} \Phi_X(J, \omega)$$

We introduce finite volume Gibbs measures $\mu_\Lambda^{J,\omega}$ in a finite volume Λ with external conditions given by configuration $\omega \in \Omega$ by :

$$\mu_{J,\Lambda}^\omega(f) \equiv \frac{1}{Z_\Lambda(J, \omega)} \int_{M^\Lambda} \mu_0^\Lambda(dx_\Lambda) e^{-\beta U_\Lambda(J, x_\Lambda \circ \omega_{\Lambda^c})} f(x_\Lambda \circ \omega_{\Lambda^c})$$

for local functions f, where μ_0^Λ denotes the restriction of the free measure μ_0 to Σ_Λ. We will denote in short $E_\Lambda[f] := \mu_{J,\Lambda}^\omega(f)$ the expectation under $\mu_{J,\Lambda}^\omega$. It has been proven in [7] that, under the assumption that the interaction has finite growth, i.e. that there exists a positive real number ξ such that :

$$\sup_X \mathbb{E} \exp\{\xi ||\Phi_X(J, \cdot)||_u\} < \infty, \tag{1}$$

and if the temperature is high enough, one can construct an infinite Gibbs measure μ_J on the whole lattice as a limit of the measures $\mu^{\omega}_{J,\Lambda_n}$ which is independent of the boundary conditions ω at the countable exhaustion Λ_n of the lattice.

Moreover, we can as well define a Markov generator $L_\Lambda \equiv L_{J,\Lambda}$ in a finite volume $\Lambda \in \mathcal{F}$ by setting

$$L_\Lambda \equiv \sum_{i \in Z^d} L_i^\Lambda$$

where

$$L_i^\Lambda \equiv L_{J,i}^\Lambda \equiv \Delta_i - \nabla_i H_{i,\Lambda} \cdot \nabla_i$$

with Δ_i denoting the Laplace-Beltrami operator and

$$H_{i,\Lambda} \equiv \sum_{\substack{X \subseteq \Lambda \\ X \ni i}} \Phi_X .$$

Then for any $J \in \mathbb{J}$, we have a well defined Markov semigroup $P_t^\Lambda \equiv P_t^{J,\Lambda} \equiv e^{tL_\Lambda}$.

It has been proven in [12] that under the moderate growth assumption (1), one can construct for almost all interaction an infinite volume semigroup $P_t \equiv P_t^J$ defined formally by an operator

$$L \equiv L_J \equiv \sum_{i \in Z^d} L_i. \tag{2}$$

as a limit of the finite volume semigroups P^{Λ_n}. It was even shown in Theorem 2.1 of [12] that (1) is enough to insure an approximation property similar to the non random case (i.e. to the case where the couplings are bounded).

In the next section, we will focus on the case where M is an open subset of $\mathbb{R}^n$ for some integer number n. We will then restrict ourselves to the case where M fulfills the assumptions of [24] and let the spins be reflected at the boundary. It is straightforward to extend the method used in [12] to this setting and prove that, if the interaction has moderate growth, the above generator with correctly modified domain generate a Feller Markov process on M^{Z^d}.

3 Decay to equilibrium in ferromagnetic random spin systems on a lattice

In this section, we restrict ourselves to ferromagnetic systems with spins taking their values in a finite interval $[-A, +A]$ of $\mathbb{R}$ (the generalization to the case where $M \subset [-A, +A]^n$, for some integer number n being straightforward) . We show that, under some high temperature type restriction on the interaction, the corresponding Langevin dynamics have a decay to equilibrium much faster than any algebraic with probability one. It is worth noting that this decay is not as fast as the stretched exponential decay proven in [12] for the two dimensional case and general type of interaction, and that there is no obvious reason for which it should be so. Our method is based on use of Logarithmic Sobolev inequality together with the usual arguments due to attractivity of ferromagnetic systems (see [6], [13], [19]...).

In this Section, we will assume that we are given a ferromagnetic random interaction $\Phi \equiv \{\Phi_X\}_{X\in\mathcal{F}}$ of finite range $R > 0$, i.e. a random interaction such that, for any $i \in \mathbb{Z}^d$,

$$-\nabla_i \sum_{X\ni i} \Phi_X$$

is an increasing function of the spins with respect to the order $\ll$ defined by :

$$x \ll y \quad \leftrightarrow \quad x_i \leq y_i \quad \forall i \in \mathbb{Z}^d.$$

Moreover, we will make the following assumption on the random functions $\Phi \equiv \{\Phi_X\}_{X\in\mathcal{F}}$

($\mathbb{H}$1) We first assume that the interaction has moderate growth, i.e. that Φ has some exponential moments and more precisely that for a real number ξ large enough, we have :

$$\mathbb{E}\exp\{\xi||\tilde{\Phi}^{(j)}(J,\cdot)||_1\} < \infty \tag{3}$$

where

$$||\tilde{\Phi}^{(j)}(J,\cdot)||_1 \equiv \sum_{\substack{X\in\mathcal{F}\\ X\ni j}} ||\Phi_X(J,\cdot)||_u$$

and that

$$\sup_k \mathbb{E}\exp\{\xi||\nabla_k\tilde{\Phi}^{(j)}(J,\cdot)||_1\} < \infty \tag{4}$$

as well as

$$\sup_{k,j} \mathbb{E}\exp\{\xi||\nabla_k\nabla_j\tilde{\Phi}^{(j)}(J,\cdot)||_1\} < \infty. \tag{5}$$

($\mathbb{H}$2) We will then assume that we are in a high temperature regime, namely that there is a small constant $J_0 > 0$ (specified later), such that

$$0 < p_1 \equiv \sup_X \mathbb{E}\{\ ||\Phi_X||_u > J_0\} < p_c^b(d,R) \tag{6}$$

where $p_c^b(d,R)$ is a clustering coefficient (see Kesten [15] and Lemma 5.4 in [12]).

We remark that, according to [12], the conditions $\mathbb{H}$1 and $\mathbb{H}$2 show that, with probability one, the exponential approximation property for the infinite volume semigroup holds up to linear time in the size of the finite volume. As a consequence, our Markov Process is Feller continuous. The condition $\mathbb{H}$2 assures that with probability one we have uniqueness and exponential cluster property for equilibrium measures.

In this Section we prove the following result.

Theorem 3.1 *Suppose the conditions $\mathbb{H}$1 - $\mathbb{H}$2 are satisfied with J_0 sufficiently small. Then there is a $\theta > 0$ and an $\epsilon > 0$ such that for any local function $f \in \mathcal{C}^1$, with $\mathbb{E}$ probability one, we have*

$$||P_t^J f - \mu_J(f)||_u \leq C(\Lambda(f),J)e^{-\epsilon\exp\{\theta(\log t)^{\frac{1}{(d-1)^2}}\}}|||f||| \tag{7}$$

for all $t \geq T(\Lambda(f),J)$, with some positive random variables $C(\Lambda(f),J)$ and $T(\Lambda(f),J)$.

This result is deduced from the following logarithmic Sobolev inequality for finite cubes :

Theorem 3.2 *Suppose the conditions of Theorem 3.1 are satisfied. Then for any cube $\Lambda_n \equiv [-n,+n]^d$, for almost all $J \in \mathbb{J}$, there exists a positive integer $N(J)$ such that, for any $n \geq N(J)$, for any boundary condition ω,*

$$\mu^{\omega}_{J,\Lambda_n} f^2 \log f^2 \leq 2c_{J,\Lambda_n} \mu^{\omega}_{J,\Lambda_n} \left(f(-\mathcal{L}_{J,\Lambda_n} f)\right) + \mu^{\omega}_{J,\Lambda_n} f^2 \log \mu^{\omega}_{J,\Lambda_n} f^2 \tag{8}$$

with

$$c_{J,\Lambda_n} \leq c(J) \exp\{\eta (\log n)^{(d-1)^2}\} \tag{9}$$

with some random variable $c(J) \in (0,\infty)$ and η both independent of n, boundary conditions and the function f.

It should be already underlined that the last theorem holds without any restriction as ferromagneticity or on the choice of the Riemannian manifold M. In the two dimensional case,Theorem 3.2 has already been proved and even extended to the discrete setting in [12]. The authors showed there as well that, if J_0 and p_1 are small enough, a stretched exponential decay to equilibrium occurs. In higher dimension, the bound (9) on the logarithmic Sobolev coefficient grows much faster than any polynomial so that the strategy followed in [12] is useless. Nevertheless, it appears that Theorem 3.2 is enough to get Theorem 3.1 for ferromagnetic systems. It is this last point that we would like to detail in this section, leaving the proof of Theorem 3.2 to the next section.

Proof of Theorem 3.1:

To prove Theorem 3.1, it is clearly enough to bound $P_t f(x) - P_t f(y)$, for any smooth local function f and uniformly in $(x,y) \in \Omega$. But because of attractivity of our spin systems, for any $(x,y) \in \Omega$ such that $x \leq y$, one can construct, (see e.g. [6]), two coupled processes $\{X(t,x), t \geq 0\}$ and $\{X(t,y), t \geq 0\}$ on the same probability space $(\tilde{\Omega}, \mathcal{A}, Q)$ with laws $P_t(x)$ and $P_t(y)$ and so that, for any time t :

$$X(t,x) \leq X(t,y) \qquad Q - a.s. \tag{10}$$

Thus, for any $(x,y) \in \Omega$ such that $x \leq y$, any time $t \geq 0$,

$$\begin{aligned}
|P_t f(x) - P_t f(y)| &= |E_Q[f(X(t,x)) - f(X(t,y))]| \\
&\leq \sum_{i \in \mathbf{Z}^d} ||\nabla_i f|| E_Q[|X_i(t,x) - X_i(t,y)|] \\
&= \sum_{i \in \mathbf{Z}^d} ||\nabla_i f|| E_Q[X_i(t,y) - X_i(t,x)] \\
&= \sum_{i \in \mathbf{Z}^d} ||\nabla_i f|| \left(P_t X_i(y) - P_t X_i(x)\right)
\end{aligned}$$

In the general case, one can always introduced a $z \in \Omega$ so that $z \le x$ and $z \le y$ so that, one gets that, for any $(x, y) \in \Omega$, any time $t \ge 0$,

$$\begin{aligned} |P_t f(x) - P_t f(y)| &\le |P_t f(x) - P_t f(z)| + |P_t f(z) - P_t f(y)| \\ &\le 2|||f||| \sup_{i \in \Lambda(f)} \sup_{x \le y} (P_t X_i(y) - P_t X_i(x)) \end{aligned} \tag{11}$$

Thus, attractivity allows us to restrict our study to the first moment of the spins. On the other hand, since we have assumed that our spins take their values in a bounded interval $[-A, +A]$ of $I\!R$, there exists two extremum states in Ω, namely the states where all the spins are equal to $+A$, resp. $-A$. We will denote in short $(+)$ and $(-)$ these two states. Then, again by attractivity (or more precisely (10)), we get :

$$|P_t f(x) - P_t f(y)| \le 2|||f||| \sup_{i \in Z^d} (P_t X_i(+) - P_t X_i(-)) \tag{12}$$

To bound the right hand side of (12), we shall localize our Markov process in order to use Theorem 3.2. This can as well be done thanks to attractivity as already underlined in the discrete setting in [19], Part I. Indeed we claim that, for any integer number N and if we denote $P_t^{N,+}$ (resp. $P_t^{N,-}$) the Markov Processes on the cube Λ_N with spins equal to $+A$ (resp. $-A$) at the boundary of Λ_N, then we have, for any $i \in \Lambda_N$,

$$P_t X_i(+) - P_t X_i(-) \le P_t^{N,+} X_i(+) - P_t^{N,-} X_i(-) \tag{13}$$

To see this last point, we consider the Markov semigroups $\{\mathcal{P}_t, t \ge 0\} = \{\mathcal{P}_t^J, t \ge 0\}$ (resp. $\{\bar{\mathcal{P}}_t^N, t \ge 0\} = \{\bar{\mathcal{P}}_t^{J,N}, t \ge 0\}$) on the space of smooth functions on Ω^2 and with generators $\mathcal{L}$ (resp. $\bar{\mathcal{L}}^N$) defined on the space of local twice differentiable function with null inner derivative at the boundary and formally given by :

$$\begin{aligned} \mathcal{L} &= L_1 + L_2 \\ &= \sum_{i \in Z^d} \Delta_{i,1} + \Delta_{i,2} - \nabla_{i,1} U_{i,1}(J).\nabla_{i,1} - \nabla_{i,2} U_{i,2}(J).\nabla_{i,2} \\ \bar{\mathcal{L}}^N &= \sum_{i \in Z^d} \Delta_{i,1} + \Delta_{i,2} - \nabla_{i,1} \bar{U}_{i,1}^N(J).\nabla_{i,1} - \nabla_{i,2} \bar{U}_{i,2}^N(J).\nabla_{i,2} \end{aligned}$$

where $\Delta_{i,\epsilon}$ (resp. $\nabla_{i,\epsilon}$) denote, for ϵ equal 1 or 2, the Laplacian (resp. the gradient) on the i composent of the ϵ variable. Moreover, $U_{i,\epsilon}(J, x) = U_i(J, x^\epsilon)$ so that $\mathcal{L}$ is the in fact the sum of two copies of the generator L. Furthermore, we choose $\bar{U}_{i,\epsilon}^N$ so that :

$$\bar{U}_{i,\epsilon}^N(x) := U_i(J, x_{\Lambda_N}^\epsilon.((-1)^{\epsilon+1}A)_{\partial_R \Lambda_N}.x_{\Lambda_N^c}^\epsilon) = \sum_{X \ni i} \Phi_X(J, x_{\Lambda_N}^\epsilon.((-1)^{\epsilon+1}A)_{\partial_R \Lambda_N}.x_{\Lambda_N^c}^\epsilon),$$

or, more explicitely we choose $\bar{U}^N$ to be the energy of our system forced to be at the extremal states on the boundary of the cube Λ_N. It is not hard to see that the Markov semigroups $\{\mathcal{P}_t, t \geq 0\}$ (resp. $\{\bar{\mathcal{P}}_t^N, t \geq 0\}$) can be well defined under assumption ($\mathbb{H}1$) since they are nothing but the joint laws of independent Markov processes we are able to build. Moreover, it is clear that $\{\mathcal{P}_t, t \geq 0\}$ has both marginals equal to P_t and that the restriction of $\{\bar{\mathcal{P}}_t^N, t \geq 0\}$ on Λ_N^2 has for first (resp. second) marginal $\{P_t^{N,+}, t \geq 0\}$ (resp. $\{P_t^{N,-}, t \geq 0\}$). We can now observe as in [5] that if, for any $i \in \Lambda_N$, we denote f_i the function on Ω^2 such that :

$$f_i(x^1, x^2) = x_i^1 - x_i^2,$$

then :

$$\begin{aligned}
& P_t X_i(+) - P_t X_i(-) - P_t^{N,+} X_i(+) + P_t^{N,-} X_i(-) = \\
= \; & \mathcal{P}_t(f_i)(+,-) - \bar{\mathcal{P}}_t^N(f_i)(+,-) \\
= \; & \int_0^t \frac{d}{ds} \bar{\mathcal{P}}_{t-s}^N \mathcal{P}_s(f_i)(+,-) ds = \\
= \; & \int_0^t \bar{\mathcal{P}}_{t-s}^N \left(\bar{\mathcal{L}}^N - \mathcal{L} \right) \mathcal{P}_s(f_i)(+,-) ds
\end{aligned} \tag{14}$$

But, by definition, we have :

$$\bar{\mathcal{L}}^N - \mathcal{L} = \sum_i \Big((\nabla_{i,1} U_{i,1}(J) - \nabla_{i,1} \bar{U}_{i,1}^N(J)).\nabla_{i,1} - (\nabla_{i,2} \bar{U}_{i,2}^N(J) - \nabla_{i,2} U_{i,2}(J)).\nabla_{i,2} \Big)$$

so that, for any $(z, y) \in \Omega^2$:

$$\begin{aligned}
& \left(\bar{\mathcal{L}}^N - \mathcal{L} \right) \mathcal{P}_s(f_i)(z,y) \qquad (15) \\
= \; & \left(\bar{\mathcal{L}}^N - \mathcal{L} \right) (P_s X_i(z) - P_s X_i(y)) \\
= \; & \sum_i \Big((\nabla_{i,1} U_{i,1}(J) - \nabla_{i,1} \bar{U}_{i,1}^N(J)).\nabla_i P_s X_i(z) - (\nabla_{i,2} \bar{U}_{i,2}^N(J) - \nabla_{i,2} U_{i,2}(J)).\nabla_{i,2} P_s X_i(y) \Big)
\end{aligned}$$

Now, because of ferromagneticity, we can deduce from Theorem 1.2 proved by Chen and Wang [5] that $x \to P_s X_i(x)$ is monotone for every $s \geq 0$. Moreover, again by ferromagneticity and for any $x \in \Omega$, we know that :

$$\nabla_{i,1} U_{i,1}(J, x) - \nabla_{i,1} \bar{U}_{i,1}^N(J, x) \leq 0$$

and

$$\nabla_{i,2} \bar{U}_{i,2}^N(J, x) - \nabla_{i,2} U_{i,2}(J, x) \geq 0.$$

Thus, (14) and (15) give the desired bound (13).

We now turn to the bound of the right hand side of (13). This last step does not use ferromagneticity but Theorem 3.2 (and the induced hypercontractivity property) as well as the exponential cluster property of equilibrium measures. To deal with this point, let us first write :

$$\begin{aligned}
& P_t^{N,+}X_i(+) - P_t^{N,-}X_i(-) \\
\leq \; & |P_t^{N,+}X_i(+) - \mu^{N,+}(X_i)| + |\mu^{N,+}(X_i) - \mu^{N,-}(X_i)| + |P_t^{N,-}X_i(-) - \mu^{N,-}(X_i)| \\
\leq \; & |P_t^{N,+}X_i(+) - \mu^{N,+}(X_i)| + |P_t^{N,-}X_i(-) - \mu^{N,-}(X_i)| + e^{-Md(i,\Lambda_N^c)} \qquad (16)
\end{aligned}$$

where we have estimated the middle term using the result of [7] (see also [9], [8]), with some constant $M > 0$ independent of J and i. We thus need only to focus on the two other terms, or, more generally one shall bound

$$\left| P_t^{N,\omega}X_i(x) - \mu^{N,\omega}(X_i) \right|$$

for arbitrary boundary condition ω at Λ_N and initial condition $x \in \Omega$. One can do it using Theorem 3.2 applying a general strategy of [14], [25]-[27] and [29]. For the sake of completeness, we nevertheless introduce a short proof. Let us first denote $\bar{X}_i = X_i - \mu^{N,\omega}(X_i)$. Then, letting run the Markov process for a unit time and using Hölder inequality we get, for any $q > 1$,

$$\left| P_t^{N,\omega}X_i(x) - \mu^{N,\omega}(X_i) \right| = \left| P_1^{N,\omega}P_{t-1}^{N,\omega}\bar{X}_i(x) \right| \leq \left(P_1^{N,\omega}[P_{t-1}^{N,\omega}\bar{X}_i]^q(x) \right)^{\frac{1}{q}} \qquad (17)$$

In the case of non reflected processes, it has been proven in Lemma 2.3 of [25] that there exists a constant C such that, for any non negative function f,

$$||P_1^{N,\omega}[f]||_u \leq \exp\{W_N\}\mu^{N,\omega}[f]$$

with

$$W_N = 4||U^{N,\omega}||_u + 2\sum_{k\in\Lambda_N}(||\nabla_k U^{N,\omega}||_u^2 + ||\Delta_k U^{N,\omega}||_u^2).$$

The same result holds here since the kernel of the reflected Brownian motion at time 1 is uniformly bounded. Indeed, Girsanov Theorem shows that :

$$||P_1^{N,\omega}[f]||_u \leq \exp\{\frac{1}{2}W_N\}||P_1^{\otimes N}[f]||_u,$$

where P_1 is the law of the brownian motion reflected at $\{-A\}$ and $\{+A\}$ at time 1. But it is easy to see that there exists a finite constant c (namely the supremum norm of the kernel of the reflected Brownian motion at time 1) such that, for any non negative function f, we have :

$$||P_1[f]||_u \leq c\int_{-A}^{A} f(x)dx.$$

Inserting again the energy gives the result, i.e. :

$$||P_1^{N,\omega}[f]||_u \leq \exp\{W_N\}\mu^{N,\omega}[f].$$

Of course, this result is enough to conclude, according to (17) that, for any $q > 1$,

$$\left|P_t^{N,\omega}X_i(x) - \mu^{N,\omega}(X_i)\right| \leq \exp\{\frac{1}{q}W_N\}\left(\mu^{N,\omega}[P_{t-1}^{N,\omega}\bar{X}_i]^q\right)^{\frac{1}{q}} \tag{18}$$

We then use the hypercontractivity for a small fraction of our time, therefore we choose with some small $\delta \in (0,1)$

$$q \equiv 1 + e^{2(\delta t - 1)/c_N} \tag{19}$$

with $c_N = c_{J,\Lambda_N}$.

This leads us to :

$$\begin{aligned}\left|P_t^{N,\omega}X_i(x) - \mu^{N,\omega}(X_i)\right| &\leq \exp\{\frac{1}{q}W_N\}\left(\mu^{N,\omega}[P_{(1-\delta)t}^{N,\omega}\bar{X}_i]^2\right)^{\frac{1}{2}} \\ &\leq \exp\{\frac{1}{q}W_N\}\exp\{-\frac{(1-\delta)t}{c_N}\}\end{aligned} \tag{20}$$

where we have finally used the fact that the logarithmic Sobolev inequality with a coefficient $c_N = c_{J,\Lambda_N}$ implies a spectral gap at the bottom of the operator $-L_{\Lambda_N}$ of size not smaller than the inverse of that coefficient.

As a consequence, (16) and (20) give :

$$P_t^{N,+}X_i(+) - P_t^{N,-}X_i(-) \leq e^{-Md(i,\Lambda_N^c)} + 2\exp\{\frac{1}{q}W_N\}\exp\{-\frac{(1-\delta)t}{c_N}\} \tag{21}$$

Combining this last result with (11) and (13) leads to :

$$\begin{aligned}|P_tf(x) - P_tf(y)| &\leq |||f|||\sup_{i\in\Lambda(f)}\left(P_t^{N,+}X_i(+) - P_t^{N,-}X_i(-)\right) \\ &\leq |||f|||\left(e^{-Md(\Lambda(f),\Lambda_N^c)} + 2\exp\{\frac{1}{q}W_N\}\exp\{-\frac{(1-\delta)t}{c_N}\}\right)\end{aligned} \tag{22}$$

and (22) holds for any integer number N so that $\Lambda(f) \subset \Lambda_N$. Therefore, we can choose N as a function of the time and, more precisely to be the biggest integer number such that :

$$c_N d(\Lambda(f), \Lambda_N^c) \leq t.$$

Since this condition means $d(\Lambda(f),\Lambda_N^c)\leq t/c_N$ and $d(\Lambda(f),\Lambda_N^c)>N/2$, for large N's, it is easy to see that for our choice of q we have

$$\lim_{N\to\infty}\frac{1}{q}W_N=0 \qquad I\!\!E.a.s.$$

To find such the integer N of interest to us, we use Theorem 3.2 which tells us that there exists a positive real number η such that, for almost all J, there exists an integer number $N(J)$ such that, for any $N\geq N(J)$,

$$c_N\leq c(J)\exp\{\eta(\log N)^{(d-1)^2}\}. \tag{23}$$

Therefore for some positive real number $\theta>\eta$ big enough, there exists a finite integer $N(\Lambda(f),J)$, which only depends on the interaction and on the set $\Lambda(f)$ where f is localized, such that for any $N\geq N(\Lambda(f),J)$, we have :

$$c_N d(\Lambda(f),\Lambda_N^c)\leq c(J)\exp\{\theta(\log N)^{(d-1)^2}\}.$$

Thus, if we choose

$$N=[\exp(\frac{1}{\theta}\log\frac{t}{c(J)})^{\frac{1}{(d-1)^2}})],$$

then (22) implies that there exists a positive real number ϵ, a finite time $T(\Lambda(f),J)$ and a constant $C(J)\in(0,\infty)$ such that, for any time $t\geq T(\Lambda(f),J)$,

$$|P_tf(x)-P_tf(y)| \;\leq\; |||f|||C(J)e^{-\epsilon\exp(\frac{1}{\theta}\log t)^{\frac{1}{(d-1)^2}}}$$

This ends the proof of Theorem 3.1. $\diamondsuit$

4 Logarithmic Sobolev Inequality for Random Continuous Spin Systems

In this section, we prove an almost sure logarithmic Sobolev inequality for finite (but randomly large enough) cubes stated in Theorem 3.2, i.e. that, for $I\!\!E$ almost all interaction, any boundary condition ω and any integer number N large enough, the smallest constant c_{J,Λ_N} such that :

$$\mu^\omega_{J,\Lambda_N}f^2\log f^2\leq 2c_{J,\Lambda_N}\mu^\omega_{J,\Lambda_N}\left(f(-L_{\Lambda_N}f)\right)+\mu^\omega_{J,\Lambda_N}f^2\log\mu^\omega_{J,\Lambda_N}f^2 \tag{24}$$

is bounded by :

$$c_{J,\Lambda_N}\leq c(J)\exp\{\eta(\log N)^{(d-1)^2}\} \tag{25}$$

for some $\eta > 0$ and finite random variable $c(J)$.

In the non random interaction setting (or when $p_1 = 0$), this type of problem has been intensively studied (see [25]-[27], [19], [17], [16]) and it is well known that, if J_0 is small enough, the logarithmic Sobolev coefficient c_{J,Λ_N} can be bounded independently of the size N of the cube Λ_N. This result is strongly related to the exponential decay of correlation observed at high temperature. In the random case, one has to be more careful since the decay of correlation only holds for sites which are at a distance larger than some random length (see [7]). It is the main reason why our bound (25) on c_{J,Λ_N} depends on N. Nevertheless, the approach proposed in [25] can be still applied and is actually the key step to our proof of Theorem 3.2.

We thus want to recall briefly the main assumptions and results obtained in [25] and, in particular, to specify them for non translation invariant interaction. The method proposed in [25] to prove logarithmic Sobolev inequalities relies on the observation that sweeping out relations allow to deduce the logarithmic Sobolev coefficient for a set Λ from the logarithmic Sobolev coefficients of a well chosen covering of this set.

To make these heuristics more precise, we first recall what we mean by sweeping out relations. We will say that a conditional expectation E_Λ associated to a set Λ satisfies a sweeping out relation with parameters (α, A, r) and will denote this fact in short by $SOR(\alpha, A, r)$, if there exists a family $(\alpha^\Lambda_{i,j})_{i,j\in Z^d}$ of non negative real numbers such that :

$$\alpha^\Lambda_{i,j} \leq \alpha^{|i-j|} \quad \text{if } |i-j| \geq r \tag{26}$$

$$\sum_{j\in\Lambda} \alpha^\Lambda_{i,j} \leq A \quad \text{for any } i \in \partial_R\Lambda \tag{27}$$

and that, for all i in $\mathbb{Z}^d$:

$$\left|\nabla_i \left(E_\Lambda[f^2]\right)^{\frac{1}{2}}\right| \leq \left(E_\Lambda\left[|\nabla_i f|^2\right]\right)^{\frac{1}{2}} + \mathbb{1}_{i\in\partial_R\Lambda} \sum_{j\in\Lambda} \alpha^\Lambda_{i,j} \left(E_\Lambda\left[|\nabla_j f|^2\right]\right)^{\frac{1}{2}} \tag{28}$$

In the case where the above coefficients only satisfy (27) we will say that Λ satisfies $SOR(A)$. We will recall further (see subsection 4.1) how the sweeping out relations can be related to the decay of correlations. The above relations for a family of finite sets a given integer number r and an $\alpha \ll 1$, are the main step towards logarithmic Sobolev inequalities as proved in [25]-[27]. Indeed, let r be an integer number (to be specified later) and define :

$$Y_r(0) = \{k \in \mathbb{Z}^d, -r - R \leq k_i \leq r + R \quad 1 \leq i \leq d\},$$

and, for $k \in \mathbb{Z}^d$, $Y_r(k) = k + Y_r(0)$.
Moreover, let

$$\phi_{K,r}(\gamma) = \gamma + \frac{K}{1-\gamma^2} \sum_{l\geq r} (2l+1)^{d-1} \gamma^{2lr/(9R+6r)},$$

and, for $m \geq 2$, $\phi^m_{K,r}(\gamma) = \phi_{K,r}\left(\phi^{m-1}_{K,r}(\gamma)\right)$. Finally, if C_d denotes the volume of the sphere with radius one in $\mathbb{Z}^d$, we define for $m \in [1, d-1]$, any $r > 0$, $\alpha > 0$:

$$\xi_m(\alpha, A, r) = \phi^m_{A,r}\left(C_d \sum_{n \geq r} (2n+1)^{d-1}\alpha^n\right).$$

Then, it is proven in [25] that :

Theorem 4.1 *Let Λ be a subset of $\mathbb{Z}^d$. If there exist finite constants $(A_\Lambda, \alpha_\Lambda, r_\Lambda)$ such that*
1) $\xi_{d-1}(\alpha_\Lambda, A_\Lambda, r_\Lambda) < 1$,
2) For any $k \in [(r_\Lambda + 2R)\mathbb{Z}]^d \cap \bar{\Lambda}^R$, the conditional expectations associated to $Y_{r_\Lambda}(k) \cap \Lambda$ satisfy $SOR(\alpha_\Lambda, A_\Lambda, r_\Lambda)$,
3) If we denote

$$Y^l_r(k) = \cup_{n_1,\cdots,n_l \in Z^l}\{Y_r(k + n_1(r+2R)e_1 + \cdots n_l(r+2R)e_l)\},$$

for any $l \in [1, d-1]$, for any $k \in \{0\}^l \times [(r+2R)\mathbb{Z}]^{d-l} \cap \bar{\Lambda}^R$, the conditional expectations associated to $Y^l_{r_\Lambda}(k) \cap \Lambda$ satisfy $SOR(A_\Lambda)$,
then the logarithmic Sobolev coefficient of E_Λ can be bounded, for any $r \geq r_\Lambda$, by :

$$c(\Lambda) \leq \prod_{l=0}^{d-1}\left(\frac{2A_\Lambda + 3}{1 - \xi_l(\alpha_\Lambda, A_\Lambda, r)^2}\right) \max_{k \in [(r+2R)Z]^d \cap \bar{\Lambda}^R} c(Y_r(k \cap \Lambda)). \tag{29}$$

Thus, our task is reduced to show that, for $\mathbb{E}$ almost all interactions, the sweeping out relations of Theorem 4.1 are verified when Λ is a large cube centered at the origin. Indeed, we will prove that :

Theorem 4.2 *Let $\Lambda_n = [-n, n]^d$. If $\mathbb{H}1$ and $\mathbb{H}2$ are satisfied with positive J_0 small enough, then there exist a positive real number $\alpha < 1$ ($\alpha \to 0$ when $J_0 \to 0$), and finite constants τ and τ', ($\tau \to 0$ when $J_0 \to 0$), such that for $\mathbb{E}$ almost every J there exists a finite integer number $N(J)$ for which for any integer number $n \geq N(J)$,*

a) for any $k \in [(2R + \tau(\log n)^d)\mathbb{Z}]^d \cap \bar{\Lambda}^R_n$, the conditional expectation associated to $Y_{\tau(\log n)^d}(k) \cap \Lambda_n$ satisfies the condition $SOR\left(\alpha, \exp\{\tau'(\log n)^{(d-1)}\}, \tau(\log n)^{(d-1)}\right)$,

b) for any $l \in [1, d-1]$, any $k \in \{0\}^l \times [(2R + \tau(\log n)^d)\mathbb{Z}]^{d-l} \cap \bar{\Lambda}^R_n$, the conditional expectation associated to $Y^l_{\tau(\log n)^{d-1}}(k) \cap \Lambda_n$ satisfies the corresponding condition $SOR\left(\exp\{\tau'(\log n)^{(d-1)}\}\right)$.

As a consequence, if α is small enough so that

$$\epsilon(d, \alpha) \equiv \sup_n \xi_{d-1}(\alpha, \exp\{\tau'(\log n)^{(d-1)}\}, \tau(\log n)^{d-1}) < 1,$$

(29) implies that there exists a finite integer number $N(J)$ for which for any integer number $n \geq N(J)$,

$$c(\Lambda) \leq \left(\frac{2\exp\{\tau'(\log n)^{d-1}\}+3}{1-\epsilon(d,\alpha)^2}\right)^{d-1} \max_{k\in[(\tau(\log n)^{d-1}+2R)Z]^d\cap\bar{\Lambda}_n^R} c(Y_{\tau(\log n)^{d-1}}(k)\cap\Lambda_n).$$

Thus, it is enough to bound the logarithmic coefficients of the $Y_{\tau(\log n)^{d-1}}(k)$'s to find Theorem 3.2. This last step can be easily performed as e.g. in [12] (see Lemma 5.5) and one can find :

Lemma 4.3 *For any $\tau > 0$, there exists a finite constants τ", satisfying $\tau" \to 0$ when $\tau \to 0$, and such that for $\mathbb{E}$ almost every J, there exists an integer number $N(J)$ such that, for any $n \geq N(J)$, for any $k \in [(\tau(\log n)^{d-1}+2R)\mathbb{Z}]^d$, we have*

$$c(Y_{\tau(\log n)^d}(k)\cap\bar{\Lambda}_n^R) \leq \exp\{\tau"(\log n)^{(d-1)^2}\}.$$

It is then easy to check that Lemma 4.3 and Theorem 4.2 imply Theorem 3.2. As a remark, we would like to mention that, in the case where the interaction is bounded but possibly large, the proofs of Theorem 4.2.b) and of Lemma 4.3 are much easier. Nevertheless, we do not see how this additional assumption would improve Theorem 3.2.

In the following we will prove Theorem 4.2 by first bounding the sweeping out coefficients for a given interaction and then deducing an almost sure bound.

4.1 Sweeping out relations

For every realization of the interaction, the following bound holds :

Property 4.4 *There exist finite nonnegative numbers $\left(\alpha_{i,j}^\Lambda, i\in\Lambda^c,\ j\in\Lambda\right)$ such that, for any smooth function f in $L^2(\mu_\Lambda^{J,\omega})$, for any i in Λ^c,*

$$\left|\nabla_i\left(E_\Lambda[f^2]\right)^{\frac{1}{2}}\right| \leq \left(E_\Lambda\left[|\nabla_i f|^2\right]\right)^{\frac{1}{2}} + 1\!\!I_{i\in\partial_R\Lambda}\sum_{j\in\Lambda}\alpha_{i,j}^\Lambda\left(E_\Lambda\left[|\nabla_j f|^2\right]\right)^{\frac{1}{2}} \tag{30}$$

Moreover, for any integer number D, $D>R$, the following bound holds :

$$\alpha_{i,j}^\Lambda \leq \nu_{i,j}^\Lambda + \sum_{l=1}^{|\Lambda|-1}\sum_{\{k_1,\cdots,k_l\}\in G_{l,D}^\Lambda(i)}\nu_{i,k_1}^\Lambda\left(\prod_{p=1}^{l-1}\nu_{k_p,k_{p+1}}^\Lambda\right)\nu_{k_l,j}^\Lambda \tag{31}$$

where

$$\begin{aligned} G_{l,D}^\Lambda(i) \quad = \quad & \Big\{\{k_1,\cdots,k_l\}\in\Lambda | k_i\neq k_{i+1} \text{ for } i\in[1,l-1] \text{ and} \\ & d(k_p,k_{p+1})>D \quad \text{for } p\in[1,l-1] \quad d(k_1,i)>\max\{D,|i-j|\}\Big\}, \end{aligned}$$

Furthermore, if one denotes, for any subset Y of $\mathbb{Z}^d$, $m(Y)$ the spectral gap coefficient :

$$m(Y) = \inf_{\omega\in\Omega} \inf_{\substack{f\in L_2(\mu^\omega_{J,Y}) \\ f\perp 1}} \frac{\mu^\omega_{J,Y}|\nabla_Y f|^2}{\mu^\omega_{J,Y}(f,f)},$$

and :

$$m_\Lambda(D) = \inf_{i\in\Lambda} \inf_{X\subset\Lambda} m\,(X\cap B(i,D)),$$

then one can find a finite constant C, which only depends on M and ν_M, such that, for any Λ :

$$\nu^\Lambda_{k,k'} = \begin{cases} C||\nabla_k\tilde\Phi^{(k)}(J,.)||_1 m_\Lambda(D)^{-\frac{1}{2}} & \text{if } |k-k'|\le D, \\ C||\nabla_k\tilde\Phi^{(k)}(J,.)||_1 \exp\{3||\tilde\Phi^{(k')}(J,.)||_1\} \sum_{\gamma\in\mathcal{W}^\Lambda_{k,k'}} \prod_{Y\in\gamma}\rho_Y & \text{otherwise,} \end{cases}$$

where

$$\rho_Y = \begin{cases} \exp\{4J_0\}-1 & \text{if } \quad ||\Phi_Y(J,.)||_u \le J_0 \\ 1 & \text{otherwise,} \end{cases}$$

and $\mathcal{W}^\Lambda_{k,k'}$ denotes the set of connected paths $\gamma = (X_1,\cdots,X_p)$ of sets intersecting Λ with diameter less than R and such that $k\in X_1$, $X_i\cap X_{i+1}\neq\emptyset$, $k'\in X_p$.

Proof of Property 4.4.

We shall prove Property 4.4 by induction on the volume of Λ.

We first prove the property when $|\Lambda| = 1$.

In that case, since we consider integer numbers $D > R$, we only need to check that the sweeping out coefficients can be bounded roughly speaking by the inverse of $m_\Lambda(D)$. But we have :

$$\begin{aligned} \nabla_i\left(E_\Lambda[f^2]\right)^{\frac{1}{2}} &= \frac{1}{2\,(E_\Lambda[f^2])^{\frac{1}{2}}}\nabla_i E_\Lambda[f^2] \\ &= \frac{1}{2\,(E_\Lambda[f^2])^{\frac{1}{2}}}\left(2E_\Lambda[f\nabla_i f] + E^i_\Lambda[f^2]\right) \end{aligned} \tag{32}$$

where we have denoted :

$$E^i_\Lambda[f^2] = E_\Lambda\left[f^2\left(E_\Lambda[\nabla_i H_\Lambda] - \nabla_i H_\Lambda\right)\right] \tag{33}$$

It is clear that $E^i_\Lambda[f^2]$ is null unless $d(i,\Lambda)\le R$ since H_Λ is localized in $\Lambda\cup\partial_R\Lambda$. Thus, using Cauchy Schwarz inequality, the right hand side of (32) is bounded by :

$$\left|\nabla_i\left(E_\Lambda[f^2]\right)^{\frac{1}{2}}\right| \;\le\; \left(E_\Lambda\left[|\nabla_i f|^2\right]\right)^{\frac{1}{2}} + 1\!\!I_{i\in\partial_R\Lambda}\frac{1}{2\,(E_\Lambda[f^2])^{\frac{1}{2}}}|E^i_\Lambda[f^2]| \tag{34}$$

To bound the last term in the right hand side of (34), we first notice that :

$$||\nabla_i H_\Lambda||_u \leq ||\nabla_i \tilde{\Phi}^{(i)}(J,.)||_1, \tag{35}$$

so that :

$$\begin{aligned}
|E^i_\Lambda[f^2]| &= |E_\Lambda\left[\nabla_i H_\Lambda\left(E_\Lambda[f^2]-f^2\right)\right]| \\
&= |E^{\otimes 2}_\Lambda\left[\nabla_i H_\Lambda(x)\left(f^2(x)-f^2(\tilde{x})\right)\right]| \\
&\leq ||\nabla_i H_\Lambda||_u |E^{\otimes 2}_\Lambda\left[(f(x)+f(\tilde{x}))(f(x)-f(\tilde{x}))\right]| \\
&\leq 2||\nabla_i \tilde{\Phi}^{(i)}(J,.)||_1 \left(E_\Lambda[f^2]\right)^{\frac{1}{2}} \left(E^{\otimes 2}_\Lambda\left[(f(x)-f(\tilde{x}))^2\right]\right)^{\frac{1}{2}}
\end{aligned} \tag{36}$$

where we have finally used Cauchy Schwarz inequality to get the right hand side of (36). Moreover, by definition of $m_\Lambda(D)$ we have that :

$$E^{\otimes 2}_\Lambda\left[(f(x)-f(\tilde{x}))^2\right] \leq \frac{2}{m_\Lambda(D)} E_\Lambda\left[(\nabla_\Lambda f)^2\right]$$

so that (36) gives :

$$|E^i_\Lambda[f^2]| \leq 2\sqrt{\frac{2}{m_\Lambda(D)}}||\nabla_i \tilde{\Phi}^{(i)}(J,.)||_1 \left(E_\Lambda[f^2]\right)^{\frac{1}{2}} \left(E_\Lambda\left[(\nabla_\Lambda f)^2\right]\right)^{\frac{1}{2}} \tag{37}$$

As a consequence (34) and (37) show :

$$\begin{aligned}
\left|\nabla_i\left(E_\Lambda[f^2]\right)^{\frac{1}{2}}\right| \leq\ & \left(E_\Lambda\left[|\nabla_i f|^2\right]\right)^{\frac{1}{2}} \\
&+ 1\!\!I_{i\in\partial_R\Lambda}\sqrt{\frac{2}{m_\Lambda(D)}}||\nabla_i \tilde{\Phi}^{(i)}(J,.)||_1 \left(E_\Lambda\left[(\nabla_\Lambda f)^2\right]\right)^{\frac{1}{2}},
\end{aligned} \tag{38}$$

which proves that Property 4.4 holds when $|\Lambda| = 1$, as well as the bound (31) with $C = \sqrt{2}$ since $d(\Lambda, i) \leq R \leq D$ when $i \in \partial_R\Lambda$.

The induction hypotheses : We now let m be an integer number and assume Property 4.4 is true with any Λ with volume smaller or equal to $m-1$, as well as the control (31).

Let now Λ be a subset of $\mathbb{Z}^d$ such that $|\Lambda| = m$ and i be a site in $\partial_R\Lambda$. We then consider an enumeration $\{j_1, j_2, \cdots, j_m\}$, of Λ such that $|j_n - i| \geq |j_{n+1} - i|$, for any $n \in [1, m-1]$. We write $\Lambda_0 = \Lambda$ and, for $n \geq 1$, $\Lambda_n = \Lambda_{n-1} - \{j_n\}$. We also denote $f_n^2 = E_{\Lambda_n}[f^2]$.

According to (32), we essentially need to bound $\nabla_i E_\Lambda[f^2]$. Let N be an integer number (to be specified latter) smaller than m. One notices that, by induction and with the notations of (33) :

$$\nabla_i E_\Lambda[f^2] \;=\; \sum_{n=0}^{N-1} E_\Lambda\left[E^i_{\Lambda_n}[f^2_{n+1}]\right] + E_\Lambda\left[E^i_{\Lambda_N}[f^2]\right] + E_\Lambda[\nabla_i f^2] \tag{39}$$

One can now bound the first sum of terms in the right hand side of (39) using cluster properties in the small couplings regions as developed in [7] and the induction hypotheses.

Indeed, we first have :

$$\begin{aligned} E^i_{\Lambda_n}\left[f^2_{n+1}\right] &= \frac{1}{2}E^{\otimes 2}_{\Lambda_n}\left[\left(f^2_{n+1}(x) - f^2_{n+1}(\tilde{x})\right)\left(\nabla_i H_{\Lambda_n}(x) - \nabla_i H_{\Lambda_n}(\tilde{x})\right)\right] \\ &= \frac{1}{2}E^{\otimes 2}_{\Lambda_n}[\,\left(f^2_{n+1}(x) - f^2_{n+1}(\tilde{x})\right) \\ &\qquad \left(E_{\Lambda_{n+1}}[\nabla_i H_{\Lambda_n}](\omega \bullet_{j_{n+1}} x) - E_{\Lambda_{n+1}}[\nabla_i H_{\Lambda_n}](\omega \bullet_{j_{n+1}} \tilde{x})\right)] \end{aligned} \tag{40}$$

where in the last step step we have used that f_{n+1} is $\mathcal{F}_{\Lambda^c_{n+1}}$ measurable. We now make the following crude bond in the right hand side of (40) :

$$\begin{aligned} &\left|E^i_{\Lambda_n}\left[f^2_{n+1}\right]\right| \\ \leq\; &\frac{1}{2}\left|\left|E_{\Lambda_{n+1}}[\nabla_i H_{\Lambda_n}](\omega \bullet_{j_{n+1}} x) - E_{\Lambda_{n+1}}[\nabla_i H_{\Lambda_n}](\omega \bullet_{j_{n+1}} \tilde{x})\right|\right|_u E^{\otimes 2}_{\Lambda_n}\left[\left|f^2_{n+1}(x) - f^2_{n+1}(\tilde{x})\right|\right] \\ \leq\; &\left|\left|E_{\Lambda_{n+1}}[\nabla_i H_{\Lambda_n}](\omega \bullet_{j_{n+1}} x) - E_{\Lambda_{n+1}}[\nabla_i H_{\Lambda_n}](\omega \bullet_{j_{n+1}} \tilde{x})\right|\right|_u \\ &\times E^{\otimes 2}_{\Lambda_n}\left[\left(f_{n+1}(x) - f_{n+1}(\tilde{x})\right)^2\right]^{\frac{1}{2}} E_{\Lambda_n}\left[f^2_{n+1}\right]^{\frac{1}{2}} \end{aligned} \tag{41}$$

Moreover, since $f_{n+1}(x)$ only depends on the site j_{n+1} in Λ_n, we have that :

$$\begin{aligned} &E^{\otimes 2}_{\Lambda_n}\left[\left(f_{n+1}(x) - f_{n+1}(\tilde{x})\right)^2\right]^{\frac{1}{2}} \leq \\ \leq\; &\exp\{4||\tilde{\Phi}^{(j_{n+1})}(J,.)||_1\}\int \left(f_{n+1}(x) - f_{n+1}(\tilde{x})\right)^2 d\nu_M(x_{j_{n+1}})d\nu_M(\tilde{x}_{j_{n+1}}) \end{aligned}$$

And we know that there exists m_0 such that :

$$\int \left(f_{n+1}(x) - f_{n+1}(\tilde{x})\right)^2 d\nu_M(x_{j_{n+1}})d\nu_M(\tilde{x}_{j_{n+1}}) \leq \frac{2}{m_0}\int \left(\nabla_{j_{n+1}} f_{n+1}\right)^2 d\nu_M(x_{j_{n+1}}) \tag{42}$$

And that we have again :

$$\int \left(\nabla_{j_{n+1}} f_{n+1}\right)^2 d\nu_M(x_{j_{n+1}}) \leq \exp\{2||\tilde{\Phi}^{(j_{n+1})}(J,.)||_1\} E_{\Lambda_n}\left[\left(\nabla_{j_{n+1}} f_{n+1}\right)^2\right]$$

As a consequence, we get that :

$$E^{\otimes 2}_{\Lambda_n}\left[(f_{n+1}(x) - f_{n+1}(\tilde{x}))^2\right]^{\frac{1}{2}} \leq \sqrt{\frac{2}{m_0}} \exp\{3||\tilde{\Phi}^{(j_{n+1})}(J,.)||_1\} E_{\Lambda_n}[|\nabla_{j_{n+1}} f_{n+1}|^2]^{\frac{1}{2}} \tag{43}$$

But $f^2_{n+1} = E_{\Lambda_{n+1}}[f^2]$ and $|\Lambda_{n+1}| \leq m-1$ so that we can use the induction hypotheses and find :

$$E_{\Lambda_n}[|\nabla_{j_{n+1}} f_{n+1}|^2]^{\frac{1}{2}} \quad \leq \quad E_{\Lambda_n}[|\nabla_{j_{n+1}} f|^2]^{\frac{1}{2}} + 1\!\!I_{k\in\partial_R\Lambda_{n+1}} \sum_{k\in\Lambda_{n+1}} \alpha^{\Lambda_{n+1}}_{j_{n+1},k} E_{\Lambda_n}[|\nabla_k f(x)|^2]^{\frac{1}{2}} \tag{44}$$

Moreover, using cluster properties in the small couplings regions as developped in [7] (see Theorem 2.1), one finds that, for any $n \in [0, N-1[$;

$$\left|\left|E_{\Lambda_{n+1}}[\nabla_i H](\omega \bullet_{j_{n+1}} x) - E_{\Lambda_{n+1}}[\nabla_i H](\omega \bullet_{j_{n+1}} \tilde{x})\right|\right|_u \quad \leq \quad 2||\nabla_i \tilde{\Phi}^{(i)}(J,.)||_1 \sum_{\gamma\in\mathcal{W}^{\Lambda_{n+1}}_{i,j_{n+1}}} \prod_{X\in\gamma} \rho_X$$

with the notations given in property 4.4. Thus we get that, for any integer number n in $[0, N-1]$ so that $|i - j_{n+1}| \geq D$ and if $C \geq 4$, $\nu^{\Lambda_{n+1}}_{i,j_{n+1}}$ has been defined so that :

$$\left|\left|E_{\Lambda_{n+1}}[\nabla_i H](x_{j_{n+1}}) - E_{\Lambda_{n+1}}[\nabla_i H](\tilde{x}_{j_{n+1}})\right|\right|_u \quad \leq \quad \frac{1}{2C} \exp\{-3||\nabla_{j_{n+1}} \tilde{\Phi}^{(j_{n+1})}(J,.)||_1\} \nu^{\Lambda_{n+1}}_{i,j_{n+1}} \tag{45}$$

Then (41), (43), (44) and (45) show that :

$$\left|E^i_{\Lambda_n}\left[f^2_{n+1}\right]\right| \leq$$

$$\leq \nu^{\Lambda_{n+1}}_{i,j_{n+1}} \left(E_{\Lambda_n}\left[f^2\right]\right)^{\frac{1}{2}} \left(E_{\Lambda_n}[|\nabla_{j_{n+1}} f|^2]^{\frac{1}{2}} + \sum_{k\in\Lambda_{n+1}} \alpha^{\Lambda_{n+1}}_{j_{n+1},k} E_{\Lambda_n}[|\nabla_k f|^2]^{\frac{1}{2}}\right) \tag{46}$$

Quite similarly, we can bound the second term $E_\Lambda\left[E^i_{\Lambda_N}[f^2]\right]$ in (39). Indeed, following (33) we have :

$$\begin{aligned} E^i_{\Lambda_N}[f^2] &= E_{\Lambda_N}\left[\nabla_i H_{\Lambda_N}\left(E_{\Lambda_N}[f^2] - f^2\right)\right] \\ &\leq 2||\nabla_i \tilde{\Phi}^{(i)}(J,.)||_1 (E_{\Lambda_N}[f^2])^{\frac{1}{2}} E_{\Lambda_N}\left[(E_{\Lambda_N}[f] - f)^2\right]^{\frac{1}{2}} \end{aligned} \tag{47}$$

But, by definition of the spectral gap,

$$E_{\Lambda_N}\left[(E_{\Lambda_N}[f]-f)^2\right]^{\frac{1}{2}} \leq \frac{1}{\sqrt{m(\Lambda_N)}}E_{\Lambda_N}\left[|\nabla_{\Lambda_N}f|^2\right]^{\frac{1}{2}} \tag{48}$$

Plugging (46) and this last inequality in (39), yields :

$$\begin{aligned}\left|\nabla_i E_\Lambda[f^2]\right| &\leq \left(E_\Lambda\left[f^2\right]\right)^{\frac{1}{2}}\sum_{n=0}^{N-1}\nu_{i,j_{n+1}}^{\Lambda_{n+1}}\times\left(E_\Lambda[|\nabla_{j_{n+1}}f|^2]^{\frac{1}{2}}+\sum_{k\in\Lambda_{n+1}}\alpha_{j_{n+1},k}^{\Lambda_{n+1}}E_\Lambda[|\nabla_k f|^2]^{\frac{1}{2}}\right)\\ &\quad +2m(\Lambda_N)^{-\frac{1}{2}}||\nabla_i\tilde{\Phi}^{(i)}(J,.)||_1\left(E_\Lambda\left[f^2\right]\right)^{\frac{1}{2}}E_\Lambda\left[|\nabla_{\Lambda_N}f|^2\right]^{\frac{1}{2}}+E_\Lambda[\nabla_i f^2]\end{aligned} \tag{49}$$

Let us now choose N such that :

$$N=\inf\{n\in[0,m-1[\;|\;\Lambda_n\subset B(i,D)\}.$$

Then, by definition of $m_\Lambda(D)$, we have :

$$\frac{1}{m(\Lambda_N)}\leq\frac{1}{m_\Lambda(D)},$$

so that reordering the right hand side of (50) after the obvious remark that $\nu_{i,j}^{\Lambda}\leq\nu_{i,j}^{\Lambda'}$ when $\Lambda\subset\Lambda'$, gives :

$$\begin{aligned}\left|\nabla_i E_\Lambda[f^2]^{\frac{1}{2}}\right| &\leq \sum_{n=0}^{N-1}\nu_{i,j_{n+1}}^{\Lambda_{n+1}}\left(E_\Lambda[|\nabla_{j_{n+1}}f|^2]^{\frac{1}{2}}+\sum_{k\in\Lambda_{n+1}}\alpha_{j_{n+1},k}^{\Lambda_{n+1}}E_\Lambda[|\nabla_k f|^2]^{\frac{1}{2}}\right)\\ &\quad +2m_\Lambda(D)^{-\frac{1}{2}}||\nabla_i\tilde{\Phi}^{(i)}(J,.)||_1\sum_{k\in\Lambda_N}E_\Lambda[|\nabla_k f|^2]^{\frac{1}{2}}+E_\Lambda[|\nabla_i f|^2]^{\frac{1}{2}}\\ &\leq \sum_{k\in\Lambda}\nu_{i,k}^{\Lambda}E_\Lambda[|\nabla_k f|^2]^{\frac{1}{2}}+\sum_{j\in\Lambda}\left(\sum_{n\leq N-1:j\in\Lambda_n}\nu_{i,j_n}^{\Lambda}\alpha_{j_n,j}^{\Lambda_n}\right)E_\Lambda[|\nabla_j f|^2]^{\frac{1}{2}}\\ &\quad +E_\Lambda[|\nabla_i f|^2]^{\frac{1}{2}}\end{aligned} \tag{50}$$

Thus, it remains to prove that the bound (31) holds. To this end, first notice that (50) shows that we can take :

$$\alpha_{i,j}^{\Lambda} \leq \nu_{i,j}^{\Lambda}+\sum_{n\leq N-1:j\in\Lambda_n}\nu_{i,j_n}^{\Lambda}\alpha_{j_n,j}^{\Lambda_n} \tag{51}$$

But, for $n \geq 1$, $|\Lambda_n| \leq m-1$ so that we have, according to our induction hypotheses :

$$\alpha^{\Lambda_n}_{j_n,j} \leq \nu^{\Lambda}_{j_n,j} + \sum_{l=1}^{|\Lambda_n|-1} \sum_{\{k_1,\cdots,k_l\}\in G^{\Lambda_n}_{l,D}(j_n)} \nu^{\Lambda}_{j_n,k_1} \left(\prod_{p=1}^{l-1} \nu^{\Lambda}_{k_p,k_{p+1}} \right) \nu^{\Lambda}_{k_l,j}.$$

Moreover, by construction, we know that $|j_n - i| > D$ for $n \leq N-1$ and that $\Lambda_n \subset \Lambda \backslash \{j_n\}$. Furthermore, $|i-j_n| \geq |i-j|$ for any n such that $j \in \Lambda_n$. Thus, the relation (51) shows that

$$\alpha^{\Lambda}_{i,j} \quad \leq \quad \nu^{\Lambda}_{i,j} + \sum_{l=1}^{|\Lambda|-1} \sum_{\{k_1,\cdots,k_l\}\in G^{\Lambda}_{l,D}(i)} \nu^{\Lambda}_{i,k_1} \prod_{p=1}^{l-1} \nu^{\Lambda}_{k_p,k_{p+1}} \nu^{\Lambda}_{k_l,j}$$

which ends the proof of Property 4.4.

$\diamondsuit$

4.2 Almost sure bound on the sweeping out coefficients

We are now ready to finish the proof of Theorem 4.2, and thus the one of Theorem 3.2.

Let us first notice that, in the high temperature regime (J_0 small), the interaction is expected to be weak in general. This feature can be expressed by means of the following percolation result :

Lemma 4.5 *Let*

$$\lambda(i,j) = \inf_{\gamma \in \mathcal{W}^{\mathbb{Z}^d}_{j,i}} \frac{\#\{X \in \gamma : \ ||\Phi_X(J,\cdot)||_u \leq J_0\}}{\#\{X \in \gamma\}},$$

Then, for any $\eta \in (0,1)$, there exists a positive function $\bar{r}^{\eta}_0(p_1,\eta,d)$, $\bar{r}^{\eta}_0(p_1,\eta,d) \to 0$ when $p_1 \to 0$, such that, for any $r_0 \geq \bar{r}^{\eta}_0(p_1,\eta,d)$:

$$\sum_n \mathbb{E}\left\{ \exists (i,j) \in [-n,n]^d \mid \lambda(i,j) \leq \eta \text{ and } |i-j| \geq r_0 \log n \right\} < \infty,$$

so that, for almost all interaction, there exists a finite integer number $N(J)$ so that, for any $n \geq N(J)$, for any (i,j) in $[-n,n]^d$ such that $|j-i| \geq r_0 \log n$,

$$\lambda(i,j) \geq \eta.$$

This result is proven in [15] in the nearest neighbors interaction setting and in [12](Lemma 5.4) in the general setting. On the other hand, we are going to see that :

Property 4.6 *Let Ξ be a finite subset of $\mathbb{Z}^d$, such that there exists a real number $\eta > 0$ and an integer number N such that, for any (i,j) in Ξ satisfying $|i-j| \geq N$, we have :*

$$\lambda(i,j) \geq \eta \tag{52}$$

Then, for any $\Lambda \subset \Xi$, for any (i,j) in Ξ, there exists a random variable $C_{i,j}^{\Lambda,N}(J)$ such that :

$$\alpha_{i,j}^{\Lambda} \leq C_{i,j}^{\Lambda,N}(J) m_{\Xi}(N)^{-\frac{1}{2}} \alpha_0^{|i-j|-N},$$

with $\alpha_0 = (\exp\{4J_0\} - 1)^{\eta}$.

Furthermore, if the interaction satisfies $I\!H1$ and $I\!H2$ with J_0 small enough, there exist an integer number $N_0(\eta, J_0)$ and a finite constant $C(\eta, J_0)$ such that, for any set Ξ which satisfies (52) with an integer $N \geq N_0(\eta, J_0)$, we have :

$$\sup_{\Lambda \subset \Xi} \sup_{(i,j) \in \bar{\Xi}_R} I\!E[C_{i,j}^{\Lambda,N}] < C(\eta, J_0).$$

Let us assume for a while that Property 4.6 is true and check that Lemma 4.5 and Property 4.6 give the sweeping out relations stated in Theorem 4.2. We first prove Theorem 4.2.2), namely we show that :

Corollary 4.7 *If the interaction satisfies $I\!H1$ and $I\!H2$ with J_0 small enough, there exist a finite positive constants τ and α, for which we have $\alpha < 1$ and (α, τ) go to zero when J_0 goes to zero, such that, for almost every coupling J, there exists an integer number $N(J)$, so that for any integer $n \geq N(J)$, any $k \in [(\tau(\log n)^{d-1} + 2R) Z\!\!Z]^d \cap \bar{\Lambda}_n^R$, all $i \in \partial_R Y_{\tau(\log n)^{d-1}}(k)$, all $j \in Y_{\tau(\log n)^{d-1}}(k)$ we have*

$$\alpha_{i,j}^{Y_{\tau(\log n)^{d-1}}(k)} \leq \alpha^{|i-j|}$$

provided that $|i-j| \geq \tau(\log n)^{d-1}$.

<u>*Proof.*</u> Let η and τ be positive real numbers. Furthermore, assume that J_0 is small enough so that $e^{4J_0} - 1 < 1$ and choose $\alpha \in]\alpha_0 = (e^{4J_0} - 1)^{\eta}, 1[$. To simplify the notation let us set $Y_n(k)$ for $Y_{\tau(\log n)^{d-1}}(k)$ and $C_{i,j}^k(r_0, J)$ for $C_{i,j}^{Y_n(k), r_0 \log n}(J)$. Then according to Borel Cantelli lemma, it is enough to show that the following sum is finite :

$$S := \sum_n P\left(\cup_{k \in [(2R + \tau(\log n)^{d-1})Z]^d \cap \bar{\Lambda}_n^R} \{\exists (i,j) | \alpha_{i,j}^{Y_n(k)} > \alpha^{|i-j|} \text{ and } |i-j| \geq \tau(\log n)^{d-1}\}\right) \quad (53)$$

For this, let us notice that Property 4.6 (with $N = r_0 \log n$) implies that, for any positive integer number r_0, for any finite constant C and any n large enough :

$$\left\{\cup_{k \in [(2R + \tau(\log n)^{d-1})Z]^d \cap \bar{\Lambda}_n^R} \{\alpha_{i,j}^{Y_n(k)} > \alpha^{|i-j|}, |i-j| \geq \tau(\log n)^{d-1}\}\right\} \subset A_n \cup B_n \cup C_n \quad (54)$$

where

$$A_n \equiv \{\exists (i,j) \in \Lambda_n \mid \lambda(i,j) \leq \eta \text{ and } |i-j| \geq r_0 \log n\},$$

$$B_n \equiv \left\{\cup_{k \in [(2R + \tau(\log n)^{d-1})Z]^{d-l} \cap \bar{\Lambda}_n^R} \{C_{i,j}^k(r_0, J) > C^{-(\log n)^{d-1}} (\frac{\alpha}{\alpha_0})^{|i-j|}, |i-j| \geq \tau(\log n)^{d-1}\}\right\},$$

and :

$$C_n \equiv \left\{ m_{[-n,n]^d}(r_0 \log n) \le \alpha_0^{-2r_0 \log n} C^{-2(\log n)^{d-1}} \right\},$$

for some finite constant C. Hence,

$$S \le \sum_n P(A_n) + \sum_n P(B_n) + \sum_n P(C_n). \tag{55}$$

But, for any $r_0 > \bar{r}_0(p_1, \eta, d)$, Lemma 4.5 implies that $\sum_n P(A_n)$ is finite. Let us now focus on the second term in the right hand side of (55) and notice that the number of cubes $Y_n(k)$'s is less than $2^{d-1}\left(n/\tau(\log n)^{d-1}\right)^d$, that their volume is $(2(\tau(\log n)^{d-1} + R) + 1)^d$ and the volume of their boundary is of order $(\log n)^{(d-1)^2}$. Therefore, Chebyshev inequality gives :

$$P\left(B_n\right) \quad \le \quad \text{const.} C(\eta, J_0) n^{2d} e^{(\log C + \tau \log \frac{\alpha_0}{\alpha})(\log n)^{d-1}}$$

Thus, if (C, τ, α) is chosen so that $\log C + \tau \log \frac{\alpha_0}{\alpha} < 0$, we get that :

$$\sum_n P\left(B_n\right) < \infty \tag{56}$$

We finally show that $\sum_n P\left(C_n\right)$ is finite if C is large enough. This can be proven using a standard spectral gap lower bound. Indeed, it is known that, for any set Y we have :

$$m(Y) \ge \left(|Y| \sup_{i \in Y} \exp\{8 \sum_{X \cap V^i(Y) \neq \emptyset} ||\Phi_X||_u\} \right)^{-1} \tag{57}$$

where $V^i(Y) = \{j \in Y : |j_1 - i_1| \le R\}$. This result implies that

$$\begin{aligned} m_{[-n,n]^d}(r_0 \log n) \quad &= \quad \inf_{i \in [-n,n]^d} \inf_{X \subset [-n,n]^d} m(X \cap B(i, r_0 \log n)) \\ &\ge \quad \left((2r_0 \log n + 1)^d \sup_{i,j \in [-n,n]^d} \exp\{8 \sum_{X \cap V^i(B(j, r_0 \log n)) \neq \emptyset} ||\Phi_X||_u\} \right)^{-1} \end{aligned}$$

Thus, Chebyshev inequality implies that :

$$\begin{aligned} P\left(C_n\right) \quad &\equiv \quad P\left(m_{[-n,n]^d}(r_0 \log n) \le \alpha_0^{-2r_0 \log n} C^{-2(\log n)^{d-1}} \right) \\ &\le \quad \text{const.} (\log n)^d n^{2d} C^{-2(\log n)^{d-1}} \mathbb{E}\left[\exp\{8||\tilde{\Phi}^{(j)}||_1)\} \right]^{(2R+1)(2r_0 \log n + 1)^{d-1}} \end{aligned}$$

where we used in the last line the fact that $|V^{i,n}| \le (2R+1)(2r_0 \log n + 1)^{d-1}$ and the invariance by translation of the law of $||\tilde{\Phi}^{(j)}||_1$. As a consequence, it is clear that if C is large enough so that $-2\log C + (2R+1)(2r_0)^{d-1} \log \mathbb{E}\left[\exp\{8||\tilde{\Phi}^{(j)}||_1)\}\right] < 0$, we get :

$$\sum_n P\left(C_n\right) < \infty.$$

As a conclusion, for any $\eta \in]0,1[$, we can choose $r_0 > \bar{r}_0(p_1,\eta,d)$ so that $\sum_n P(A_n)$ is finite, then a constant C large enough so that $\sum_n P(C_n)$ is finite and finally two constants $\alpha < 1$ and $\tau > 0$ so that $P(B_n)$ is finite. Then, we get (53), i.e. Corollary 4.7. Moreover, it is clear that we can take α and τ as small as we want when α_0 goes to zero, i.e. J_0 does.

$\diamondsuit$

Similarly, we can prove the second part of Theorem 4.2, i.e. :

Lemma 4.8 *If $\mathbb{H}1$ and $\mathbb{H}2$ are satisfied with J_0 small enough, there exists a finite constant τ', such that, for $\mathbb{E}$ almost every J, there exists an integer number $\tilde{N}(J)$ for which we have for any integer $n \geq N(J)$, for any $l \in [0,d-1]$, any $k \in \{0\}^l \times [(2R+\tau(\log n)^{d-1})\mathbb{Z}]^{d-l} \cap [-n,n]^d$, any $i \in \partial_R Y^l_{\tau(\log n)^{d-1}}(k)$, the following bound*

$$\sum_{j \in Y^l_{\tau(\log n)^{d-1}}} \alpha_{i,j}^{Y^l_{\tau(\log n)^{d-1}}(k)} \leq \exp\{\tau'(\log n)^{d-1}\}.$$

Proof.

Let us first define, for an integer n in $\mathbb{N}$, a point $i \in \mathbb{Z}^d$ and a finite set Y of $\mathbb{Z}^d$, a measurable set $\mathcal{B}(n,i,Y)$ by :

$$\mathcal{B}(n,i,Y) = \left\{ \sum_{j \in Y} \alpha_{i,j}^Y > \exp\{\tau'(\log n)^{d-1}\} \right\}.$$

Then, for any set Y in $[-n,n]^d$ and any $i \in [-n-R, n+R]^d$, Property 4.6 shows that :

$$\mathcal{B}(n,i,Y) \subset A_n \cup C_n \cup D_n(i,Y),$$

where A_n and C_n are defined as in the previous proof with coefficients (r_0, C) such that $\sum_n P(A_n)$ and $\sum_n P(C_n)$ are finite. Moreover

$$D_n(i,Y) = \left\{ \sum_{j \in Y} C_{i,j}^Y > C^{-2(\log n)^{d-1}} \alpha_0^{-2r_0 \log n} \exp\{\tau'(\log n)^{d-1}\} \right\}.$$

But Chebyshev inequality gives that :

$$P\left(D_n(i,Y)\right) \;\; \leq \;\; C(\eta, J_0)|Y|\alpha_0^{-r_0 \log n} \exp\{(\log C - \tau')(\log n)^{d-1}\}$$

Applying this result with $Y = Y^l_{\tau(\log n)^{d-1}}(k)$ for $l \in [0,d-1]$, $k \in \{0\}^l \times [(2R+\tau(\log n)^{d-1})\mathbb{Z}]^{d-l}$ $\cap[-n,n]^d$ and $i \in \partial_R Y^l_{\tau(\log n)^{d-1}}(k)$ after the obvious remark that the number of such sets is of order n^{2d} and their volume of order $(\log n)^{(d-1)d}$ allows us to deduce Lemma 4.8 from Borel Cantelli Lemma provided that τ' is large enough.

$\diamondsuit$

We finally turn to the proof of Property 4.6.

Let (Ξ, N, η) satisfy (1), $\Lambda \subset \Xi$, (i,j) be an element of $(\Xi - \Lambda)\times\Lambda$ and let $\alpha_0 = (\exp\{4J_0\} - 1)^\eta$. Property 4.4 (with $D = N$) implies that there exists a finite constant c such that :

$$\begin{aligned}
\alpha_{i,j}^{\Lambda} \leq\ & c||\nabla_i \tilde{\Phi}^{(i)}(J,.)||_1 \exp\{3||\tilde{\Phi}^{(i)}(J,.)||_1\} \alpha_0^{|i-j|-N} m_\Lambda(N)^{-1} \Big(1 + \\
& + \sum_{l=1}^{|\Lambda|-1} \sum_{\{k_1,\cdots,k_l\}\in G_{l,N}^{\Lambda}(i)} \prod_{p=1}^{l} c||\nabla_{k_p} \tilde{\Phi}^{(k_p)}(J,.)||_1 \exp\{3||\tilde{\Phi}^{(k_p)}(J,.)||_1\} \times \\
& \times \alpha_0^{|i-k_1|-|j-i|+\sum_{p=1}^{l-1}|k_p-k_{p+1}|} \Big)
\end{aligned} \tag{58}$$

As a consequence, the first part of Property 4.6 is proved, with :

$$\begin{aligned}
C_{i,j}^{\Lambda,N}(J) \equiv\ & c||\nabla_i \tilde{\Phi}^{(i)}(J,.)||_1 \exp\{3||\tilde{\Phi}^{(i)}(J,.)||_1\} \Big(1 + \\
& + \sum_{l=1}^{|\Lambda|-1} \sum_{\{k_1,\cdots,k_l\}\in G_{l,N}^{\Lambda}(i)} \prod_{p=1}^{l} c||\nabla_{k_p} \tilde{\Phi}^{(k_p)}(J,.)||_1 \exp\{3||\tilde{\Phi}^{(k_p)}(J,.)||_1\} \times \\
& \times \alpha_0^{|i-k_1|-|j-i|+\sum_{p=1}^{l-1}|k_p-k_{p+1}|} \Big)
\end{aligned} \tag{59}$$

We now bound the expectation of $C_{i,j}^{\Lambda,N}(J)$ by bounding the expectation of the right hand side of (59). To do so, we need a probability estimate, i.e. we have to bound the expectation of terms of the form :

$$\prod_{p=1}^{l} ||\nabla_{k_p} \tilde{\Phi}^{(k_p)}(J,.)||_1 \exp\{3||\tilde{\Phi}^{(k_p)}(J,.)||_1\}.$$

We claim that :

Lemma 4.9 *Under assumption $\mathbb{H}1$, there exist positive numbers ϵ and a ($\epsilon \to 0$ when p_1 and J_0 go to zero), such that, for any family $\{k_1, \cdots, k_l\}$ of different points, we have*

$$\mathbb{E}\left[\prod_{p=1}^{l} ||\nabla_{k_p} \tilde{\Phi}^{(k_p)}(J,.)||_1 \exp\{3||\tilde{\Phi}^{(k_p)}(J,.)||_1\} \right] \leq a\epsilon^l \tag{60}$$

Before turning to the proof of Lemma 4.9, we first end the proof of Property 4.6. For this, let us notice that each element of $G_{l,N}^{\Lambda}(i)$ is made of different points so that we can apply

Lemma 4.9 in (59) and find already that :

$$\mathbb{E}[C_{i,j}^{\Lambda,N}(J)] \;\; \leq \;\; Cc\epsilon\Big(1+\sum_{l=1}^{|\Lambda|-1} a\epsilon^l c^l \sum_{\{k_1,\cdots,k_l\}\in G_{l,N}^{\Lambda}(i)} \alpha_0^{|i-k_1|-|j-i|+\sum_{p=1}^{l-1}|k_p-k_{p+1}|}\Big) \qquad (61)$$

But, since $|i-k_1|\geq|j-i|$, it is clear that :

$$\sum_{\{k_1,\cdots,k_l\}\in G_{l,N}^{\Lambda}(i)} \alpha_0^{|i-k_1|-|j-i|+\sum_{p=1}^{l-1}|k_p-k_{p+1}|} \leq \left(\sum_{n\geq 0}(2l+1)^{d-1}\alpha^l\right)\left(\sum_{n\geq N}(2l+1)^{d-1}\alpha^l\right)^{l-1},$$

and, if α is smaller than 1 and N large enough so that $c\epsilon\sum_{n\geq N}(2l+1)^{d-1}\alpha^l<1$, we can conclude that :

$$\mathbb{E}[C_{i,j}^{\Lambda,N}(J)] \;\; \leq \;\; A\epsilon\left(1+\frac{1\epsilon}{1-c\epsilon\sum_{n\geq N}(2l+1)^{d-1}\alpha^l}\right) \qquad (62)$$

where $A=Cc\epsilon\left(\sum_{n\geq 0}(2l+1)^{d-1}\alpha^l\right)$.

Proof of Lemma 4.9.

We want to get a bound independently of the choice of our distinct points $\{i,j,k_1,\cdots,k_l\}$ of the following quantity :

$$A:\equiv \mathbb{E}\Big[\prod_{p=1}^{l}||\nabla_{k_p}\tilde{\Phi}^{(k_p)}(J,.)||_1\exp\{3||\tilde{\Phi}^{(k_p)}(J,.)||_1\}\Big]$$

But using the definition of $||\nabla_i\tilde{\Phi}^{(i)}(J,.)||_1$ and expanding the product, we find :

$$A \;\; = \;\; \sum_{X_p\ni k_p}\mathbb{E}\Big[\prod_{p=1}^{l}||\Phi_{X_p}(J,.)||_u\exp 3\left\{\sum_{p=1}^{l}\sum_{X_p\ni k_p}||\Phi_{X_p}(J,.)||_u\right\}\Big]$$

Then, by Cauchy Schwarz inequality, we get that :

$$A \;\; \leq \;\; \sum_{X_p\ni k_p}\left(\mathbb{E}\left[\prod_{p=1}^{l}||\nabla_i\Phi_{X_p}(J,.)||_u^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\exp 6\left\{\sum_{p=1}^{l}\sum_{X_p\ni k_p}||\Phi_{X_p}(J,.)||_u\right\}\right]\right)^{\frac{1}{2}} \qquad (63)$$

To bound the terms in the right hand side of (63), we first make the basic observation that, since $\{k_1,\cdots,k_l\}$ are different points, each set X in the above sum can appear at most $|X|$ (

$\leq (2R+1)^d$) times. Thus, if $C(R,d)$ is the number of sets containing a point i and included in $B(i,R)$, we get :

$$\mathbb{E}\left[\exp 6\left\{\sum_{p=1}^{l}\sum_{X_p\ni k_p}||\Phi_{X_p}(J,.)||_u\right\}\right] \leq \left(\sup_X \mathbb{E}[\exp\{6|X|||\Phi_X(J,.)||_u\}]\right)^{C(R,d)l}$$

Similarly, we have :

$$\mathbb{E}\left[\prod_{p=1}^{l}||\nabla_i\Phi_{X_p}(J,.)||_u^2\right] \leq \left(\sup_X \max_{k\leq(2R+1)^{d+1}} \mathbb{E}\left[||\nabla_i\Phi_X(J,.)||_u^k\right]^{\frac{1}{2}}\right)^{2l}.$$

Hence, (63) can be bounded by :

$$A \;\leq\; B\left(C(R,d)\sup_X \max_{k\leq(2R+1)^{d+1}}\left(\mathbb{E}\left[||\nabla_i\Phi_X(J,.)||_u^k\right]\right)^{\frac{1}{2}}\sup_X\left(\mathbb{E}[\exp\{6|X|||\Phi_X(J,.)||_u\}]\right)^{\frac{C(R,d)}{2}}\right)^l$$

where B denotes some finite constant. But, when $\mathbb{H}1$ is satisfied, it is obvious that

$$\sup_X \max_{k\leq(2R)^{d+1}}\left(\mathbb{E}\left[||\nabla_i\Phi_X(J,.)||_u^k\right]\right)^{\frac{1}{2}}$$

goes to zero when p_1 and J_0 do, therefore

$$C(R,d)\sup_X \max_{k\leq(2R)^{d+1}}\left(\mathbb{E}\left[||\nabla_i\Phi_X(J,.)||_u^k\right]\right)^{\frac{1}{2}}\sup_X\left(\mathbb{E}[\exp\{6|X|||\Phi_X(J,.)||_u\}]\right)^{\frac{C(R,d)}{2}}$$

goes to zero with p_1 and J_0, which achieves the proof of Lemma 4.9.

$\diamondsuit$

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References

[1] AIZENMAN M. ; HOLLEY R. ; Rapid Convergence to Equilibrium of Stochastic Ising Models in the Dobrushin-Shlosman Regime.

pp. 1–11 in Percolation Theorey and Ergodic Theory of Infinite Particle Sytems, ed. Kesten H., Springer-Verlag 1987

[2] BEN AROUS G. ; GUIONNET A.; Large Deviations for Langevin Spin Glass Dynamics. Probab. Theory Relat. Fields Vol 102, 455–509 (1995).

[3] BEN AROUS G. ; GUIONNET A.; Symmetric Langevin spin glass dynamics.
Submitted to Annals of Probab.

[4] BERETTI A.; Some Properties of Random Ising models.
J. Stat. Phys. Vol 38, 438–496 (1985)

[5] CHEN M.F. ; WANG F.Y ; Order-preservation and positive correlations.
Probab. Theory Relat. Fields Vol 95, 421-428 (1993).

[6] DEUSCHEL J-D. ; Algebraic L^2 decay of attractive critical processes on the lattice.
Annals of Prob. Vol 22 (1994) 264-283

[7] VON DREYFUS, H. ; KLEIN, A. ; PEREZ, J. F. ; Taming Griffiths singularities : Infinite differentiability of quenched correlation functions.
Comm. Math. Phys. Vol 170 (1995) 21–39

[8] DOBRUSHIN R. L.; KASSALYGO L. A. ; Uniqueness of a Gibbs Field with random potential- An elementary approach.
Theory Probab. Appl. Vol 31, No 4.

[9] FRÖHLICH J, ; IMBRIE J.Z.; Improved Perturbation Expansion for Disordered Systems: Beating Griffiths Singularities
Commun. Math. Phys. Vol 96, 145–180 (1984)

[10] GELIS G.; MAES C.; Percolation Techniques in Disordered Spin Flip Dynamics: Relaxation to the Unique Invariant Measure.
K.U. Leuven, Preprint 1995

[11] GRIFFITHS R.; Non - Analytic Behaviour Above the Critical Point in a Random Ising Ferromagnet.
Phys. Rev. Lett. Vol 23, 17–19 (1969)

[12] GUIONNET A. ; ZEGARLINSKI B. ; Decay to Equilibrium in Random Spin Systems on a Lattice.
Submitted to Commun. Math. Phys.

[13] HOLLEY R. ; Possible rates of convergence in finite range attraction spin systems.
Contemp. Math. Vol 41, 215-234 (1985)

[14] HOLLEY R.; STROOCK D. Logarithmic Sobolev inequalities and stochastic Ising models.
J. Stat. Phys. Vol 46, 1159–1194 (1987)

[15] KESTEN H. ; Aspect of first passage percolation.
Ecole d'ete de St Flour, LNM 1180, p. 125–264 (1986).

[16] LAROCHE E.; Hypercontractivité pour des systèmes de spin de portée infinie
Probab. Theor. Relat. Fields Vol 101, 89–132 (1995).

[17] SHENG LIN LU; HORNG - TZER YAU; Spectral Gap and Logarithmic Sobolev Inequality for Kawasaki and Glauber Dynamics.
Commun. Math. Phys. Vol 156, 399–433 (1993)

[18] MAJEWSKI A.W. ; ZEGARLINSKI B; Quantum Stochastic Dynamics I: Spin Systems on a Lattice.
Math. Phys. Electr. J. Vol 1, Paper 2 (1995)

[19] MARTINELLI F.; OLIVIERI E.; Approach to Equilibrium of Glauber Dynamics in the One Phase Region: I. The Attractive case/ II. The General Case.
Commun. Math. Phys. Vol 161, 447–486 / 487–514 (1994)

[20] MARTINELLI F.; OLIVIERI E.; SHONMANN R.H.; For 2 - D Lattice Spin Systems Weak Mixing Implies Strong Mixing.
Commun. Math. Phys. Vol 165, 33–47 (1994)

[21] PFISTER C.-E.; Translation Invariant Equilibrium States of Ferromagnetic Abelian Lattice Systems.
Commun. Math. Phys. Vol 86, 375–390 (1982)

[22] RUELLE D. Statistical Mechanics: Rigorous Results.
W.A. Benjamin Inc. 1969

[23] SCHONMANN R.H.; SHLOSMAN S.B. ; Complete Analyticity for 2D Ising Completed.
Preprint 1995

[24] STROOCK D. W. ; VARADHAN S. R. S ; Diffusion Processes with boundary conditions.
Comm. Pure Appl. Math. XXIV, 147-225

[25] STROOCK D. W. ; ZEGARLINSKI B. ; The Logarithmic Sobolev inequality for Continuous Spin Systems on a Lattice.
J. Funct. Anal. Vol 104, 299–326 (1992)

[26] STROOCK D. W. ; ZEGARLINSKI B. ; The Equivalence of the Logarithmic Sobolev Inequality and the Dobrushin – Shlosman Mixing Condition.
Commun. Math. Phys. Vol 144, 303–323 (1992)

[27] STROOCK D. W. ; ZEGARLINSKI B. ; On the ergodic properties of Glauber dynamics.
J. Stat. Phys. Vol 81, 1007-1019(1995)

[28] ZEGARLINSKI B. ; Interactions and Pressure Functionals for Disordered Lattice Systems.
Commun. Math. Phys. Vol 139, 305–339 (1991)

[29] ZEGARLINSKI B. ; Strong Decay to Equilibrium in One Dimensional Random Spin Systems.
J. Stat. Phys. Vol 77, 717–732 (1994)

[30] ZEGARLINSKI B.; Spin Systems with Long-Range Interactions.
Rev. Math. Phys. Vol 6, 115–134 (1994)

[31] ZEGARLINSKI B.; Ergodicity of Markov Semigroups.
pp. 312 - 337, in Proc. of the Conference: Stochastic Partial Differential Equations, Edinburgh 1994, Ed. A. Etheridge, LMS Lecture Notes 216, Cambridge University Press 1995