First order correction for the hydrodynamic limit of asymmetric simple exclusion processes in dimension $d \geq 3$.

C. Landim[1], S. Olla[2] and H. T. Yau[3]

Summary: It is well known that the hydrodynamic limit of the asymmetric simple exclusion is governed by a viscousless Burgers equation in the Euler scale [R]. We prove that, in the same scale, the next order correction is given by a viscous Burgers equation up to a fixed time T for dimension $d \geq 3$, provided that the corresponding viscousless Burger equation has a smooth solution up to time T. The diffusion coefficient was characterized via a variation of Green-Kubo formula by [V, X, EMY]. Within the framework of asymmetric simple exclusion, this provides a rigorous verification for the interpretation of analogous phenomena that the correction to the Euler equation is given by the Navier–Stokes equation if the time scale is within the Euler scale.

Keywords: Infinite interacting particle systems, Navier–Stokes equations, hydrodynamic limit, non-gradient methods.

[1]CNRS URA 1378, Université de Rouen, 76138 Mont Siant Aignan Cedex, France. Research partially supported by U. S. National Science Foundation grant DMS 9112654

[2]Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau Cedex, France and Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy.

[3]Courant Institute, 251 Mercer Street, New York NY 10012, USA. Research partially supported by U. S. National Science Foundation grant 9403462 and David and Lucile Packard Foundation Fellowship.

Introduction.

Consider the configuration space consisting of particles in $\mathbf{Z}^d$. Suppose that there is a hard core interaction so that two particles can not occupy the same site. By imposing some probability distribution on the configuration space one has a model of lattice gases in equilibrium. This lattice gas model can be turned into a dynamical one if some dynamical rules are given. A very simple example is the asymmetric simple exclusion process in which a particle waits for an exponential time then hops to a nearest neighbor chosen with probability p_e, where e denotes the unit vector from the particle to its nearest neighbor. With any choice of p_e the Bernoulli measures are the invariant measures. If all p_e are equal, it is called symmetric simple exclusion.

Since particles can never be created or destroyed in the process, the total number of particle is conserved. Therefore we have an interacting particle systems with a conservation law. If the p_e are not identical, there is a net drift and the dynamics is nonreversible. This model can then be viewed as a very simple toy model for Hamiltonian systems. It oversimplifies many important features of Hamiltonian systems and its dynamics is stochastic. Nevertheless it is not exactly solvable and it reflects some basic features of the Hamiltonian systems.

Our main interest is the evolution of the local density of particles. For this purpose, let us fix the scale of lattice to be N^{-1}. In order to have the particle moving a distance of order one in macroscopic scale, one has to wait time of order N in the asymmetric case and order N^2 in the symmetric case. The symmetric case is very special and one can check easily that the density follows the linear heat equation. We shall focus on the asymmetric case. Suppose for simplicity of exposition the particle can only jump to the "right" with rate one, i.e. $p_e = 0$ unless e points to the positive axis direction and $p_e = 1$ in this case. The current of this dynamics is not hard to compute: it is given by $\rho(1-\rho)$ if the local density is ρ. Therefore one has the equation

$$\partial_t \rho + \sum_{i=1}^{d} \partial_i (\rho(1-\rho)) = 0 \tag{0.1}$$

where e denotes positive coordinate directions. This equation does not have a unique solution and an entropy condition has to be supplied. This was proved under the most general condition in [R] in the hydrodynamic limit $N \to \infty$. This equation is hyperbolic and it resembles the Euler equation. Indeed, one can derive the the full Euler equation (before the first singularity appeared) from Hamiltonian systems if some ergodic assumptions are made or some random noise is introduced [OVY].

A fundamental question in nonequilibrium statistical physics is the derivation and interpretation of the Navier-Stokes equation. The Navier-Stokes equation is not scaling invariant and cannot be obtained from a scaling limit. This leads to confusion as to how to interpret the NS equation correctly. Since a study of NS equation on Hamiltonian systems is beyond the current mathematical technology, let us focus on similar phenomena in the context of ASEP. The equation that corresponds to the NS equation in this context is

$$\partial_t \rho + \sum_{i=1}^{d} \partial_i(\rho(1-\rho)) = N^{-1} \partial_i(a^{i,j}(\rho)\partial_j \rho) \tag{0.2}$$

where $a^{i,j}(\rho)$ is the diffusion coefficient depending on ρ. This leads to the following three interpretations of NS equation:

(1) The NS equation (0.2) is the next order correction to the Euler equation (0.1) . In other words, equation (0.2) describes the evolution of the density up to order N^{-1}.

(2) The incompressible limit explanation. Consider $\rho = \rho_0 + N^{-1}u$ and the diffusive scale N^2. For simplicity, take $\rho_0 = 1/2$. Otherwise a uniform motion has to be taken out. One finds that u satisfies the equation

$$\partial_t u + 2\sum_e \partial_e u^2 = \partial_i(a^{i,j}(1/2)\partial_j u) \tag{0.3}$$

Note that the diffusion coefficient is computed at the equilibrium density 1/2 and is a constant in the equation.

(3) The NS equation (0.2) describes the evolution of density up to time scale N^2. More precisely, let us rescale the equation (0.2) in diffusive scale:

$$\partial_t \rho + N\sum_{i=1}^{d} \partial_i(\rho(1-\rho)) = \partial_i(a^{i,j}(\rho)\partial_j \rho) \tag{0.4}$$

This equation has no limit as $N \to \infty$ unless the initial data is constant in the direction of the drift. In this case, the drift term vanishes. For other general cases, the interpretation of NS equation is much more complicated and we shall not discuss it further here.

The incompressible limit interpretation (2) was studied in [EMY] for dimension $d \geq 3$. In this paper we shall prove the interpretation (1) holds also for dimension $d \geq 3$ before the time the corresponding Euler equation (0.1) develops singularities (shocks). The interpretation (3) will be studied in a forthcoming paper. The interpretation (1) has been advocated in particular by Dobrushin (cf. [D]). One might argue that, practically, the next order correction to equation (0.1) is not important because it is of order $10^{-23/d}$. It is interesting, however, to determine if the correction in Euler scale is governed by the same diffusion coefficient as the one that appeared in the diffusive scale or in the incompressible limit. Furthermore, there is no a priori reason why the next order correction to (0.1) can be described by a differential equation at all. Indeed, this may not be true in dimension $d \leq 2$. Our result also provides a rare example for which the next order correction to the hydrodynamic limit can be computed.

The diffusion coefficient can be characterized using the method of [EMY]. Even though in equations (0.2) and (0.3), only the symmetric part of the diffusion matrix is relevant, we believe that the diffusion matrix as we define it in ♣*adef can be non-symmetric.

Rather than the hydrodynamic limit considered here, one can derive the Euler equations and the NS equations from the Boltzmann equation. In this context the interpretation (1) was proved by [C] among others. The corresponding incompressible limit was established in [DEL]. Various cases related to the interpretation (3) were also considered by [ELM-1] and [ELM-2]. Compared with the hydrodynamic limit considered here, the Boltzmann equation is an integral-differential equation and is not a many-body interacting particle system. It can be derived from the Hamiltonian system in the kinetic limit [La] for which the average collisions per particle are kept finite. Because the average collisions per particle are fixed, the kinetic limit is not a good approximation for time scale much longer a few *mean free times*. In the hydrodynamic limit the average collisions per particle are of order N in the Euler scale and of order N^2 in the diffusive scale. It provides more relevant scale for long time behavior of interacting particle systems.

Finally, let us comment briefly on the technique. The basic ingredients in [EMY] are a nongradient system method [V, X], a multiscale analysis and a relative entropy argument [Y1].

In general the relative entropy method works as follow. One first selects a local equilibrium state describing a state with macroscopic density ρ solving the correct equation (0.1), (0.2) or (0.3) depending on the cases interested. One then bounds the specific entropy (per particle) relative to this slowly varying Gibbs state. For the usual hydrodynamic limit, one needs the specific relative entropy to vanish in the limit $N \to \infty$. To get accuracy of order N^{-1}, one needs the specific relative entropy of order $O(N^{-2})$. Should one follow the argument of [EMY], one can only bound the specific relative entropy by order $O(N^{-1})$, which is not sufficient to identify the correction term. This is not surprising because most entropy related methods work better when applied to processes near the equilibrium as in the incompressible limit. Our strategy is as follow. We first use the approach of [EMY] to bound the specific relative entropy by $o(N^{-1})$. Instead of using the entropy bound directly, we derive an estimate of the associated Dirichlet from this entropy bound. This bound turns out to be sufficient if more careful analysis of the equation (0.2) is used. Details will be described in sections 2 and 3.

1. Statement of the Result.

We consider the asymmetric simple exclusion process on a lattice of size N in dimension $d \geq 3$ with periodic boundary conditions. We shall denote by T_N^d the d–dimensional torus with length N. The configuration of this process are given by

$$\eta = \{\eta(x);\ x \in \mathbf{Z}^d;\ \eta(x + Ne_i) = \eta(x)\}$$

where $(e_i)^j = \delta_{ij}$, and $\eta(x) = 1$ or 0 indicating if the site x is occupied or not.

The generator of this process is given by

$$L_N f(\eta) \;=\; \sum_{x,i} \eta(x)(1 - \eta(x + e_i))\left(f(\eta^{x,x+e_i}) - f(\eta)\right) \tag{1.1}$$

where

$$\eta^{x,x+e_i}(z) \;=\; \begin{cases} \eta(x + e_i), & z = x \\ \eta(x), & z = x + e_i \\ \eta(z), & \text{otherwise.} \end{cases}$$

We are considering here the totally asymmetric case for simplicity of notations, but all our results are valid also for the case where jumps are allowed in the negative directions.

We start the process with the inhomogeneous product measure

$$\mu_0^N = Z^{-1} \exp\left[\sum_x \lambda_0(\frac{x}{N})\eta(x)\right] \tag{1.2}$$

where $\lambda_0(u)$ is a smooth function on T^d (the d-dimensional torus of volume 1).

For a positive integer N and $T_0 > 0$, denote by P_N the probability measure on the path space $D([0,T_0],\{0,1\}^{S_N^d})$ of the Markov process with generator L_N accelerated by N and starting from μ_0^N and by E_N the expectation with respect to P_N.

To the measure (1.2) correspond the initial density profile

$$m_0(u) = \frac{e^{\lambda_0(u)}}{e^{\lambda_0(u)}+1} \,. \tag{1.3}$$

Notice that the initial profile m_0 is bounded away from 0 and 1: there exists $\delta > 0$ such that $\delta \le m_0(u) \le 1-\delta$.

Denote by m the solution of the quasilinear hyperbolic equation

$$\begin{cases} \partial_t m + \mathbf{1}\cdot\nabla_u F(m) = 0 \\ m(0,\cdot) = m_0(\cdot) \end{cases} \tag{1.4}$$

with $F(m) = m(1-m)$.

Throughout this article we restrict our attention on a time interval $[0,T_0]$ where there exists a classical smooth solution of this equation.

This equation describes the macroscopic behavior of the asymmetric simple exclusion process. More precisely, if $q^N(t,u)$ is defined as:

$$q^N(t,u) := E_N\Big[\eta_t([Nu])\Big] \,,$$

it follows from [R] that, for each fixed time t, $q^N(t,\cdot)$ converges weakly to $m(t,\cdot)$. In fact a much stronger statement is proved there as a law of large numbers. Later it was proved [L] that this convergence is pointwise at the continuity points of $m(t,\cdot)$.

Our result describe the first order correction in this limit:

For each integer N consider the quasilinear parabolic equation

$$\begin{cases} \partial_t \rho^N + \mathbf{1}\cdot\nabla_u F(\rho^N) = (1/N)\,\nabla_u\big(a(\rho^N)\nabla_u\rho^N\big) \\ \rho^N(0,\cdot) = m_0(\cdot) \end{cases} \tag{1.5}$$

Here $a(m) = (a^{i,j}(m))_{1\le i,j\le d}$ is the diffusion matrix formally defined as

$$a^{i,j}(m) \;=\; \lim_{k\to\infty}(2k+1)^{-d}\left\langle \sum_{|y|\le k-1}\tau_y(\eta(e_i)-\eta(0)),(-\tilde{L}_k)^{-1}\sum_{|x|\le k-1}\tau_x(\eta(e_j)-\eta(0))\right\rangle_{k,m}$$

where $\tilde{L}_k$ is the generator L_N restricted to the box $\Lambda_k = \{-k,\dots,k\}^d$ (i.e., $\tilde{L}_k$ consists of jumps involving only particles inside $\Lambda - k$) and $< \cdot,\cdot >_{k,m}$ denotes the inner product in $L^2(\nu_{k,m})$ with $\nu_{k,m}$ being the canonical measure on $\{0,1\}^{\Lambda_K}$ with density m. The diffusion matrix is precisely defined in section 10 by the variational formula (cf. [EMY]) presented in Theorem 10.2. We can only prove that $a(m)$ is a continuous matrix valued function of m bounded below by $(1/2)I$ in the sense of matrices: $a(m) \ge (1/2)I$, if I denotes the identity (cf. Lemma 10.3 in section 10).

A function $\rho^N(t,u)$ is a weak solution of (1.5) if is bounded,

$$\int_0^{T_0} dt \int du |\nabla_u \rho^N|^2 < +\infty$$

and for any smooth function $J(t,u)$ with compact support on $[0,T_0)\times T^d$

$$\begin{aligned}&\int dt \int du\,\big(\rho^N\,\partial_t J + F(\rho^N)\,\mathbf{1}\cdot\nabla_u J\big)\\ &\qquad - N^{-1}\int dt\int du \nabla_u J\cdot a(\rho^N)\nabla_u\rho^N + \int du J(0,u)m_0(u) = 0.\end{aligned}$$

The existence of a weak solution with Holder continuity (depending on N) is well known. We do not know the uniqueness. Nevertheless, since there is a small parameter N^{-1} in the diffusive term, one can understand (1.5) as a perturbation of (1.4) as follows.

For any solution m of (1.4) fixed, consider the linear equation

$$\begin{cases} \partial_t R \;+\; \mathbf{1}\cdot\nabla_u\big[F'(m)R\big] \;=\; \nabla_u\cdot[a(m)\cdot\nabla_u m] \\ R(0,\cdot) \;=\; 0\,. \end{cases} \tag{1.6}$$

Since $a(m)$ is only continuous, we can consider the equation (1.6) in the following weak sense:

For $t \le T_0$ there exists a distribution $R(t)$ in $H_{-1}(T^d)$ such that for any function $J(u) \in H_1(T^d)$ (i.e. with $\nabla J \in L^2$) we have

$$\langle R(t), J\rangle \;=\; -\int_0^t ds \int du \nabla J(s,u)\cdot a(m(s,u))\nabla m(s,u)$$

where $J(s,u)$ is the solution of the equation

$$\partial_s J(s,u) \;+\; F'(m(s,u))\mathbf{1}\cdot\nabla_u J(s,u) \;=\; 0 \tag{1.7}$$

with final condition $J(t,u) = J(u)$. We call such family of distributions $\{R(t),\ 0\le t\le T_0\}$ a solution of the equation (1.6).

We will prove in section 2 that, for any solution ρ^N of (1.5), $N[\rho^N - m]$ tends to R weakly as $N\to\infty$. Hence the uniqueness of (1.5) does not affect our problem as far as the next order correction of (1.4) is concerned. The main result of this paper is the following Theorem.

Theorem 1.1 . *Consider a solution ρ^N of (1.5). Then $\lim_{N\to\infty} N(\rho^N - q^N) \;=\; 0$ weakly in H_{-1}*

Granting the claim $N[\rho^N - m] \rightharpoonup R$, we only have to prove the following Theorem 1.2 in order to prove Theorem 1.1.

Theorem 1.2 . *For any $t\le T_0$ we have*

$$N\left(q^N(t,\cdot) - m(t,\cdot)\right) \;\rightharpoonup R(t)$$

weakly in H_{-1}.

We will prove the claim $N[\rho^N - m] \rightharpoonup R$ in section 2. This will give us an idea about the strategy of the proof of Theorem 1.2. A key point in the argument of section 2 is to show that the L^2 norm of $\rho^N - m$ goes to zero faster than $N^{-1/2}$. In section 3 we will repeat the computation but with q^N in place of ρ^N. Besides other problems that arise, one needs to prove, as for the deterministic case, that the L^2 norm of $q^N - m$ goes to zero faster than $N^{-1/2}$. This can be achieved only with a very strong control of the entropy of the distribution of the particles at the macroscopic time t relative to the inhomogeneous product measure parametrized by $\lambda(m(u,t))$.

2. The strategy of the proof.

In order to present the strategy of the proof we consider in this section the problem of convergence of solutions of quasilinear parabolic equations with diffusion coefficient decreasing to 0 to solutions of hyperbolic equations in regions where these solutions are smooth.

Let m, R and ρ^N the solution respectively of (1.4), (1.6) and (1.5) . We would expect ρ^N to be well approximated by $m + N^{-1}R$ or the difference $N[\rho^N - m]$ to converge to R. This is the content of the main result of this section stated below. We will use the notation $m_T(u) = m(T,u)$.

Proposition 2.1 . *For each* $0 \le T \le T_0$ *the function*

$$N[\rho^N - m] \rightharpoonup R$$

weakly.

Proof. Fix a time $0 < T \le T_0$. For each smooth function $J_0 : T^d \to \mathbf{R}$, let $J(t,u)$ be the smooth solution of the first order linear equation

$$\begin{cases} \partial_t J \;+\; F'(m)\mathbf{1}\cdot\nabla_u J \;=\; 0 \\ J(T,\cdot) \;=\; J_0(\cdot)\,. \end{cases}$$

Recall that the entropy solution m of equation (1.4) has L^∞ norm bounded by the L^∞ norm of the initial data. In particular it is bounded in absolute value by 1. Moreover, by the maximum principle, the same is true for ρ_t^N, the solution of the Navier–Stokes equation (1.5). We have to show that

$$\lim_{N\to\infty} \int du\, J(T,u)\, N\Big[\rho^N(T,u) - m(T,u) \;-\; N^{-1}R(T,u)\Big] \;=\; 0\,.$$

In order to prove this statement, we start by computing the time derivative of the integral at time t. A careful computation, since ρ^N is a weak solution of equation (1.6) and not a classical one, shows that after some integration by parts this time derivative is equal to

$$N\int du\,(\nabla_u J_t)\Big[F(\rho_t^N) - F(m_t) \;-\; F'(m_t)[\rho_t^N - m_t]\Big] \\ +\int du \sum_{i,j}\Big[A^{i,j}(\rho_t^N) - A^{i,j}(m_t)\Big]\partial_{u_i}\partial_{u_j} J_t\,.$$

where $A(m)$ is the matrix valued function such that $(A^{i,j})'(m) = a^{i,j}(m)$ for $1 \le i,j \le d$. Observe that $A(m)$ is *increasing* as a function of m in the sense of matrices since $a(\cdot)$ is bounded below by $(1/2)I$.

Since $F(b)-F(a)-F'(a)(b-a) = -(b-a)^2$ and since A is uniformly Lipschitz, to conclude the proof we have to show that

$$\lim_{N\to\infty} N \int_0^T dt \int du \left[\rho_t^N - m_t\right]^2 = 0 . \tag{2.1}$$

and

$$\lim_{N\to\infty} \int_0^T dt \int du \left|\rho_t^N - m_t\right| = 0 . \tag{2.2}$$

Let $\psi: \mathbf{R} \times T^d \to \mathbf{R}_+$ be a smooth approximation of the identity :

$$\psi(t,u) \geq 0 , \qquad \text{supp}\, \psi \subset [-1,1]^{d+1} \quad \text{and} \quad \int dt \int du\, \psi(t,u) = 1 .$$

For each positive δ, define ψ_δ as

$$\psi_\delta(t,u) = \delta^{-(d+1)} \psi(t\delta^{-1}, u\delta^{-1}) .$$

Fix $0 < t_0 \leq T$. Let $*$ denote the convolution. Observe that for each fixed N, ρ^N is bounded and $\nabla_u \rho^N$ belongs to $L^2([0,t_0]\times T^d)$. Define $w_\delta^N(t,u) = \psi_\delta * (\rho_t^N - m_t)$, then even though $\lim_{\delta\to 0} \partial_t w_\delta^N(t,u)$ may not exists, one can show that

$$\begin{aligned}
&\lim_{\delta\to 0} \frac{1}{2}\frac{d}{dt} \int du \left[w_\delta^N(t,u)\right]^2 \\
&\quad = \int du\, \mathbf{1}\cdot\nabla_u \left[\rho_t^N - m_t\right] \left[F(\rho_t^N) - F(m_t)\right] \\
&\qquad - \frac{1}{N} \int du\, \nabla_u(\rho_t^N - m_t)\cdot a(\rho_t^N)\nabla_u(\rho_t^N) \\
&\quad = \int du\, \mathbf{1}\cdot\nabla_u \left[\rho_t^N - m_t\right] \left[F(\rho_t^N) - F(m_t) - F'(m_t)(\rho_t^N - m_t)\right] \\
&\qquad + \int du\, \mathbf{1}\cdot\nabla_u \left[\rho_t^N - m_t\right] F'(m_t) \left[\rho_t^N - m_t\right] \\
&\qquad - \frac{1}{N} \int du\, \nabla_u(\rho_t^N - m_t)\cdot a(\rho_t^N)\nabla_u(\rho_t^N - m_t) \\
&\qquad - \frac{1}{N} \int du\, \nabla_u \left[\rho_t^N - m_t\right]\cdot a(\rho_t^N)\nabla_u m_t .
\end{aligned}$$

Since $F(b)-F(a)-F'(a)(b-a) = -(b-a)^2$ the first term of the right hand side vanishes because the integration is over the torus. Integrate the second term by parts,

$$-\frac{1}{2} \int du \left[\rho_t^N - m_t\right]^2 \mathbf{1}\cdot\nabla_u F'(m_t) .$$

After a simple computation we can rewrite the last term of the right hand side as

$$\begin{aligned} -\frac{1}{N}\int du\, \nabla_u \Big[\rho_t^N - m_t\Big] \cdot a(\rho_t^N)\nabla_u m_t \\ = \frac{1}{N}\int du \sum_{i,j} \Big[A^{i,j}(\rho_t^N) - A^{i,j}(m_t)\Big] \partial_{u_i}\partial_{u_j} m_t \\ + \frac{1}{N}\int du\, \nabla_u m_t \cdot \Big[a(\rho_t^N) - a(m_t)\Big] \nabla_u m_t\, . \end{aligned}$$

Since m is smooth with first and second space derivatives bounded, integrating in time and taking the limit $\delta \to 0$ one obtains

$$\begin{aligned} \frac{1}{2}\int du\, \Big[\rho_{t_0}^N - m_{t_0}\Big]^2 \; + \; \frac{1}{N}\int_0^{t_0} dt \int du\, \nabla_u(\rho_t^N - m_t)\cdot a(\rho_t^N)\nabla_u(\rho_t^N - m_t) \\ \le \; C_0 \int_0^{t_0} dt \int du\, \Big[\rho_t^N - m_t\Big]^2 \; + \; \frac{C_1}{N}\int_0^{t_0} dt \int du\, \Big\| A(\rho_t^N) - A(m_t)\Big\| \\ + \; \frac{C_1}{N}\int_0^{t_0} dt \int du\, \Big| a(\rho_t^N) - a(m_t)\Big|\, . \end{aligned}$$

In this formula C_0 and C_1 are constants depending uniquely on m and $\|A\|$ denotes the norm of the matrix $A:\ \|A\| = \sum_{i,j} |A^{i,j}|$.

By Gronwal's inequality we obtain that

$$\begin{aligned} N\int du\, \Big[\rho_{t_0}^N - m_{t_0}\Big]^2 \; + \; 2 \int_0^{t_0} dt \int du\, \nabla_u(\rho_t^N - m_t)\cdot a(\rho_t^N)\nabla_u(\rho_t^N - m_t) \\ \le \; C_2 e^{C_2 T} \int_0^T dt \int du\, \Big\{ \Big\| A(\rho_t^N) - A(m_t)\Big\| \; + \; \Big| a(\rho_t^N) - a(m_t)\Big| \Big\} \end{aligned}$$

for every $0 \le t_0 \le T$. In particular, since A and a are bounded, ρ_t^N converges to m_t in $L^2(T^d)$ as $N \uparrow \infty$. This proves (2.2). Moreover, since ρ_t^N converges to m_t in $L^2(T^d)$ and since $A(\cdot)$ and $a(\cdot)$ are continuous, the right hand side of the last expression converges to 0 as $N \uparrow \infty$. This proves (2.1). $\square$

3. Outline of the proof of Theorem 1.2.

In this section, following the strategy presented in the last section we outline the proof of Theorem 1.2.

Denote by $R\colon [0, T_0] \times T^d \to \mathbf{R}$ the weak solution of the linear equation (1.6). Fix a time $0 \le T < T_0$ and a smooth function $J_0\colon T^d \to \mathbf{R}$. Denote by $J\colon \mathbf{R}_+ \times T^d \to \mathbf{R}$ the solution of the linear equation (1.7). We have to prove that

$$\lim_{N\to\infty} E_N \left[N^{1-d} \sum_x J(T,x/N) \Big[\eta_T(x) - m(T,x/N) \Big] \right] \; = < R_T, J_T > \quad .$$

Following the steps of the last section, for $\sigma > 0$, let a_σ be a smooth function on $[0,1]$ converging to a uniformly on each compact subset of $(0,1)$. This is possible since, by [LOY1], a is continuous in $(0,1)$. Denote by $R_\sigma \colon \mathbf{R}_+ \times T^d \to \mathbf{R}$ the smooth solution of the linear equation (1.6) (with the corresponding smoothed diffusion a_σ). We start computing the time derivative of

$$E_N \left[N^{1-d} \sum_x J(T,x/N) \Big[\eta_t(x) - m(T,x/N) \; - \; N^{-1} R_\sigma(T,x/N) \Big] \right]$$

A simple computation shows that it is equal to

$$\begin{aligned}
& N^{-d} \sum_{x,i} J(t,x/N) \Big\{ \partial_{u_i} [F'(m) R_\sigma] - \partial_{u_i} \Big(\sum_j a_\sigma^{i,j}(m(t,x/N)) \partial_{u_j} m(t,x/N) \Big) \Big\} \\
& \quad + \; N^{1-d} \sum_{x,i} F'(m(t,x/N)) (\partial_{u_i} J)(t,x/N) \Big[m(t,x/N) \; + \; N^{-1} R_\sigma(t,x/N) \Big] \\
& \qquad + \; N^{1-d} \sum_{x,i} J(t,x/N) \, \partial_{u_i} F\big(m(t,x/N)\big) \\
& \qquad\quad + \; E_N \left[N^{1-d} \sum_x J(t,x/N) \, N L_N \eta_t(x) \right] \\
& \qquad\qquad - E_N \left[N^{1-d} \sum_{x,i} F'(m(t,x/N)) (\partial_{u_i} J)(t,x/N) \, \eta_t(x) \right] \quad .
\end{aligned}$$

At the end of this section, in Remark 3.3, we prove that the error we obtain when making a summation by parts in the above deterministic sums are of order $o_N(1)$. Therefore, summing by parts, computing the generator applied to the occupation variables and using Taylor's expansion, we obtain that this last expression is equal to

$$\begin{aligned}
& E_N \Bigg[N^{1-d} \sum_{x,i} (\partial_{u_i} J)(t,x/N) \\
& \qquad\qquad \Big\{ \tau_x W_i(\eta_t) - F\big(m(t,x/N)\big) - F'(m(t,x/N)) \big[\eta_t(x) - m(t,x/N) \big] \Big\} \Bigg] \\
& \quad + \; E_N \left[2^{-1} N^{-d} \sum_{x,i} (\partial_{u_i}^2 J)(t,x/N) \tau_x W_i(\eta_t) \right] \\
& \qquad + \; N^{-d} \sum_{x,i,j} (\partial_{u_i} J)(t,x/N) \, a_\sigma^{i,j}(m(t,x/N)) (\partial_{u_j} m)(t,x/N) \; + \; o_N(1) \, .
\end{aligned} \tag{3.1}$$

Here $W_i(\eta) = \eta(0)[1-\eta(e_i)]$. Notice that this formula corresponds to the one obtained in the last section since, by conservation of local equilibrium, the expected value of $\tau_x W_i(\eta_t)$ converges to $F(m(t,x/N))$. The second and third lines are the corrections needed for the first line to converge to 0. By the law of large numbers these lines converge, as N increases to ∞, to

$$\begin{aligned} -\frac{1}{2}\sum_i \int_0^T dt \int du\, (\partial_{u_i} J)(t,u)\big(\partial_{u_i} F(m)\big)(t,u) \\ + \sum_{i,j} \int_0^T dt \int du\, (\partial_{u_i} J)(t,u)\, a_\sigma^{i,j}(m(t,u))(\partial_{u_j} m)(t,u)\ . \end{aligned} \tag{3.2}$$

We now concentrate on the first line of (3.1). Recall that $F(m) = m(1-m)$. Since the linear terms of the first line in (3.1) cancel, we may rewrite it as

$$\begin{aligned} - E_N \Big[N^{1-d} \sum_{x,i} (\partial_{u_i} J)(t,x/N) \\ \Big\{ \eta_t(x)\eta_t(x+e_i) - m^2(t,x/N) - 2m(t,x/N)\big[\eta_t(x) - m(t,x/N)\big] \Big\} \Big]\ . \end{aligned} \tag{3.3}$$

At this point, following the approach of [GPV], we would like to replace the cylinder function $\eta(x)\eta(x+e_i)$ by a function of the empirical measure. In order to do it we need some notation.

For an integer ℓ, denote by $\eta^\ell(x)$ the empirical density on a hypercube of length $2\ell+1$ centered at x :

$$\eta^\ell(x) \;=\; \frac{1}{|\Lambda_\ell|} \sum_{y;y-x\in\Lambda_\ell} \eta(y)\ ,$$

$$\Lambda_\ell = \Big\{ y \in \mathbf{Z}^d;\ -\ell \le y_i \le \ell\ , \quad \text{for} \quad 1 \le i \le d \Big\}\ .$$

Furthermore, denote by $S_\ell(\eta)$ the conditional expectation of $\eta(0)\eta(e_i)$ given the density of particles on Λ_ℓ :

$$S_\ell(\eta) \;=\; E\Big[\eta(0)\eta(e_i) | \eta^\ell(0)\Big]\ . \tag{3.4}$$

Recall that the canonical measure on finite boxes is the uniform measure. In particular $S_\ell(\eta)$ can be computed explicitly. It is given by

$$S_\ell(\eta) \;=\; \Big(\eta^\ell(0)\Big)^2 \;-\; \frac{1}{|\Lambda_\ell|-1}\eta^\ell(0)\Big(1-\eta^\ell(0)\Big)\ . \tag{3.5}$$

Since $S_\ell(\cdot)$ is a function of $\eta^\ell(0)$ for convenience we denote it sometimes by $S_\ell(\eta^\ell(0))$. Define the sequence $K(N)$ by

$$K = BN^{1/d} ,$$

where B is a positive integer independent of N and that will increase to ∞ after N.

A summation by parts permits us to rewrite (3.3) as

$$\begin{aligned} - E_N \Big[N^{1-d} \sum_{x,i} (\partial_{u_i} J)(t, x/N) \Big\{ \Big[\eta_t(x)\eta_t(x+e_i) \ - \ S_K(\eta_t^K(x)) \Big] \\ + \ (\eta_t^K(x))^2 - m^2(t,x/N) - \ 2m(t,x/N) \Big[\eta_t^K(x) - m(t,x/N) \Big] \Big\} \Big] \ + \ o_{N,B}(1) \ . \end{aligned} \tag{3.6}$$

Here $o_{N,B}(1)$ represents a term vanishing as $N \uparrow \infty$ and then $B \uparrow \infty$. The error made in the summation by parts is small since $K \ll \sqrt{N}$ and m and J are smooth functions. Moreover, we could ignore the term $(|\Lambda_K| - 1)^{-1} \eta^K(0)[1 - \eta^K(0)]$ because $N^{1/d} \ll K$.

To keep notation simple, for each $1 \le i \le d$ and each positive integer ℓ, denote by $V_{i,\ell}(\eta)$ the cylinder function defined by

$$V_{i,\ell}(\eta) \ = \ \eta(0)\eta(e_i) \ - \ S_\ell(\eta) \ . \tag{3.7}$$

With these notation we may rewrite the last expectation as

$$-E_N \Big[N^{1-d} \sum_{x,i} (\partial_{u_i} J)(t, x/N) \Big\{ \tau_x V_{i,K}(\eta_t) \ + \ \Big[\eta_t^K(x) - m(t,x/N) \Big]^2 \Big\} \Big] \ + \ o_{N,B}(1) \ . \tag{3.8}$$

In view of (3.2), since m is bounded away from 0 and 1 and since $a_\sigma^{i,j}$ converges uniformly to $a^{i,j}$ on each compact subsets of $(0,1)$, to conclude the proof of the Theorem 1.2 it remains to prove the following two lemmas. The second one corresponds to the proof of (2.1) in the deterministic case considered in last section.

Lemma 3.1 . *Recall that $K = BN^{1/d}$. For each $1 \le i \le d$,*

$$\begin{aligned} &\lim_{N\to\infty} \int_0^T dt\, E_N \Big[N^{1-d} \sum_{x,i} (\partial_{u_i} J)(t, x/N) \tau_x V_{i,K}(\eta_t) \Big] \\ &\quad = \sum_{i,j} \int_0^T dt \int du\, (\partial_{u_i} J)(t,u) \Big[a^{i,j}(m(t,u)) - ((1/2) - m(t,u))\delta_{i,j} \Big] (\partial_{u_j} m)(t,u) \ . \end{aligned}$$

In this formula $\delta_{i,j}$ stands for the delta of Kroenecker and is equal to 1 if $i = j$ and 0 otherwise.

Lemma 3.2 . *Let K_N a sequence increasing to ∞ faster than $N^{1/d}$ and slower that N ($N^{1/d} \ll K_N \ll N$). Then,*

$$\lim_{N\to\infty} \int_0^T dt\, E_N \left[N^{1-d} \sum_x \Big(\eta_t^{K_N}(x) - m(t, x/N) \Big)^2 \right] \;=\; 0 \;.$$

It is this last lemma that forced us to choose the sequence $K(N)$ much larger than $N^{1/d}$. The proof of Theorem 1.2 follows then directly from Lemma 3.1 and Lemma 3.2.

We conclude this section by proving a statement that justifies the summation by parts used throughout this section.

Remark 3.3 (Summation by parts formula). *Let G, H: $T^d \to \mathbf{R}$ be two smooth functions. Then,*

$$\lim_{N\to\infty} N^{1-d} \sum_x \Big[(\partial_{u_i} G)(x/N) H(x/N) + (\partial_{u_i} H)(x/N) G(x/N) \Big] \;=\; 0 \;.$$

Proof. Denote by J the product of the functions GH. With this notation the sum in the statement can be rewritten as

$$\begin{aligned} & N^{1-d} \sum_x \Big[(\partial_{u_i} J)(x/N) \Big] \\ & \quad = \; N^{1-d} \sum_x \Big[(\partial_{u_i} J)(x/N) - N \Big\{ J\big(x/N + e_i/N\big) - J(x/N) \Big\} \Big] \;. \end{aligned}$$

Taylor's expansion at the second order gives that this expression is equal to

$$-2^{-1} N^{-d} \sum_x (\partial^2_{u_i} J)(x/N) \;+\; O(N^{-1}) \;.$$

As N increases to ∞ this expression converges to

$$-2^{-1} \int du\, (\partial^2_{u_i} J)(u) \;=\; 0 \;.$$

This concludes the proof of the Remark. $\square$

4. Bounds on Entropy and Dirichlet Form.

The strategy in proving the three lemmas of the previous section is based on the study of the time evolution of the relative entropy. Fix a reference product invariant measure ν_ρ on the configurations space $\{0,1\}^{T_N^d}$. For each time $t \ge 0$, let $\psi_t^N = \psi_t$ be the density of the slowly varying product measure $\nu^N_{m(t,\cdot)}$ with respect to ν_ρ :

$$\psi_t(\eta) \;=\; \frac{d\nu^N_{m(t,\cdot)}}{d\nu_\rho} \;=\; \frac{1}{Z_t^N}\,\exp\Big\{\sum_x \lambda(t,x/N)\eta_x\Big\}\;,$$

where

$$\lambda(t,x/N) \;=\; \log\left\{\frac{m(t,x/N)\,[1-\rho]}{\rho\,[1-m(t,x/N)]}\right\} \tag{4.1}$$

and Z_t^N is the renormalizing constant

$$Z_t^N \;=\; \exp\left\{-\sum_x \log\frac{1-m(t,x/N)}{1-\rho}\right\}\;.$$

Denote by f_t^N the density of the state of the process $\nu^N_{m(0,\cdot)}P_t^N$ with respect to ν_ρ. P_t^N denotes the semigroup of the Markov process with generator L_N accelerated by N. It is easy to see that the density f_t^N is the solution of

$$\begin{cases} \partial_t f_t^N \;=\; N L_N^* f_t^N \\[1ex] f_0^N \;=\; \psi_0\;. \end{cases} \tag{4.2}$$

Here L_N^* represents the adjoint of L_N in $L^2(\nu_\rho^N)$. From now on we will omit the superindex N of f_t^N.

Denote by $H_N(t)$ the entropy of $\nu^N_{m(0,\cdot)}P_t^N$ with respect to $\nu^N_{m(t,\cdot)}$:

$$H_N(t) \;=\; H\Big(\nu^N_{m(0,\cdot)}P_t^N \;\big|\; \nu^N_{m(t,\cdot)}\Big)\;.$$

It is well known that the entropy is given by

$$H_N(t) \;=\; \int f_t^N\,\log\frac{f_t^N}{\psi_t}\,d\nu_\rho\;.$$

Notice that $H_N(0)=0$.

For each density $f: S_N^d \to \mathbf{R}_+$, denote by $D_N(f)$ the convex lower semi continuous functional defined by

$$D_N(f) \;=\; -\int \sqrt{f(\eta)} L_N \sqrt{f(\eta)}\, \nu_\rho(d\eta) \;=\; \sum_{|x-y|=1} I_{x,y}(f)$$

where

$$I_{x,y}(f) \;=\; (1/4)\int \Big[\sqrt{f(\eta^{x,y})} - \sqrt{f(\eta)}\Big]^2\, \nu_\rho(d\eta)\;.$$

Recall that T_0 is defined as the largest time T such that equation (1.4) has a smooth classical solution in the interval $[0,T]$. In section 7 we prove the following result.

Proposition 4.1 (First entropy bound). *There exists a constant C_0 such that for every $t < T_0$,*

$$\limsup_{N\to\infty} \Big\{ N^{1-d} H_N(t) \;+\; \int_0^t N^{2-d} D_N(f_s)\, ds \Big\} \;\le\; C_0\;.$$

The proof of Lemma 3.1 is based on this result. However, to prove Lemma 3.2 we need more: a bound on the entropy of order $o(N^{d-1})$ instead of $O(N^{d-1})$. To obtain such a bound we need to consider corrections of order $1/N$ of the density ψ_t.

In order to state the result we need to introduce some notation. Denote by $\mathfrak{F}$ the space of functions $F: [0,1]\times\{0,1\}^{\mathbf{Z}^d} \to \mathbf{R}$ such that

(i) For each $\rho\in[0,1]$, $F(\rho,\cdot)$ is a cylinder function with uniform support. That is there exists a finite set Λ such that for each ρ in $[0,1]$ the support of $F(\rho,\cdot)$ is contained in Λ.

(ii) For each configuration η, $F(\cdot,\eta)$ is a smooth function.

(iii) For each density ρ the cylinder function $F(\rho,\cdot)$ has mean zero with respect to ν_ρ and the derivative with respect to the parameter θ of the expected value $\nu_\theta[F(\rho,\eta)]$ vanishes at $\theta=\rho$:

$$\nu_\rho\Big[F(\rho,\eta)\Big] \;=\; 0 \qquad \text{and} \qquad \frac{d}{d\theta}\,\nu_\theta\Big[F(\rho,\eta)\Big]\Big|_{\theta=\rho} \;=\; 0\;.$$

A simple computation shows that the second condition of (iii) is equivalent to imposing that the covariance under ν_ρ between $F(\rho,\cdot)$ and the formal sum $\sum\eta(x)$ is equal to 0 :

$$\sum_{x\in Z^d} \nu_\rho\Big[F(\rho,\eta)\eta(x)\Big] \;=\; 0\;.$$

There are only a finite number of non zero terms in the last sum since $F(\rho,\eta)$ is a mean zero cylinder function.

Denote by $(\partial_1 F)(\rho,\cdot)$ the derivative with respect to the first variable of $F(\rho,\eta)$ which exists in virtue of (ii). It follows from conditions (iii) that the cylinder function $(\partial_1 F)(\rho,\cdot)$ has mean zero with respect to the invariant measure ν_ρ.

For functions F_i in $\mathfrak{F}$, $1\le i\le d$, a time $0\le t<T_0$ and an integer N define the density $\psi_t^{N,F}(\eta)=\psi_t^F(\eta)$ with respect to the reference measure ν_ρ by

$$\psi_t^F(\eta) \;=\; \frac{1}{Z_t^F}\exp\left\{\sum_x \lambda(t,x/N)\eta_x \;-\; N^{-1}\sum_{x,i}\big(\partial_{u_i}\lambda(t,x/N)\big)F_i\big(m(t,x/N),\tau_x\eta\big)\right\}$$

where Z_t^F is a renormalizing constant.

Denote by $H_N(f_t|\psi_t^F)$ the entropy of the probability measure $\nu_{m(0,\cdot)}P_t^N$ with respect to the measure $(Z_t^F)^{-1}\psi_t^F(\eta)\nu_\rho(d\eta)$. Furthermore, for each density $f\colon S_N^d\to\mathbf{R}_+$ and each $t\ge 0$, denote by $D_N(f\,|\,\psi_t^F)$ the positive convex lower semicontinuous functional defined by

$$D_N\Big(f\,\big|\psi_t^F\Big) \;=\; \int \psi_t^F\left\{L_N\frac{f}{\psi_t^F} \;-\; \frac{f}{\psi_t^F}L_N\log\frac{f}{\psi_t^F}\right\}\,\nu_\rho(d\eta)\,. \tag{4.3}$$

With these notation we may state the main theorem on the time behavior of the entropy

Proposition 4.2 (Second entropy bound). *For every $t<T_0$,*

$$\inf_{F\in\mathfrak{F}}\lim_{N\to\infty}\left\{N^{1-d}H_N\Big(f_t\big|\psi_t^F\Big) \;+\; \int_0^t N^{2-d}D_N\Big(f_s\,\big|\psi_s^F\Big)\,ds\right\} \;=\; 0\,.$$

The correction $N^{-1}\sum_{x,i}\big(\partial_{u_i}\lambda(t,x/N)\big)F_i\big(m(t,x/N),\tau_x\eta\big)$ is introduced to control the time derivative of the entropy but not the entropy itself. In fact, from Propositions 4.1 and 4.2 it follows that the entropy of f_t with respect to ψ_t is of order $o(N^{d-1})$:

$$\lim_{N\to\infty} N^{1-d}H_N(t) \;=\; 0\,. \tag{4.4}$$

Indeed, it follows from Proposition 4.1 that for every function F in $\mathfrak{F}$, for every smooth function $J\colon T^d\to\mathbf{R}$ and for every function $\rho\colon T^d\to[0,1]$, there exists a constant $C_1 =$

$C_1(C_0, F, J, \rho)$ such that

$$\sqrt{N} \left| E_{f_t} \left[N^{-d} \sum_x J(x/N) F\big(\rho(x/N), \tau_x \eta\big) \right] \right.$$
$$\left. - \; E_{\psi_t} \left[N^{-d} \sum_x J(x/N) F\big(\rho(x/N), \tau_x \eta\big) \right] \right| \;\le\; C_1 \; .$$

In the above definition of C_1, C_0 is the constant appearing in the statement of Proposition 4.1. The proof of this claim is omitted since it is similar to the one of Lemma 2.2 in [EMY].

Furthermore, from this last assertion and the entropy inequality for each family of functions $\{F_i \in \mathfrak{F};\ 1 \le i \le d\}$, there exists a constant $C_2 = C_2(C_0, m, F)$ such that

$$N^{\frac{3}{2}-d} \Big| H_N(t) \;-\; H_N\Big(f_t \big| \psi_t^F\Big) \Big| \;\le\; C_2 \; . \tag{4.5}$$

The proof of this claim is also omitted since it is similar to the one of Lemma 3.3 in [EMY]

From these two statements it is easy to see that we may replace $H_N\Big(f_t \,\big|\, \psi_t^N\Big)$ by $H_N(t)$ in Proposition 4.2, proving (4.4).

5. A Bound from the Entropy..

In this section we will prove a bound that is controlled by the relative entropy. Lemma 3.2 is simple corollary of this estimate and Proposition 4.2.

The result presented below is a consequence of the entropy inequality and of the central limit theorem.

Lemma 5.1 . *Let K_N a sequence that increases to ∞ as $N \uparrow \infty$ and small if compared to N : $K_N \ll N$. There exist a $\gamma > 0$ such that*

$$E_N \left[N^{1-d} \sum_x \Big(\eta_t^{K_N}(x) - m(t, x/N) \Big)^2 \right] \;\le\; \frac{H_N(t)}{\gamma N^{d-1}} \;+\; C(\gamma)\, O\left(\frac{N}{K_N^d}\right) .$$

Proof. By the entropy inequality for any positive constant γ

$$E_N \left[N^{1-d} \sum_x \Big(\eta_t^{K_N}(x) - m(t, x/N) \Big)^2 \right] \;=\; E_{f_t} \left[N^{1-d} \sum_x \Big(\eta^{K_N}(x) - m(t, x/N) \Big)^2 \right]$$

is bounded above by

$$\frac{H_N(t)}{\gamma N^{d-1}} \;+\; \frac{1}{\gamma N^{d-1}} \log E_{\psi_t} \left[\exp \left\{ \gamma \sum_x \left(\eta^{K_N}(x) - m(t, x/N) \right)^2 \right\} \right] .$$

By Hölder inequality, since ψ_t is a product measure, the limit as N increases to ∞ of the second term of right hand side is bounded above by

$$\limsup_{N\to\infty} \frac{1}{\gamma N^{d-1} \bar{K}_N^d} \sum_x \log E_{\psi_t} \left[\exp \left\{ \gamma \bar{K}_N^d \left(\eta^{K_N}(x) - m(t, x/N) \right)^2 \right\} \right] .$$

Here $\bar{K}_N$ stands for $2K_N + 1$. For γ sufficiently small and since $K_N \ll N$, by the central limit theorem,

$$\frac{1}{\gamma N^d} \sum_x \log E_{\psi_t} \left[\exp \left\{ \gamma \bar{K}_N^d \left(\eta^{K_N}(x) - m(t, x/N) \right)^2 \right\} \right] .$$

converge, as N increases to ∞, to

$$\frac{1}{\gamma} \int du \, \log E_{m(t,u)} \left[\exp \left\{ \gamma X^2 \right\} \right] .$$

In this formula for m in $[0,1]$, E_m represents the expectation with respect to a gaussian law with mean 0 and variance $m(1-m)$. □

6. Multiscale Estimates.

In this section we recall some multiscale estimates from [EMY] and extend them to our setting. These estimates replace the usual one block–two blocks estimates and will be used later on to prove Lemma 3.1 and Propositions 4.1 and 4.2. Our estimates are similar to [EMY] except some extra work is needed because our system is not near equilibrium. In particular, we need estimates uniform with respect to the density ρ. We start with some notation.

Denote by $\mathcal{C}$ the space of cylinder functions. Recall from the previous sections that for each positive integer K and m in $\{0, 1/(\bar{K}^d), \dots, 1\}$ we denote by $\nu_{K,m}$ the canonical measure on $\{0,1\}^{\Lambda_K}$ with density m. Let $\mathcal{G}$ be the linear space of cylinder functions that have mean zero with respect to all canonical measures on a sufficiently large box Λ_K :

$$\mathcal{G} \;=\; \left\{ g \in \mathcal{C};\ \nu_{K,m}[g] = 0 \text{ for some } K > 0 \text{ and all } m \right\} . \tag{6.1}$$

Moreover, for a density $0 \leq m \leq 1$, let $\mathcal{G}_m$ the space of cylinder functions such that :

$$\tilde{g}(m) \;=\; \nu_m[g] \;=\; 0 \qquad \text{and} \qquad \partial_\rho \nu_\rho[g]\big|_{\rho=m} \;=\; \tilde{g}'(m) \;=\; 0\;.$$

Note that the second condition is equivalent to impose that the covariance, with respect to the measure ν_m, of g and the formal sum $\sum_x \eta(x)$ vanishes :

$$\sum_z \nu_m\big[g(\eta)\,;\,\eta(z)\big] \;=\; 0\;.$$

In this formula and below, for two cylinder functions f and g, $\nu_m[f;g]$ denotes the covariance of f and g : $\nu_m[f;g] = \nu_m[fg] - \nu_m[f]\nu_m[g]$. Notice that $\mathcal{G} \subset \mathcal{G}_m$ for all m in $[0,1]$. The following definition is taken from [EMY]

Definition 6.1 . *Let g be a cylinder function and denote by $s(g)$ its support :*

$$s(g) \;=\; \min\Big\{\ell \in \mathbf{N};\; supp\; g \subset \{-\ell,\dots,\ell\}^d\Big\}\;.$$

For each and $\ell \geq s(g)$ and m in $\{0, 1/\bar{\ell}^d, \dots, 1\}$, define the "variance" $V_\ell(g,m)$ of g with respect to $\nu_{\ell,m}$ by

$$V_\ell(g,m) \;=\; (2\ell(g)+1)^{-d}\left\langle \left[\sum_{|x|\leq \ell(g)} (\tau_x g - \tilde{g}_\ell(m))\right] (-L_\ell^s)^{-1} \left[\sum_{|x|\leq \ell(g)} (\tau_x g - \tilde{g}_\ell(m))\right]\right\rangle_{\nu_{\ell,m}} . \tag{6.2}$$

In this formula $\ell(g)$ denotes the integer $\ell - s(g)$ so that $\sum_{|x|\leq \ell(g)} \tau_x g$ is measurable with respect to $\{\eta(x);\; x \in \Lambda_\ell\}$. Moreover L_ℓ^s is the restriction to Λ_ℓ of the symmetric part of the generator L and $\tilde{g}_\ell(m)$ is the expected value of g with respect to the canonical measure $\nu_{\ell,m}$.

If $g \in \mathcal{G}_m$ we define also the "variance" of g by

$$\mathbf{V}_m(g) \;=\; \limsup_{\ell\to\infty} \nu_m\Big[V_\ell(g, \eta^\ell(0))\Big]\;.$$

Notice that for $g \in \mathcal{G}$ the subtraction in (6.2) is unnecessary for ℓ sufficiently large.

We need the following two results. The first one is proved in [LOY1] and the second one can be found in [EMY].

Lemma 6.2 . *For each cylinder function h in $\mathcal{G}$, the finite volume variance of h converges uniformly in $[0,1]$ to the infinite volume variance :*

$$\lim_{\ell\to\infty} V_\ell(h,m) \;=\; \mathbf{V}_m(g)$$

uniformly in $[0,1]$.

Lemma 6.3 (Integration by parts formula). *Let $g \in \mathcal{G}_m$ be a cylinder function. Denote by ℓ the smallest integer such that Λ_ℓ contains the support of g.*

Then there exists a family of functions $\{\Phi_b(x,g);\; x\in \mathbf{Z}^d, b\in\mathbf{Z}^d\}$ such that

$$\begin{aligned}\Big\langle \tau_x g, u\Big\rangle_m \;&=\; \sum_b \Big\langle \Phi_b(x,g), \nabla_b u\Big\rangle_m \\ \textit{and} \quad \sum_b |b-x|^{d+(1/2)}&\Big\langle \big(\Phi_b(x,g)\big)^2\Big\rangle_m \;\le\; C(g)\end{aligned}$$

for some constant $C(g)$ depending only on g.

The following result is a one-block estimate whose proof, up to very minor change, is taken from [V]. For this reason we shall only sketch it.

Lemma 6.4 (One Block Estimate). *Let f_t be the solution of the forward equation (4.2). There exists an universal constant C_1 such that for any smooth function J, positive γ and $h\in\mathcal{G}$,*

$$\begin{aligned}&\lim_{N\to\infty}\Big\{N^{1-d}\sum_x \int J(x/N)\,\tau_x h\, f_t\, d\nu_\rho \;-\; \gamma N^{2-d} D_N(f_t)\Big\} \\ &\quad\le\; C_1\gamma^{-1}\int J(u)^2 \mathbf{V}_{m(t,u)}(h)\,du\;.\end{aligned} \tag{6.3}$$

In this formula $m(t,u)$ is the solution of the Burgers equation (1.4).

Proof. Fix a positive integer ℓ independent of N and that will increase to ∞ after N. Since J is smooth, the summation on the left hand side of the statement of the lemma can be rewritten as

$$N^{1-d}\sum_x \int J(x/N)\,(2\ell+1)^{-d}\sum_{|y-x|\le\ell}\tau_y h \;+\; O(\ell^2/N)\;.$$

From the standard perturbation theorem on the largest eigenvalue of a symmetric operator, (cf. [RS] or [V]) one has

$$\lim_{N\to\infty}\left\{N^{1-d}\sum_x\int J(x/N)(2\ell+1)^{-d}\sum_{|y-x|\le\ell}\tau_y h f_t\,d\nu_\rho\;-\;\gamma N^{2-d}D_N(f_t)\right\}$$
$$\le\;C_1\gamma^{-1}\lim_{N\to\infty}N^{-d}\sum_x J(x/N)^2\int V_\ell(h,\eta^\ell(x))f_t d\nu_\rho\,.$$

In this formula V_ℓ stands for the finite volume variance defined in (6.2). Note that from the first to the second line we got an extra factor N^{-1} so that the expression on the right hand side is of order 1. By the law of large numbers, the right hand side of the last expression is equal to

$$C\gamma^{-1}\int du\,J^2(u)\nu_{m(t,u)}\Big[V_\ell(h,\eta^\ell(0))\Big]\,.$$

To conclude the proof it remains to invoke Lemma 6.2 which states that the finite volume variance converges uniformly to the infinite volume variance. $\square$

A slight modification of this proof gives that for every density f the left hand side of (6.3) is bounded above by

$$C_1\gamma^{-1}\int J(u)^2\,du\sup_{0\le m\le 1}\mathbf{V}_m(h)\,.$$

For any local function h and any integer $K\ge s(h)$ fixed, consider the decomposition

$$h\;=\;\Big\{h-\nu_\rho\big[h\mid\eta^L(0)\big]\Big\}\;+\;\nu_\rho\big[h\mid\eta^L(0)\big]\;:=\;h_{(L)}\;+\;\tilde h_L(\eta^L(0))\,. \tag{6.4}$$

Notice that for each L, $h_{(L)}$ belongs to $\mathcal{G}$ since it has mean zero with respect to all canonical measures on boxes of length larger than $\bar L$.

The following theorem, up to changes due to the nonequilibrium nature of our problem, is taken form [EMY]. Because more careful analysis is needed in our setting and some details were skipped in [EMY] we shall give a detailed proof.

Theorem 6.5 (Multiscale Estimates). *Recall the decomposition introduced in (6.4). Fix a cylinder function h and a sequence $K=K(N)$ such that*

$$\lim_{N\to\infty}\frac{K^{2+(1/2)+a}}{N}\;=\;0(6.5)$$

for some constant $a > 0$. Let f_t be the solution of the forward equation (4.2). There exists an universal constant C_1 and a function $C(h, L)$ vanishing as $L \uparrow \infty$ ($\lim_{L\to\infty} C(h, L) = 0$) such that for any smooth function J and any positive γ,

$$\lim_{N\to\infty} \Big\{ N^{1-d} \int \sum_x J(x/N) \Big[\tau_x h - \tilde{h}_K(\eta^K(x)) \Big] f_t \, d\nu_\rho - \gamma N^{2-d} D_N(f_t) \Big\} \\ \le \; C_1 \gamma^{-1} \int J(u)^2 \mathbf{V}_{m(t,u)}(h_{(L)}) \, du \; + \; C(h, L) \; .$$

Proof. Fix a positive integer L independent of N. We first divide the summation over x appearing in the left hand side of the statement of the theorem in two pieces :

$$N^{1-d} \sum_x J(x/N) \Big[\tau_x h - \tilde{h}_L(\eta^L(x)) \Big] \\ + \; N^{1-d} \sum_x J(x/N) \Big[\tilde{h}_L(\eta^L(x)) - \tilde{h}_K(\eta^K(x)) \Big] \; .$$

Since $h_{(L)} = h - \tilde{h}_L(\eta^L(0))$ belongs to $\mathcal{G}$, we may apply Lemma 6.4 to obtain that

$$\lim_{N\to\infty} \Big\{ N^{1-d} \sum_x \int J(x/N) \tau_x h_{(L)}(\eta) f_t \, d\nu_\rho \; - \; (1/2)\gamma N^{2-d} D_N(f_t) \Big\} \\ \le \; C_1 \gamma^{-1} \int J(u)^2 \mathbf{V}_{m(t,u)}(h_{(L)}) \, du \; .$$

To conclude the proof of the theorem it remains to show that there exists a function $C(h, L)$ vanishing as $L \uparrow \infty$ such that

$$\lim_{N\to\infty} \Big\{ N^{1-d} \int \sum_x J(x/N) \Big[\tilde{h}_L(\eta^L(x)) - \tilde{h}_K(\eta^K(x)) \Big] f_t d\nu_\rho \; - \; (1/2)\gamma N^{2-d} D_N(f_t) \Big\} \\ \le \; C(h, L) \; . \tag{6.6}$$

For the rest of the proof, we shall assume that $h = \tilde{h}_L$. Fix an integer $R = L^{1/(8d)}$. The exponent $1/8d$ is chosen for convenience. Certainly the choice is not unique and the number $1/8d$ does not have special significance. Divide the cube $\tau_x \Lambda_K$ into disjoint subcubes of size L. For the rest of the argument, x will be fixed and omitted and all results will be uniform with respect to x. Define a sequence of grids on Λ_K as follows. For each n fixed, divide the cube Λ_K into disjoint subcubes of size $L_n = LR^n$. The grids are compatible in the sense that each cube in a level is a proper subset of some cube in the next level. More precisely, denote by $A(n)$ the number of cubes at level n and, for $1 \le \alpha \le A(n)$, by Λ^n_α the α-th cube at level n. We require that

(i) For each $1 \le \alpha \le A(n)$ there exist x^n_α in $\mathbf{Z}^d$ such that $\Lambda^n_\alpha = \tau_{x^n_\alpha} \Lambda_{L_n}$. We shall call x^n_α the center of the cube Λ^N_α;

(ii) $\Lambda^N_\alpha \cap \Lambda^N_\beta = \emptyset$ for $\alpha \neq \beta$;

(iii) $\cup_{\alpha=1}^{A(n)} \Lambda^N_\alpha = \Lambda_K$;

(iv) if $n \ge 1$, for each $1 \le \alpha \le A(n)$, there exists $\alpha_1, \dots, \alpha_{A(n-1)/A(n)}$ such that $\Lambda^n_\alpha = \cup_{1 \le i \le A(n-1)/A(n)} \Lambda^{n-1}_{\alpha_i}$. Notice that $A(n-1)/A(n) = R^d$.

To keep notation simple, for $n \ge 0$ and $1 \le \alpha \le A(n)$, denote by $\eta^{n,\alpha}$ the average number of particles on box Λ^n_α :

$$\eta^{n,\alpha} \;=\; \frac{1}{(2L_n + 1)^d} \sum_{x \in \Lambda^n_\alpha} \eta(x) \; .$$

Moreover, denote by $\mathcal{F}_n$ the σ-algebra generated by $\{\eta^{n,\alpha};\ 1 \le \alpha \le A(n)\}$.

The first step in proving (6.6) consists in rewriting the sum as

$$N^{1-d} \sum_x J(x/N) \tau_x \Big\{ \frac{1}{A(0)} \sum_{\alpha=1}^{A(0)} h(\eta^{0,\alpha}) - \tilde{h}_K(\eta^K(x)) \Big\} \;+\; O(K^2/N) \; .$$

Here we used the convention that $h = \tilde{h}_L$ and the factor $O(K^2/N)$ come from the summation by parts.

Recall the definition of $\tilde{h}_{L_n}$ given in (6.4). To keep notation simple we shall denote $\tilde{h}_{L_n}$ simply by $\tilde{h}_n$. For $1 \le \alpha \le A(n)$, define the cylinder function $h_{n,\alpha}$ as the conditional expectation of $\tau_{x^n_\alpha} h$ given the density on Λ^n_α :

$$h_{n,\alpha}(\eta) \;=\; \nu_\rho \Big[\tau_{x^n_\alpha} h \mid \eta^{n,\alpha} \Big] \;=\; \tilde{h}_n(\eta^{n,\alpha}) \; .$$

In particular $h_{0,\alpha} = \tau_{x^0_\alpha} h$. Denote by $h^{(n)}$ the average of $h_{n,\alpha}$:

$$h^{(n)} \;=\; \frac{1}{A(n)} \sum_{1 \le \alpha \le A(n)} h_{n,\alpha}$$

so that $h^{(0)} = A(0)^{-1} \sum_{1 \le \alpha \le A(0)} h(\eta^{0,\alpha})$. Notice that $h^{(n)}$ is the conditional expectation of $h^{(0)}$ given $\mathcal{F}_n$: $h^{(n)} = \nu_\rho \Big[h^{(0)} \mid \mathcal{F}_n \Big]$. In particular $h^{(n)}$ is discrete time reversed martingale.

We may rewrite the covariance

$$\nu_\rho\Big[\big\{h^{(0)}(\eta) - \tilde h_K(\eta^K(0))\big\} f_t\Big] \quad \text{as} \quad \sum_{n=0}^{\infty} \nu_\rho\Big[\big\{h^{(n)} - h^{(n+1)}\big\} f_t\Big] \ .$$

For each n fixed, from the definition of conditional expectation,

$$\nu_\rho\Big[f_t\big\{h^{(n)} - h^{(n+1)}\big\}\Big] \ = \ \nu_\rho\Big[\nu_\rho\Big[f_t\big\{h^{(n)} - h^{(n+1)}\big\}|\mathcal{F}_{n+1}\Big]\Big] \ .$$

For each cube $\Lambda^{n+1}_{\alpha_0}$, consider the eigenvalue problem

$$\mathcal{E}_N(f_t) \ = \ N \int R^{-d} \sum_{\beta\subset\alpha_0} \Big[h_{n,\beta} - h_{n+1,\alpha_0}\Big] f_t d\nu_{n+1,\alpha_0} \ - \ N^2 L_{n+1}^{-2-d-a} L_{n+1}^2 D_{n+1,\alpha_0}(f_t) \ .$$

Here ν_{n+1,α_0} represents the canonical measure on $\Lambda^{n+1}_{\alpha_0}$ with density η^{n+1,α_0}, D_{n+1,α_0} is the Dirichlet form with respect to ν_{n+1,α_0} and the sum is carried over all indices β such that $\Lambda^n_\beta \subset \Lambda^{n+1}_{\alpha_0}$. By standard arguments and the convexity of the Dirichlet form, in order to prove (6.6) it is enough to show that for each fixed n and α_0

$$\limsup_{N\to\infty} \mathcal{E}_N(f_t) \ \le \ C(h,L)\, b_n \tag{6.7}$$

for some summable positive sequence b_n.

From the logarithmic Sobolev inequality (cf. [Y2]), the Dirichlet form $L^2_{n+1} D_{n+1,\alpha_0}(f_t)$ is bounded below by the entropy $H(f_t\nu_{n+1,\alpha_0}|\nu_{n+1,\alpha_0})$. The last expression is therefore bounded above by

$$N \int R^{-d} \sum_{\beta\subset\alpha_0} \Big[h_{n,\beta} - h_{n+1,\alpha_0}\Big] f_t d\nu_{n+1,\alpha_0} \ - \ N^2 L_{n+1}^{-2-d-a} H(f_t\nu_{n+1,\alpha_0}|\nu_{n+1,\alpha_0}) \ .$$

By the entropy inequality this difference is bounded above by

$$N^2 L_{n+1}^{-2-d-a} \log \nu_{n+1,\alpha_0}\Big[\exp\Big\{N^{-1} L_{n+1}^{2+d+a} R^{-d} \sum_{\beta\subset\alpha_0} \Big[h_{n,\beta} - h_{n+1,\alpha_0}\Big]\Big\}\Big] \ . \tag{6.8}$$

We claim that this expression is bounded above by $C(h,L)b_n$ for some summable positive sequence b_n and some constant $C(h,L)$ vanishing in the limit as $L\uparrow\infty$.

The exponent can be rewritten as

$$N^{-1} \frac{L_{n+1}^{2+d+a}}{L_n^d R^d} L_n^d \sum_{\beta\subset\alpha_0} \Big[h_{n,\beta} - h_{n+1,\alpha_0}\Big] \ .$$

By (6.5) we have

$$N^{-1}\frac{L_{n+1}^{2+d+a}}{L_n^d R^d} = N^{-1}L_{n+1}^{2+a} \le L_n^{-1/4-8\epsilon}$$

because $R^d L_n^d = L_{n+1}^d$ and $L_n \le K$. Therefore, using corollary ♣rn with $\beta = N^{-1}L_{n+1}^{2+a}$ we obtain that (6.7) is bounded by

$$C_1\left(\frac{L_{n+1}^{2+a+d}}{NL_n^d}\right)^2 N^2 L_{n+1}^{-2-d-a} \;=\; C_1\frac{L_{n+1}^{2+a+d}}{L_n^{2d}} = C_1\frac{L^{2+d+a}}{L^{2d}}\frac{R^{(n+1)(2+d+a)}}{R^{2dn}}\;.$$

Since $2+a<d$, the first term vanishes as $L\uparrow\infty$ and the last term is summable. This concludes the proof of the theorem. □

Since Theorem 6.5 holds for all L, we can take the limit $L\uparrow\infty$. We would like to have that

$$\lim_{L\to\infty}\mathbf{V}_m(h_{(L)}) \;=\; \mathbf{V}_m(h)$$

uniformly on compact subsets of $(0,1)$. Unfortunately, since h may not be in the space $\mathcal{G}_m$, $\mathbf{V}_m(h)$ may not even be defined. For this reason we introduce the function h^m given by

$$h^m = h - \langle h\rangle_m - \tilde h'(m)\big[\eta(0)-m\big]$$

where $\tilde h(\rho) =< h >_\rho$ and $\tilde h'(\cdot)$ is its derivative. Hence h^m belongs to $\mathcal{G}_m$. From Definition 6.1

$$\lim_{\ell\to\infty}V_\ell(h_{(L)}-(h^m)_{(L)},m) \;\le\; \ell^d\Big\{\Big\langle Av_{|x|\le\ell}(\eta_x-\tau_x\eta^L)\,,\; Av_{|x|\le\ell}((\eta_x-\tau_x\eta^L))\Big\rangle.$$

Hence

$$\lim_{\ell\to\infty}V_\ell(h_{(L)}-(h^m)_{(L)},m) \;=\; 0$$

uniformly in $[0,1]$. Hence we can assume, without loss of generality, that h is in $\mathcal{G}_m$. We claim that under this assumption

$$\lim_{L\to\infty}\mathbf{V}_m(h_{(L)}) \;=\; \mathbf{V}_m(h)\;.$$

To prove this it suffice to check that $\lim_{L\to\infty}\lim_{\ell\to\infty}V_\ell(\tilde h_L,m) = 0$. From the variational formula for the finite volume variance,

$$V_\ell(\tilde h_L,m) \;=\; \sup_u\,(2\ell(\tilde h_L)+1)^{-d}\Big\{2\Big\langle\sum_{|x|\le\ell(\tilde h_L)}(\tau_x\tilde h_L-\tilde h_\ell)\,,\,u\Big\rangle \;-\; D_{\ell,m}(u)\Big\}\;.$$

In this formula the supremum is carried over all functions u in $L^2(\nu_{\ell,m})$, $< \cdot, \cdot >$ represents the inner product of $L^2(\nu_{\ell,m})$ and $D_{\ell,m}$ is the Dirichlet form with respect to the measure $\nu_{\ell,m}$.

We now apply the integration by parts formula stated at Lemma 6.3, to the function $\tilde{h}_L$ to obtain that the expression inside the supremum is bounded above by

$$(2\ell' + 1)^{-d} \Big\{ \sum_{|x| \le \ell'} \sum_{b \in \Lambda_\ell} 2 \Big\langle \Phi_b(x, \tilde{h}_L) \, , \, \nabla_b u \Big\rangle \; - \; D_{\ell,m}(u) \Big\} \; .$$

In this formula, to keep notation simple we denoted $\ell(\tilde{h}_L)$ simply by ℓ'.

From the integration by parts lemma, we can derive a bound on $\Phi_b(x, \tilde{h}_L)$. This bound turns out to be not sufficient for our purpose. Our function $\tilde{h}_L$ is very special and it depends only on $\eta^L(0)$. using this special feature, one can check that, from the proof in [EMY], $\Phi_b(x, \tilde{h}_L) \; = \; \Phi_b(x, h)$ for all bonds b not in Λ_L and $\Phi_b(x, \tilde{h}_L) \le L^{-d-1/2}$ for bonds b in Λ_L. From this, Schwarz inequality and the bound on the L^2 norm of $\Phi_b(x, h)$ of the above integration by parts lemma, it is not difficult to prove that

$$\lim_{L \to \infty} \lim_{\ell \to \infty} V_\ell(\tilde{h}_L, m) \; = \; 0$$

uniformly on $[0, 1]$. We have thus showed that for every cylinder function h,

$$\lim_{L \to \infty} \mathbf{V}_m(h_{(L)}) \; = \; \mathbf{V}_m(h^m) \; .$$

We summarize these last conclusions in the following corollary.

Corollary 6.6 . *Fix a cylinder function h and a sequence $K = K(N)$ such that*

$$\lim_{N \to \infty} \frac{K^{2+a}}{N} \; = \; 0$$

for some constant $a > 0$. Let f_t be the solution of the forward equation (4.2). There exists an universal constant C_1 such that for any positive constant γ,

$$\lim_{N \to \infty} \Big\{ N \int \sum_x J(x/N) \Big[\tau_x h - \tilde{h}_K(\eta^K(x)) \Big] \, f_t d\nu_\rho - \gamma N^{2-d} D_N(f_t) \Big\}$$
$$\le \; C_1 \gamma^{-1} \int J(u)^2 \mathbf{V}_{m(t,u)}(h^m) \, du \; .$$

Furthermore,

$$\lim_{L \to \infty} \mathbf{V}_m(h_{(L)}) \; = \; \mathbf{V}_m(h^m) \; .$$

7. Proof of the first entropy bound.

The following lemma will be useful in the proof of Proposition 4.1. It shows that $D_N(f\,|\psi)$ and $D_N(f)$ are not too far apart.

Lemma 7.1 . *Let $\beta\colon T^d \to \mathbf{R}$ be a smooth function. Define the density χ with respect to the product measure ν_ρ by*

$$\chi(\eta) \;=\; \frac{1}{Z}\exp\Big\{\sum_x \beta(x/N)\eta(x)\Big\}\;,$$

where Z is a normalizing constant. Let $\tilde D_N(\cdot)$ be the functional defined on the positive densities by

$$\tilde D_N(f) \;=\; -\int f L_N \log f\, d\nu_\rho\;.$$

Recall the definition of the Dirichlet form $D_N(f|\chi)$ given in (4.3). Then

$$D_N(f|\chi) - \tilde D_N(f) \;=\; O(N^{d-2})\;.$$

Proof. The proof relies on the explicit formula of the Dirichlet forms. Indeed, by definition we have that

$$\begin{aligned} D_N(f|\chi) - \tilde D_N(f) \;&=\; \int f\Big\{\frac{L_N^*\chi}{\chi} \;+\; L_N\log\chi\Big\}d\nu_\rho \\ &=\; \int f\Big\{\frac{L_N^*\chi}{\chi} \;-\; L_N^*\log\chi \;+\; 2L_N^s\log\chi\Big\}d\nu_\rho\;. \end{aligned}$$

In this formula L_N^* represents the adjoint of L_N in $L^2(\nu_\rho)$ and L_N^s the symmetric part. Computing the expression inside braces we obtain that $L_N^s\log\chi$ is of order N^{d-2} and that the terms of order N^{d-1} in $\chi^{-1}L_N^*\chi - L_N^*\log\chi$ cancel. $\square$

We now turn to the proof of Proposition 4.1. Recall the definition of the entropy $H_N(t)$ and the one of the Dirichlet form introduced in (4.3). A simple and by now classical computation (cf. [Y1]) shows that

$$\partial_t H_N(t) \;=\; -N D_N\Big(f_t\,|\psi_t\Big) \;+\; \int \psi_t^{-1}\Big(NL_N^* - \partial_t\Big)\psi_t\, f_t\, d\nu_\rho \tag{7.1}$$

where L_N^* represents the adjoint of L_N in $L^2(\nu_\rho)$. We start computing the second term of the right hand side. From the definition of λ given in (4.1) we obtain that it satisfies the equation

$$\partial_t\lambda \;+\; (1-2m)\sum_{i=1}^d \partial_{u_i}\lambda \;=\; 0$$

thus we obtain that $\psi_t^{-1}\Big(NL_N^* - \partial_t\Big)\psi_t$ is equal to

$$
\begin{aligned}
&\sum_{x,i} (\partial_{u_i}\lambda)(t,x/N)\Big\{\eta(x)\eta(x+e_i) - m^2(t,x/N) - 2m(t,x/N)\big[\eta(x) - m(t,x/N)\big]\Big\} \\
&\quad - \sum_{x,i} (\partial_{u_i}\lambda)(t,x/N)\Big\{\eta(x+e_i) - \eta(x)\Big\} \\
&\quad\quad - \sum_{x,i} (\partial_{u_i}\lambda)(t,x/N) m(t,x/N)\big[1 - m(t,x/N)\big] \\
&\quad\quad + (2N)^{-1} \sum_{x,i} \Big\{(\partial_{u_i}\lambda)^2(t,x/N) - (\partial_{u_i}^2\lambda)(t,x/N)\Big\}\eta(x+e_i)\big[1-\eta(x)\big] \;+\; O(N^{d-2})\;.
\end{aligned}
\tag{7.2}
$$

Since $(\partial_{u_i}\lambda)m(1-m) = \partial_{u_i}m$ the third line is of order $o(N^{d-1})$ by summation by parts (cf. Remark 3.3). By the law of large numbers the expected value with respect to the measures $f_t d\nu_\rho$ of the fourth line divided by N^{d-1} converges to

$$
\sum_{i=1}^d \int_{T^d} \frac{1-m}{m(1-m)}\big[\partial_{u_i}m\big]^2\, du\;. \tag{7.3}
$$

By a summation by parts the second line divided by N^{d-1} and integrated with respect to $f_t d\nu_\rho$ converges, as $N\uparrow\infty$, to

$$
-\sum_{i=1}^d \int_{T^d} \frac{1}{m(1-m)}\big[\partial_{u_i}m\big]^2\, du\;. \tag{7.4}
$$

In turn, to compute the expected value of the first line, recall the definition given in section 3 of K, Λ_ℓ and $V_{i,\ell}$ and the computations made just after (3.3). We showed there that the expected value with respect to $f_t d\nu_\rho$ of the first line divided by N^{d-1} is equal to

$$
\begin{aligned}
&N^{1-d}\sum_{x,i}(\partial_{u_i}\lambda)(t,x/N)\int \tau_x V_{i,K}(\eta) f_t d\nu_\rho \;+\; o_{N,B}(1) \\
&\quad + \; N^{1-d}\sum_{x,i}(\partial_{u_i}\lambda)(t,x/N)\int \Big[\eta_t^K(x) - m(t,x/N)\Big]^2 f_t d\nu_\rho\;.
\end{aligned}
$$

By Lemma 5.1, there exists $\gamma_0 > 0$ such that the second line is bounded above by

$$
\gamma_0^{-1} N^{1-d} H_N(t) \;+\; o_N(1)\;.
$$

Keep in mind that the entropy $H_N(t)$ vanishes at time 0. From Lemma 7.1 there exists a constant $C_0 = C_0(m)$ such that

$$\int_0^{t_0} N^{2-d}\Big\{\tilde D_N(f_t) - D_N\Big(f_t|\psi_t\Big)\Big\}dt \;\le\; C_0 t_0 \; .$$

On the other hand a simple computation relying on the elementary inequality $a\log(b/a) \le 2\sqrt{a}(\sqrt{b}-\sqrt{a})$ shows that $D_N(f_t) \le \tilde D_N(f_t)$. Therefore from (7.1) and the previous bounds we get that

$$\begin{aligned}
N^{1-d}H_N(t_0) \;+\; &\int_0^{t_0} (1/2)N^{2-d}D_N(f_t)\,dt \\
&\le (1+C_0)t_0 \;+\; C_1\int_0^{t_0} N^{1-d}H_N(t)\,dt \\
&\quad+ \int_0^{t_0} dt\,\Big\{N^{1-d}\sum_{x,i}\int(\partial_{u_i}\lambda)(t,x/N)\tau_x V_{i,K}(\eta)f_t d\nu_\rho - (1/4)N^{2-d}D_N(f_t)\Big\}
\end{aligned} \tag{7.5}$$

for some finite constant C_1.

To conclude the proof of the proposition it remains to show that the third line is bounded and then apply Gronwal's lemma. But this follows directly from Corollary 6.6 applied to the functions $h_i(\eta) = \eta(0)\eta(e_i)$.

8. Proof of Lemma 3.1.

Recall that a_σ is a smooth approximation of the diffusion coefficient a converging uniformly to a on each compact subset of $(0,1)$. For a positive integer ℓ, $1\le i\le d$ and a family $\{F_i,\ 1\le i\le d\}$ of functions in $\mathfrak{F}$, let $W_{i,\ell}(\eta)$ be given by

$$\begin{aligned}
W_{i,\ell}(\eta) \;=\; &\Big\{V_{i,\ell}(\eta) \;+\; \Big[(1/2)-\eta^\ell(0)\Big](2\ell'+1)^{-d}\sum_{|y|\le\ell'}\nabla_{e_i}\eta(y)\Big\} \\
&- \sum_{j=1}^d a_\sigma^{i,j}(\eta^\ell(0))\Big\{(2\ell'+1)^{-d}\sum_{|y|\le K'}\nabla_{e_j}\eta(y)\Big\} \;+\; L_N F_i(\eta^\ell(0),\eta)\; .
\end{aligned}$$

Here, as in the previous sections, $\ell' = \ell-1$ and, for $1\le j\le d$ and $y\in\mathbf{Z}^d$, $\nabla_{e_j}\eta(y)$ stands for $\eta(y+e_j)-\eta(y)$. For a function $F(\rho,\eta)$ in $\mathfrak{F}$ we establish the following convention. Whenever appears the expression $LF(\eta^\ell(x),\tau_y\eta)$ for some generator L, it is understood that L is acting only on the second coordinate and not on the first one.

The time integral on the left hand side of the statement of Lemma 3.1 is equal to

$$
\begin{aligned}
&\int_0^T dt\, E_N\Big[N^{1-d}\sum_{x,i}(\partial_{u_i}J)(t,x/N)\tau_x W_{i,K}(\eta_t)\Big] \\
&\quad + \int_0^T dt\, E_N\Big[N^{1-d}\sum_{x,i}(\partial_{u_i}J)(t,x/N)\sum_{j=1}^{d}\Big[a_\sigma^{i,j}(\eta_t^K(x)) \;-\; \big[(1/2)-\eta_t^K(x)\big]\delta_{i,j}\Big]\times \\
&\qquad\qquad\qquad\qquad \times\ (2K'+1)^{-d}\sum_{|y-x|\le K'}\nabla_{e_j}\eta_t(y)\Big] \\
&\qquad - \int_0^T dt\, E_N\Big[N^{1-d}\sum_{x,i}(\partial_{u_i}J)(t,x/N)L_N F_i(\eta_t^K(x),\tau_x\eta_t)\Big] \;:=\Omega_1+\Omega_2+\Omega_3.
\end{aligned}
\tag{8.1}
$$

We claim that Ω_3 vanishes in the limit as $N\uparrow\infty$. First observe that, since L_N is acting only on the second coordinate of F, that $L_N F_i(\eta^K(x),\tau_x\eta)$ is in fact equal to

$$
\sum_{|b|\le B_F} L_{b+x}F_i(\eta^K(x),\tau_x\eta)\ .
$$

Here B_F is a large integer so that $\{-B_F,\dots,B_F\}^d$ contains the common support of all cylinder function $F_i(m,\cdot)$. Moreover, for a bond $b=(b_1,b_2)$, L_b is the piece of generator corresponding to jumps over the bond b. Notice furthermore that for K sufficiently large ($K\ge B_F+1$) it does not make any difference if L_{b+x} is acting on the second coordinate or on both.

The first step in the proof that Ω_3 vanishes as $N\uparrow\infty$ is to replace $\eta_t^K(x)$ by $m(t,x/N)$. This is the content of the next result.

Lemma 8.1 . *Fix $1\le i\le d$ a bond $b=(b_1,b_1+e_i)$ in $\mathbf{Z}^d$. Denote by L_b the piece of the generator L_N corresponding to this bond and by L_b^* its adjoint in $L^2(\nu_\rho)$. For every $t>0$, every smooth function function $J\colon[0,t]\times T^d\to\mathbf{R}$ and every function F in $\mathfrak{F}$*

$$
\lim_{\ell\to\infty}\lim_{N\to\infty}\int_0^t ds\, E_N\Big[N^{1-d}\sum_x J(s,x/N)L_{b+x}^*\Big\{F(\eta_s^\ell(x),\tau_x\eta_s)-F(m(s,x/N),\tau_x\eta_s)\Big\}\Big] \;=\; 0\ .
$$

Here we assume that L_b^* is acting on both coordinates.

Proof. Denote b_1+e_i by b_2 and fix a positive ℓ such that $|b_1|+1<\ell$. Since by definition L_b^* is the adjoint of L_b in $L^2(\nu_\rho)$, the expected value appearing in the statement of the lemma may be rewritten as

$$N^{1-d}\sum_x J(s,x/N)\int\Big\{F(\eta^\ell(x),\tau_x\eta)-F(m(s,x/N),\tau_x\eta)\Big\}\times$$
$$\times\ \eta(b_1+x)[1-\eta(b_2+x)]\Big[f_s(\eta^{b_1+x,b_2+x})-f_s(\eta)\Big]\nu_\rho(d\eta)\,.$$

By Schwarz inequality this expression is bounded above by

$$N^{-d}A^{-1}\sum_x J^2(x/N)\int\sup_\xi\Big|F(\eta^\ell(x),\xi)-F(m(s,x/N),\xi)\Big|^2 f_s(\eta)\nu_\rho(d\eta)$$
$$+\ 2AN^{2-d}D_N(f_s)$$

for every positive A. Since $F(\cdot,\xi)$ is uniformly smooth, the first line is bounded by

$$N^{-d}C(J,F)A^{-1}\sum_x E_N\Big[\Big|\eta_s^\ell(x)-m(s,x/N)\Big|^2\Big]$$

which converges to 0 as $N\uparrow\infty$ and then $\ell\uparrow\infty$ by the law of large numbers. We conclude the proof letting $A\downarrow 0$ in the penultimate formula and invoking the content of Proposition 4.1. $\square$

Notice that we may let ℓ depend on N in the statement of the last lemma. In this case we just need to require that $\ell_N\ll N$.

We now return to the proof of the claim that Ω_3 vanishes in the limit $N\to\infty$. First of all, from Lemma 8.1, we may replace $\eta_t^K(x)$ by $m(t,x/N)$. On the other hand by the martingale property we have the following identity :

$$\int_0^T dt\,E_N\Big[N^{-d}\sum_{x,i}NL_N\Big\{(\partial_{u_i}J)(t,x/N)F_i(m(t,x/N),\tau_x\eta_t)\Big\}\Big]$$
$$=\ -\int_0^T dt\,E_N\Big[N^{-d}\sum_{x,i}\partial_t\Big\{(\partial_{u_i}J)(t,x/N)F_i(m(t,x/N),\tau_x\eta_t)\Big\}\Big]$$
$$+\,E_N\Big[N^{-d}\sum_{x,i}(\partial_{u_i}J)(T,x/N)F_i(m(T,x/N),\tau_x\eta_T)\Big]$$
$$-\ E_N\Big[N^{-d}\sum_{x,i}(\partial_{u_i}J)(0,x/N)F_i(m(0,x/N),\tau_x\eta_0)\Big]\,.$$

By the law of large numbers, as $N\uparrow\infty$, the right hand side of the last expression converges to

$$
\begin{aligned}
&-\sum_i \int_0^T dt\,(\partial_t \partial_{u_i} J)(t,u) E_{\nu_{m(t,u)}} \Big[F_i(m(t,u),\eta) \Big] \\
&\qquad -\sum_i \int_0^T dt\,(\partial_{u_i} J)(t,u)(\partial_t m)(t,u) E_{\nu_{m(t,u)}} \Big[F_{1,i}(m(t,u),\eta) \Big] \\
&\qquad\qquad + \sum_i \int du\,(\partial_{u_i} J)(T,u) E_{\nu_{m(T,u)}} \Big[F_i(m(T,u),\eta) \Big] \\
&\qquad\qquad\qquad - \sum_i \int du\,(\partial_{u_i} J)(0,u)\, E_{\nu_{m(0,u)}} \Big[F_i(m(0,u),\eta) \Big] \;.
\end{aligned}
$$

In this formula, $F_{1,i}$ denotes the partial derivative of F_i with respect to the first coordinate. This expression is equal to 0 by property (iii) of functions in $\mathfrak{F}$.

We turn now to the second integral of (8.1) that we denoted by Ω_2. Using similar argument to the ones presented in the proof of Lemma 8.1 we may replace $a_\sigma^{i,j}\left(\eta_t^K(x)\right)$ by $a_\sigma^{i,j}\left(m(t,x/N)\right)$ and $\eta_t^K(x)$ by $m(t,x/N)$. Since a_σ is smooth, we can use summation by parts so that the difference operator ∇ will act on a smooth function. Since $N\nabla$ is of order one, by the law of large numbers, as $N\uparrow\infty$, the second integral of (8.1) converges to

$$
\sum_{i,j} \int_0^T dt \int du\,(\partial_{u_i} J) \Big\{ a_\sigma^{i,j}(m) - \big[(1/2) - m\big]\delta_{i,j} \Big\} (\partial_{u_j} m)
$$

Letting $\sigma \downarrow 0$, since a_σ converges to a uniformly on compact subsets of $(0,1)$, we obtain that this expression converges to the expression appearing on the right hand side of the statement of the Lemma 3.1.

Finally, by Proposition 4.1 and Corollary 6.6 the limit, as $N\uparrow\infty$, of the first integral of (8.1) is bounded above by

$$
\frac{C_1}{\delta} \sum_i \int_0^T ds \int du\,(\partial_{u_i} J)^2(s,u) \mathbf{V}_{m(s,u)} \Big(W_{i,m(s,u),\sigma} \Big) \;+\; C_2 \delta \;,
$$

where

$$
W_{i,m,\sigma}(\eta) \;=\; (\eta(0)\eta(e_i))^m \;+\; \big[(1/2) - m\big] \nabla_{e_i}\eta(0) \;-\; \sum_j a_\sigma^{i,j}(m) \nabla_{e_j}\eta(0) \;+\; LF_i(m,\eta)\;.
$$

Here we adopted the notation introduced in section 6 and, for a cylinder function h, denoted by h^m the projection of h on $\mathcal{G}_m$: $h^m(\eta) = h - \tilde{h}(m) - \tilde{h}'(m)[\eta(0) - m]$. In particular

$$
(\eta(0)\eta(e_i))^m \;=\; [\eta(e_i) - m][\eta(0) - m] \;+\; m\nabla_{e_i}\eta(0)
$$

and

$$W_{i,m,\sigma}(\eta) \;=\; w_i(m) - \sum_j a_\sigma^{i,j}(m)\nabla_{e_j}\eta(0) \;+\; LF_i(m,\eta)$$
$$w_i(m) \;=\; (1/2)\nabla_{e_i}\eta(0) + \;[\eta(e_i) - m][\eta(0) - m]\;.$$

As $\sigma \downarrow 0$ the last integral converges to

$$\frac{1}{\delta}\sum_i \int_0^t ds \int du\,(\partial_{u_i} J)^2(s,u)\mathbf{V}_{m(s,u)}\Big(W_{i,m(s,u),0}\Big)\;.$$

Since the solution m of the Burger's equation (1.4) is bounded away from 0 and 1, to conclude the proof of Lemma 3.1 it is enough to prove the following result, which proof can be found in [LOY1].

Lemma 8.2 . *For every $\delta > 0$,*

$$\inf_{F\in\mathfrak{F}}\sup_{\delta\le m\le 1-\delta}\mathbf{V}_m\Big(W_{i,m,0}\Big) \;=\; 0$$

for $1 \le i \le d$.

This concludes the proof of Lemma 3.1. □

9. Proof of Proposition 4.2.

The proof of this result follows closely the one of Proposition 4.1. For this reason we will omit some details.

For $1 \le i \le d$, fix functions $F_i(\rho,\eta)$ in $\mathfrak{F}$ and denote $(F_1,\dots,F_d)$ by F. Recall the definition of ψ_t^F given in section 4. We start computing the time derivative of the entropy of f_t with respect to ψ_t^F which is equal to

$$-ND_N\Big(f_t\,|\psi_t^F\Big) \;+\; \int\Big(\psi_t^F\Big)^{-1}\Big(NL_N^* - \partial_t\Big)\psi_t^F\, f_t\,d\nu_\rho\;. \tag{9.1}$$

A careful calculation, taking into account computations already done in the proof of Proposition 4.1 in section 7, shows that $\Big(\psi_t^F\Big)^{-1}\Big(NL_N^* - \partial_t\Big)\psi_t^F$ is equal to

$$
\begin{aligned}
&\sum_{x,i}(\partial_{u_i}\lambda)(t,x/N)\Big\{\eta(x+e_i)\eta(x) - m^2(t,x/N) - 2m(t,x/N)\big[\eta(x)-m(t,x/N)\big]\Big\}\\
&\quad - \sum_{x,i}(\partial_{u_i}\lambda)(t,x/N)L_N^* F_i(m(t,x/N),\tau_x\eta)\\
&\quad\quad - \sum_{x,i}(\partial_{u_i}\lambda)(t,x/N)\Big\{\eta(x+e_i)-\eta(x)\Big\}\\
&\quad\quad + (2N)^{-1}\sum_{x,i}\eta(x+e_i)\big[1-\eta(x)\big]\,\times\\
&\quad\quad\quad \times\ \Big\{(\partial_{u_i}\lambda)(t,x/N)\ +\ \nabla_{x,x+e_i}\sum_{y,j}(\partial_{u_j}\lambda)(t,y/N)F_j(m(t,y/N),\tau_y\eta)\Big\}^2\\
&\quad\quad\quad - (2N)^{-1}\sum_{x,i}(\partial^2_{u_i}\lambda)(t,x/N)\eta(x+e_i)\big[1-\eta(x)\big]\\
&\quad\quad\quad + N^{-1}\sum_{x,i}\partial_t\Big\{(\partial_{u_i}\lambda)(t,x/N)F_i(m(t,x/N),\tau_x\eta)\Big\}\\
&\quad\quad\quad - \sum_{x,i}(\partial_{u_i}\lambda)(t,x/N)m(t,x/N)\big[1-m(t,x/N)\big]\ +\ \partial_t\log\Big\{Z_t^F/Z_t\Big\}
\end{aligned}
\tag{9.2}
$$

plus a small error of order $o(N^{d-1})$. In this formula Z_t^F and Z_t are normalizing constants associated to the densities ψ_t^F and ψ_t respectively. The last five lines are of order N^{d-1}. We now compute separately the limit of the time integral of the expected value of each of these terms divided by N^{d-1}.

The second term of the last line integrated in time and divided by N^{d-1} is equal to the difference of

$$
N^{1-d}\log E_{\psi_t}\Big[\exp\Big\{-N^{-1}\sum_{x,i}(\partial_{u_i}\lambda)(t,x/N)F_i(m(t,x/N),\tau_x\eta)\Big\}\Big]
\tag{9.3}
$$

and a similar term taken at time 0. In this formula E_{ψ_t} represents expectation with respect to the measure $\psi_t(\eta)\nu_\rho(d\eta)$. Since each F_i is a cylinder function with uniform support and since $\psi_t(\eta)\nu_\rho(d\eta)$ is a product measure, by Hölder's inequality we obtain that the last expression is bounded above by

$$
\frac{N^{1-d}}{d\bar\ell^d}\sum_{x,i}\log E_{\psi_t}\Big[\exp\Big\{-N^{-1}d\bar\ell^d(\partial_{u_i}\lambda)(t,x/N)F_i(m(t,x/N),\tau_x\eta)\Big\}\Big]\,.
$$

Here ℓ is a positive integer such that the support of each cylinder function $F_i(\rho, \cdot)$ is contained in $\{-\ell, \dots, \ell\}^d$. From the elementary inequalities $\log(1+u) \le u$ and $e^u - 1 \le u + 2^{-1}u^2 e^{|u|}$ and since $\mathrm{div}\lambda$ and F_i are bounded functions, we obtain that the last sum is bounded above by

$$-N^{-d} \sum_{x,i} E_{\psi_t} \Big[(\partial_{u_i}\lambda)(t, x/N) F_i(m(t, x/N), \tau_x \eta) \Big] \; + \; O(N^{-1}) \; .$$

From the definition of ψ_t, as $N \uparrow \infty$, this expression converges to

$$-\sum_i \int (\partial_{u_i}\lambda)(t,u) \nu_{m(t,u)} \Big[F_i(m(t,u), \eta) \Big] \, du \;=\; 0$$

since from property (iii) of functions in $\mathfrak{F}$ the expected value of $F(m, \cdot)$ with respect to ν_m is equal to 0 for every $0 \le m \le 1$.

On the other hand, by Jensen's inequality, expression (9.3) is bounded below by

$$-N^{-d} \sum_{x,i} E_{\psi_t} \Big[(\partial_{u_i}\lambda)(t, x/N) F_i(m(t, x/N), \tau_x \eta) \Big]$$

and we have just seen that this expression converges to 0. Therefore, the second term of the last line of (9.2) integrated in time and divided by N^{d-1} which is equal to

$$N^{1-d} \Big\{ \log \frac{Z_t^F}{Z_t} \;-\; \log \frac{Z_0^F}{Z_0} \Big\}$$

converges to 0 as $N \uparrow \infty$.

Since $\partial_{u_i}\lambda = [m(1-m)]^{-1}\partial_{u_i} m$, by the integration by parts formula stated in Remark 3.3, the first term of the last line of (9.2) divided by N^{d-1} vanishes in the limit when $N \uparrow \infty$.

The expected value of the next to the last line of (9.2) divided by N^{d-1} is equal to

$$\begin{aligned} &(1/2) N^{-d} \sum_{x,i} E_N \Big[(\partial_t \partial_{u_i}\lambda)(t, x/N) F_i(m(t, x/N), \tau_x \eta_t) \Big] \\ &\quad + (1/2) N^{-d} \sum_{x,i} E_N \Big[(\partial_{u_i}\lambda)(t, x/N)(\partial_t m)(t, x/N) F_{1,i}(m(t, x/N), \tau_x \eta) \Big] \; . \end{aligned}$$

Here $F_{1,i}$ stands for the derivative with respect to the first coordinate of F_i. By the law of large numbers this expression converges, as $N \uparrow \infty$, to

$$\begin{aligned} &(1/2) \sum_i \int (\partial_t \partial_{u_i}\lambda)(t,u) \nu_{m(t,u)} \Big[F_i(m(t,u), \eta) \Big] \, du \\ &\quad + (1/2) \sum_i \int (\partial_{u_i}\lambda)(t,u)(\partial_t m)(t,u) \nu_{m(t,u)} \Big[F_{1,i}(m(t,u), \eta) \Big] \, du \; . \end{aligned}$$

This expression is equal to 0 from property (iii) of cylinder functions in $\mathfrak{F}$. Indeed, from property (iii) we have that $\nu_m[F(m,\eta)]$ as a function of the density m is identically equal to 0. Taking derivatives with respect to m, we obtain that

$$0 \;=\; \partial_\rho \nu_\rho[F(m,\eta)]\Big|_{\rho=m} \;+\; \nu_m[F_1(m,\eta)]\;.$$

Since by property (iii) the first member of the right hand side vanishes, the second one also vanishes.

By the law of large numbers the time integral of the expectation of the fifth term (sixth line) of (9.2) divided by N^{d-1} converges to (after an integration by part)

$$(1/2)\sum_i \int_0^t ds \int du\,[m(1-m)]^{-1}(1-2m)(\partial_{u_i} m)^2\;. \tag{9.4}$$

Furthermore, by the law of large numbers, the time integral of the expected value of the fourth term (line 4 and 5) of (9.2) converges as $N\uparrow\infty$ to

$$\begin{aligned}(1/2)\sum_i \int_0^t ds \int du\, \nu_{m(s,u)}\Big[\eta(e_i)[1-\eta(0)]\times\\ \times\;\Big\{(\partial_{u_i}\lambda)(s,u) \;+\; \sum_j (\partial_{u_j}\lambda)(s,u)\nabla_{0,e_i}\Gamma_{F_j(m(s,u),\cdot)}\Big\}^2\Big]\end{aligned}$$

where, for a cylinder function Ψ, Γ_Ψ represents the formal infinite sum

$$\Gamma_\Psi(\eta) \;=\; \sum_{y\in\mathbf{Z}^d} \Psi(\tau_y\eta)\;.$$

We have also used here the smoothness of F, λ and m.

We turn now to the third line of (9.2). We take one half of this expression to compensate part of the integral (9.4). The other part counterbalance a piece of the first line of (9.2). Thus, by a discrete integration by parts, the expected value of the time integral of one half of the third line divided by N^{d-1} converges to

$$-(1/2)\sum_i \int_0^t ds \int du\,[m(1-m)]^{-1}(\partial_{u_i} m)^2\;.$$

The time integral of the expected value of the other half is equal to

$$-(1/2)N^{1-d}\int_0^t ds\, E_N\Big[\sum_{x,i}(\partial_{u_i}\lambda)(t,x/N)(\bar{K}')^{-d}\sum_{|y-x|\le K'}\Big\{\eta_s(y+e_i)-\eta_s(y)\Big\}\Big] \;+\; o_N(1)\;,$$

where $K' = K - 1$ and $K = K(N)$ is the sequence defined in section 3.

Keep in mind the convention that in the expression $L_N F_i(\eta^\ell(0), \eta)$ the generator is acting only on the second coordinate.

Lemma 8.1 permits us to replace $m(s, x/n)$ by $\eta_s^{K_N}(x)$ in the second line of (9.2) for any sequence K_N increasing to infinity and small compared to N.

We will deal with the first term of (9.2) as we have done in the proof of proposition 4.1. Recall the definition of $V_{i,\ell}(\eta)$ given by (3.7). As shown in the proof of 4.1, there exists $\gamma_0 > 0$ such that the time integral of the expected value of the first line of (9.2) divided by N^{d-1} is bounded above by

$$\int_0^t ds\, N^{1-d} \sum_{x,i} (\partial_{u_i}\lambda)(s, x/N) \int \tau_x V_{i,K}(\eta) f_s d\nu_\rho \\ + \gamma_0 \int_0^t ds\, N^{1-d} H_N(f_s) \; + \; o_{N,B}(1)$$

where $K = K(N)$ is the sequence defined just after (3.5).

Since by (4.5) the entropy $N^{1-d}H_N(f_s)$ is bounded by $N^{1-d}H\Big(f_s \,|\psi_s^F\Big) + O(N^{-(1/2)})$, up to this point we have shown that the entropy $N^{1-d}H\Big(f_t \,|\psi_t^F\Big)$ is bounded above by

$$\begin{aligned} & N^{1-d}H\Big(f_0 \,|\psi_0^F\Big) \; + \; \gamma_0 \int_0^t ds\, N^{1-d}H\Big(f_s \,|\psi_s^F\Big) \; - \; \int_0^t ds\, N^{2-d} D_N\Big(f_s \,|\psi_s^F\Big) \\ & + \int_0^t ds\, E_N\Big[N^{1-d}\sum_{x,i}(\partial_{u_i}\lambda)(s,x/N)\tau_x V_{i,K}(\eta_s)\Big] \\ & \quad - (1/2)N^{1-d}\int_0^t ds\, E_N\Big[\sum_{x,i}(\partial_{u_i}\lambda)(s,x/N)(\bar{K}')^{-d}\sum_{|y-x|\le K'}\Big\{\eta_s(y+e_i) - \eta_s(y)\Big\}\Big] \\ & \quad - \int_0^t ds\, E_N\Big[N^{1-d}\sum_{x,i}(\partial_{u_i}\lambda)(s,x/N)L_N^* F_i(\eta_s^K(x), \tau_x\eta_s)\Big] \\ & \quad + (1/2)\sum_i \int_0^t ds \int du\, \nu_{m(s,u)}\Big[\eta(e_i)[1-\eta(0)] \times \\ & \qquad \times \Big\{(\partial_{u_i}\lambda)(s,u) \; + \; \sum_j (\partial_{u_j}\lambda)(s,u)\nabla_{0,e_i}\Gamma_{F_j(m(s,u),\cdot)}\Big\}^2\Big] \\ & \quad - \sum_i \int_0^t ds \int du\, \frac{m}{m(1-m)}(\partial_{u_i}m)^2 \end{aligned} \tag{9.5}$$

plus a small error of order $o_N(1)$.

We now concentrate our attention on the second, third and fourth line of this sum. Recall that a_σ is a smooth approximation of the diffusion coefficient a. Let $W^*_{i,K}(\eta)$ be the cylinder function defined by

$$W^*_{i,\ell}(\eta) \;=\; V_{i,\ell}(\eta) \;-\; \Big[(1/2)+\eta^K(0)\Big](\bar{K}')^{-d}\sum_{|y|\le K'}\nabla_{e_i}\eta(y)$$
$$+\;\sum_{j=1}^{d} a^{i,j}_\sigma(\eta^K(0))\Big\{(\bar{K}')^{-d}\sum_{|y|\le K'}\nabla_{e_j}\eta(y)\Big\} \;-\; L^*_N F_i(\eta^K(0),\eta)\;.$$

In this formula K' stands for $K-1$. Notice also that to reduce notation we omitted the dependence of $W^*_{i,K}$ on σ and F.

The sum of the second, third and fourth line of (9.5) is equal to

$$\int_0^t ds\, E_N\Big[N^{1-d}\sum_{x,i}(\partial_{u_i}\lambda)(s,x/N)\tau_x W^*_{i,K}(\eta_s)\Big]$$
$$-\;\int_0^t ds\, E_N\Big[N^{1-d}\sum_{x,i,j}(\partial_{u_i}\lambda)(s,x/N)a^{i,j}_\sigma(\eta^K_s(0))(\bar{K}')^{-d}\sum_{|y-x|\le K'}\Big[\eta_s(y+e_j)-\eta_s(y)\Big]\Big]$$
$$+\;N^{1-d}\int_0^t ds\, E_N\Big[\sum_{x,i}(\partial_{u_i})(s,x/N)\eta^K_s(0)(\bar{K}')^{-d}\sum_{|y-x|\le K'}\Big[\eta_s(y+e_j)-\eta_s(y)\Big]\Big]\;.$$

We have seen in last section that the second line of this expression converges, as $N\uparrow\infty$, to

$$-\sum_{i,j}\int_0^t ds\int du\,(\partial_{u_i}\lambda)(s,u)a^{i,j}_\sigma(m(s,u))(\partial_{u_j}m)(s,u)\;.$$

By similar reasons the third line converges to

$$\int_0^t ds\int du\,\frac{m}{m(1-m)}(\partial_{u_i}m)^2\;.$$

Notice that this term cancels the last one of (9.5).

Furthermore, from Corollary 6.6 and Proposition 4.1 the limit, as $N\uparrow\infty$, of the first line is bounded above by

$$\frac{1}{\delta}\sum_i\int_0^t ds\int du\,(\partial_{u_i}\lambda)^2(s,u)\mathbf{V}_{m(s,u)}\Big(W^*_{i,m(s,u),\sigma}\Big)\;+\;C_0 t\delta\;,$$

for some universal constant C_0 and where

$$W^*_{i,m,\sigma}(\eta)\;=\;[\eta(0)\eta(e_i)]^m\;-\;\Big[(1/2)+m\Big]\nabla_{e_i}\eta(0)+\sum_{j=1}^{d}a^{i,j}_\sigma(m)\nabla_{e_j}\eta(0)\;-\;L^*F_i(m,\eta)\;.$$

Since $[\eta(0)\eta(e_i)]^m = [\eta(0) - m][\eta(e_i) - m] + m\nabla_{e_i}\eta(0)$, we have that

$$W^*_{i,m,\sigma}(\eta) \;=\; -w^*_i \;+\; \sum_{j=1}^{d} a^{i,j}_\sigma(m)\nabla_{e_j}\eta(0) \;-\; L^* F_i(m,\eta)$$
$$w^*_i \;= (1/2)\nabla_{e_i}\eta(0) \;-\; [\eta(e_i) - m][\eta(0) - m]\;.$$

Define $W^*_{i,m,0}(\eta)$ to be the cylinder function $W^*_{i,m,\sigma}$ with $a^{i,j}(m)$ replacing $a^{i,j}_\sigma(m)$ in the above formula. From the variational formula (10.1) for $\mathbf{V}_m$, the triangular inequality and the identity $\mathbf{V}_m(\nabla_{e_i}\eta(0), \nabla_{e_j}\eta(0)) = \delta_{i,j}2m(1-m)$, one obtain that for every $0 \le m \le 1$,

$$\mathbf{V}_m\Big(W^*_{i,m,\sigma}\Big) \;\le\; 2\mathbf{V}_m\Big(W^*_{i,m,0}\Big) \;+\; 4\sum_{j=1}^{d}\Big[a^{i,j}_\sigma(m) - a^{i,j}(m)\Big]^2 m(1-m)\;.$$

Since a_σ converges pointwise to a, letting $\sigma \downarrow 0$, we obtain that the sum of the second, third, fourth and sixth line of (9.5) is bounded above by

$$\begin{aligned}
&\frac{1}{\delta}\sum_i \int_0^t ds \int du\, (\partial_{u_i}\lambda)^2(s,u)\mathbf{V}_{m(s,u)}\Big(W^*_{i,m(s,u),0}\Big) \;+\; C_0\delta t\\
&\qquad -\sum_{i,j}\int_0^t ds \int du\, (\partial_{u_i}\lambda)(s,u) a^{i,j}(m(s,u))(\partial_{u_j}m)(s,u) \;+\; o_N(1)
\end{aligned}$$

for every positive δ.

In conclusion, from (9.5), the above bound and Gronwal's inequality, we proved that there exists $\gamma_0 > 0$ such that

$$N^{1-d}H\Big(f_t\,\big|\psi^F_t\Big) \;+\; \int_0^t ds\, N^{2-d}D_N\Big(f_s\,\big|\psi^F_s\Big)$$

is bounded above by

$$\begin{aligned}
e^{\gamma_0 t}\Big\{ & o_N(1) + C_0\delta t + \frac{1}{\delta}\sum_i\int_0^t ds \int du\, (\partial_{u_i}\lambda)^2(s,u)\mathbf{V}_{m(s,u)}\Big(W^*_{i,m(s,u),0}\Big)\\
& -\sum_{i,j}\int_0^t ds\int du\, (\partial_{u_i}\lambda)(s,u)a^{i,j}(m(s,u))(\partial_{u_j}m)(s,u)\\
& + (1/2)\sum_i\int_0^t ds\int du\, \nu_{m(s,u)}\Big[\eta(e_i)[1-\eta(0)] \;\times\\
&\qquad \times\; \Big((\partial_{u_i}\lambda)(s,u) \;+\; \sum_j (\partial_{u_j}\lambda)(s,u)\nabla_{0,e_i}\Gamma_{F_j(m(s,u),\cdot)}\Big)^2\Big]\Big\}\;.
\end{aligned} \tag{9.6}$$

Developing the square of the last line observe that the cross product gives a null term since

$$\nu_{m(s,u)}\Big[\eta(e_i)[1-\eta(0)]\nabla_{0,e_i}\Gamma_{F_j(m(s,u),\cdot)}\Big] \;=\; 0$$

Furthermore observe that

$$\begin{aligned}\nu_{m(s,u)}&\Big[\eta(e_i)[1-\eta(0)]\Big(\sum_j(\partial_{u_j}\lambda)(s,u)\nabla_{0,e_i}\Gamma_{F_j(m(s,u),\cdot)}\Big)^2\Big]\\ &=\frac{1}{2}\nu_{m(s,u)}\Big[\Big(\sum_j(\partial_{u_j}\lambda)(s,u)\nabla_{0,e_i}\Gamma_{F_j(m(s,u),\cdot)}\Big)^2\Big]\end{aligned}$$

It follows then that the third and fourth lines of (9.6) are equal to

$$\begin{aligned}&(1/2)\sum_i\int_0^t ds\int du\,\frac{1}{m(1-m)}(\partial_{u_i}m)^2(s,u)\\ &\quad+(1/4)\sum_i\int_0^t ds\int du\,\frac{1}{m^2(1-m)^2}\nu_{m(s,u)}\Big[\Big\{\nabla_{0,e_i}\sum_j(\partial_{u_j}m)(s,u)\Gamma_{F_j(m(s,u),\cdot)}\Big\}^2\Big]\,.\end{aligned}$$

By assumption there exists $\delta_0>0$ such that the solution m of the hyperbolic equation is bounded below by δ_0 and above by $1-\delta_0$.

To conclude the proof of the proposition, for each $1\le i\le d$, consider a sequence of function $H_k^i(m,\eta)$ such that

$$\lim_{k\to\infty}\sup_{\delta_0\le m\le 1-\delta_0}\mathbf{V}_m\Big(w_i^*-\sum_j a^{i,j}(m)\nabla_{e_j}\eta(0)+L^*H_k^i(m,\eta)\Big) \;=\; 0\,.$$

The existence of such a sequence is proved in [LOY1]. Moreover, it is shown there that this sequence is such that

$$\begin{aligned}\lim_{k\to\infty}&\frac{1}{4m(1-m)}\sum_i\nu_m\Big[\Big\{\nabla_{0,e_i}\sum_j v_j\Gamma_{H^j(m,\cdot)}\Big\}^2\Big]\\ &=\sum_{i,j}v_ia^{i,j}(m)v_j-(1/2)|v|^2\end{aligned}$$

for every $\delta_0\le m\le 1-\delta_0$ and $v\in\mathbf{R}^d$. This concludes the proof of Proposition 4.2. $\square$

10. The diffusion coefficient.

In this section we state all the properties of the diffusion coefficient matrix a used in the previous sections. All these result are proven in [LOY1].

Recall that for $0 \le m \le 1$, $\mathcal{G}_m$ denotes the space of cylinder functions g with two properties. We require that $\nu_m[g] = 0$ and that the derivative with respect to ρ of $\nu_\rho[g]$ vanishes at m :

$$\tilde g(m) \;=\; \nu_m[g] \;=\; 0 \qquad \text{and} \qquad \partial_\rho \nu_\rho[g]\Big|_{\rho=m} \;=\; \tilde g'(m) \;=\; 0\;.$$

Note that the second condition is equivalent to impose that the covariance, with respect to the measure ν_m, of g and the formal sum $\sum_x \eta(x)$ vanishes :

$$\sum_z \nu_m\big[g(\eta)\eta(z)\big] \;=\; 0\;.$$

Denote by $\mathcal{C}$ the space of cylinder functions. For each function g in $\mathcal{G}_m$ we define $\mathbf{V}_m(g)$ by

$$\begin{aligned} \mathbf{V}_m(g) \;&=\; \sum_{i=1}^d \sup_{\alpha\in\mathbf{R}} \Big\{ 2\alpha t_i(g) - \frac{\alpha^2}{4}\Big\langle (\nabla_{e_i}\eta(0))^2\Big\rangle \Big\} \\ &\quad + \sup_{h\in\mathcal{C}} \sum_{i=1}^d \Big\{ 2 < g, h >_0 - \frac{1}{4}\Big\langle \Big(\nabla_{e_i} \sum_x \tau_x h\Big)^2 \Big\rangle \Big\}\;. \end{aligned} \tag{10.1}$$

In this formula, for $1\le i\le d$, g in $\mathcal{G}_m$ and h in $\mathcal{C}$, $t_i(g)$ and $< g, h >_0$ are given by

$$t_i(g) \;=\; \Big\langle g, \sum_x x_i \eta(x) \Big\rangle\;, \qquad \Big\langle g, h\Big\rangle_0 \;=\; \Big\langle g, \sum_x \tau_x h\Big\rangle$$

and $< \cdot, \cdot >$ denotes expectation with respect to ν_m. By Lemma 6.2, the finite volume variance converges to the infinite volume variance uniformly in $[0, 1]$: $\mathbf{V}_m(g) = \lim_{\ell\to\infty} V_\ell(g, m)$, where $V_\ell(g, m)$ is defined by (6.2). The slight difference between this formula for the variance and the one obtained in [EMY] comes from the fact that their generator is accelerated by 2. It is clear that we may replace $\mathcal{C}$ by $\mathcal{G}_m$ in the second supremum.

From the definition of $\mathbf{V}_m$ we may introduce the bilinear form $\mathbf{V}_m(\cdot, \cdot)$ on $\mathcal{G}_m$ by polarization

$$\mathbf{V}_m(g, h) \;=\; \frac{1}{4}\Big\{ \mathbf{V}_m(g+h) \;-\; \mathbf{V}_m(g-h)\Big\}\;.$$

Denote by $\overline{\mathcal{G}_m}$ the closure of $\mathcal{G}_m$ with respect to $\mathbf{V}_m$ and by $\mathcal{N}_m$ the kernel of $\mathbf{V}_m$. Then $(\overline{\mathcal{G}_m}/_{\mathcal{N}_m}, \mathbf{V}_m)$ is a Hilbert space.

For $1\le i\le d$, let w_i^* and w_i be the cylinder function given by

$$\begin{aligned} w_i^* &= (1/2)\nabla_{e_i}\eta(0) \;-\; [\eta(e_i) - m][\eta(0) - m]\;, \\ w_i &= (1/2)\nabla_{e_i}\eta(0) \;+\; [\eta(e_i) - m][\eta(0) - m]\;. \end{aligned} \tag{10.2}$$

From Theorem 5.9 (i) of [EMY] there exists a unique matrix $a(m)$ such that

$$w_i^* \; - \; \sum_{j=1}^d a^{i,j}(m) \nabla_{e_j} \eta(0) \; \in \; \overline{L^* \mathcal{G}_m} \; . \tag{10.3}$$

$a(m)$ is the diffusion coefficient of equations (1.6) and (1.5).

In [LOY1] we prove that the diffusion coefficient is continuous and is bounded below by 1/2 in the matrix sense :

$$(1/2)|v|^2 \; \le \; v^* a v \tag{10.4}$$

for every $v \in \mathbf{R}^d$. Here and below, for a vector v in $\mathbf{R}^d$, v^* represents the transposition of v.

The followings are results contained in [LOY1].

Corollary 10.1 . *For each $1 \le i \le d$, and $\delta > 0$,*

$$\inf_{F \in \mathcal{F}} \sup_{\delta \le m \le 1-\delta} \mathbf{V}_m \Big(w_i^* - \sum_{j=1}^d a^{i,j}(m) \nabla_{e_i} \eta(0) - L^* F \Big) \; = \; 0 \; .$$

Theorem 10.2 . *For each cylinder function h denote by $P(h) = \big(P_{i,j}(h)\big)_{1 \le i,j \le d}$ the symmetric positive definite matrix with entries $\mathbf{V}_m(\sigma_i + Lh, \sigma_j + Lh)$. For all v in $\mathbf{R}^d$,*

$$v^*[a + (1/2)I]^{-1} v \; = \; \chi \sup_{h \in \mathcal{G}_m} \Big\{ v^* P(h)^{-1} v \Big\} \; .$$

In this formula and below v^* denotes the transposition of the vector v.

Lemma 10.3 . *For every $v \in \mathbf{R}^d$,*

$$v^*[a + (1/2)I] v \; \le \; C_1 \chi^{-1} |v|^2$$

for some universal constant C_1.

Lemma 10.4 . *For $\delta > 0$ and $1 \le i \le d$, let $\{H_k^i(m, \eta);\; k \ge 1\}$ be a sequence of functions in $\mathcal{F}$ such that*

$$\lim_{k \to \infty} \sup_{\delta \le m \le 1-\delta} \mathbf{V}_m \Big(w_i^* - \sum_{j=1}^d a^{i,j}(m) \nabla_{e_i} \eta(0) - L^* H_i(m) \Big) \; = \; 0 \; .$$

Such sequence exists in virtue of Corollary 10.1.

For every $v \in \mathbf{R}^d$ and $\delta \le m \le 1 - \delta$,

$$\lim_{k \to \infty} \frac{1}{2\chi} \sum_i \Big\langle \Big\{ \nabla_{e_i} \sum_j v_j \sum_x \tau_x H_j \Big\}^2 \Big\rangle \; = \; \sum_{i,j} v_i a^{i,j}(m) v_j \; - \; (1/2)|v|^2 \; .$$

11. Appendix.

In this appendix, we prove (6.8). We need to introduce some notation. Recall that Λ_K is a cube of length $2K+1 = \bar{K}$. For $0 \le n \le \bar{K}^d$, denote by $\mu_{K,n}$ the canonical measure on Λ_K with n particles. Notice that in our context $\mu_{K,n}$ is the uniform measure with n particles. Divide the cube Λ_K into $R^d = (\bar{K}/\bar{L})^d$ disjoint subcubes of size L as we did in section 6 and assume that Λ_L is a cube of this decomposition. We shall assume further that $\bar{L}^{1+(1/4d)} \ge \bar{K}$. The exponent $1/4d$ is convenient but not essential. Recall that we denote the density in Λ_L by $\eta^L(0)$. Moreover, to keep notation as simple as possible, we denote $n/\bar{K}^d$ by ρ.

Lemma 11.1 . *Recall that $\bar{L}^d \le \bar{K}^d/2$. There is an universal constant C_0 such that*

$$\mu_{K,n}\Big[\, \big|\eta^L(0) - \rho\big| \;\ge\; a\Big] \;\le\; C_0 \exp\Big\{ -(1/4)\bar{L}^d a^2 \Big\}\, . \tag{11.1}$$

Proof. Recall that we denoted $n/\bar{K}^d$ by ρ. Denote by $\mu_{K,L,n}$ the restriction of $\mu_{K,n}$ to Λ_L. We shall prove at the end of this appendix that the Radon–Nikodym derivative of $\mu_{K,L,n}$ with respect to ν_ρ (the product measure with density ρ) is bounded by an universal constant C_0 provided $\bar{L}^d \le \bar{K}^d/2$. In particular the left hand side of (11.1) is bounded above by

$$C_0 \nu_\rho \Big[\, \big|\eta^L(0) - \rho\big| \;\ge\; a\Big]\, .$$

From Chebyshev exponential inequality, for $y > \rho$, we have that

$$\bar{L}^{-d} \log \nu_\rho \Big[\eta^L(0) \ge y\Big] \;\le\; -h(y,\rho) \tag{11.2}$$

and

$$\bar{L}^{-d} \log \nu_\rho \Big[\eta^L(0) \le y\Big] \;\le\; -h(y,\rho) \tag{11.3}$$

for $y < \rho$, where $h(y,\rho)$ is the rate functional of the large deviations principle and is given by

$$h(y,\rho) \;=\; y \log\Big\{\frac{y(1-\rho)}{(1-y)\rho}\Big\} \;-\; \log\Big\{\frac{1-\rho}{1-y}\Big\}\, .$$

It is easy to show that $h(y,\rho)$ is bounded below by $(1/4)(y-\rho)^2$. This concludes the proof of the lemma. $\square$

Lemma 11.2 . *Let g be a polynomial in $[0,1]$ vanishing at ρ : $g(\rho)=0$. Denote by $\tilde g_{K,L}(\rho)$ the expected value of $g(\eta^L(0))$ with respect to the measure $\mu_{n,K}$: $\tilde g_{K,L}(\rho) = \mu_{K,n}[g(\eta^L(0))]$. For any positive ϵ and for sufficiently large L we have that*

$$\Big| \tilde g_{K,L}(\rho) - g(\rho) \Big| \;\le\; C(g) L^{-d+2\epsilon}$$

for some constant $C(g)$ depending only on g.

Proof. Fix $\epsilon > 0$. From the definition of $\tilde g_{K,L}$, the difference $\tilde g_{K,L}(\rho) - g(\rho)$ is equal to $\mu_{K,n}[g(\eta^L(0)) - g(\rho)]$. Since ρ is the expected number of particles under $\mu_{K,n}$, we may rewrite this expression as $\mu_{K,n}[g(\eta^L(0)) - g(\rho) - g'(\rho)\{\eta^L(0)-\rho\}]$. For this reason denote by $q(\cdot)$ the function $q(y) = g(y) - g(\rho) - g'(\rho)\{y-\rho\}$. To prove the lemma we have to show that $\mu_{K,n}[q(\eta^L(0))] \le L^{-d+2\epsilon}$.

Since q is a polynomial vanishing at ρ as well as its first derivative, there exists a constant $C = C(g)$ such that $|q(y)| \le C(y-\rho)^2$. Therefore $\mu_{K,n}[q(\eta^L(0)]$ is bounded above by

$$\begin{aligned} &C\mu_{K,n}\Big[(\eta^L(0)-\rho)^2 \mathbf{1}\Big\{ |\eta^L(0)-\rho| \ge \bar L^{-(d/2)+\epsilon} \Big\}\Big] \\ &\qquad + \; C\mu_{K,n}\Big[(\eta^L(0)-\rho)^2 \mathbf{1}\Big\{ |\eta^L(0)-\rho| \le \bar L^{-(d/2)+\epsilon} \Big\}\Big] \,. \end{aligned}$$

In this formula, for a set A, $\mathbf{1}\{A\}$ represents the indicator of the set A. From Lemma 11.1, the first line is bounded above by $C(g)\exp\{-(1/4)\bar L^{2\epsilon}\}$. On the other hand the second line is trivially bounded by $C(g)\bar L^{-d+2\epsilon}$. This concludes the proof of the lemma. □

The next result follows from direct computations on hypergeometric distributions (cf. Lemma ♣rn) or from the local central limit theorem.

Lemma 11.3 . *For every cylinder function g there exists a constant C depending only on g such that*

$$\Big| \mu_{K,n}[g] \;-\; \nu_\rho[g] \Big| \;\le\; \frac{C(g)}{\bar K^d}$$

for all K large enough and all $0 \le n \le \bar K^d$. In this formula $\rho = n/\bar K^d$.

Corollary 11.4 . *For every polynomial qthere exists a constant C depending only on the degree deg q of q such that*

$$\Big|\mu_{K,n}[q(\eta^L(0))] \;-\; \nu_\rho[q(\eta^L(0))]\Big| \;\le\; \frac{C}{\bar{L}^d}$$

for all $K \ge 2$deg q and all $0 \le n \le \bar{K}^d$. Here again $\rho = n/\bar{K}^d$.

Proof. We just need to prove the corollary for monomials. In this case, $q(\eta^L(0))$ is equal to $\bar{L}^{-d}\sum \eta(x_{i_1})\cdots\eta(x_{i_r})$ plus terms of order smaller than $\bar{L}^{-d}$. Here the sum is carried over all distinct sites $x_{i_1},\dots,x_{i_r}$ in Λ_L and r stands for the degree of the monomial q. Since the measure $\mu_{K,n}$ is homogeneous, to conclude the proof of the corollary, it is enough to show that

$$\Big|\mu_{K,n}[\eta(x_{e_1})\cdots\eta(x_{re_1})] \;-\; \nu_\rho[\eta(x_{e_1})\cdots\eta(x_{re_1})]\Big| \;\le\; \frac{C}{\bar{K}^d}\,.$$

This follows by the previous theorem. □

From now on ϵ shall denote a given small positive constant.

Lemma 11.5 . *Let g be a polynomial in $[0,1]$ vanishing at ρ : $g(\rho) = 0$. Recall the notation introduced in the previous lemma. There exists a constant $C = C(g)$ such that*

$$\sup_{0\le n\le \bar{K}^d} \log \mu_{K,n}\Big[\exp \beta L^d\Big\{g(\eta^L(0)) - \tilde{g}_{K,L}(n/\bar{K}^d) - g'(n/\bar{K}^d)(\eta^L(0) - n/\bar{K}^d)\Big\}\Big] \;<\; C\beta^2 \tag{11.4}$$

for $\beta < L^{-8\epsilon}$.

Proof. Fix $0 \le n \le \bar{K}^d$. Denote by $\tilde{q}(y)$ the function defined by $\tilde{q}(y) = g(y) - \tilde{g}_{K,L}(\rho) - g'(\rho)(y-\rho)$ and notice that the expression inside braces on the left hand side of (11.4) may be rewritten as $\tilde{q}(\eta^L(0))$.

We consider first the case $\beta L^{3d} \le 1$. In this situation we may expand the exponential up to third order and obtain that the expectation in (11.4) is bounded above by

$$1 \;+\; \mu_{K,n}\Big[\beta L^d \tilde{q}(\eta^L(0))\Big] \;+\; \beta^2\mu_{K,n}\Big[\Big(L^d\tilde{q}(\eta^L(0))\Big)^2\Big] \;+\; C(g)\beta^2$$

since $\beta L^{3d} \le 1$. By definition of $g_{K,L}$, the linear term vanishes. Moreover, since g is a polynomial, by the previous corollary and Lemma ♣rn, the second order term is bounded by a constant depending only on g :

$$\mu_{K,n}\Big[\Big(L^d\tilde q(\eta^L(0))\Big)^2\Big] \;\le\; C(g)\;.$$

Since $\log(1+u)\le u$ the left hand side of (11.4) is bounded above by $C(g)\beta^2$.

We turn now to the case $\beta > L^{-3d}$. In order to obtain an upper bound for $\tilde q(y)$ we shall decompose it as

$$\tilde q(y) \;=\; g(y) \;-\; g(\rho) \;-\; g'(\rho)(y-\rho) \;+\; g(\rho) - g_{K,L}(\rho)\;.$$

Since g is a polynomial, from Lemma 11.2, $\tilde q(\eta^L(0))$ is bounded above by

$$C_1\Big\{(\eta^L(0)-\rho)^2 \;+\; L^{-d+2\epsilon}\Big\}$$

for some constant $C_1 = C_1(g)$.

We shall decompose the left hand side of (11.4) according to the value of $|\eta^L(0)-\rho|$. We first consider the case in which $|\eta^L(0)-\rho| \ge \bar L^{-(d/2)+\epsilon}$. In this case the expectation on the left hand side of (11.4) is equal to

$$\sum_{j;\,(j/\bar L^d-\rho)\ge \bar L^{-(d/2)+\epsilon}} \exp\Big\{\beta L^d\tilde q(j/\bar L^d)\Big\}\mu_{K,n}(\eta^L(0)=j/\bar L^d)$$
$$+\sum_{j;\,(j/\bar L^d-\rho)\le -\bar L^{-(d/2)+\epsilon}} \exp\Big\{\beta L^d\tilde q(j/\bar L^d)\Big\}\mu_{K,n}(\eta^L(0)=j/\bar L^d)\;.$$

Since both sums are handled in the same way, we shall consider only the first one. By Lemma 11.1 $\mu_{K,n}(\eta^L(0)=j/\bar L^d)$ is bounded above by $C_0\exp[-(1/4)L^d(j/\bar L^d-\rho)^2]$. The first sum is thus less than or equal to

$$C_0\sum_{j;\,(j/\bar L^d-\rho)\ge \bar L^{-(d/2)+\epsilon}} \exp\Big\{\beta L^d\tilde q(j/\bar L^d)\Big\}\exp\Big\{-(1/4)L^d(j/\bar L^d-\rho)^2\Big\}\;.$$

A simple computation, taking advantage of the upper bound on $\tilde q$ obtained above, shows that this expression is bounded above by $\exp\{-C(g)L^{2\epsilon}\}$ for L sufficiently large.

We now consider the case in which $|\eta^L(0)-\rho|\le L^{-(d/2)+\epsilon}$. In this region there exists a constant $C_2 = C_2(g)$ depending only on g such that

$$\beta L^d\tilde q(\eta^L(0)) \;\le\; C_2\beta L^{2\epsilon} \;<\; L^{-6\epsilon} \quad\text{and}\quad \beta\Big\{L^d\tilde q(\eta^L(0))\Big\}^3 \;\le\; C_2\beta L^{6\epsilon} \;<\; C_2 L^{-2\epsilon}$$

since, by assumption, $\beta < L^{-8\epsilon}$.

Expanding the exponential up to third order, we obtain that in the case $|\eta^L(0) - \rho| \le L^{-(d/2)+\epsilon}$ the expectation on the left hand side of (11.4) is bounded above by

$$1 \;+\; \mu_{K,n}\left[\beta L^d \tilde{q}(\eta^L(0)) \mathbf{1}\Big\{ |\eta^L(0) - \rho| \le L^{-(d/2)+\epsilon} \Big\}\right] \\ +\; \beta^2 \mu_{K,n}\left[\Big(L^d \tilde{q}(\eta^L(0))\Big)^2\right] \;+\; C_2 \beta^2 L^{-2\epsilon} \;.$$

We may rewrite the first order term as

$$\mu_{K,n}\left[\beta L^d \tilde{q}(\eta^L(0))\right] \;+\; \mu_{K,n}\left[\beta L^d \tilde{q}(\eta^L(0)) \mathbf{1}\Big\{ |\eta^L(0) - \rho| > L^{-(d/2)+\epsilon} \Big\}\right] \;.$$

The first expectation vanishes by definition of $\tilde{q}$. The second one is exponentially small by Lemma 11.1 : it is bounded by $C(g)\exp\{-C(g)L^{2\epsilon}\}$ for sufficiently large L. Moreover, we have already seen in the beginning of the proof that the second order term is bounded by $C(g)\beta^2$. Therefore, since $\log(1+u) \le u$, the left hand side of (11.4) is bounded above by

$$C(g)\exp\{-C(g)L^{2\epsilon}\} \;+\; C(g)\beta^2 \;\;\le\; C(g)\beta^2$$

for sufficiently large L because $\beta > L^{-3d}$. This concludes the proof of the lemma. □

Recall that we divided the cube Λ_K in $R^d = (\bar{K}^d / \bar{L}^d)$ disjoint subcubes of length $\bar{L}$. For $1 \le \alpha \le R^d$, denote by x_α the center of the α–th cube. In particular $\eta^K(0) = R^{-d} \sum_\alpha \eta^L(x_\alpha)$. Recall further that we assumed $\bar{K}/\bar{L} \le \bar{L}^{(1/4d)}$. In particular $R^d \le \bar{L}^{1/4}$.

Corollary 11.6 . *Under the same notation and condition of Lemma 11.5 there exists an universal constant C_1 such that*

$$\sup_{0 \le n \le K^d} \log \mu_{K,n}\left[\exp\Big\{\beta \sum_\alpha L^d [g(\eta^L(x_\alpha)) - \tilde{g}_{K,L}(n/\bar{K}^d)]\Big\}\right] \;<\; C_1 (K/L)^{2d} \beta^2$$

for $\beta \le L^{-(1/4)-8\epsilon}$.

Proof. Since $\sum_\alpha \{\eta^L(x_\alpha) - \eta^K(0)\} = 0$, we can add $\sum_\alpha g\prime(n/\bar{K}^d)\big[\eta^L(x_\alpha) - \eta^K(0)\big]$ to the exponential without changing the expectation in the statement of the lemma. By Hölder inequality we have

$$\log \mu_{K,n}\left[\exp\Big\{\beta \sum_\alpha L^d [g(\eta^L(x_\alpha)) - \tilde{g}_{K,L}(n/\bar{K}^d)]\Big\}\right] \\ \le\; \log \mu_{K,n}\Big[\exp\Big\{\beta R^d L^d [g(\eta^L(0)) - \tilde{g}_{K,L}(n/\bar{K}^d) - g\prime(n/\bar{K}^d)[\eta^L(0) - n/\bar{K}^d]\Big\}\Big] \;.$$

By the previous lemma and since $\beta R^d \le \beta \bar{L}^{1/4} \le L^{-8\epsilon}$, this expression is bounded above by $C(g)R^{2d}\beta^2 = C(g)(K/L)^{2d}\beta^2$. This concludes the proof of the lemma. $\square$

We conclude this section by proving that the canonical measure $\mu_{K,n}$ restricted to Λ_L has a bounded Radon–Nikodym derivative with respect to the grand canonical measure ν_ρ.

Lemma 11.7 . *There exists an universal constant C_0 such that*

$$\sup_{\xi \in \{0,1\}^{\Lambda_L}} \frac{\mu_{K,n}\Big[\{\eta;\ \eta(x) = \xi(x) \quad \mathit{for} \quad x \in \Lambda_L\}\Big]}{\nu_\rho\Big[\{\eta;\ \eta(x) = \xi(x) \quad \mathit{for} \quad x \in \Lambda_L\}\Big]} \ \le\ C_0$$

provided $\bar{L}^d \le \bar{K}^d/2$.

Proof. We may assume that $1 \le n \le \bar{K}^d - 1$, since for $n = 0$, $\bar{K}^d$ the result is trivial.

Fix ξ in $\{0,1\}^{\Lambda_L}$ and denote by $j = j(\xi)$ the total number of particles of ξ : $j = \sum_{x\in\Lambda_L} \xi(x)$. Denote by Λ_L^c the subset $\Lambda_K - \Lambda_L$ and set $M = \bar{K}^d$, $J = \bar{L}^d$. The ratio on the left hand side of the statement is equal to

$$\frac{\nu_\rho\Big[\sum_{x\in\Lambda_L^c} \eta(x) = n - j\Big]}{\nu_\rho\Big[\sum_{x\in\Lambda_K} \eta(x) = n\Big]} \ \le\ \frac{\nu_\rho\Big[\sum_{x\in\Lambda_L^c} \eta(x) = \rho(M-J)\Big]}{\nu_\rho\Big[\sum_{x\in\Lambda_K} \eta(x) = n\Big]} \ ,$$

since $\nu_\rho\Big[\sum_{x\in\Lambda_L^c} \eta(x) = m\Big]$ attains its maximum value at $m = \rho(M-J)$ provided that $\rho(M-J)$ is an integer. In this case for $4 \le n \le M-4$, Stirling's formula shows that there exists an universal constant C_0 such that the last expression is bounded above by

$$C_0 \, \frac{\sqrt{M}}{\sqrt{M-J}} \ .$$

Which proves the lemma since $J \le M/2$. A simple computation gives a similar bound in the case where $\rho(M-J)$ is not an integer.

The cases $1 \le n \le 4$ and $M-4 \le n \le M-1$ follows by inspection. $\square$

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