Dynamical ensembles equivalence in fluid mechanics*

G. Gallavotti
Fisica, Università di Roma La Sapienza, P.le Moro 2, 00185, Roma, Italia.

Abstract: Dissipative Euler and Navier Stokes equations are discussed with the aim of proposing several experiments apt to test the equivalence of dynamical ensembles and the chaotic hypothesis.

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§1 Reversible dissipation in Euler and Navier Stokes equations.

In [Ga6] one finds an analysis leading to a conjecture on the equivalence between the irreversible NS equation and a reversible equation that was called GNS ("gaussian NS equation").

The ideas discussed in [Ga6] are, however, much more general and it is worth pointing out some other applications, as well as several possible tests that seem under reach of present day experimental (numerical *and* "real") techniques.

I shall focus on fluid mechanics problems considering a fluid that:

(1) is enclosed in a periodic box Ω with side L, possibly with a few disks ("obstacles") removed so that no infinite straight path can be found in Ω that avoids the obstacles,

(2) is incompressible with density ρ.

I shall consider four distinct evolution equations for this fluid, all of dissipative nature.

$$\begin{aligned}
\dot{\underline u} + \underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline u &= -\frac{1}{\rho}\underline\partial p + \underline g + \nu\Delta\underline u, &\qquad \underline\partial\cdot\underline u = 0 &\qquad \text{NS}\\
\dot{\underline u} + \underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline u &= -\frac{1}{\rho}\underline\partial p + \underline g + \beta\Delta\underline u, &\qquad \underline\partial\cdot\underline u = 0 &\qquad \text{GNS}\\
\dot{\underline u} + \underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline u &= -\frac{1}{\rho}\underline\partial p + \underline g - \chi\,\underline u, &\qquad \underline\partial\cdot\underline u = 0 &\qquad \text{ED}\\
\dot{\underline u} + \underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline u &= -\frac{1}{\rho}\underline\partial p + \underline g - \alpha\,\underline u, &\qquad \underline\partial\cdot\underline u = 0 &\qquad \text{GED}
\end{aligned}\tag{1.1}$$

In the case Ω contains obstacles a "no friction" boundary condition will be imposed on $\partial\Omega$, i.e. $\underline u\cdot\underline n = 0$ if $\underline n$ is the normal to $\partial\Omega$. The first equation is the well known Navier Stokes equation with ν being the *viscosity*.

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* *mp_arc@math.utexas.edu*, #96-???, and *chao-dyn@xyz.lanl.gov*,#960????

The second equation, introduced in [Ga6] and called GNS, has a multiplier β defined so that the total vorticity $\eta L^3 = \rho \int \underline{\omega}^{\,2}\, dx$, with $\underline{\omega} = \partial \wedge \underline{u}$ being the *vorticity*, is a constant of motion; this means that:

$$\beta(\,\underline{u}\,) = \frac{\int \left(\underline{\partial} \wedge \underline{g} \cdot \underline{\omega} + \underline{\omega} \cdot (\underline{\omega} \cdot \underline{\partial}\ \underline{u}) \right) d\,\underline{x}}{\int \underline{\omega}^{\,2}\, d\,\underline{x}} \tag{1.2}$$

The third equation will be called the *Euler dissipative* equation, ED: it represents a non viscous ideal fluid moving in a "sticky background": χ is a "sticky" viscosity. The model is, as far as I know, not a good model for any physical situation, but it is interesting to consider it mainly for comparison purposes.

The fourth equation will be called GED equation, *gaussian dissipative Euler* equation and here α is a multiplier defined so that the total (kinetic) energy $\varepsilon L^3 = \frac{\rho}{2} \int \underline{u}^{\,2}\, dx$ is a constant of motion *in spite* of the action of the force $\underline{g}$; this means that α is given by:

$$\alpha(\,\underline{u}\,) = \frac{\int \underline{g} \cdot \underline{u}}{\int \underline{u}^{\,2}} \tag{1.3}$$

A similar equation, with a constraint on the energy contained in each "momentum shell" to be constant was considered in [ZJ], which is the first paper in which the idea of a reversible Navier Stokes equation is advanced and studied. The energy content of each "momentum shell" was fixed to be the value predicted by Kolmogorov theory, [LL].

Note that both the GED and the GNS equations have a symmetry in $\underline{u}$, so that they are *reversible* in the sense that, if V_t is the flow describing the equation solution (so that $t \to V_t\, \underline{u} = \underline{u}(t)$ is the solution with initial data $\underline{u}$), then the transformation $i : \underline{u} \to -\underline{u}$ *anti-commutes* with the time evolution V_t:

$$i\, V_t = V_{-t}\, i \tag{1.4}$$

We shall avoid (as it is, unfortunately, always the case in the current literature) considering the problem of proving the global existence of solution to the equations (1.1) (the problem is in fact *open*, see [Ga1]) and we shall consider the *truncated equations* with momentum cut off K.

The truncation will be performed on a suitable orthonormal basis for the *divergenceless* fields in Ω: given the boundary conditions we consider it natural to use the basis generated by the *minimax* principle applied to the Dirichlet quadratic form $\int_\Omega (\underline{\partial}\ \underline{u})^2 d\,\underline{x}$ defined on the space of the $C^\infty(\Omega)$ divergenceless fields $\underline{u}$ with $\underline{u} \cdot \underline{n} = 0$ on $\partial\Omega$. The basis fields $\underline{u}_j$ will verify: $\Delta\, \underline{u}_j = -E_j\, \underline{u}_j + \underline{\partial}_j \mu$, for a suitable multiplier μ_j, with $\underline{u}_j, \mu_j \in C^\infty$ and E_j are eigenvalues).

For instance in the case of *no obstacles* let the $\underline{u} = \sum_{\underline{k} \neq \underline{0}} \underline{\gamma}_{\,\underline{k}}\, e^{i\,\underline{k}\cdot\underline{x}}$ be the velocity field represented in Fourier series with $\underline{\gamma}_{\,\underline{k}} = \overline{\underline{\gamma}}_{-\underline{k}}$ and $\underline{k} \cdot \underline{\gamma}_{\,\underline{k}} = 0$ (incompressibility condition); here $\underline{k}$ has components that are integer multiples of the "lowest momentum" $k_0 = \frac{2\pi}{L}$. Then consider the equation:

$$\dot{\underline{\gamma}}_{\underline{k}} = -\vartheta(\underline{k})\underline{\gamma}_{\underline{k}} - i \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} (\underline{\gamma}_{\underline{k}_1} \cdot \underline{k}_2)\,\Pi_{\underline{k}}\,\underline{\gamma}_{\underline{k}_2} + \underline{g}_{\underline{k}} \tag{1.5}$$

where the $\underline{k}$'s take only the values $0 < |\underline{k}| < K$ for some *momentum cut–off* $K > 0$ and $\Pi_{\underline{k}}$ is the projection on the plane orthogonal to $\underline{k}$. This is an equation that defines a "truncation on the momentum sphere with radius K of the equations (1.1)" if:

$$\begin{aligned} \vartheta(\underline{k}) &= -\nu\,\underline{k}^2 && \text{NS case}\\ \vartheta(\underline{k}) &= -\beta\,\underline{k}^2 && \text{GNS case}\\ \vartheta(\underline{k}) &= -\chi && \text{ED case}\\ \vartheta(\underline{k}) &= -\alpha && \text{GED case} \end{aligned} \tag{1.6}$$

For simplicity we may suppose, in this no obstacles cases, that the mode $\underline{k} = \underline{0}$ is *absent, i.e.* $\underline{\gamma}_{\underline{0}} = \underline{0}$: this can be done if, as we suppose, the external force $\underline{g}$ does not have a zero mode component (*i.e.* if it has zero average).

In order that the resulting cut–off equations be physically acceptable, and supposing that $\underline{g}_{\underline{k}} \neq \underline{0}$ only for $|\underline{k}| \sim k_0$, one shall have to fix K large. For instance in the NS case it should be much larger than the *Kolmogorov scale* $K = (\eta\nu^{-2})^{1/4}$, where $\nu\eta$ is the average dissipation of the solutions to (1.6) with $K = +\infty$ (determined on the basis of heuristic dimensional considerations by $\eta \sim |\underline{g}|^2 L^2 \nu^{-2}$): see [LL].

We shall use the same cut off for the other equations with $\underline{k}$ replaced by the basis label j and $|\underline{k}|$ replaced by $\sqrt{E_j}$, which is certainly a natural choice for the GNS equation.

For the ED and GNE equations the choice of K should be made by developing a theory analogous to Kolmogorov's theory. We only attempt a preliminary analysis in §6 as the latter equations have no physical interpretation (of which I am aware) and they are used here only for the purpose of illustrating some interesting mechanisms and theories. Below we always refer to the truncated equations, unless otherwise stated.

It is easy, in the no obstacles cases, to express the coefficients α, β for the cut off equations:

$$\begin{aligned} \alpha &= \frac{\sum_{0<|\underline{k}|<K} \overline{\underline{g}}_{\underline{k}} \cdot \underline{\gamma}_{\underline{k}}}{\sum_{0<|\underline{k}|<K} \underline{\gamma}_{\underline{k}}^2} \qquad \beta = \beta_i + \beta_e \\ \beta_e &= \frac{\sum_{\underline{k}\neq\underline{0}} \underline{k}^2\, \underline{g}_{\underline{k}} \cdot \overline{\underline{\gamma}}_{\underline{k}}}{\sum_{\underline{k}} \underline{k}^2 |\underline{\gamma}_{\underline{k}}|^2} \\ \beta_i &= \frac{-i \sum_{\underline{k}_1+\underline{k}_2+\underline{k}_3=\underline{0}} \underline{k}_3^2 (\underline{\gamma}_{\underline{k}_1} \cdot \underline{k}_2)(\underline{\gamma}_{\underline{k}_2} \cdot \underline{\gamma}_{\underline{k}_3})}{\sum_{\underline{k}} \underline{k}^2 |\underline{\gamma}_{\underline{k}}|^2} \end{aligned} \tag{1.7}$$

where the $\underline{k}$'s take only the values $0 < |\underline{k}| < K$ for some *momentum cut–off* $K > 0$ and $\Pi_{\underline{k}}$ is the orthogonal projection on the plane perpendicular to $\underline{k}$.

The cases in which the region Ω contains obstacles is very similar but we cannot write simple expressions for the basis fields and therefore the equations, although formally very similar to (1.5), (1.7) cannot be written very explicitly.
The solutions of the equations (1.5), or to the corresponding ones in the obstacles cases, will be denoted $V_t^{\nu,ns}\,\underline{u}\,, V_t^{\eta,gns}\,\underline{u}\,, V_t^{\chi,ed}\,\underline{u}\,, V_t^{\varepsilon,ged}\,\underline{u}$ when the initial datum is $\underline{u}$. Or in general:

$$V_t^{\xi}\,\underline{u}\,,\quad \xi=(\nu,ns),\,(\eta,gns),\,(\chi,ed),\,(\varepsilon,ged) \tag{1.8}$$

Keeping the forcing $\underline{g}$ constant we shall admit that for each equation, *i.e.* for each choice of ξ, there is a unique stationary distribution μ_ξ describing the statistics of all initial data $\underline{u}$ that are randomly chosen with a "Liouville distribution", *i.e.* (in the no obstacle cases, to fix the ideas) with a distribution $\mu_0(d\underline{\gamma})$ proportional to the volume measure $\delta_\xi\,\prod_{|\,\underline{k}\,|<K} d\underline{\gamma}_{\,\underline{k}}$, where the delta function is present only in the case of the reversible equations and fixes the constants of motion to the value prescribed by the first label in ξ.
This means that given "any observable" F on the phase space $\mathcal{F}$ (of the velocity fields with momentum cut–off K) it is:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F(V_t^{\xi}\,\underline{\gamma})dt=\int_{\mathcal{F}}F(\underline{\gamma}\,')\mu_\xi(d\underline{\gamma}\,')\stackrel{def}{=}\langle F\rangle_\xi \tag{1.9}$$

for all choices of $\underline{\gamma}$ except a set of zero Liouville measure. The distribution μ_{ns} will be called the SRB distribution for the Eq. (1.5),(1.6) (see the "zero-th law" in [UF],[GC2]).
A particular role will be plaid by the averages $\langle\varepsilon\rangle_\xi,\langle\eta\rangle_\xi$ as well as by the averages of $\langle\alpha\rangle_\xi,\langle\beta\rangle_\xi$ and of the *entropy production rate* $\sigma(\underline{\gamma})$ that is *defined by the divergence of the r.h.s. of the cut off equations.*
Consider explicitly only the no obstacles case: if D_K is the number of modes $\underline{k}$ with $0<|\,\underline{k}\,|<K$ then the number of (independent) components of $\{\underline{\gamma}_{\,\underline{k}}\}$ is $2D_K$ and, see (1.5), setting $2\overline{D}_K=\sum_{|\,\underline{k}\,|<K}2\,\underline{k}^{\,2}$ (which in the case with obstacles become $2\overline{D}_K=\sum_{\sqrt{E_j}<K}\sqrt{E_j})$), one finds that σ is given by:

$$\begin{aligned}
\sigma=&2\overline{D}_K\nu && \xi=(\nu,ns)\\
\sigma=&2\overline{D}_K\beta-\overline{\beta}_e-\overline{\beta}_i && \xi=(\eta,gns)\\
\sigma=&2D_K\chi && \xi=(\chi,ed)\\
\sigma=&2D_K\alpha-\alpha && \xi=(\varepsilon,ged)
\end{aligned} \tag{1.10}$$

where $\overline{\beta}_i,\overline{\beta}_e$ are suitably defined, *e.g.* in the no obstacles cases:

$$\overline{\beta}_e=\frac{\sum_{\underline{k}}\,\underline{k}^{\,4}\overline{\underline{g}}_{\,\underline{k}}\cdot\underline{\gamma}_{\,\underline{k}}}{\sum_{\underline{k}}\,\underline{k}^{\,2}|\underline{\gamma}_{\,\underline{k}}|^2}-2\frac{(\sum_{\underline{k}}\,\underline{k}^{\,2}\overline{\underline{g}}_{\,\underline{k}}\cdot\underline{\gamma}_{\,\underline{k}})(\sum_{\underline{k}}\,\underline{k}^{\,4}\underline{\gamma}_{\,\underline{k}}^{\,2})}{(\sum_{\underline{k}}\,\underline{k}^{\,2}|\underline{\gamma}_{\,\underline{k}}|^2)^2} \tag{1.11}$$

so that $\sigma\simeq2\overline{D}_K\beta$ for $\xi=(\eta,gns)$ and $\sigma\simeq2D_K\alpha$ for $\xi=(\varepsilon,ged)$.

The *equivalence of dynamical ensembles conjecture*, [Ga6], is the following:[1]

Conjecture NS: The statistics $\mu_{\nu,ns},\mu_{\eta,gns}$ of the NS equations and of the GNS equations respectively are equivalent in the limit of large Reynolds number provided the parameters η and ν are so related that $\langle\sigma\rangle_{\nu,ns}=\langle\sigma\rangle_{\eta,gns}$.

Here *equivalent* means that the ratios of the averages of the same observables with respect to the two distributions approaches 1 as $R\to\infty$. The Reynolds number is defined here to be $R=\varepsilon^{1/3}L^{4/3}\nu^{-1}$.

A corresponding conjecture can be formulated for the ED and GED equations:

Conjecture ED: The statistics $\mu_{\chi,ed},\mu_{\varepsilon,ged}$ of the ED equations and of the GED equations respectively are equivalent in the limit of large Reynolds number provided the parameters ε and χ are so related that $\langle\sigma\rangle_{\chi,ed}=\langle\sigma\rangle_{\varepsilon,ged}$.

The above stated conjectures are closely analogous to the familiar statements on the equivalence of thermodynamic ensembles, with the thermodynamic limit replaced by the limit $R\to\infty$ of infinite Reynolds number. They can be substantially weakened for the purposes of possible applications.

The idea of non equilibrium ensembles and their possible equivalence is not really new: the recent literature contains many hints in such direction. The clearest is perhaps [ZJ]. See also [Ga2] (§4) and [MR].

On heuristic grounds, the conjectures would be justified if one did accept that the entropy creation rate reaches its average on a time scale that is fast compared to the hydrodynamical scales. The coefficients $\alpha\simeq(2D_K)^{-1}\sigma$, and $\beta\simeq(2\overline{D}_K)^{-1}\sigma$, see (1.10), would be confused with their time averages $\langle\alpha\rangle_{\varepsilon,gne}$ or $\langle\beta\rangle_{\eta,gns}$ and *identified with the viscosity* constants ν or χ.

In this way the GNS and the NS equations would be equivalently good: both being the macroscopic manifestation of two *equivalent microscopic dissipation mechanisms*: one explicitly specified by the Gaussian constraint of constant dissipation and the other with dissipation *unspecified a priori* but phenomenologically modeled by a constant viscosity. Likewise one can view the GED and the ED equations as macroscopically equivalent: one with constant energy and the other with constant sticky viscosity χ.

The interest of the above conjectures is that the same physical system in which irreversible dissipation occurs (the NS or ED equations) can be described equivalently by a reversibly dissipative system (the GNS or GED equations).

For reversible systems a general principle, *the chaotic hypothesis*, can be reasonably assumed to hold and to imply consequences that seem to be non trivial, see [GC1], [GC2], [Ga3], [Ga4],[Ga5].

The next section is devoted to a quick discussion of some of the established consequences of the principle and §3%§6 will deal with *heuristic* ideas and with describing a possible *scenario* for the phenomenology of the equations (1.1). The scenario will be developed *without any pretension of rigor* and it

[1] ... è tanto nuova e, nella prima apprensione, remota dal verisimile, che quando non si avesse modo di dilucidarla e renderla più chiara che'l Sole meglio sarebbe il tacerla che'l pronunziarla; però, già che me la son lasciata scappare di bocca..., [G], p. 231.

will present what will appear as the *simplest* among many other possibilities. It leads (implicitly) to several possible experimental tests of the chaotic hypothesis and of the other ideas involved in its development: the tests can also be viewed, independently, just as interesting experiments proposals.

In the following we shall always consider the NS equations and the GNS equations with parameters fixed so that $\mu_{\nu,ns}$ and $\mu_{\eta,gns}$ are equivalent by the conjecture NS, and likewise we shall always consider the ED and GED equations with parameters fixed so that $\mu_{\chi,ed}$ and $\mu_{\varepsilon,ged}$ are equivalent by the conjecture ED.

§2 The fluctuation theorems.

In reference [GC1],[GC2] the *chaotic hypothesis* was presented as a reformulation of an older principle due to Ruelle, [R1]. It gave us the possibility of some quantitative parameterless "predictions", in various cases, see also [Ga3], [Ga4], [Ga5]. The hypothesis is:

Chaotic hypothesis: A chaotic many particle system or fluid in a stationary state can be regarded, for the purpose of computing macroscopic properties, as a smooth dynamical system with a transitive Axiom A global attractor. In reversible systems it can be regarded, for the same purposes, as a smooth transitive Anosov system.

The main result of [GC1] is the *fluctuation theorem* that gives a property of the variable $p = p(\underline{\gamma})$ defined in terms of the contraction rate σ_0 of the attractor surface elements by:

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \sigma_0(V_t\,\underline{\gamma}\,)\,dt = \langle\sigma_0\rangle_+ p \tag{2.1}$$

which can be regarded as a random variable with the distribution $\pi_\tau(p)dp$ that it inherits from $\mu_{\eta,gns}$ or $\mu_{\varepsilon,ged}$; here $\langle\sigma_0\rangle_+$ is the average of σ_0 with respect to the distribution $\mu_{\eta,gns}$ or $\mu_{\varepsilon,ged}$.

Note that σ_0 should *not be confused* with σ, (1.8): thinking of the attractor as a smooth surface σ_0 is the contraction rate of its surface elements, *which is different* from the contraction rate σ of the phase space volume elements, see §4%§6.

If the conjectures of §1 are accepted $\langle\sigma_0\rangle_+$ is also the $\mu_{\nu,ns}$ or $\mu_{\chi,ed}$ average of σ_0 or at least tends to it as $R\to\infty$.

If $\langle\sigma_0\rangle_+ > 0$, see [R2] for a discussion of the conditions for this inequality ("Ruelle's H-theorem"), and if $\zeta(p) = \lim_{\tau\to\infty}\frac{1}{\tau}\log\pi_\tau(p)$ then the *fluctuation theorem* of [GC1] gives the following *large deviation* relation, see also [Ga3], [Ge], for the equations GNS and GED:

$$\frac{\zeta(p)-\zeta(-p)}{\langle\sigma_0\rangle_+ p} = 1, \qquad \text{for all } p \tag{2.2}$$

which, in the case of nonequilibrium statistical mechanics, has been interpreted as an *extension of the fluctuation dissipation theorem* to large forcing fields,

[Ga5]. Here "for all" p means for all possible values of p (which is in general a bounded quantity).

The fluctuation theorem (2.2) says that the distribution of p is *multifractal*, not surprisingly since $\zeta(p)$ can be regarded as a kind of *generalized sum of Lyapunov exponents* in the sense of [FP], [BJPV], and the *odd part* of $\zeta(p)$ is linear.

A more general fluctuation theorem concerns the *joint* distribution of the variable p and of any other variable $q=q(\underline{\gamma})$ that is similarly defined in terms of an observable Q which is *odd* under the time reversal operation that is defined on the attractor, *i.e.* :

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt\, Q(V^\xi\,\underline{\gamma}) = \langle Q\rangle_+\, q \tag{2.3}$$

If $\pi_\tau(p,q)$ denotes the joint probability density of the observables p,q and if $\zeta(p,q)=\lim_{\tau\to\infty}\frac{1}{\tau}\log\pi_\tau(p,q)$ then it follows from the chaotic hypothesis that the distributions for p,q, with respect to the statistics μ_ξ, $\xi=(\beta,gns)$ or $\xi=(\alpha,ged)$, verify:

$$\frac{\zeta(p,q)-\zeta(-p,-q)}{\langle\sigma_0\rangle_+ p}=1, \qquad \text{for all } p,q \tag{2.4}$$

which, in the case of nonequilibrium statistical mechanics, has been interpreted as an *extension of the Onsager's reciprocity* to large forcing fields, [Ga5].

In the case of the equations (1.1), second and fourth, the above relations are applicable when the motions are chaotic, *i.e.* have at least one positive Lyapunov exponent (which should happen as soon as R is large enough, excepted possibly very special cases, see [Ma]); and it gives an interesting parameterless prediction *if* the contraction rate σ_0 can be related to the contraction rate σ of the GNS equations.

If the conjecture in §1 holds in the strong sense of complete asymptotic equivalence between the "ensembles" $\mu_{\eta,gns}$ and $\mu_{\nu,ns}$ (or $\mu_{\varepsilon,ged}$ and $\mu_{\chi,ed}$) then (2.2) *should also hold for other models* of the viscous stationary states, like the one given by the *classical NS equation in particular*: note the quantities σ in (1.10) are *still* fluctuating variables even when the evolution considered is given by the NS or ED equation. A check of this property might be under reach of present day experimental techniques.

A check of (2.2) or (2.4) might be more accessible in the case of fluid systems because they often show chaotic motions with relatively few degrees of freedom so that the large fluctuations, that must occur in order to make possible testing the fluctuation theorems, are more likely occur and be observable.

But this also means that the attractors have a very small size, compared to that of phase space: hence a problem in the interpretation of (2.2) is the fact that the quantity σ_0 that appears in the theorem does coincide with the easily determined (see (1.10)) contraction of volume in the phase space *only if* the attractor is dense in phase space.

This is a property that can be expected in the case of nonequilibrium statistical mechanics under small or moderately strong forcing, see [BGG] for a discussion of this point, but it cannot be expected at large forcing or in fluid systems. In the latter case one should use the contraction rate of the volume elements on the attractor, see [BGG], [BG]. In the GNS systems it is likely that the property never really holds.

This might render (2.2) quite useless as it is usually unrealistic to hope to determine the attractor equation with accuracy sufficient to compute its area contraction per unit time (assuming, as the chaotic hypothesis implies, that the attractor can be regarded as a smooth surface).

Nevertheless in nonequilibrium statistical mechanics further analysis is possible, based on an important symmetry of the Lyapunov exponents spectrum, and one can relate easily the area contraction on the surface defined by the attractor and the total phase space volume contraction. Hence one can try to push further the analysis of [BGG], [BG] to see if one can say something also in the case of the GNS equations.

§3 A scenario for experimental checks of the fluctuation theorem. The Lyapunov spectrum of ED and GED equations. Pairing rule and Axiom C.

What follows is a very *heuristic analysis* aimed at giving an argument for the explicit form that the fluctuation theorem (2.2) will take in the case of the GED equations: and therefore by the equivalence conjecture *also* for the ED equations. No pretention of mathematical rigor is present and the idea is to illustrate the *simplest possible scenario* that I consider possible. The interest is (apart from the subjective feeling of a certain beauty) that the discussion suggests experiments and checks that have intrinsic interest and that do not seem to have yet been considered in the literature.

We consider first the case of (1.1) *in a domain Ω with obstacles*: in spite of the appearances this is an easier case because in this case we can imagine forcing the system with a *locally conservative force* which is *not* globally conservative, like a field roughly parallel to one axis and tangent to the obstacles (one can imagine a uniformly charged fluid under an electromotive constant electric field).

Note that in order to have a non trivial forcing the forcing field must be non globally conservative: otherwise its effect would be just that of altering the pressure.

The Euler equations can, in general, be regarded as *hamiltonian equations* for a system whose configurations are the diffeomorphisms of the box Ω (in our case a torus with, possibly, a few holes) containing the fluid: they are not directly in hamiltonian form in the same sense as the (closely analogous) Euler equations for a rigid body with a fixed point are not immediately hamiltonian (*e.g.* they involve half the number of actual equations of motion).

In this way the GNS or GED equations can be regarded as hamiltonian equations (approximately so, because the cut–off K destroys this property) *modified* by the action of a non conservative force $\underline{g}$ and by the gaussian constraint that the total vorticity or the total energy are constants.

Of course we exploit the "slip" (*i.e.* no friction) boundary conditions in order to be able to conclude the hamiltonian nature of the Euler equations.

The phase space will then consist of a space larger than the above $\mathcal{F}$, see (1.9): its points $(\underline{u}, \underline{\delta})$ will be (cut–off) *velocity fields* and (cut–off) *displacement fields* describing the positions of the fluid particles with respect to a reference configuration. We call this the *"full phase space"* of the equations (1.1).

The equations for the displacements will be in all models (1.1):

$$\dot{\underline{\delta}} = \underline{u}(\underline{\delta}, t), \qquad \underline{\delta}(\underline{x}, 0) = \underline{\delta}_0(\underline{x}) \tag{3.1}$$

which, once $\underline{u}$ is known from (1.3), (1.2), permit us to compute the physical fluid flow and the positions $\underline{\delta}(\underline{x}, t)$ of the fluid particles that at time 0 were at the points $\underline{\delta}_0(\underline{x})$, away from the reference configuration position $\underline{x} \in \Omega$.

In the case in which the $\underline{u}$ verify truncated equations also (3.1) have to be truncated, for instance by replacing each $e^{i\underline{k}\cdot\underline{\delta}(\underline{x})}$ in $\underline{u}(\underline{\delta}, t)$ by its truncated Fourier expansion.

The system motions (describing velocity and displacement fields) can be regarded as motions with $2D_K$ degrees of freedom where, for instance in the no obstacles case, D_K is the number of non zero modes $\underline{k}$ with $0 < |\underline{k}| < K$ (because each $\underline{\gamma}_{\underline{k}}$ has two complex components but $\underline{\gamma}_{-\underline{k}} = \overline{\underline{\gamma}}_{\underline{k}}$). This means that $4D_K$ coordinates are necessary to describe the motion.

Hence there are $4D_K$ Lyapunov exponents, $2D_K$ from the velocity equations (1.2) and $2D_K$ from the displacements equations (3.1).

In view of the equivalence conjectures we study the equations GNS and GED when convenient and the NS or ED when convenient.

Out of the $4D_K$ exponents one has to extract, in the GNS or GED cases, one exponent that is trivially 0 because of the conservation of the dissipation rate and one exponent that is trivially zero and corresponds to the vector field given by the r.h.s. of the GNS equation. Furthermore in the GNS or GED cases two more vanishing Lyapunov exponents are associated with other constants of motion.[1]

The other $2N = 4D_K - 4$ exponents, or in the ED, NS cases all the $2N = 4D_K$ exponents, can be ordered in two groups the first containing the first N exponents in decreasing order and the second the remaining N ones in increasing order.

The exponents of the first group are denoted λ_j^+, $j = 1, \ldots, N$ and the ones in the second group are denoted λ_j^-, $j = 1, \ldots, N$. We call the two exponents $(\lambda_j^+, \lambda_j^-)$ a *pair.*

We consider first in detail the ED and GED equations. In the above context it seems reasonable that in the full phase space of the GED and ED equations a *pairing rule* holds:

$$\frac{\lambda_j^+ + \lambda_j^-}{2} = const \tag{3.2}$$

[1] In the case of the GNS equations the *helicity* $\int \underline{\omega} \cdot \underline{u}\, dx$ is a constant of the motion and such is $\int \underline{\delta}^2 dx$ for the displacement equations.

at least when the forcing is locally conservative as we suppose from now on unless otherwise stated. The value of the constant will be called the "*pairing level*" or "*pairing constant*", which must be $\frac{1}{2}\langle\sigma\rangle_+$, see (1.10).

The pairing rule, in fact, formally holds in the present ED case. One can easily adapt the proof in [Dr]: this is discussed in the Appendix A1.

The rule then holds also for GED as a consequence of the conjecture ED. A direct proof can probably be made along the lines of the work [DM]. In fact the constraint imposed by the definition of the multiplier α, (1.7), is a constraint of the type called *isokinetic* in [DM] and their proof seems to apply "without change", although I did not check the details (the appendix A1 should give the background for such an analysis).

In the cases in which (3.2) has been proved, [DM], [Dr], it holds also in a far *stronger* sense: the *local Lyapunov exponents*,[2] of which the Lyapunov exponents are the averages, are paired as in (3.2) to a constant that is j independent but, of course, is dependent on the point in phase space. We call this the *strong pairing rule.* See the final comments.

Note that the Lyapunov exponents of the full system can also be easily divided into *velocity exponents, i.e.* the ones of the GED or ED equations, and the *displacement exponents, i.e.* the others (which cannot be measured from the GED or ED evolution alone but require also (3.1)). In fact if we denote symbolically by (x,y) the pair $(\underline{u}\,,\,\underline{\delta})$ then the jacobian matrix of the equations is described by a matrix having the form $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ where A,B,C are operators.

For a further classification of the exponents we shall think that the Lyapunov exponents are divided into three classes that we call *viscous*, *inertial* and *slow*. The following scenario will be again summarized and enriched in the figure in §4.

(1) The slow exponents ("*slow pairs*") consist of M pairs of exponents the largest of which is ≤ 0 and it is a velocity exponent corresponding to slow motions of the velocity field, while the other (necessarily < 0) exponent of the pair is a displacement exponent and corresponds to a fast approach to the stationary state of some of the displacement variables.

(2) The viscous exponents ("*viscous pairs*") consist of V negative velocity exponents describing the fast approach to the stationary state of the viscous degrees of freedom of the velocity field: their paired positive exponents are displacement exponents associated with chaotic motions of the displacement variables.

(3) The remaining $2P = N - M - V$ pairs ("*inertial pairs*") have one > 0 and one < 0 Lyapunov exponents: P of the *pairs* are pairs of velocity exponents and the P *other* pairs are displacement exponents. The P pairs of velocity exponents are the only pairs of exponents of the equations for the velocity field that contain one positive and one negative element: they describe the

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[2] *i.e.* the logarithms of the eigenvalues of $\sqrt{J_t^T J_t}$, if J_t is the local jacobian matrix of the evolution operator V_t, other than those relative to the directions of the flow or of the imposed constraints or of other constants of motion

gross characteristics of the chaotic motion on the attractor. It follows that the three types of exponents can in principle be *uniquely identified* among the N exponents of the velocity field equations, see also below.
The existence of a certain number denoted P above of pairs of exponents, for the velocity field evolution, that are pairs of exponents of opposite sign does not follow simply from the fact that we are collecting togheter pairs containing a >0 exponent. In principle the >0 exponents of the velocity field could be paired with negative displacement exponents. We think that it is natural that the >0 Lyapunov exponents for the velocity field are paired with <0 exponents of the velocity field because we associate such pairs with the motions on the attractor. Since the GED equations are reversible it follows from [BG] that *if the motions are also supposed to verify a geometric property called in* [BG] *Axiom C property* (a simple extension of the paradigm of turbulent behavior, see [R1], that is the Axiom A property) *then there must be an equal number P of positive and negative exponents* for the restriction of the GED equations to their attractor. It seems therefore natural to think that they form P pairs.
The equality of the number of >0 and <0 exponents for the motion on the attractor for the velocity fields is due to the existence, in reversible Axiom C systems, of a *local time reversal* map i^* that transforms the attractor into itself anti-commuting with the time evolution, even when (and this is the rule in fluid dynamics) the attractor itself is *not* time reversal invariant: see [BG]. We proceed under the assumption that the Axiom C property is verified: for a complete discussion of the property we must unfortunately refer to [BG].
In Axiom C systems the time reversal symmetry "cannot be lost": when it is spontaneously broken (because the attractor is not time reversal invariant) it is replaced by a "weaker" symmetry, good enough to make "effectively reversible" all the motions on the attractor, a relation similar to the one in fundamental Physics between T and TCP (the latter being the "real" time reversal as the first is not a symmetry of the world we see).
(4) The other P pairs should consist of displacement exponents exhibiting a rather symmetric behavior with respect to that of the GED exponents. Below we are going to suggest that very similar properties hold for the NS and GNS equations: in that case this further appealing symmetry seems compatible with (and in fact it was suggested by) the data on the velocity Lyapunov spectrum for models ("GOY shell models") whose behavior is "believed" to be related to NS equations: see [BJPV], figure in p. 71, taking into account that the pairing level in such data is very small because the viscosity is very small.
In the above scenario the existence of the other P pairs of displacement exponents is assumed in order to make the total count of the number of exponents correct and is not based on evidence of any other kind. The displacement exponents do not seem to have been considered in the literature, at least not in conjunction with the velocity field exponents (not surprisingly in view of the difficulty of the measurements).

Thus if we measure the Lyapunov exponents for the GED equations alone we expect to find P pairs of opposite sign exponents paired at the value $\frac{1}{2N}\langle\sigma\rangle_+$ for some $P \le N$.

It seems reasonable that the P pairs of displacement exponents *coincide* with the P pairs of inertial exponents for the velocity field equations: but this is not really necessary in order that an unambiguous identification of the three type of exponents be possible. They are already identified by the above properties.

However if the P pairs of velocity inertial exponents and the P pairs of displacement inertial exponents do coincide we see that, by the pairing rule, the knowledge of the Lyapunov spectrum for the velocity equations *implies that all the displacement exponents* are knwon as well: no need to compute them.

§4 Predictions of the fluctuation theorem for ED and GED.

With the scenario developed in §3 we reconsider the fluctuation theorem and note that it is easy to check, by evaluating the divergence of the r.h.s. of the equations (1.2), (3.1), that the volume in the full ($2N$ dimensional) phase space contracts at the *same* rate σ at which the volume in velocity space does. Fluid incompressibility, and absence of displacement variables in the equations for the velocity field, imply this property.

Furthermore if *the strong pairing rule* is assumed the total volume contraction in the full phase space, including the displacement variables, will be $\sigma(\underline{\gamma})$ and *it will be related* to the contraction $\sigma_0(\underline{\gamma})$ of the area on the attractor surface by $\sigma_0(\underline{\gamma}) = \frac{2P}{2N}\sigma(\underline{\gamma})$, see [BGG] where the same mechanism was first exploited. This gives proportionality between the "apparent" contraction rate σ and the "true" contraction rate σ_0 on the attractor for the GNS equations.

As discussed at the end of §2 *the fluctuation theorem holds for the fluctuations of* σ_0 so that the fluctuations of σ will verify (2.2) *but with a r.h.s* in which 1 is replaced by $\frac{P}{N}$ where P is the number of pairs of Lyapunov exponents for the GED equations with one positive element.

If the number of degrees of freedom is increased by increasing K one should expect, therefore, that the constant $\frac{P}{N}\langle\sigma\rangle_+$ approaches $P\langle\alpha\rangle_+$ because the number of "true exponents" (*i.e.* inertial exponents) *should not change* as soon as K is so large that the motion is well described by the truncated equations: in fact it is believed that the attractor dimension does not depend on the truncation scale K (as long as it is large enough).

Since the conjecture of §2 implies $\langle\alpha\rangle_+ = \chi$, the constant *should approach* $P\chi$, at least if R (*i.e.* the Reynolds number) is large. The fluctuation theorem will thus take the form:

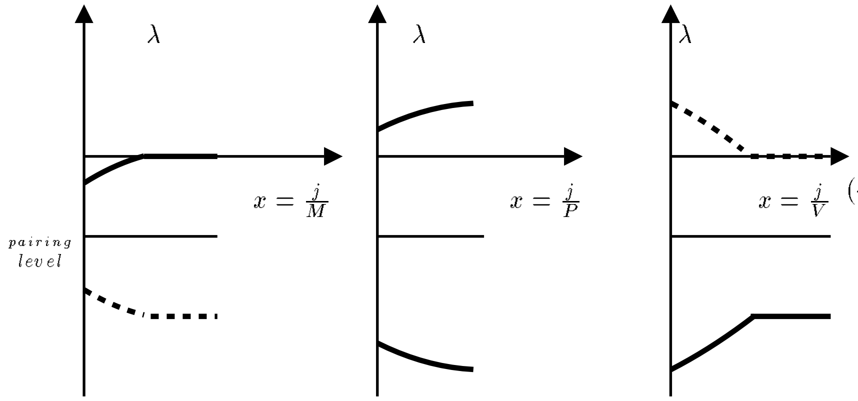
$$\frac{\zeta(p) - \zeta(-p)}{p} = P\,\chi \tag{4.1}$$

if the variable p is defined as in (2.1) *but with the measurable* σ replacing the *a priori* difficult to measure σ_0, and $\zeta(p)$ is the limit of $\tau^{-1}\log\pi_\tau(p)$ with $\pi_\tau(p)$ being the $\mu_{\varepsilon,ged}$ distribution of p. The number P is accessible by measurements performed *only* on the GED equations and not involving the displacement variables (being the number of positive Lyapunov exponents of the GED equations).

The above analysis is *somewhat conjectural* but experiments, at least numerical ones, are possible to check the picture; *e.g.* one could attempt at:

(1) checking the just derived slope $P\chi$ or

(2) checking the following picture, representing the above classification of the exponents:



(4.2)

The continuous line in the first graph gives the value (≤ 0) of the j–th (among M) slow Lyapunov exponent (as a function of $\frac{j}{M}$) of the GED equations; the dashed line is the graph of the paired exponents (of the displacement equations) and the intermediate line is the pairing constant. The exponents are defined only for $x = \frac{j}{M}$ but the graphs give, instead, continuous (or dashed) lines for visual aid.

The second graph gives the values of the j-th (among P) pair of inertial exponents of the GED equation (one positive and one negative per pair): here too we use the continuous curves even though the number of such exponents will ususally be much smaller than the total number and therefore a discrete representation would be more appropriate.

The negative curve in the third graph is the graph of the j-th viscous exponent (out of V) of the GED equation; the corresponding positive curve (dashed) is the curve of the companion exponents which correspond to displacement exponents. A fourth graph giving the other P displacement exponents (in pairs of one negative and one positive) would be qualitatively equal to the second graph (with the curves dashed for consistence of notation).

The graphs *are not experimental data*: they are just sketches illustrating the "simplest" picture that I considered reasonably possible. They should be taken as a conjecture, and they suggest performing experimental evaluation of the exponents for a check of the ideas of the present paper.

What do we imagine to happen when the equations are changed by enlarging the cut off (in the velocity as well as in the displacement variables) or by changing the forcing? Suppose that the cut off is already so large that adding one extra mode does not really affect the qualitative and quantitative features of the motion. Then adding one mode, *i.e.* increasing the total dimension of the system by 2, should add one pair of viscous exponents at the end of the spectrum respectively equal to 0 and $-\chi$, as drawn in the third graph in the figure. While changing the forcing should, *from time to time* as the forcing changes, change the category of some exponents. Namely the simplest picture would be that one of the vanishing slow GED exponents "becomes" positive and one of the viscous "becomes" inertial and paired with it; a symmetric evolution should take place with the displacement exponents. Or vice-versa. The attractor changes dimension by 2 units, see [BG],[BGG], at each of such events.

In order that the latter picture be possible one needs that *at a transition* the viscous spectrum bottom consists of a pair of a > 0 displacement exponent and a negative viscous exponent coinciding with the inertial exponents top pair; and at the same should happen for the bottom pair of the inertial and the top pair of the slow spectra.

The case of periodic boundary conditions does not fit in the above analysis of the pairing rule because on the torus there is no way of forcing the system with a locally conservative but globally non conservative force field with 0 average. Nevertheless some kind of pairing might still occurr under simple non conservative forcing acting only on some large scale modes, see §5.

It has been pointed out to me by F. Bonetto that consistency of the above picture *requires that the sum of the displacement exponents be exactly* 0: the two of us have indeed been able to verify that this property is formally exactly verified in the ED equations. And this led to a correction of the graphs drawn in the figure above that I had originally drawn without taking such property into account. We shall come back on this point in a future study. Note also that the fact that the sum of the displacement exponents vanishes provides a natural test that the truncation that one is using is actually large enough for having reached cut-off independence of the asymptotic properties ot the motions: this happens at the cut off value where the sum of the displacements exponents vanishes: further addition of modes only makes longer the flat part in the third graph of the figure above.

§5 The NS and GNS equations. Extension of the pairing rule.

We turn to the NS and GNS equations, whose interest is far greater than the just studied ED or GED equations.

One is tempted to say that the scenario should be the same. However the pairing rule analysis, which is essential for the physical interpretation of the results, is no longer naively possible, not even at a heuristic level.

A pairing rule, first pointed out in special *non constant* friction cases in [EM], p. 281, has been proved only in the case of systems subject to special gaussian constraints, see [DM], but it has apparently a much wider validity, see [EM], [ECM1], [SEM], [DPH] and it is likely to hold also in the cases GNS and NS, *at least in some sense.*

Also the argument in [Dr] implies the existence of pairing in systems that are obtained from hamiltonian systems by adding to them an irreversible constant friction term *proportional to the momenta in a system of canonical coordinates.* And the argument in [DM] is restricted to "isokinetic" constraints *precisely* because they are reversible constraints that are obtained by adding to a hamiltonian system a suitable force proportional to the canonical momenta. Since this is an essential feature for the validity of (3.2) the latter becomes doubtful in the cases (that include NS and GNS equations) in which the friction or thermostat forces are proportional to the canonical momenta via a matrix C which is not the identity (it is the laplacian in the case of the NS or GNS equations).[3]

In such cases one could envisage that (3.2) is replaced, in the GNS equations case, by a relation like:

$$\frac{\lambda_j^+ + \lambda_j^-}{2} = \langle\beta\rangle\, c_j \tag{5.1}$$

where $\langle\beta\rangle$ is the $\mu_{\eta,gns}$ average (in the case of the NS equations one would write ν instead of $\langle\beta\rangle$); and c_j is some suitable function of j, that might be related to the spectrum of the matrix C. However attempting at establishing such a connection would lead to too many too detailed assumptions at this stage and one would like not to rely on them. And from the proofs in [Dr], [DM] it seems unlikely that a pairing rule can hold in a strong form, *i.e.* that (5.1) holds for the local exponents if $\langle\beta\rangle$ is replaced by β.

We therefore *define* c_j by the (5.1) *without linking* c_j *to the matrix* C. However we shall suppose that (5.1) holds in a "almost local" form *in the sense that on a rapid time scale (5.1) becomes true also for the local exponents.* This means that, *up to an error that tends to zero very quickly with the time* τ, the logarithms of the eigenvalues of the matrix $(J_\tau^T(x)J_\tau(x))^{1/2\tau}$, with $J_\tau(x)$ being the jacobian matrix for the evolution operator V_τ at x, divided by τ verify $\frac{1}{2}(\lambda_j^+ + \lambda_j^-) = c_j\beta_\tau(x)$ with $\beta_\tau(x)$ denoting the average $\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\beta(V_t x)\,dt$.

This property, together with the Axiom C assumption, suffices to extend, in a suitable form, the validity of the predictions (*i.e.* conjectures) discussed in this section for the ED and GED equations to the case of the NS and GNS equations as follows.

We first remark that the really relevant feature of the pairing rule, as far as the applications in [BGG] and above are concerned, is not the constancy of the pairing *but, rather, the fact that some kind of pairing takes place on a fast enough time scale.* secondly we assume that this is actually the case for the GNS equations. These are the main remark and the main assumption on which we base the analysis of the fluctuation theorem predictions in the NS and GNS equations.

If a relation (5.1) holds the constants c_j will have to add up to the sum of the Lyapunov exponents that can be derived as the average value of σ: this means that $\langle\sigma\rangle_+ = 2\overline{D}_K\langle\beta\rangle_{\eta,gns} = (\sum_{j=1}^{2D_K} c_j)\langle\beta\rangle_{\eta,gns}$, up to terms neglegible as the Reynolds number tends to ∞, see (1.10) and comments preceding it.

[3] I owe to F. Bonetto the clarification of this point.

Furthermore let I be the set of P inertial pairs (*i.e.* of pairs of Lyapunov exponents λ^+_j, λ^-_j with one positive element) and suppose that the (5.1) becomes valid on a sufficiently fast time scale, then the values of $\langle\sigma\rangle_+$ and $\langle\sigma_0\rangle_+$ would have ratio (see [BGG]) $(\sum_{j\in I} c_j)/(\sum_j c_j)$ so that:

$$\langle\sigma_0\rangle = \frac{\sum_{j\in I} c_j}{\sum_j c_j}\langle\sigma\rangle_+ = (\sum_{j\in I} c_j)\,\langle\beta\rangle_{\eta,gns} = (\sum_{j\in I} c_j)\nu \stackrel{def}{=} \overline{P}\,\nu \tag{5.2}$$

having used the conjecture NS of §1 equating $\langle\beta\rangle_{\eta,gns}$ to ν.

Then if a local time reversal exists on the attractor (*i.e.* if the geometric Axiom C is assumed as well, [BG], for the dynamics generated by the GNS equations) the fluctuations of the observable σ will have a "free energy" (or a "generalized sum of Lyapunov exponents" to adhere to the terminology in [FP], [BJPV]) $\zeta(p)$, in the sense of (4.1), with an odd part $p\,\overline{P}\,\nu$, with $\overline{P}$ defined in (5.2). This is a property whose validity can be conceivably tested in, real or numerical, moderately turbulent systems. At least the linearity in p of $\zeta(p)-\zeta(-p)$ should be observable.

Note also that, in all cases, the pairing rule is trivially valid in the case of no forcing: in fact the equivalence criterion in the conjecture in §2 requires that in absence of forcing one has to take $\eta=0$ or $\varepsilon=0$: *i.e.* the stationary state is, in that case, the trivial (non chaotic) flow $\underline{u}=\underline{0}$, $\underline{\delta}=const.$

The assumption that the forcing be locally conservative *has not been used and disappears* togheter with the constancy of the pairing: the above more general pairing hypothesis (see (5.1) and the comment following it) is more "flexible" and does not require the special hypothesis of local conservativity of the the forcing.

§6 *Relation between the NS and ED equations. The barometric formula.*

Finally we discuss another main point of our analysis.

In reference [Ga6] the argument leading to the conjecture NS above can be interpreted as saying that NS and ED are *also* in some sense equivalent.

The argument is based on the constancy of the dissipation rate ε in a stationary flow at high Reynolds number and on the *microscopic* reversibility. In some sense the GNS equations emerge as *even more natural* than the NS equations.

A criticism can be raised, however. In fact one can argue that the energy is *also* constant in a stationary state and one could develop the argument in [Ga6] to imply that the GED equations are also a good model for a fluid motion.

Since clearly one should not expect NS and ED to be equivalent this looks at first as an unsoluble logical contradiction. Which can furthermore be conceivably easily checked to occurr.

However on further thought the contradiction can be resolved and one should think that all what has been deduced is that there should be a relation between the stationary states of ED (or GED equivalently) and of NS (or GNS). The relation to which I think is the kind of relation that one also finds in equilibrium statistical mechanics in gases in a strongly varying external field of intensity g, like the gravity field.

Locally a gas in a field looks just like a homogeneous gas in equilibrium, but globally over a length scale H over which the external potential really changes ($\beta mgH \sim 1$, if β is the inverse temperature and m the particles mass) one will see that pressure and the density are not constants and one gets the *barometric formula*, see [MP].

Likewise we can expect that the stationary states of ED (or equivalently of GED) are *also* "locally" equivalent to stationary states for NS (or GNS): in the sense that if we only look at observables depending on field components $\underline{u}_{\underline{k}}$ with modes $\underline{k}$ on a certain scale $|\underline{k}| \sim \kappa$ whose size depends on the dissipation then we should see essentially no difference. The precise relation that determines κ will be called *barometric formula*: it should be easy to determine the formula on the basis of dimensional considerations. *Locality* is here to be interpreted in momentum space rather than in coordinate space.

The determination of the barometric formula amounts essentially at a development of the analogous of the Kolmogorov theory for the ED equations.

We now attempt at a development of such theory, in the no obstacles case for simplicity, on the basis of a few assumptions that we did not subject to as much criticism as they certainly deserve: a criticism that will be undertaken in a separate publication. We follow closely the ideas (and imitate the assumptions) of the exposition of Kolmogorov's theory in [LL]. We set $\rho = 1$.

It seems reasonable to suppose that in the ED case the stationary distribution equipartitions the energy among the modes, *i.e.* $\langle|\underline{\gamma}_{\underline{k}}|^2\rangle = \gamma^2$ for all $\underline{k}$ in the "inertial range" $L^{-1} \ll |\underline{k}| \ll k_\chi$ where k_χ is the "Kolmogorov scale", to be determined below. Hence $\gamma^2(k_\chi L)^3 = \varepsilon$.

Then a velocity variation characteristic of the momentum scale κ is $v_\kappa^2 = \langle(\sum_{|\underline{k}|\in[\kappa/2,\kappa]} \underline{\gamma}_{\underline{k}})^2\rangle$ and, assuming statistical independence of the distribution of the various $\underline{\gamma}_{\underline{k}}$, we get $v_\kappa^2 = (\kappa L)^3\gamma^2$ up to a constant factor.

The scale k_χ can be determined because it has to be a momentum scale that only depends on ε and χ so that $k_\chi = (\frac{\varepsilon}{\chi^2})^{1/2}$ up to a constant factor: this is a momentum scale analogous to the Kolmogorov scale k_ν of the NS equations, which is $k_\nu = (\frac{\eta}{\nu^2})^{1/4}$.

For purposes of comparison we note that the quantity called ε in the Kolmogorov's theory ("K41-theory"), see [LL], corresponds to $\eta\nu$ of the present paper.

In this case the energy distribution (*i.e.* the amount $K(k)dk$ of energy per unit volume and between k and $k+dk$) is $K(k) = \frac{3\varepsilon}{4\pi}\frac{k^2}{k_\chi^3}$, for $k < k_\chi$: very different from the Kolmogorov's $k^{-5/3}$ law.

In the K-41 theory a key role is plaid by the quantity $v_\kappa^3\kappa$ which is *identical to $\eta\nu$ for all $k_0 \ll \kappa \ll k_\nu$*. Therefore we compute the value of $v_\kappa^3\kappa$ in our case and we find:

$$\frac{v_\kappa^3\kappa}{\varepsilon\chi} = \frac{((\kappa L)^3\gamma^2)^{3/2}\kappa}{\varepsilon\chi} = \frac{\big((k_\chi L)^3\gamma^2\big)^{3/2}k_\chi}{\varepsilon\chi}\Big(\frac{\kappa}{k_\chi}\Big)^{11/2} = \frac{\varepsilon^{3/2}k_\chi}{\varepsilon\chi}\Big(\frac{\kappa}{k_\chi}\Big)^{11/2} \quad (6.1)$$

and we see that the quantity $v_\kappa^3\kappa$ does depend on κ in the ED case. Given κ the SRB statistics for the ED equations driven with a total energy ε gives to this quantity the same value that it has in the SRB statistics for the NS equation driven with a total vorticity η if:

$$\frac{\varepsilon\chi}{\eta\nu}=\big(\frac{\kappa}{k_\chi}\big)^{-\frac{11}{2}} \tag{6.2}$$

provided (of course) κ is greater than the Kolmogorov scales k_ν, k_χ.

The "barometric formula" is then the statement of equivalence between NS and ED *on scale* κ, *i.e.* if one only looks at field properties depending on $\underline{\gamma}_{\,\underline{k}}$ for $\frac{1}{2}\kappa<|\,\underline{k}\,|<\kappa$, if (6.2) holds and $\kappa\gg k_\nu,k_\chi$.

If we look at a different scale $\kappa'=2^n\kappa$ for some (large) n then we can expect equivalence between ED (or GED) and NS (or GNS) *but* the pairs ε,η should now be such that (6.2) holds on the new scale: the analogy with the usual barometric formula for the Boltzmann Gibbs distribution in the gravity field justifies the name given to (6.2). We see that $\eta\nu$ plays the role of the gravity, $\varepsilon\nu$ plays the role of the chemical potential and κ/k_χ plays the role of the height.

The above analysis seems to be fully consistent with the numerical results in [ZJ] who first proposed, in a different context, a picture very close to the one developed here.

It is clear that this point of view has several consequences: for instance in particular it tells us that that the shape of the pairing curve in (5.1) cannot be arbitrary (*i.e.* $\langle\beta\rangle c_j\sim\nu\,\underline{k}_j^{\,2}$ if the modes are ordered in increasing order). This is a point on which I hope to return in a later analysis.

Also: the equivalence between NS and ED on a given momentum scale makes more interesting the ideas in [Ga4] and a test of the Onsager reciprocity derived in the latter paper seems now quite feasible and seems also to have conseqiences for the real NS equations.

A further remark is that although (6.2) depends on the validity of the K-41 theory and of the corresponding theory for ED equations the barometric formula can be developed *independently* of such theories: hence any modification of the K-41 theory (and of the corresponding theory for ED) will lead to a barometric formula, with a relation between ε,η,κ, possibly more complicated than (6.2).

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Appendix A1: The hamiltonian formalism for Euler equations and Dressler's theorem for ED.

To check the applicability of the results on pairing of [Dr] to the ED equations we must check that the equations can be written, in canonical coordinates for some hamiltonian function H, in the form:

$$\begin{aligned}\dot{\underline{q}} &= \underline{\partial}_{\,\underline{p}} H\\ \dot{\underline{p}} &= -\,\underline{\partial}_{\,\underline{q}} H + \underline{F} - \chi\,\underline{p}\end{aligned} \tag{A1.1}$$

where $\underline{F}$ is such that $\partial_{q_j} F_i = \partial_i F_j$ without being necessarily $\underline{F} = -\underline{\partial}\, V$ for some *globally defined* V (the latter would be a trivial case). The labels for the components of $\underline{q}$ are $\underline{x}\,, i$ with $\underline{x} \in \Omega$ and $i = 1,2,3$. The partial derivatives are, correspondingly, functional derivatives; we shall ignore this because a "formally proper" analysis is easy and leads to the same results. By "formal" we do not mean rigorous, but *only* rigorous if the functions we consider have suitably strong smoothness properties: a fully rigorous treatment is of course impossible for want of reasonable existence, uniqueness and regularity theorems for the Euler equations or the Navier Stkes equations in 3 dimensions.
Consider first the Euler equations. They can be derived from the lagrangian:

$$\mathcal{L}_0(\dot{\underline{\delta}}\,, \underline{\delta}\,) = \frac{\rho}{2}\int \dot{\underline{\delta}}^{\,2}\, d\,\underline{x} \tag{A1.2}$$

(ρ = density) *defined on the space* $\mathcal{D}$ *of the diffeomorphisms* $\underline{x} \longrightarrow \underline{\delta}\,(\,\underline{x}\,)$ *of the box* Ω, by imposing the *ideal* constraint:

$$\det J \equiv \det \frac{\partial\,\underline{\delta}}{\partial\,\underline{x}}(\,\underline{x}\,) = \underline{\partial}\,\delta_1 \wedge \underline{\partial}\,\delta_2 \cdot \underline{\partial}\,\delta_3 \equiv 1 \tag{A1.3}$$

In fact, if Q is a Lagrange multiplier, the stationarity condition for:

$$\mathcal{L}(\dot{\underline{\delta}}\,, \underline{\delta}\,) = \frac{\rho}{2}\int \dot{\underline{\delta}}^{\,2}\, d\,\underline{x} + \int Q(\,\underline{x}\,) \det J(\,\underline{\delta}\,)(\,\underline{x}\,)\, d\,\underline{x} \tag{A1.4}$$

leads to:

$$\rho\,\ddot{\underline{\delta}} = -(\det J)\, J^{-1}\, \underline{\partial}\, Q = -\det J\, \frac{\partial\,\underline{x}}{\partial \underset{\sim}{\delta}} \cdot \underset{\sim}{\partial}\; Q \tag{A1.5}$$

so that setting $\underline{u}\,(\,\underline{\delta}\,(\,\underline{x}\,)) = \dot{\underline{\delta}}\,(\,\underline{x}\,)$, $p(\,\underline{\delta}\,) = Q(\,\underline{x}\,)$ if $\underline{\delta} = \underline{\delta}\,(\,\underline{x}\,)$, we see that:

$$\frac{d\,\underline{u}}{dt} = -\frac{1}{\rho}\,\underline{\partial}\, p \tag{A1.6}$$

which are the Euler equations. And the multiplier $Q(\,\underline{x}\,)$ can be computed as:

$$Q(\,\underline{x}\,) = p(\,\underline{\delta}\,) = -\big[\Delta^{-1}(\,\underline{\partial}\,\underset{\sim}{\underline{u}}\;\cdot \underset{\sim}{\underline{\partial}}\;\;\underline{u}\,)\big]_{\,\underline{\delta}} \tag{A1.7}$$

where the functions in square brackets are regarded as functions of the variable $\underline{\delta}$ and the differential operators also differentiate over such variable. After the computation the variable $\underline{\delta}$ has to be set equal to $\underline{\delta}\,(\,\underline{x}\,)$.

Therefore by using the lagrangian:

$$\mathcal{L}_i(\dot{\underline{\delta}},\underline{\delta})=\int\big(\frac{\rho\dot{\underline{\delta}}(\underline{x})^2}{2}-\big[\Delta^{-1}(\underline{\partial}\,\underline{u}\cdot\underline{\partial}\;\underline{u})\big]_{\underline{\delta}(\underline{x})}(\det J(\underline{\delta})|_{\underline{x}}-1)\big)\,d\underline{x}\quad(A1.8)$$

we generate lagrangian equations for which the "surface" Σ of the *incompressible diffeomorphisms* in the space $\mathcal{D}$ is *invariant*: these are the diffeomorphisms $\underline{x}\to\underline{\delta}(\underline{x})$ such that $J(\underline{\delta})=\underline{\partial}\,\delta_1\wedge\underline{\partial}\,\delta_2\cdot\underline{\partial}\,\delta_3\equiv 1$ at every point $\underline{x}\in\Omega$.

Then Σ is invariant in the sense that the solution to the lagrangian equations with initial data "on Σ", *i.e.* such that $\underline{\delta}\in\Sigma$ and $\underline{\partial}\cdot\dot{\underline{\delta}}(\underline{x})=0$, evolve remaining "on Σ".

The hamiltonian for the lagrangian (A1.8) is obtained by computing the canonical momentum $\underline{p}(\underline{x})$ and the hamiltonian as:

$$\begin{aligned}\underline{p}(\underline{x})=&\frac{\delta\mathcal{L}_i}{\delta\,\dot{\underline{\delta}}(\underline{x})}=\rho\dot{\underline{\delta}}(\underline{x})+\ldots\\ H(\underline{p},\underline{q})=&\frac{1}{2}\big(G(\underline{q})\underline{p},\underline{p}\big)\end{aligned}\qquad(A1.9)$$

where $G(\underline{q})$ is a suitable quadratic form that can be read directly from (A1.8) (but it has a somewhat involved expression of no interest for us), and the $\ldots$ (that can also be read from (A1.8)) *are terms that vanish if* $\underline{\delta}\in\Sigma$ and $\underline{\partial}\cdot\dot{\underline{\delta}}=0$, *i.e.* they vanish on the incompressible motions.

The above is well known and shows that the Euler flow can be interpreted as a gedodesic flow on the surface Σ of the incompressible diffeomorphisms of the box Ω enclosing the fluid, see appendix 2 in [A].

Modifying the Euler equations by the addition of a force $\underline{f}(\underline{x})$ such that *locally* $\underline{f}(\underline{x})=-\underline{\partial}\,\Phi(\underline{x})$ means modifying the equations into:

$$\frac{d\,\underline{u}}{dt}=-\underline{\partial}\,p-\underline{\partial}_{\underline{x}}\Phi\qquad(A1.10)$$

which can be derived from a lagrangian:

$$\mathcal{L}_i^{\Phi}(\dot{\underline{\delta}},\underline{\delta}))=\mathcal{L}_i(\dot{\underline{\delta}},\underline{\delta}))-\int\Phi(\underline{\delta}(\underline{x}))\,d\underline{x}\qquad(A1.11)$$

which leads to the equations:

$$\dot{\underline{u}}(\underline{\delta}(\underline{x}))=-\frac{1}{\rho}\underline{\partial}_{\underline{\delta}}\,p(\underline{\delta}(\underline{x}))+\underline{\partial}_{\underline{\delta}}\,\Phi(\underline{\delta}(\underline{x}))\qquad(A1.12)$$

Hence the ED equations have the form:

$$\begin{aligned}\dot{\underline{q}}=&\partial_{\underline{p}}H\\ \dot{\underline{p}}=&-\partial_{\underline{q}}H-\underline{F}-\chi\,\underline{p}\end{aligned}\qquad(A1.13)$$

at least as far as the motions which have an incompessible initial datum are concerned. This is true because the ED equations which have an incompressible initial datum evolve it by keeping it incompressible.

The Lyapunov exponents of the equation (A1.13) verify the pairing rule by the analysis in [Dr]. However the pairing takes place in the *full* phase space of the diffeomorphisms of Ω, including the incompressble ones.

It is not difficult to see, by using that the constraint to stay on the surface Σ is *holonomous*, that one can find canonical coordinates $\underline{\pi}$, $\underline{\kappa}$, $\underline{\pi}^{\perp}$, $\underline{\kappa}^{\perp}$ describing the motions on Σ or, respectively, transversally to it. And the equations for $\underline{\pi}^{\perp}$, $\underline{\kappa}^{\perp}$ are, *near* Σ and for $\underline{\pi}^{\perp}$ small, $\dot{\underline{\pi}}^{\perp} = -\chi\,\underline{\pi}^{\perp}$ and $\dot{\underline{\kappa}}^{\perp} = \underline{\pi}^{\perp}$ so that the corresponding Lyapunov exponents are trivially paired in pairs $0, -\chi$ with pairing sum $-\frac{\chi}{2}$.

Since we have seen above that *all the exponents are paired* at the level $\frac{\chi}{2}$ this means that *all the physically intereting exponents* (relative to the incompressible motions, *i.e.* relative to the $\underline{\pi}$, $\underline{\kappa}$ coordinates) are *also* paired at the same level, as claimed in §4.

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