Nonlinear Stability of One-Layer Geostrophic Fronts *

November 30, 1994

Jinqiao Duan and Stephen Wiggins
California Institute of Technology, Applied Mechanics 104-44
Pasadena, CA 91125, USA.
E-mail: duan@galcit.caltech.edu
Fax: (818) 568-2719

Abstract

We study the nonlinear stability of one-layer geostrophic fronts described by frontal geostrophic equations. We have showed that the β-effect plays a crucial role in the stability of one-layer geostrophic fronts. Especially, we have proved that the class of nonlinearly stable fronts is more restricted when the β-effect is present than when it is absent. This indicates an important difference between the β-plane approximation (of the earth's surface) and the f-plane approximation.

Key words: frontal geostrophic dynamics, β-effect, stability

*This research was supported by the ONR grant No. N00014-89-J-3023.

1 Introduction

The complete mathematical analysis of the primitive equations governing geophysical (oceanic and atmospheric) systems is not possible at the present time. Simplified equations which describe only the essential aspects of large scale geostrophic motions have been developed. These simplified equations apply to different regimes, e.g., the quasi-geostrophic (QG) regime (Pedlosky [1987] and references therein), the intermediate geostrophic (IG) regime (Charney and Flierl [1981], Yamagata [1982]), the planetary geostrophic (PG) regime (Williams and Yamagata [1984]), and the frontal geostrophic (FG) regime (Cushman-Roisin [1986]). The unification and generalization of these equations are discusseed in Hukuda and Yamagata [1988] and Cushman-Roisin and Yang [1990]. The investigation of these simplified equations should provide some insight to the original geophysical systems governed by primitive equations.

In this paper, we consider the frontal geostrophic equations (Cushman-Roisin [1986]). These are simplified equations derived from a $1\frac{1}{2}$ layer shallow-water, reduced-gravity and rigid lid model (the upper layer is active while the lower layer is at rest) on the β plane. This reduction or formalism eliminates inertial motions. Specifically, these equations govern the motion of the upper layer depth $h(x,y,t)$, i.e., the location of the interface.

From Cushman-Roisin [1986], when $\beta \sim \epsilon$, the equation is

$$h_t - J(h, h \triangle h + \frac{1}{2} \bigtriangledown h \cdot \bigtriangledown h + \alpha y h) = 0, \tag{1.1}$$

where $\alpha = \frac{\beta}{\epsilon}$ with β the meridional gradient of the Coriolis parameter ($0 < \beta << 1$) and $\epsilon(0 < \epsilon << 1)$ the Rossby number, and $J(f,g) = f_x g_y - f_y g_x$ is the Jacobian operator or determinant.

When the β-effect is zero, the equation is

$$h_t - J(h, h \triangle h + \frac{1}{2} \bigtriangledown h \cdot \bigtriangledown h) = 0. \tag{1.2}$$

When $\beta >> \epsilon$, the equation is

$$h_t - h h_x = 0. \tag{1.3}$$

The equations (1.1) and (1.3) were also derived by Williams and Yamagata [1984].

We consider the cases where the interfaces intersect with the surface of the top layer. A line along which $h(x,y,t) = 0$ is called an outcrop or front, which separates two different regions in geophysical fluids. We are interested in cases with closed outcrops. Example for such cases are Gulf Stream cyclonic rings formed by Gulf Stream meanders (e.g. Spence and Legeckis [1981]). It is important to understand the evolution and stability of fronts. In this paper, we study the formal and nonlinear stability of stady states (i.e. steady fronts) of equations (1.1), (1.2), and (1.3). Especially, we will check the role of the β-effect in the stability conditions.

2 Hamiltonian Formalism

We use $\mathcal{R}$ to denote the two dimensional bounded domain on which the frontal geostrophic motion occurs. We denote $\Omega \subset \mathcal{R}$ as the two-dimensional domain where $h > 0$. The boundary of Ω is an outcrop or front. In the following, all double integrals are on the domain Ω unless specified otherwise.

To use the Arnold method to study the stability of fronts, as in Swaters [1993], we write the equation (1.1) or (1.2) as a Hamiltonian system (Olver [1986])

$$h_t = \mathcal{D}\frac{\delta H}{\delta h}, \tag{2.1}$$

where

$$H(h) = -\frac{1}{2}\int\int h\nabla h \cdot \nabla h dxdy + \frac{1}{2}\int\int \alpha y h^2 dxdy, \tag{2.2}$$

and $\mathcal{D}$ is a skew-symmetric operator defined by

$$\mathcal{D}\cdot = J(h, \cdot). \tag{2.3}$$

In the Arnold method we need to find more constants of motion (so-called Casimir functions). This can be easily done using a Poisson bracket

$$[H_1, H_2] = (\frac{\delta H_1}{\delta h}, \mathcal{D}\frac{\delta H_2}{\delta h}), \tag{2.4}$$

where $(\cdot, \cdot)$ is the usual scalar product in $L^2(\Omega)$, we can also rewrite the equation (1.1) as

$$h_t = [h, H], \tag{2.5}$$

as long as we think h as

$$h = \int\int \delta(x - x')\delta(y - y')h(x', y', t)dx'dy'. \tag{2.6}$$

The Casimir functions $C(h)$ can be found by solving

$$0 = [F, C(h)] = (\frac{\delta F}{\delta h}, \mathcal{D}\frac{\delta C}{\delta h}), \tag{2.7}$$

for all $F(h) \in L^2(\Omega)$. Hence we have

$$J(h, \frac{\delta C}{\delta h}) = 0, \tag{2.8}$$

which means that $\frac{\delta C}{\delta h}$ needs to be an arbitrary (smooth) function of h, or

$$C(h) = \int\int g(h) dx dy \tag{2.9}$$

for arbitrary function $g(h)$.

The steady states $h_0(x, y)$ of the equation (1.1) satisfy

$$J(h_0, h_0 \triangle h_0 + \frac{1}{2} \bigtriangledown h_0 \cdot \bigtriangledown h_0 + \alpha y h_0) = 0, \tag{2.10}$$

or

$$h_0 \triangle h_0 + \frac{1}{2} \bigtriangledown h_0 \cdot \bigtriangledown h_0 + \alpha y h_0 = f(h_0), \tag{2.11}$$

for arbitrary (smooth) function $f(h)$. In particular, all zonal steady flow $h_0(y)$ are steady states for the equation (1.1).

For the equation (1.3), we can also write it in Hamiltonian form as above with

$$H(h) = \frac{1}{2}\int\int y h^2 dx dy. \tag{2.12}$$

The Casimir functions are still as in (2.9) and all steady states satisfy

$$J(h_0, y h_0) = 0, \tag{2.13}$$

or

$$y h_0 = f(h_0) \tag{2.14}$$

for arbitrary (smooth) function $f(h)$.

3 Formal and Linear Stability

To study formal stability of a steady state $h_0(x, y)$ of equation (1.1) (Holm et al. [1985]), we need to show that there exists a constant of motion $\tilde{H}$ such that $\delta \tilde{H}(h_0) = 0$ and $\delta^2 \tilde{H}(h_0)$ to be sign-definite.

We consider the constrained Hamiltonian function

$$\tilde{H}(h) = H(h) + C(h), \tag{3.1}$$

with the Casimir $C(h)$ satisfying

$$0 = \delta \tilde{H}(h_0) = \int\int (h_0 \triangle h_0 + \frac{1}{2} \nabla h_0 \cdot \nabla h_0 + \alpha y h_0 + g'(h_0)) dxdy, \tag{3.2}$$

i.e.

$$g(h) = -\int^h f(\xi) d\xi, \tag{3.3}$$

$$C(h) = -\int\int (\int^h f(\xi) d\xi) dxdy. \tag{3.4}$$

With this chosen $C(h)$, we calculate

$$\delta^2 \tilde{H}(h_0) = \int\int [(\triangle h_0 + \alpha y - f'(h_0))(\delta h)^2 - h_0 \nabla(\delta h_0) \cdot (\delta h_0)] dxdy. \tag{3.5}$$

We note that $f'(h_0)$ can be obtained by differentiating the equation (2.11). Hence we have

$$\delta^2 \tilde{H}(h_0) = -\int\int h_0 h_{0y}^2 \nabla(\frac{\delta h_0}{h_{0y}}) \cdot \nabla(\frac{\delta h_0}{h_{0y}}) dxdy - \alpha \int\int \frac{h_0}{h_{0y}} (\delta h)^2 dxdy. \tag{3.6}$$

We know that $h_0 > 0$ on the domain Ω. Therefore, for meridionally increasing fronts ($h_{0y} > 0$), $\delta^2 \tilde{H}(h_0)$ is negative-definite. Hence we have the following result:

Theorem 1 *(i) When the β-effect is at the order of the Rossby number ϵ, all meridionally increasing steady fronts are formally stable, and hence it is linearly stable in the sense of Lyapunov in the norm $||h|| = \sqrt{-\delta^2 \tilde{H}(h)}$.*

(ii) When the ϵ-effect is zero, all steady fronts are formally and hence linearly stable in the sense of Lyapunov.

We remark that the second statement above was proved in Swaters [1993] and can be easily seen from (3.6). The statement (i) in Theorem 1 also implies the result by Benilov [1994] who showed that when the β- effect is present, all meridionally increasing steady fronts are spectrally stable (no positive eigenvalue). Our result is generally stronger than Benilov's result.

For the equation (1.3), we also consider a constrained Hamiltonian function as in (3.1) but with $H(h)$ defined by (2.12). We choose $C(h)$ such that

$$0 = \delta\tilde{H}(h_0) = \int\int [yh_0 + g'(h_0)]\delta h dxdy \tag{3.7}$$

or

$$g(h) = -\int^h f(\xi)d\xi, \tag{3.8}$$

$$C(h) = -\int\int (\int^h f(\xi)d\xi)dxdy. \tag{3.9}$$

We calculate

$$\begin{aligned}\delta^2\tilde{H}(h_0) &= \int\int [y - g'(h_0)](\delta h)^2 dxdy \\ &= -\int\int \frac{h_0}{h_{0y}}(\delta h)^2 dxdy, \end{aligned}\tag{3.10}$$

which is negative-definite for meridionally increasing steady fronts ($h_{0y} > 0$), and positive-definite for meridionally decreasing steady fronts ($h_{0y} < 0$).

Hence we have the following result:

Theorem 2 *When the β- effect is much larger than the Rossby number ϵ, both meridionally increasing and decreasing steady fronts are formally stable, and hence linearly stable in the sense of Lyapunov, in the norm $||h|| = \sqrt{-\delta^2\tilde{H}(h)}$ or $||h|| = \sqrt{\delta^2\tilde{H}(h)}$, respectively.*

4 Nonlinear Stability

To show nonlinear stability of steady fronts, we need to derive global bounds (in some norm) of the disturbances in terms of known data. To this end we consider the conserved function as in Swaters [1993]

$$\mathcal{L}(h) = H(h_0 + h) - H(h_0) + C(h_0 + h) - C(h_0), \tag{4.1}$$

which becomes, after using (2.2) and (3.4)

$$\mathcal{L}(h) = \frac{1}{2}\int\int[\triangle h_0 h^2 - (h_0+h)\nabla h\cdot\nabla h + \alpha y(2h_0 h + h^2)]dxdy$$
$$+\int\int[\int_{h_0+h}^{h_0} f(\xi)d\xi + f(h_0)h]dxdy. \qquad (4.2)$$

We assume that

$$a_1\xi < f(\xi) < b_1\xi, \qquad (4.3)$$
$$\sup_{\mathcal{R}}(\triangle h_0 + \alpha y) < a_1, \qquad (4.4)$$

where a_1, b_1 are finite real numbers. Hence we have

$$\mathcal{L}(h) < \frac{1}{2}\int\int(\triangle h_0 + \alpha y - b_1)h^2 - (h_0+h)\nabla h\cdot\nabla h$$
$$+\int\int[\alpha y h_0 - b_1 h_0 + f(h_0)]h$$
$$< \frac{1}{2}\int\int(\triangle h_0 + \alpha y - b_1)h^2 + A(\int\int h^2)^{\frac{1}{2}}, \qquad (4.5)$$

where $A = \{\int\int[\alpha y h_0 - b_1 h_0 + f(h_0)]^2\}^{\frac{1}{2}}$, and in the last step above, we have used the Hölder inequality in $L^2(\Omega)$. Therefore

$$\mathcal{L}(h) - A(\int\int h^2)^{\frac{1}{2}} < \Gamma\int\int[h^2 + (h_0+h)\nabla h\cdot\nabla h]dxdy, \qquad (4.6)$$

where $\Gamma = \max\{\frac{1}{2}\sup_\Omega(\triangle h_0 + \alpha y - b_1), -1\} < 0$. Moreover, we denote $\hat{h}(x,y) = h(x,y,0)$, and we get

$$\mathcal{L}(\hat{h}) > \frac{1}{2}\int\int(\triangle h_0 + \alpha y - a_1)\hat{h}^2 - (h_0+\hat{h})\nabla\hat{h}\cdot\nabla\hat{h}$$
$$+\int\int[\alpha y h_0 - a_1 h_0 + f(h_0)]\hat{h}dxdy$$
$$> \hat{\Gamma}\int\int[\hat{h}^2 + (h_0+\hat{h})\nabla\hat{h}\cdot\nabla\hat{h} - B(\int\int\hat{h}^2)^{\frac{1}{2}}, \qquad (4.7)$$

where $B = \{\int\int[\alpha y h_0 - a_1 h_0 + f(h_0)]^2\}^{\frac{1}{2}}$, and $\hat{\Gamma} = \min\{\frac{1}{2}\inf_\Omega(\triangle h_0 + \alpha y - a_1), -1\} < 0$.

Note that $\int\int h^2 dxdy$ is also a constant of motion (Cushman-Roisin [1986]). So

$$\mathcal{L}(h) - A(\int\int h^2)^{\frac{1}{2}} = \mathcal{L}(\hat{h}) - A(\int\int\hat{h}^2)^{\frac{1}{2}}. \qquad (4.8)$$

Thus from (4.6), (4.7) and (4.8), we get

$$\begin{aligned}
\int\int h^2 &\leq \int\int h^2 + (h_0+h)\nabla h \cdot \nabla h \\
&< \Gamma^{-1}[\mathcal{L}(h) - A(\int\int h^2)^{\frac{1}{2}}] \\
&= \Gamma^{-1}[\mathcal{L}(\hat{h}) - A(\int\int \hat{h}^2)^{\frac{1}{2}}]
\end{aligned}$$

$$< \Gamma^{-1}\hat{\Gamma}\int\int[\hat{h}^2 + (h_0+\hat{h})\nabla\hat{h}\cdot\nabla\hat{h}] - \Gamma^{-1}(A+B)(\int\int \hat{h}^2)^{\frac{1}{2}}. \qquad (4.9)$$

Note that the right hand side is nonnegative and depends only on known data. This shows the nonlinear stability of $h_0(x,y)$ and we have the following result.

Theorem 3 *Let $h_0(x,y)$ be a steady front for the equation (1.1), i.e., it satisfies*

$$h_0 \triangle h_0 + \frac{1}{2} \bigtriangledown h_0 \cdot \bigtriangledown h_0 + \alpha y h_0 = f(h_0), \qquad (4.10)$$

for an arbitrary (smooth) function $f(h)$. If $f(h)$ satisfies the following condition

$$a_1\xi < f(\xi) < b_1\xi, \qquad (4.11)$$

$$\sup_{\mathcal{R}}(\triangle h_0 + \alpha y) < a_1, \qquad (4.12)$$

for some finite real numbers a_1, b_1, then, $h_0(x,y)$ is nonlinearly stable in $L^2(\Omega)$ (in the usual norm).

We remark that, if the β-effect is not zero, i.e., $\alpha = \frac{\beta}{\epsilon} \neq 0$, then the nonlinear stability condition directly involves the meridional stretch or size of the frontal region; see above condition (4.12) in the statement of Theorem 3. When $\beta = 0$, the stability condition involves h_0 itself. This also indicates that the set of stable fronts is generally smaller when $\beta \neq= 0$.

Now we consider the equation (1.3). We follow the scheme in Holm et al. [1985]. From (2.12) and (3.9), we calculate

$$\begin{aligned}
H(h_0+h) - H(h_0) - DH(h_0)\cdot h &= \frac{1}{2}\int\int[y(2h_0h + h^2) - 2yh_0h] \\
&= \frac{1}{2}\int\int yh^2 \equiv Q_1(h), \quad (4.13)
\end{aligned}$$

$$C(h_0+h) - C(h_0) - DC(h_0)\cdot h = \int\int[\int_{h_0+h}^{h_0} f(\xi)d\xi + f(h_0)h]. \quad (4.14)$$

We assume that

$$a_2 < f'(\xi) < b_2, \ on\ \mathcal{R} \tag{4.15}$$

for some real finite numbers a_2 and b_2. Then from (4.14) we have

$$C(h_0 + h) - C(h_0) - DC(h_0) \cdot h > -\frac{1}{2} b_2 \int\int h^2 \equiv \hat{Q}_2(h); \tag{4.16}$$

or

$$C(h_0 + h) - C(h_0) - DC(h_0) \cdot h < -\frac{1}{2} a_2 \int\int h^2 \equiv \check{Q}_2(h). \tag{4.17}$$

From Holm et al. [1985], the nonlinear stability of $h_0(x, y)$ is ensured by either

$$Q_1(h) + \hat{Q}_2(h) = \int\int (y - b_2) h^2 > 0, \tag{4.18}$$

i.e.,

$$\inf_{\mathcal{R}}(y) > b_2, \tag{4.19}$$

or

$$Q_1(h) + \check{Q}_2(h) = \int\int (y - a_2) h^2 < 0, \tag{4.20}$$

i.e.,

$$\sup_{\mathcal{R}}(y) < a_2. \tag{4.21}$$

Thus we have proved the following conclusion.

Theorem 4 *Let $h_0(x, y)$ be a steady solution of the equation (1.3), i.e.,*

$$y h_0 = f(h_0) \tag{4.22}$$

for an arbitrary (smooth) function $f(h)$. If $f(h)$ satisfies

$$a_2 < f'(\xi) < b_2, \ on\ on\ \mathcal{R} \tag{4.23}$$

for some real finite numbers a_2 and b_2, and if either

$$\inf_{\mathcal{R}}(y) > b_2, \tag{4.24}$$

or

$$\sup_{\mathcal{R}}(y) < a_2 \tag{4.25}$$

then, $h_0(x,y)$ is nonlinearly stable under the norm $||h|| = \sqrt{Q_1(h) + \hat{Q}_2(h)}$ or $||h|| = \sqrt{-Q_1(h) - \check{Q}_2(h)}$, respectively.

We remark that, when the β- effect is large comparing with the Rossby number ϵ, the nonlinear stability condition also depends on the meridional stretch or size of the frontal region (Theorem 4).

5 Discussions

In this paper, we have studied the formal and nonlinear stability of one-layer geostrophic fronts. We have seen that when the β- effect is present, the class of formally and linearly stable fronts is more restricted than when the β- effect is absent. When the β- effect is presnt, the nonlinear stability condition involves with the meridional stretch or size of the frontal region.

We should remark that the models used here do not include stratification or baroclinicity. The result may not directly be applied to real geophysical fronts (such as Gulf Stream rings) without the analysis of the effect of baroclinicity.

The fronts studied here are density fronts. Constantin et al. [1994] recently studied temperature fronts.

References

[1] E. Benilov, A note on the stability of one-layer geostrophic fronts, *Geophys. Astrophys. Fluid Dynamics* **66** (1992), 81-86.

[2] J. G. Charney and G. R. Flierl, Oceanic analogues of large-scale atmospheric motions, In *Evolution of Physical Oceanography* (eds. B. A. Warren and C. Wunsch), 514-548, Massachusetts: MIT Press, 1981.

[3] P. Constantin, A. Majda and E. G. Tabak, Singular front formation in a model for quasigeostrophic flow, *Phys. Fluids* **6** (1994), 9-11.

[4] B. Cushman-Roisin, Frontal geostrophic dynamics, *J. Phys. Oceanography* **16** (1986), 132-143.

[5] B. Cushman-Roisin and B. Yang, Geostrophic turbulence and emergence of eddies beyond the radius of deformation, *J. Phys. Oceanography* **20** (1990), 97-113.

[6] D. D. Holm, J. E. Marsden, T. Ratiu and A. Weinstein, Nonlinear stability of fluid and plasma equilibria, *Phys. Rept.* **123** (1985),1-116.

[7] H. Hukuda and T. Yamagata, A unified geostrophic equation with application to a cold core ring, *Tellus* **40A** (1988), 407-418.

[8] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[9] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, 2nd edition, 1987.

[10] T. W. Spence and R. Legeckis, Satellite and hydrographic observations of low-frequency wave motions associated with a cold core Gulf Stream ring, *J. Geophys. Res.* **86** (1981), No. C3, 1945-1953.

[11] G. E. Swaters, On the baroclinic dynamics, hamiltonian formulation and general stability characteristics of density-driven surface currents and fronts over a sloping continental shelf, *Phil. Trans. R. Soc. Lond. A* **345**(1993), 295-325.

[12] G. P. Williams and T. Yamagata, Geostrophic eddies, intermediate solitary vortices and Jovian eddies, *J. Atmos. Sci.* **41**(1984), 453-478.

[13] T. Yamagata, On nonlinear planetary waves: a class of solutions missed by the quasi-geostrophic approximation, *J. Oceanography Soc. Japan* **38** (1982), 236-244.