TRANSACTION COSTS

AND NON-MARKOVIAN DELTA HEDGING

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Abstract. We consider the problem of hedging and pricing European and American derivatives in the continuous time formalism. The underlying security is a stock whose trading involves a small relative transaction cost k. If $k = 0$, the Black and Scholes optimal trading strategy is Markovian, satisfies the self-financing condition and assures a perfect portfolio replication. If $k > 0$, transactions occur at random but discrete times. We find an optimal trading strategy that minimizes total transaction costs for a given degree of risk aversion. Since the calculation of rehedging times is part of the problem in the continuous time setting, optimal strategies are non-Markovian. They also break the self-financing constraint because hedge slippages are risky. We compute the leading term in $\sqrt{k}$ in an asymptotic expansion in the limit of small transaction costs. We express the rehedging thresholds in terms of the Black and Scholes solution and evaluate the total transaction cost by solving a final value problem for a parabolic equation of the Black and Scholes type.

Work partially supported by the National Sciences and Engineering Council of Canada under grants number OGP-0171149 and OGP-0121848.

Typeset with TEX.

1. Introduction

The problem of transaction costs in dynamic asset pricing theory is attracting considerable attention [1,2,5-8,10,12,13,15]. The popular Black and Scholes option pricing model [4] assumes that markets are frictionless. The corresponding optimal trading strategy is to *continuously* rebalance a portfolio including the underlying security and a riskless bond. If transaction costs are non-zero, this strategy fails because the total transaction cost becomes infinite with probability one [14]. A way to regularize the problem is to fix rehedging times exogenously. This simplifies the mathematical problem by introducing constraints and by generating a suboptimal solution as in [10,13]. In this article we do not make this approximation. We provide a quantitative explanation for the fact that the optimal frequency of rehedgings varies with time: it is low if the gamma of the position is small and rises as gamma rises. A numerical method to find the optimal rehedging strategy would be to discretize time by generating random scenarios with a Montecarlo algorithm and to reduce the stochastic control problem to a problem in linear programming [9,10,15]. We use the intuition provided by the Montecarlo method to study the continuous time problem directly, i.e. without any discretization. Our goal is to solve the continuous time problem at the leading order in an asymptotic expansion in the transaction cost k. We substantially simplify the problem by means of analytic methods and give a transparent and numerically efficient way of evaluating various interesting quantities, such as the expected transaction costs and the value of the rehedging thresholds.

We find that the optimal trading strategy is non-Markovian, i.e. decisions depend not only on the current stock price but also on its past evolution. Non-Markovian strategies occur naturally in this context because, to keep costs finite, one has to specify not only the number of shares in the replicating portfolio but also the optimal rehedging times at which trades occur. If transaction costs are non-zero, decisions are irreversible and timing is crucial. Mathematically, the problem is challenging, as most technical tools available in stochastic control theory concern the Markovian case. Our solution reduces the non-Markovian hedging problem to the calculation of the thresholds at which trades occur. At the leading order in the relative transaction cost k, we show that the optimal rehedging thresholds depend only on time and on the current value of the stock. Our trading strategy minimizes the sum of the expected transaction costs plus the cost of financing an insurance policy against the risk associated with hedge slippages. The idea of assigning a cost to an unbalanced position is intuitive and is commonly used by traders when deciding whether the size of a hedge slippage justifies the costs of rehedging. We express the small k asymptotics for this problem explicitly in terms of the Black and Scholes solution and show that the maximum hedge slippages allowed are proportional to $\sqrt{k}$. The fact that perturbation theory is not analytic is the only trace remaining of the singular character of the limit $k \to 0$.

We discuss in detail the case of a derivative security of European type whose payoff $f_0(S)$ at expiration T is contingent to the value of a single stock S. American securities can be treated in a similar fashion with only a few minor modifications, which we specify below. The stock price process is given by a stochastic differential equation of the form

$$\frac{dS}{S} = \rho dt + \nu \, dW, \tag{1.1}$$

where $\rho > 0$ is the expected payoff rate of the stock, $\nu > 0$ is the volatility and W is the standard Wiener process. Pure discount bonds are also available for trade and their price is

$$B(t) = \exp(-r(T-t)) \tag{1.2}$$

where r is the riskless interest rate. We assume that

$$\rho = r - \frac{\nu^2}{2} \tag{1.3}$$

so that the expected payoff of the stock is equal to the payoff of a riskless bond of the same value. This amounts to assuming that the stock price is set by investors who can perfectly diversify their portfolio. This restriction simplifies our treatment but can easily be lifted.

A trading strategy is described by a pair of adapted processes $(a(t), b(t))$, where $a(t)$ is the number of shares held at time t and $b(t)$ is the number of bonds. The face value of the portfolio held at time t is

$$\Pi(t) = a(t)S(t) + b(t)B(t). \tag{1.4}$$

The cost of this trading strategy is the adapted process $c(t)$ such that

$$dc = S da + B db + \frac{k}{2} S |da|, \tag{1.5}$$

where k is the relative transaction cost for a round trip transaction. The goal is to find the trading strategy of lowest cost which replicates the payoff $f_0(S)$ at expiration, subject to constraints that quantify the degree of risk aversion of the writer of the derivative contract.

In the limit of zero transaction costs, $k = 0$, the problem was solved by Black and Scholes [4], who proved that the price process $f_{BS}(S,t)$ satisfies the equation

$$\frac{\partial}{\partial t} f_{BS}(S,t) + rS \frac{\partial}{\partial S} f_{BS}(S,t) + \frac{1}{2}\nu^2 S^2 \frac{\partial^2}{\partial S^2} f_{BS}(S,t) = r f_{BS}(S,t). \tag{1.6}$$

In this case, the optimal trading strategy is Markovian and consists of holding a number of shares equal to $a(t) = a_{BS}(S(t), t)$, where

$$a_{BS}(S,t) = \frac{\partial f_{BS}(S,t)}{\partial S}. \tag{1.7}$$

The Black and Scholes trading strategy is riskless for two reasons: the payoff is exactly replicated at expiration, i.e. $\Pi(T)$ almost surely equals the payoff $f_0(S(T))$, and the self-financing condition $dc = 0$ is fulfilled. These are the reasons why risk preferences do not matter and the expected payoff rate of the stock ρ does not enter the Black and Scholes equation. If transaction costs are non-zero, neither perfect portfolio replication nor the self-financing condition can be achieved at finite cost, and risk preferences become relevant. A solution of the transaction cost problem requires us to set up a mathematical formalism in which risk preferences are specified in a way that is both natural and convenient. To treat the risk involved in hedge slippages we use the notion of regret function. The risk due to the failure of the self-financing condition is quantified in terms of a path integral whose evaluation requires us to solve a final value problem for a parabolic equation of the Black and Scholes type.

The formalism we propose in this article is described in section 2, while in section 3 we give a precise statement of the results. There are two appendices: the first contains the derivation of the small $\sqrt{k}$ asymptotics and in the second we explain the reasons why dynamic programming techniques cannot be used to solve a problem in stochastic control theory in the class of non-Markovian strategies.

2. Risk Aversion

The problem can be split in two parts: hedging and pricing. Each part requires a different quantification of risk aversion, as hedge slippages and random total transaction costs are related but distinct causes of risky fluctuations. We formulate the hedging problem in terms of two thresholds $A_-(S,t)$, $A_+(S,t)$, which specify the values of $a(t)$ at which a rehedging to the Black and Scholes value takes place and

$$\delta a(t) \;=\; a(S(t),t) \;-\; a_{BS}(t) \tag{2.1}$$

shares are sold.

If the rehedging decision occurs whenever the hedge slippage is δa, the cost of the transaction is equal to $\frac{1}{2}k|\delta a|$. Let us suppose that at time t the position is perfectly hedged and that the next rehedge occurs at time $t+\delta t$. Since the stock price is assumed to perform a diffusion process, we have that $\sqrt{\delta t} \;\approx\; |\delta a|$. The rate of the cash flow to cover transaction costs is

$$\frac{k|\delta a|}{\delta t} \approx \frac{k}{|\delta a|}. \tag{2.2}$$

Evidently, $|\delta a|$ cannot be chosen arbitrarily small. On the other hand, $|\delta a|$ cannot be too large, as a large hedge slippage involves risk.

A convenient way of quantifying hedge slippage risk is to think in terms of insurance premiums. During the time period when the writer has to hedge her position, her portfolio can be thought of as split into two different parts: a perfectly hedged part $(a_{BS}(t), b_{BS}(t))$ satisfying the self-financing condition $S da_{BS} \;+\; B db_{BS} \;=\; 0$, and a risky remainder

$(\delta a(t), \delta b(t))$. The number of shares $\delta a(t)$ in the remainder is given by equation (2.1) above. The number of extra bonds $\delta b(t)$ is continuously adjusted in such a way that the risky portfolio has a gain to regret ratio higher than the market ratio set by investors that hold a diversified portfolio.

More precisely, we define the following expected gain function measuring the risk to which the writer is exposed because she does not rehedge:

$$\delta G(t) \;=\; E_t\Big[\Big(\delta a(t) S(T) \;+\; \delta b(t)\Big)_+\Big]. \tag{2.3}$$

The expected gain for holding shares of the given stock S as it is set by the market price of S is

$$\delta G_0(t) \;=\; E_t\Big[\Big(\delta a(t)(S(T) \;-\; e^{r(T-t)} S(t))\Big)_+\Big]. \tag{2.4}$$

We also define regret functions. The one for the hedger is

$$\delta M(t) \;=\; E_t\Big[\Big(\delta a(t) S(T) \;+\; \delta b(t)\Big)_-\Big] \tag{2.5}$$

and the one set by the market is

$$\delta M_0(t) \;=\; E_t\Big[\Big(\delta a(t)(S(T) \;-\; e^{r(T-t)} S(t))\Big)_-\Big]. \tag{2.6}$$

The values of the gain and regret functions δG_0 and δM_0 are set by the market at the lowest levels at which agents who can diversify the risk are willing to invest. Whenever the risk cannot be diversified or otherwise compensated, one has to resort to the delta hedging techniques that we study in this article. Hence it is natural to assume that the investor who implements delta hedging wants to charge a higher premium for holding an unbalanced portfolio. We require that in the time interval $[t, t+\delta t]$ between two rehedgings, the number of bonds $\delta b(t+\tau)$ must be continuously adjusted so that

$$\frac{\delta G(t+\tau)}{\delta M(t+\tau)} \;=\; J_0\, \frac{\delta G_0(t+\tau)}{\delta M_0(t+\tau)}. \tag{2.7}$$

Here, $\tau \in [0, \delta t]$ and $J_0 > 1$ is a fixed parameter measuring the writer's aversion to undiversifiable risk.

The value J_0 is fixed exogenously. Larger values of J_0 yield a higher degree of protection but generate more expensive hedging schemes with more frequent rehedgings. On the other hand, a risk neutral writer who has so many different contracts that he is confident that the risk is as diversified as possible, would compute the limit $J_0 \to 1$. As one would expect, in this limit the optimal trading strategy becomes a simple buy and hold static strategy.

If we neglect systematic risk, then the drift ρ satisfies (1.3) and $\delta G_0 = \delta M_0$. This choice is adopted here as it is appropriate for derivative valuation. For applications to portfolio theory, it may be important to study the interplay between systematic and hedging risks.

Since $J_0 > 1$, the value of the portfolio $(\delta a(t+\tau), \delta b(t+\tau))$ is strictly positive. The time value of these extra assets in a period $[t, t+\delta t]$ between two rehedgings can be interpreted as the insurance premium δi paid in this period to cover undiversifiable risk. A rough estimate for the premium is $\frac{\delta i}{\delta t} \approx r|\delta a|$. Hence, the minimum of

$$\frac{\delta c}{\delta i} + \frac{\delta i}{\delta t} \approx \frac{k}{|\delta a|} + |\delta a| \tag{2.8}$$

is achieved for $|\delta a| \approx \sqrt{k}$ and rehedgings occur at time intervals $\delta t \approx k$. This rough argument turns out to give the correct scaling laws in the limit $k \to 0$.

To compute δi, one can use the analytic expression obtained by Black and Scholes for the path integral arising in the risk neutral valuation of European calls and puts. Put call parity implies that

$$\delta G - \delta M = S\delta a + B\delta b. \tag{2.9}$$

Hence, the equation (2.5) above is equivalent to the following one:

$$S + B\frac{\delta b}{\delta a} = (J_0 - 1)\left(\frac{\delta M}{\delta a}\right). \tag{2.10}$$

If $\delta a < 0$, then $\delta b > 0$. Let

$$\gamma_- = B^{-1}\left|\frac{\delta a}{\delta b}\right|. \tag{2.11}$$

We have that

$$\frac{\delta M}{\delta a} = \gamma_-^{-1}\, f_C(S\gamma_-, T-t). \tag{2.12}$$

where $f_C(x, T-t)$ is given by the Black and Scholes formula

$$f_C(x, T-t) = x\Phi(z) - \exp(-r(T-t))\Phi\big(z - \nu\sqrt{T-t}\big). \tag{2.13}$$

Here $\Phi(z)$ is the cumulative normal distribution and

$$z = \frac{\log x + (T-t)(r + \frac{1}{2}\nu^2)}{\nu\sqrt{T-t}}. \tag{2.14}$$

The rate of insurance expenditures is

$$\frac{di}{dt} = r(S - \gamma_-^{-1})\,\delta a.$$

In case $\delta a > 0$ we have that $\delta b < 0$ and it is more convenient to use the formula

$$S \ + \ B\frac{\delta b}{\delta a} \ = \ -(J_0^{-1} - 1)\left(\frac{\delta G}{\delta a}\right). \tag{2.15}$$

By setting

$$\gamma_+ \ = \ B^{-1} \left|\frac{\delta a}{\delta b}\right|, \tag{2.16}$$

we find that

$$\frac{\delta G}{\delta a} \ = \ \gamma_+^{-1} \ f_C(S\gamma_+, T - t). \tag{2.17}$$

In this case, the rate of insurance expenditures is

$$\frac{di}{dt} \ = \ r(S \ - \ \gamma_+^{-1}) \ \delta a. \tag{2.18}$$

Once the rehedging thresholds $A_-(S,t)$ and $A_+(S,t)$ are computed, the stochastic processes $c(t)$ and $i(t)$ are defined and we are left with the problem of pricing transaction and insurance costs. The price will depend on the value and on the fluctuations of the random variable

$$\int_0^T \ dt \ \left(\frac{dc}{dt} + \frac{di}{dt}\right) e^{-rt}. \tag{2.19}$$

Again, we need a notion of risk aversion that is both natural and practical. We propose to represent the price process by means of the following formula:

$$P(S, t_0) \ = \ f_{BS}(S, t_0) \ + \ J_1^{-1} \ \log K(S, t_0, t_0). \tag{2.20}$$

The function $K(S, t_0, t)$ is defined for $t \geq t_0$ as the following path integral:

$$K(S, t_0, t) \ = \ E_t\Big[\exp\left(\int_t^T \ J_1(\frac{dc}{dt'} + \frac{di}{dt'})e^{-r(t'-t_0)}dt'\right) \ \big| \ S(t_0) = S\Big]. \tag{2.21}$$

The parameter $J_1 > 0$ is a measure of risk aversion against losses due to the failure of self-financing. A risk neutral writer would take the limit $J_1 \to 0$, which corresponds to pricing transaction and insurance costs by their discounted expectation.

The leading term as $k \to 0$ of the path integral above can be written as follows:

$$K(S, t_0, t) \ = \ E_t\Big[\exp\left(\int_t^T \ dt' J_1 \sqrt{k} \ V(S, t') e^{-r(t'-t_0)}\right)\Big]. \tag{2.22}$$

A formula for the function $\sqrt{k} \ V(S,t)$ is given in the next section. This function gives the average rate of expenditures for transaction costs and insurance premiums at the leading order in $\sqrt{k}$ as the stock price process goes through the point (S,t). Equation (2.22) implies that

$$E_t\big[dK(S, t_0, t)\big] \ = \ -J_1\sqrt{k}V(S,t)e^{-r(t-t_0)} \ K(S, t_0, t). \tag{2.23}$$

On the other hand, from Ito's formula we find

$$E_t\Big[dK(S,t_0,t)\Big] \;=\; \left(\frac{\partial K(S,t_0,t)}{\partial t} \;+\; S\rho\frac{\partial K(S,t_0,t)}{\partial S} + \frac{1}{2}S^2\nu^2\frac{\partial^2 K(S,t_0,t)}{\partial S^2}\right)dt. \tag{2.24}$$

Hence, $K(S,t_0,t)$ solves the differential equation

$$\frac{\partial K(S,t_0,t)}{\partial t} + S\rho\frac{\partial K(S,t_0,t)}{\partial S} + \frac{1}{2}S^2\nu^2\frac{\partial K(S,t_0,t)}{\partial S^2} \;=\; -J_1\sqrt{k}V(S,t)e^{-r(t-t_0)}\,K(S,t_0,t) \tag{2.25}$$

with final condition at expiration

$$K(S,t_0,T) \;=\; 1 \tag{2.26}$$

for all $S > 0$.

3. Results

Instead of working with the rehedging thresholds $A_\pm(S,t)$, it is convenient to take the following functions as unknowns:

$$\begin{aligned} d(S,t) \;&=\; A_+(S,t) \;-\; A_-(S,t), \\ \xi(S,t) \;&=\; \frac{1}{2} \;-\; \frac{1}{2d(S,t)}\big[A_+(S,t) \;+\; A_-(S,t)\big]. \end{aligned} \tag{3.1}$$

Asymptotic expansions in powers of $\sqrt{k}$ can be obtained by minimizing the sum

$$E\left[\frac{dc}{dt}\;|S(t)=S\right] \;+\; E\left[\frac{di}{dt}\;|S(t)=S\right]. \tag{3.2}$$

In Appendix A, we derive the following formulae, which are valid at the leading order in $\sqrt{k}$:

$$E\left[\;\frac{dc}{dt}\;|\;S(t)=S\right] \;=\; \frac{\sqrt{2}kS\nu^2}{d}\left|\frac{S\partial^2 f_{BS}}{\partial S^2}(S,t)\right|^2\;\phi_1(\xi) \tag{3.3}$$

$$E\left[\;\frac{di}{dt}\;|\;S(t)=S\right] \;=\; \frac{rd}{3}\;\phi_2(S,t;\xi) \tag{3.4}$$

where

$$\phi_1(\xi) \;=\; \frac{1}{\xi} \;+\; \frac{1}{1-\xi} \;-\; \sum_{j=0}^{\infty}\zeta(3+2j)\,2^{-2(j+1)}\big[\xi^{2(j+1)} \;+\; (1-\xi)^{2(j+1)}\big] \tag{3.5}$$

and

$$\phi_2(S,t;\xi) \;=\; (S \;-\; \gamma_-(S,t)^{-1})\,\xi^2 \;+\; (S \;-\; \gamma_+(S,t)^{-1})(\xi-1)^2. \qquad (3.6)$$

The functions $\gamma_\pm(S,t)$ are defined in Sect. 2. In (3.5), $\zeta(m)$ is the Riemann zeta function. A convenient way of evaluating $\zeta(m)$ is to use the formula

$$\zeta(m) \;=\; \prod \frac{1}{1-p^{-m}} \qquad (3.7)$$

where the product runs over all prime numbers $p > 1$. The minimum of the sum (3.2) is achieved at

$$d(S,t) \;=\; \sqrt{\frac{3\sqrt{2}kS\nu^2}{r}\left|S\frac{\partial^2 f_{BS}}{\partial S^2}\right|}\cdot\sqrt{\frac{\phi_1(\xi(S,t))}{\phi_2(S,t;\xi(S,t))}} \qquad (3.8)$$

where $\xi(S,t)$ minimizes the product $\phi_1(\xi)\phi_2(S,t;\xi)$. The rehedging thresholds are given by

$$A_+(S,t) \;=\; d(S,t)(1-\xi(S,t)) \qquad\qquad A_-(S,t) \;=\; -d(S,t)\xi(S,t) \qquad (3.9)$$

The rate of transaction and insurance expenditures $V(S,t)$ used in the previous section, is equal to

$$V(S,t) \;=\; \sqrt{\frac{\sqrt{2}}{3}kS\nu^2 r}\;\left|S\frac{\partial^2 f_{BS}}{\partial S^2}\right|\;\sqrt{\phi_1(\xi(S,t))\cdot\phi_2(S,t,\xi(S,t))}. \qquad (3.10)$$

In short, the procedure we propose to price a European derivative security and to compute the hedging thresholds consists of the following steps:
(i) Fix the two parameters J_0 and J_1 that quantify the degree of risk aversion;
(ii) solve the Black and Scholes equation for the given payoff function;
(iii) compute the functions $\gamma_\pm(S,t)$ by solving the equations in (2.10-15);
(iv) compute the function $\xi(S,t)$ by minimizing the product $\phi_1(\xi)\phi_2(S,t,\xi)$;
(v) compute the rehedging thresholds by means of (3.8) and (3.9);
(vi) compute the rate of expenditures to cover transaction and insurance costs by means of (3.10);
(vii) solve the partial differential equation in (2.25) with final value condition (2.26) to obtain the price process in (2.20).

The same method can also be used to price American derivatives with transaction costs with the following modifications: (i) the function $f_{BS}(S,t)$ is defined as the Black and Scholes price for an American option; (ii) all path integrals extend not up to the expiration date but up to the stopping time at which a rehedging occurs; (iii) the equation (2.25) for the function $K(S,t_0,t)$ must be integrated up to the exercise boundary and, on this line, $K(S,t_0,t)$ is equal to one.

Appendix A. Markovian Reduction

We consider $(S_0, t_0) \in \mathbb{R}_+ \times [0,T]$ and introduce the following notations:

$$\xi_0 \;=\; \xi(S_0, t_0), \qquad\qquad d_0 \;=\; d(S_0, t_0). \tag{A.1}$$

The point of coordinate

$$\eta(\tau) \;=\; \xi(S_0, t_0+\tau) \;+\; d_0^{-1}\delta a(S_0, t_0+\tau) \tag{A.2}$$

performs a Brownian motion with drift in the interval $[0,1]$. Rehedgings occur whenever $\eta(\tau)$ hits one of the edges of the interval; at this point, η is reset to the Black and Scholes value which is ξ_0. As we argued in section 2, the size d_0 of the no-transaction region between the rehedging thresholds is of order $\sqrt{k}$. Hence, to leading order in $\sqrt{k}$ we have that

$$\eta(\tau) \;\approx\; \xi_0 \;+\; d_0^{-1}\delta a(S_0, t_0+\tau). \tag{A.3}$$

Always to leading order, the volatility $d_0^{-1}\nu_0$ and the drift $d_0^{-1}\rho_0$ are given by

$$d_0^{-1}\nu_0 \;=\; d_0^{-1}\;\nu\; S\Gamma_0, \qquad\qquad d_0^{-1}\rho_0 \;=\; d_0^{-1}\;\Psi_0 \tag{A.4}$$

where the "greeks" are defined as follows:

$$\Gamma_0 \;=\; \Gamma(S_0, t_0) \;=\; \frac{\partial^2 f_{BS}}{\partial S^2}(S_0, t_0), \tag{A.5}$$

and

$$\Psi_0 \;=\; \Psi(S_0, t_0) \;=\; \frac{\partial^2 f_{BS}}{\partial t \partial S}\,(S_0, t_0) \;+\; \rho S \frac{\partial^2 f_{BS}}{\partial S^2}(S_0, t_0) \;+\; \frac{\nu^2 S^2}{2}\frac{\partial^3 f_{BS}}{\partial S^3}\,(S_0, t_0) \tag{A.6}$$

The drift term is subleading if $\Gamma_0 \gg \frac{\sqrt{d_0 \Psi_0}}{S\nu}$. Since d_0 is infinitesimal as $k \to 0$, the region where this condition is violated is a neighborhood of width $O(k^{\frac{1}{2}})$ of the gamma neutral lines of equation $\Gamma(S,t) \;=\; 0$. To leading order in k, the presence of these regions does not affect the solution and we can thus ignore them. A more detailed consideration of these lines will be required for the analysis of the subleading corrections. The gamma neutral lines are analogous to focal lines in nonlinear optics and to turning points for the Schrödinger equation in the semiclassical limit.

Let $Q(\tau; \xi_0, \eta_0)$ be the probability that the point $\eta(\tau)$ is at $\eta_0 \in (0,1)$ conditioned to the fact that $\eta(0) \;=\; \xi_0$ and that $\eta(\tau') \;\neq\; 0,1$ for $\tau' \in [0, \tau]$.

We assume that, insofar as we are interested in the leading order asymptotics, rehedgings occur at uniformly distributed random times. A heuristic argument to justify this assumption is that rehedging times are very sensitive to the past evolution. Under this hypothesis, the following approximation is legitimate:

$$E\left[\frac{dc}{dt}(t_0) \quad \middle|\; S(t) \;=\; S_0\right] \;\approx\; \frac{kSd_0}{2}\int_0^\infty \frac{d\tau}{\tau}\left[(1-\xi_0)\;Q_+(\xi_0, \tau) \;+\; \xi_0\;Q_-(\xi_0, \tau)\right]. \tag{A.7}$$

Here, $Q_+(\xi_0,\tau)$, respectively $Q_-(\xi_0,\tau)$, is the probability distribution that the right, respectively left endpoints of the interval $[0,1]$ is hit first by $\eta(\tau)$ and that this event occurs at time τ. The function $Q(\tau;\xi_0,\eta)$ satisfies the heat equation

$$\frac{\partial}{\partial\tau} Q(\tau;\xi_0,\eta) \;=\; \frac{1}{2}\, d_0^{-2}\nu_0^2\, \frac{\partial^2}{\partial\eta^2} Q(\tau;\xi_0,\eta) \tag{A.10}$$

subject to the initial condition

$$Q(0;\xi_0,\eta) \;=\; \delta(\xi_0-\eta) \tag{A.11}$$

and to the Dirichlet boundary conditions

$$Q(\tau;\xi_0,0) \;=\; Q(\tau;\xi_0,1) \;=\; 0. \tag{A.12}$$

To compute the asymptotics for the expected cost, it is convenient to use the following expression for $Q(\tau;\xi_0,\eta)$ given by the method of images, see for example [11]:

$$Q(\tau;\xi_0,\eta) \;=\; \frac{d_0}{\nu_0\sqrt{\pi\tau}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{d_0^2(\eta-\xi_0+2n)^2}{2\nu_0^2\tau}\right) \;-\; \exp\left(-\frac{d_0^2(\eta+\xi_0+2n)^2}{2\nu_0^2\tau}\right)\right]. \tag{A.13}$$

By integrating by parts, we see that

$$\frac{\partial}{\partial\tau}\int_0^1 d\eta\; Q(\tau;\xi_0,\eta) \;=\; \frac{1}{2}\, d_0^{-2}\nu_0^2 \left[\frac{\partial}{\partial\eta}\bigg|_{\eta=1} Q(\tau;\xi_0,\eta) \;-\; \frac{\partial}{\partial\eta}\bigg|_{\eta=0} Q(\tau;\xi_0,\eta)\right]. \tag{A.14}$$

Hence

$$\begin{aligned} Q_-(\xi_0,\tau) \;&=\; \frac{1}{2}\, d_0^{-2}\nu_0^2\, \frac{\partial}{\partial\eta}\bigg|_{\eta=0} Q(\tau;\xi_0,\eta) \\ &= \frac{d_0}{\nu_0\pi^{\frac{1}{2}}\tau^{\frac{3}{2}}} \sum_{n=-\infty}^{\infty} (\xi_0 \;+\; 2n)\exp\left(-\frac{d_0^2(\xi_0+2n)^2}{2\nu_0^2\tau}\right). \end{aligned} \tag{A.15}$$

A similar expression holds also for $Q_+(\xi_0,\tau)$. We have that

$$\begin{aligned} &kSd_0\int_0^\infty \frac{d\tau}{\tau}\, \xi_0\, Q_-(\xi_0,\tau) \\ &\quad= \frac{kSd_0^2\xi_0}{\nu_0\sqrt{\pi}} \sum_{n=-\infty}^{\infty} (\xi_0+2n) \int_0^\infty \frac{d\tau}{\tau^{\frac{5}{2}}} \exp\left(-\frac{d_0^2(\xi_0+2n)^2}{2\nu_0^2\tau}\right) \\ &\quad= \frac{kSd_0^2\xi_0}{2\nu_0}\left(\frac{d_0^2}{2\nu_0^2}\right)^{-\frac{3}{2}} \sum_{n=-\infty}^{\infty} \frac{\mathrm{sgn}(\xi_0+2n)}{(\xi_0+2n)^2} \\ &\quad= \frac{\sqrt{2}kS\nu_0^2}{d_0}\left[\frac{1}{\xi_0} \;-\; \sum_{j=0}^{\infty} \zeta(3+2j)\left(\frac{\xi_0}{2}\right)^{2(j+1)}\right] \end{aligned} \tag{A.16}$$

Using reflection symmetry about $\xi_0 = \frac{1}{2}$, we find (3.5).

Supposing again that the rehedging times are randomly distributed, we can evaluate the expected insurance costs in the following way

$$E\left[\frac{di}{dt}(t_0) \;\; | \; S(t) \; = \; S_0\right] \; \approx \; r \; d_0 \int_0^\infty \; d\tau \int_0^1 \; d\eta \; \pi_1(\tau, \xi_0) \; \pi_2(\tau, \xi_0, \eta) \; w(\xi_0, \eta) \; . \qquad (A.17)$$

Here, the weight $w(\xi_0, \eta)$ is given by

$$w(\xi_0, \eta) \;\; = \;\; \begin{cases} (\xi_0 - \eta)(S_0 \; - \; \gamma_-(S_0, t_0)^{-1}) & \text{if} \;\; \eta < \xi_0. \\ (\eta - \xi_0)(S_0 \; - \; \gamma_+(S_0, t_0)^{-1}) & \text{if} \;\; \eta \geq \xi_0 \end{cases} \qquad (A.18)$$

The function $\pi_1(\tau, \xi_0)$ gives the probability distribution that the last rehedging before time t occured at time $t - \tau$. We have that

$$\pi_1(\tau, \xi_0) \;\; = \;\; \frac{\int_0^1 d\eta Q(\tau; \xi_0, \eta)}{\int_0^\infty d\tau \int_0^1 d\eta Q(\tau; \xi_0, \eta)}. \qquad (A.19)$$

Finally, the function $\pi_2(\tau, \xi_0, \eta)$ gives the probability distribution for $\eta(\tau)$ conditioned to the fact that the most recent rehedging occured at time $t - \tau$. We have that

$$\pi_2(\tau, \xi_0, \eta) \;\; = \;\; \frac{Q(\tau; \xi_0, \eta)}{\int_0^1 d\eta Q(\tau; \xi_0, \eta)}. \qquad (A.20)$$

Hence

$$E\left[\frac{di}{dt}(t_0) \;\; | \; S(t) \; = \; S_0\right] \approx \; r \; d_0 \; \left(\int_0^1 \; d\eta R(\xi_0, \eta)\right)^{-1} \int_0^1 \; d\eta \; R(\xi_0, \eta) \; w(\xi_0, \eta) \; . \qquad (A.21)$$

where

$$R(\xi_0, \eta) \;\; = \;\; \int_0^\infty \; d\tau \; Q(\tau; \xi_0, \eta). \qquad (A.22)$$

This function satisfies the Poisson equation

$$\frac{\partial^2}{\partial \eta^2} \; R(\xi_0, \eta) \;\; = \;\; \int_0^\infty \; d\tau \; d_0^2 \nu_0^{-2} \frac{\partial}{\partial \tau} \; Q(\tau; \xi_0, \eta) \;\; = \;\; -d_0^2 \nu_0^{-2} \; \delta(\xi_0, \eta) \qquad (A.23)$$

subject to Dirichlet boundary conditions

$$R(\xi_0, 0) \;\; = \;\; R(\xi_0, 1) \;\; = \;\; 0. \qquad (A.24)$$

Hence, we have that

$$R(\xi_0, \eta) \;\; = \;\; \begin{cases} d_0^2 \nu_0^{-2}(1 - \xi_0)\eta & \text{if} \;\;\; \eta \leq \xi_0 \\ d_0^2 \nu_0^{-2} \xi_0 (1 - \eta) & \text{if} \;\;\; \eta > \xi_0 \end{cases} \qquad (A.25)$$

and we find

$$E\left[\frac{di}{dt}(t_0)\,\middle|\,S(t) \;=\; S_0\right] \;=\; \frac{rd_0}{3}\left[\xi_0^2(S \;-\; \gamma_-(S_0,t_0)^{-1}) \;+\; (1-\xi_0)^2(S \;-\; \gamma_+(S_0,t_0)^{-1})\right]. \tag{A.26}$$

Appendix B. Why Dynamic Programming Doesn't Work

In this appendix, we explain why dynamic programming techniques cannot be applied directly to find an optimal non-Markovian trading strategy for the delta hedging problem with transaction costs. To use dynamic programming, one needs to introduce a utility function measuring the mismatch between the payoff $f_0(S(T))$ and the final value of the replicating portfolio $\Pi(T) \;=\; a(T)S(T) \;+\; b(T)$. A possible choice would be a function of the form

$$\mathcal{U}(S(T),a(T),b(T)) \;=\; u(a(T)S(T)+b(T)-f_0(S(T))), \tag{B.1}$$

where $u(x)$ is a concave increasing function. The utility at time $t<T$ is defined as the expectation

$$\mathcal{J}(S,a,t) \;=\; E_t\big[\mathcal{U}(S(T),a(T),b(T)) \;\big|\; S(t)=S, a(t)=a\big]. \tag{B.2}$$

A non-Markovian trading strategy is described by the adapted process $n(t)=da(t)$, which gives the rate at which shares are bought at time t. The trading strategy $n(t)$ can be seen as a functional depending on t and on the values $S(t')$ for $t'\le t$. The dynamic programming algorithm determines the utility $J(S,a,t)$ and the trading strategy $n(t)$ by proceeding backwards in time, starting from the expiration date $t=T$. The algorithm maximizes the utility $J(S,a,t)$ at time t given $J(S,a,t+dt)$. The procedure can be applied as long as decisions made at time t depend only on the state variable $S(t)$ but not on $S(t')$ with $t'<t$. If transaction costs vanish, this is indeed possible. However, when transaction costs are non-zero, one needs to refrain from trading when hedge slippages are small so that total transaction costs are finite. In this case, the number of shares $a(t)$ held at time t *cannot* be considered as an independent state variable, and dynamic programming fails.

Our trading strategy is formulated in terms of hedge slippages: if the hedge slippage $\delta a(t) \;=\; a(t) - a_{BS}(t)$ is out of the no-transaction interval $(a_{BS}(t)-A_-(S(t),t), a_{BS}(t)+A_+(S(t),t))$ then one sells $\delta a(t)$ shares. Hence, $n(t)$ is determined by $\delta a(t)$. But also the hedge slippages $\delta a(t)$ are a functional of the trading strategy $n(t)$ and of the path of $S(t')$ for $t'\le t$. Therefore, $n(t)$ must satisfy a self-consistency condition depending on the decision variables $n(t')$ at earlier times $t'<t$. To solve this problem, it is necessary to compute the effect of our decisions over a time scale $\delta t \approx k$.

The authors in [9,10,15] propose an interesting numerical approach to this class of non-Markovian problems. Their idea is to generate a finite family of scenarios either by means of a Montecarlo algorithm or by using a binomial approximation. The discretized problem is a finite dimensional linear programming problem that is solved numerically. In this article, we develop similar ideas in the continuous time framework. We believe that the asymptotic formulae we find can be usefully combined with the linear programming approach.

References

[1] Avellaneda, M., Parás, A., 1994, "Optimal Hedging Portfolios for Derivative Securities in the Presence of Large Transaction Costs ", *preprint of the Courant Institute.*

[2] Bensaid, B., Lesne, J.P., Pages, H., 1992, "Derivative Asset Pricing Theory with Transaction Costs", *Mathematical Finance,* **2** , pp. 63-87 .

[3] Bertsekas, D.P., 1987, *Dynamic Programming: Deterministic and Stochastic Models.* Englewood Cliffs, NJ: Prentice-Hall.

[4] Black, F. and Scholes, M., 1973, "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy,* **81** , pp. 637-654 .

[5] Boyle, P.P., Lin, S., 1994, "Optimal Portfolio Selection with Transaction Costs", *preprint of the University of Waterloo.*

[6] Boyle, P.P., Vorst, T., 1992, "Option Replication in Discrete Time with Transaction Costs", *Journal of Finance,* **47** , pp. 271-293 .

[7] Constantinides, G., 1979, "Multiperiod Consumption and Investment Behavior with Convex Transaction Costs", *Management Sci.* , **25** , pp. 1127-1137 .

[8] Davis, M.H., Panas, V.G., Zariphopoulou, T., 1993, "European Option Pricing with Transaction Costs", *SIAM J. Control and Optimization* , **31** , pp. 470-493 .

[9] Dembo, R., Private communication.

[10] Edirisinghe, C., Naik, V., Uppal, R., 1993, "Optimal Replication of Options with Transaction Costs and Trading Restrictions", *Journal of Finance and Quantitative Analysis,* **28** , pp. 117-138 .

[11] Feller, 1971, *An Introduction to Probability Theory with Applications* . New York, NY: John Wiley & Sons.

[12] Gennotte, G., Jung, A. , 1994, "Investment Strategies under Transaction Costs: the Finite Horizon Case", *Management Sci.* , **40** , pp. 385-404 .

[13] Leland, H., 1985, "Option Prices with Transaction Costs", *Journal of Finance,* **40** , pp. 1283-1301 .

[14] Merton, R., 1990, *Continuous Time Finance.* Oxford, UK: Basil Blackwell.

[15] Naik, V., Uppal, R., 1994, "Leverage Constraints and the Optimal Hedging of Stock and Bond Options", *Journal of Finance and Quantitative Analysis,* **29** , pp. 199-222 .