Averaging under Fast Quasiperiodic Forcing

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Abstract

We consider a non autonomous system of ordinary differential equations. Assume that the time dependence is quasiperiodic with large basic frequencies, ω/ε and that the ω vector satisfies a diophantine condition. Under suitable hypothesis of analyticity, there exists an analytic (time depending) change of coordinates, such that the new vector field is the sum of an autonomous part and a time dependent remainder. The remainder has an exponentially small bound of the form $\exp(-c\varepsilon^{-a})$, where c and a are positive constants. The proof is obtained by iteration of an averaging process. An application is made to the splitting of the separatrices of a two-dimensional normally hyperbolic torus, including several aspects: formal approximation of the torus and their invariant manifolds, numerical computations of the splitting, a first order analysis using a Melnikov approach and the bifurcations of the set of homoclinic orbits.

1 Introduction and statement of results

In many problems in Mechanics appear differential equations with fast periodic forcing of the form $z' = f(z, \tau/\varepsilon)$, $z \in D \subset \mathbb{R}^n$, D compact, $t \in \mathbb{R}$ and f being $2\pi\varepsilon$-periodic in τ. The well known result of Neishtadt,[12] by doing a suitable number of averaging steps, tell us that, if f is analytic in D and continuous with respect to t, then there is an analytic change of variables, $w = z + \varphi(z, \tau/\varepsilon)$, such that the new differential equation is of the form $w' = g(w) + r(w, \tau/\varepsilon)$ and there exists $c > 0$ such that $|r| < \exp(-c/\varepsilon)$.

It is natural to ask for for an analogous result when the dependence of f with respect to t is fast quasiperiodic. The main objective of this paper is to study this problem and to give some application. Some more restrictive conditions have to be imposed then on f: analyticity with respect to the angles varying with time has also to be required and, furthermore, the frequencies have to satisfy some diophantine condition.

By introducing the new time $t = \tau/\varepsilon$, the initial equation can be written as $\dot{z} = \varepsilon f(z,\tau)$, where $\dot{}$ denotes $\frac{d}{dt}$.

First we state the main result.

Theorem A. *(Averaging theorem) Let*

$$\dot{z} = \varepsilon f(z, \theta, \varepsilon), \tag{1}$$

where f is analytic in (z,θ) for $z \in \Omega \subset \mathbb{C}^n$, $\Omega = D + \Delta$, a Δ-neighbourhood of D in $\mathbb{C}^n$, D a compact in $\mathbb{R}^n$ and $\theta \in \mathbb{T}^p + \Delta$, $p \geq 2$, where $\mathbb{T}^p$ is a p-dimensional torus. Assume f is bounded in ε for $|\varepsilon| \leq \varepsilon_0$ and $\theta = \omega t$, where $\omega \in \mathbb{R}^n$ is a vector of frequencies satisfying

$$|(k,\omega)| \geq b|k|^{-\tau}, \tag{2}$$

where $b > 0$, $\tau > p - 1$ and $|k| = \sum_{i=1}^{p} |k_i|$. Then, if ε_0 is small enough, for a fixed ε with $|\varepsilon| \leq \varepsilon_0$, there exists a change of variables $z = h(w,\theta,\varepsilon)$, analytic in (w,θ) for $w \in D + \Delta/2$, $\theta \in \mathbb{T}^p + \Delta/2$, such that the new equation is $\dot{w} = \varepsilon(g(w,\varepsilon) + r(w,\theta,\varepsilon))$ and the remainder satisfies an exponentially small bound

$$|r|_{\Delta/2} < c_1 \exp(-c_2/\varepsilon^{c_3}), \tag{3}$$

where c_1, $c_2 > 0$, $c_3 = \frac{1}{\tau+1}$. The constants c_1, c_2 depend only on $|f|_\Delta$, the dimensions and the constants in (2). Furthermore $|g|_{\Delta/2} < 2|f|_\Delta$. Here $|f|_\Delta$ denotes the supremum norm of f in $D + \Delta$, $\mathbb{T}^p + \Delta$ for the fixed value of ε.

Remark 1. If $\tau = p - 1$ (a non generic situation) one can replace $\varepsilon^{\frac{1}{\tau+1}}$ by $(\varepsilon|\ln\varepsilon|)^{1/p}$.

Remark 2. In the particular case of (1) when f has the form

$$f = f_0(z,\varepsilon) + \varepsilon f_1(z,\varepsilon) + \cdots + \varepsilon^{k-1} f_{k-1}(z,\varepsilon) + \varepsilon^k f_k(z,\theta,\varepsilon),$$

then $|f - g|_{\Delta/2} = O(\varepsilon^k)$ and c_1, c_2 depend only on $|f_j|_\Delta$, $j = 0,\ldots,k$, the dimensions and the constants in (2). An application of Remark 2 to the periodic case can be found in Broer et al.[3] Note that in the periodic case the diophantine condition uses $b = |\omega|$, $\tau = 0$. The periodic case is a particular case of Theorem A, but the hypothesis can be weakened, as said before.

Theorem B. *In the hypothesis of Theorem A, if f is the vector field associated to a Hamiltonian H, then the quasiperiodically time depending change of variables h can be chosen in a canonical way. Hence the final vector field, $g + r$, is also Hamiltonian.*

Remark 3. Theorem B has many consequences on upper bounds for the splitting of separatrices, remainders of normal forms, etc.

The proof of Theorem A is given in Section 2. The proofs of Remarks 1 and 2 and Theorem B will appear elsewhere.[13]

In order to see how good are the upper bounds given in (3) and to illustrate Remark 3, Section 3 studies the following example: Consider the equation

$$x'' = \sin(x + \sin(\tau/\varepsilon) + \sin(\gamma\tau/\varepsilon)), \tag{4}$$

where γ denotes the golden number ($\gamma = (1+\sqrt{5})/2$). The first problem is to show that (4) has a normally hyperbolic invariant two-dimensional torus, $\mathbb{T}^2$. This is easy after one step in the averaging process. The invariant $\mathbb{T}^2$ is located at an $O(\varepsilon^2)$ distance of $x = 0 \pmod{2\pi}$.

Approximations of the invariant $\mathbb{T}^2$ and their stable and unstable manifolds, $W^S_{\mathbb{T}^2}$, $W^U_{\mathbb{T}^2}$, are obtained in Section 4. The algorithm to obtain those approximations is given in detail. By symbolic manipulation those approximations are obtained to high order on a computer implementation of the algorithm. Also in Section 4 we show how to globalize $W^U_{\mathbb{T}^2}$ by starting at points given by the previous approximation and integrating (4) numerically. This is done until the points reach $\{x = \pi\}$ for the first time. By symmetry the intersection of $W^S_{\mathbb{T}^2}$ with $\{x = \pi\}$ is also obtained. As intersections two tori are obtained. They can be described by giving the values of x' as a function of two angles, φ_1 and φ_2. Let $x'_{W^{U,S}_{\mathbb{T}^2}\cap\{x=\pi\}} = m^{U,S}(\varphi_1, \varphi_2, \varepsilon)$ and let $m = m^U - m^S$. A convenient measure of the splitting, in this first intersection, is

$$d(\varepsilon) = \sup_{(\varphi_1,\varphi_2)\in\mathbb{T}^2} |m(\varphi_1, \varphi_2, \varepsilon)|. \tag{5}$$

The function (5) has been computed for several values of ε.

In Section 5 we carry out a "first order" analysis of the splitting, that is, we use formally Melnikov's method after one step of averaging. The computations give a theoretical value for the function m described before. We do not claim this approximation is rigourously justified, but the results are in good agreement with the numerical ones.

Assuming the Melnikov approach gives a good approximation to the splitting, we analyze the function m in the case of small values of ε. The splitting is given by an infinite sum, each term being, essentially, of the form

$$\alpha_k \exp\left(-\frac{|(k,\omega)|}{b\varepsilon}\right) \sin((k,\varphi)), \qquad k \in \mathbb{Z}^2, \tag{6}$$

where $\omega = (1,\gamma)^T$ and b is independent of ε and k. However, the coefficients α_k in (6) depend strongly on k and the function $d(\varepsilon)$ defined by (5) behaves as

$$\exp\left(-\frac{c(\varepsilon)|\ln\varepsilon|^{1/2}}{\varepsilon^{1/2}}\right), \tag{7}$$

when ε goes to zero, where $c(\varepsilon)$ is a bounded function: $0 < c(\varepsilon) < \hat{c}$. The values $\overline{c} = \limsup_{\varepsilon\to 0} c(\varepsilon)$ and $\underline{c} = \liminf_{\varepsilon\to 0} c(\varepsilon)$ are computed and they satisfy $0 < \underline{c} < \overline{c}$. This shows, under the assumption above, that the exponent of ε in (3) can not be improved, in general. The extra $|\ln\varepsilon|$ factor in the exponent in (7) (note that in our case $\tau = 1$ in (2) and use Remark 1) is due to the fact that *sin* is not only analytic but entire. Some extensions and improvements are studied.

Section 6 is devoted to the set of points in $W^U_{\mathbb{T}^2} \cap W^S_{\mathbb{T}^2} \cap \{x = \pi\}$ giving rise to "primary" homoclinic orbits. This is a curve in a two-torus, defined by $m(\varphi_1, \varphi_2, \varepsilon) = 0$. The topology of the curve changes when ε does. This is a very natural phenomenon. The behaviour of the set of bifurcations is described. Let ε_k, $k \in \mathbb{N}$, $\varepsilon_1 > \varepsilon_2 > \ldots$, the set of values for which a bifurcation of the set of primary homoclinic orbits is produced. Then $\lim_{k\to\infty}\varepsilon_{k+1}/\varepsilon_k=\gamma^2$. The curves close after n_1 revolutions in the φ_1 direction and n_2 in the φ_2 direction. The succesive values of the pairs (n_1,n_2) are (1,2), (2,3), (3,5), (5,8),..., i.e., given by two successive Fibonacci numbers (F_j,F_{j+1}), where $F_1=1$, $F_2=2$, $F_j=F_{j-1}+F_{j-2}$.

2 Proof of the Averaging Theorem

To prove Theorem A we have to introduce some notation. We denote by ${}^jf$, $j=1,\dots,n$, the components of f. Let $D+\Delta=\{(z_1,\dots,z_n)\in\mathbb{C}^n$ such that it exists $(x_1,\dots,x_n)\in D$, a compact in $\mathbb{R}^n$, with $\max_j|z_j-x_j|\le\Delta\}$, and $\mathbb{T}^p+\Delta=\{(\theta_1,\dots,\theta_n)$ such that $\theta_j=\rho_j+i\sigma_j$, $\rho_j\in S^1$, $\sigma_j\in\mathbb{R}$, $|\sigma_j|\le\Delta\}$. In the previous expressions $|\ \ |$ denote the modulus of a complex or the absolute value of a real. We remark that the values of Δ (in $D+\Delta$, $\mathbb{T}^p+\Delta$) could be different, but this is not essential.

For a vector f we use the supremum norm

$$|f|_\Delta=\max_{j=1,\dots,n}\ \max_{z\in D+\Delta}\ \max_{\theta\in\mathbb{T}^p+\Delta}\ \max_{|\varepsilon|\le\varepsilon_0}\ |{}^jf(z,\theta,\varepsilon)|,$$

while for a matrix $M(z,\theta,\varepsilon)=({}^{ij}M(z,\theta,\varepsilon))_{i,j=1,\dots,n}$, we use the induced norm

$$|M|_\Delta=\max_{i=1,\dots,n}\sum_{j=1}^{n}|{}^{ij}M|_\Delta. \tag{8}$$

Let us define the average of a vector, h, as $\overline{h}(z,\varepsilon)=\int_{\mathbb{T}^p}h(z,\theta,\varepsilon)\,d\theta$, and the purely quasiperiodic part as $\tilde{h}(z,\theta,\varepsilon)=h(z,\theta,\varepsilon)-\overline{h}(z,\varepsilon)$.

For convenience we relabel our initial vector, f, as f_0. So we have $f_0=\overline{f}_0+\tilde{f}_0$. We also relabel Δ as Δ_0. Let $\alpha=|f_0|_{\Delta_0}$. Hence $|\overline{f}_0|_{\Delta_0}=\overline{\alpha}_0$, $|\tilde{f}_0|_{\Delta_0}=\tilde{\alpha}_0$ satisfy

$$\overline{\alpha}_0\le\alpha,\quad \tilde{\alpha}_0\le 2\alpha. \tag{9}$$

For any real positive r let

$$\Delta_r=\Delta_0-r\delta, \tag{10}$$

where δ is a suitable constant, depending on ε, to be determined later.

We shall perform a finite number of steps of some averaging process. The number of steps depends on ε. Assume that, after j steps, we have a current variable, z_j, that we rename as z for shortness, and a current equation

$$\dot{z}=\varepsilon(\overline{f}_j(z,\varepsilon)+\tilde{f}_j(z,\theta,\varepsilon)), \tag{11}$$

where $(z,\theta)\in(D+\Delta_j)\times(\mathbb{T}^p+\Delta_j)$. We denote this domain by S_j. We assume that the bounds

$$|\overline{f}_j|_{\Delta_j}\le\overline{\alpha}_j,\quad |\tilde{f}_j|_{\Delta_j}\le\tilde{\alpha}_j \tag{12}$$

hold. Next we do one more step of the averaging process, changing to a new variable, z_{j+1}, replacing $\overline{f}_j$, $\tilde{f}_j$ by $\overline{f}_{j+1}$, $\tilde{f}_{j+1}$ and reducing a little the domain (from Δ_j to Δ_{j+1}).

In fact we shall still introduce intermediate domains $D+\Delta_{j+1/2}$. Assume $(z,\theta)\in S_j$ and let

$$z = w+\varepsilon v(w,\theta,\varepsilon), \tag{13}$$

for $(w,\theta)\in R_{j+1}$, where $R_{j+1}=(D+\Delta_{j+1/2})\times(\mathbb{T}^p+\Delta_{j+1})$. The norm in this domain will be denoted by $|\ \ |_{R_{j+1}}$. By substitution of (13) in (11) we get

$$\begin{aligned}&(Id+M_j(w,\theta,\varepsilon))\dot{w}+\varepsilon\frac{\partial v}{\partial\theta}(w,\theta,\varepsilon)(w) = \\ &\quad = \varepsilon(A_j(w,\theta,\varepsilon)+\overline{f}_j(w,\varepsilon)+B_j(w,\theta,\varepsilon)+\widetilde{f}_j(w,\theta,\varepsilon)).\end{aligned} \tag{14}$$

In the expression (14) Id denotes the n-dimensional identity matrix, and M_j, A_j, B_j are defined by

$$\begin{aligned} M_j(w,\theta,\varepsilon) &= \varepsilon\frac{\partial v}{\partial w}(w,\theta,\varepsilon), && (15)\\ A_j(w,\theta,\varepsilon) &= \overline{f}_j(w+\varepsilon v((w,\theta,\varepsilon),\varepsilon)-\overline{f}_j(w,\varepsilon), && (16)\\ B_j(w,\theta,\varepsilon) &= \widetilde{f}_j(w+\varepsilon v((w,\theta,\varepsilon),\theta,\varepsilon)-\widetilde{f}_j(w,\theta,\varepsilon). && (17)\end{aligned}$$

Let us choose v such that

$$\frac{\partial v}{\partial\theta}(w,\theta,\varepsilon)(\omega)=\widetilde{f}_j(w,\theta,\varepsilon) \text{ and } \overline{v}(w,\varepsilon)=0. \tag{18}$$

This defines v uniquely and can be obtained by equating Fourier coefficients. Let

$$\widetilde{f}_j(w,\theta,\varepsilon)=\sum_{k\in\mathbb{Z}^p\setminus\{0\}}\widetilde{f}_{j,k}(w,\varepsilon)e^{2\pi i(k,\theta)}$$

and

$$v(w,\theta,\varepsilon)=\sum_{k\in\mathbb{Z}^p\setminus\{0\}}v_k(w,\varepsilon)e^{2\pi i(k,\theta)}.$$

From (18) we get

$$v_k(w,\varepsilon)=\widetilde{f}_{j,k}(w,\varepsilon)/(2\pi i(k,\omega)).$$

We have to bound v in R_{j+1}. By analyticity of $\widetilde{f}_j$ in S_j we have $|\widetilde{f}_{j,k}(w,\varepsilon)|_{\Delta_{j+1/2}}\leq \frac{\widetilde{\alpha}_j}{2\pi}e^{-\Delta_j|k|}$. Then

$$|v|_{R_{j+1}}\leq\frac{\widetilde{\alpha}_j}{2\pi}\sum_{k\in\mathbb{Z}^p\setminus\{0\}}\frac{e^{-\Delta_j|k|}}{|(k,\omega)|}e^{-\Delta_{j+1}|k|}=\frac{\widetilde{\alpha}_j}{2\pi}\sum_{k\in\mathbb{Z}^p\setminus\{0\}}\frac{e^{-\delta|k|}}{|(k,\omega)|}. \tag{19}$$

At this point we need a technical Lemma on small divisors.

Lemma 1. *Let $\omega\in\mathbb{R}^p$ satisfy the diophantine condition (2) with $\tau>p-1$. Let $s_m=\sum_{k\in\mathbb{Z}^p\setminus\{0\},|k|\leq m}\ |(k,\omega)|^{-1}$. Then*

$$s_m\leq\Gamma m^\tau, \tag{20}$$

where Γ is a positive constant depending on ω but independent of m. If $\tau=p-1$ the previous inequality is replaced by $s_m\leq\Gamma' m^\tau\ln m$, where Γ' is also independent of m.

This result is rather well known. A complete proof can be found in Simó.[13]

Let us define $c_m = s_m - s_{m-1}$, with $s_0 = 0$. Then we can finish the bound in (19) by using (20):

$$\begin{aligned} |v|_{R_{j+1}} &\leq \frac{\tilde{\alpha}_j}{2\pi} \sum_{m\geq 1} e^{-\delta m} c_m = \frac{\tilde{\alpha}_j}{2\pi} \sum_{m\geq 1} s_m (e^{-\delta m} - e^{\delta(m+1)}) \leq \\ &\leq \frac{\tilde{\alpha}_j}{2\pi}(1-e^{-\delta})\Gamma \sum_{m\geq 1} e^{-\delta m} m^{\tau} \leq \tilde{\alpha}_j \frac{K}{\delta^{\tau}}, \end{aligned} \tag{21}$$

where K is a constant independent of $\tilde{\alpha}_j$ and δ.

If $w \in D + \Delta_{j+1/2}$, we must be sure that $z \in D + \Delta_j$. It is enough to have

$$\varepsilon \tilde{\alpha}_j \frac{K}{\delta^{\tau}} < \frac{\delta}{2}, \tag{22}$$

a requirement to be checked later by a suitable election of δ.

The equation (14) can be written as

$$\dot{w} = \varepsilon(\overline{f}_{j+1}(w,\varepsilon) + \tilde{f}_{j+1}(w,\theta,\varepsilon)) \tag{23}$$

for $(w,\theta) \in S_{j+1}$. Then (23) is like (11) after one more step of averaging, and we take w as the new variable z_{j+1}. We should relate the bounds of $|\overline{f}_{j+1}|_{\Delta_{j+1}}$, $|\tilde{f}_{j+1}|_{\Delta_{j+1}}$ to the ones of $|\overline{f}_j|_{\Delta_j}$, $|\tilde{f}_j|_{\Delta_j}$ as given by (12). We use the identity $(Id+M_j)^{-1} = Id - (Id+M_j)^{-1}M_j$ to write

$$\dot{w} = \varepsilon[(Id+M_j)^{-1}(A_j + B_j - M_j\overline{f}_j) + \overline{f}_j], \tag{24}$$

where we have used the cancellation due to (18). By comparing (23) and (24) we see that all the terms in the square brackets in (24) can give contributions to $\overline{f}_{j+1}$, and all but the last one to $\tilde{f}_{j+1}$.

First we need some preliminary estimates:

a) From the Cauchy inequality we have

$$\left|\frac{\partial({}^k v)}{\partial({}^j w)}\right|_{\Delta_{j+1}} \leq \tilde{\alpha}_j \frac{2K}{\delta^{\tau+1}}, \tag{25}$$

where we have reduced the strip by $\delta/2$ and we have used the bound (21). Then, from (15) and because of (8) and (25), we have

$$|M_j|_{\Delta_{j+1}} \leq \frac{2Kn\varepsilon\tilde{\alpha}_j}{\delta^{\tau+1}}, \tag{26}$$

b) Again by using Cauchy inequality and the mean value Theorem, we have from (16) and (17), and using again (21), the bounds

$$|A_j|_{\Delta_{j+1}} \leq \overline{\alpha}_j \varepsilon \frac{2Kn\tilde{\alpha}_j}{\delta^{\tau+1}}, \quad |B_j|_{\Delta_{j+1}} \leq \tilde{\alpha}_j \varepsilon \frac{2Kn\tilde{\alpha}_j}{\delta^{\tau+1}}. \tag{27}$$

c) Assume $|M_j|_{\Delta_{j+1}} < 1$. For this it is enough to have

$$\varepsilon \tilde{\alpha}_j \frac{K}{\delta^{\tau+1}} < \frac{1}{2n}. \tag{28}$$

This is stronger than (22). Hence we had to select δ later on to satisfy (28) and then (22) will hold. Under this assumption we have

$$|(Id + M_j)^{-1}|_{\Delta_{j+1}} \leq \left(1 - \frac{\varepsilon \tilde{\alpha}_j 2Kn}{\delta^{\tau+1}}\right)^{-1}. \tag{29}$$

From (24) we get

$$\overline{f}_{j+1}(w,\varepsilon) = \overline{f}_j(w,\varepsilon) + \overline{g(w,\theta,\varepsilon)}, \quad \tilde{f}_{j+1}(w,\theta,\varepsilon) = \widetilde{g(w,\theta,\varepsilon)}, \tag{30}$$

where

$$g(w,\theta,\varepsilon) = (Id + M_j(w,\theta,\varepsilon))^{-1}(A_j(w,\theta,\varepsilon) + B_j(w,\theta,\varepsilon) - M_j(w,\theta,\varepsilon)\overline{f}_j(w,\varepsilon)). \tag{31}$$

By using (27), (29), (26) and (12) we bound the norm in (31). To simplify the notation we introduce

$$\psi = \frac{2Kn\varepsilon}{\delta^{\tau+1}}, \tag{32}$$

and then

$$|g|_{\Delta_{j+1}} \leq (1 - \psi\tilde{\alpha}_j)^{-1}((\overline{\alpha}_j + \tilde{\alpha}_j)\psi + \overline{\alpha}_j\psi)\tilde{\alpha}_j. \tag{33}$$

We note that, now, the requirement (28) is just

$$\tilde{\alpha}_j \psi < 1. \tag{34}$$

From (30) and (33) we get

$$\begin{aligned} |\overline{f}_{j+1}|_{\Delta_{j+1}} &\leq \overline{\alpha}_j + (1 - \psi\tilde{\alpha}_j)^{-1}\psi(\tilde{\alpha}_j + 2\overline{\alpha}_j)\tilde{\alpha}_j, \\ |\tilde{f}_{j+1}|_{\Delta_{j+1}} &\leq 2(1 - \psi\tilde{\alpha}_j)^{-1}\psi(\tilde{\alpha}_j + 2\overline{\alpha}_j)\tilde{\alpha}_j. \end{aligned} \tag{35}$$

Let us denote by $\overline{\alpha}_{j+1}$ and $\tilde{\alpha}_{j+1}$, respectively, the right-hand terms in the inequality of (35).

Claim *By a suitable selection of ψ we have*

$$2\alpha > \cdots > \overline{\alpha}_j > \overline{\alpha}_{j-1} > \cdots > \overline{\alpha}_0, \;\; 2\alpha \geq \tilde{\alpha}_0 > \tilde{\alpha}_1 > \cdots > \tilde{\alpha}_j > \cdots. \tag{36}$$

Let $0 < \eta < 1/2$ and choose ψ such that

$$\psi\alpha \leq \frac{\eta}{12 + 2\eta}. \tag{37}$$

Then, if (36) holds it does (34).

We shall prove the claim by induction. If $j = 0$ it follows from (35), (37) and (9) that

$$\overline{\alpha}_0 < \overline{\alpha}_1 < \overline{\alpha}_0 + \frac{\psi 4\alpha}{1-\psi 2\alpha} 2\alpha < \alpha\left(1 + \frac{2\eta}{3}\right) < 2\alpha,$$

$$\widetilde{\alpha}_1 < \frac{2\psi 4\alpha}{1-\psi 2\alpha}\widetilde{\alpha}_0 < \frac{2\eta}{3}\widetilde{\alpha}_0 < \widetilde{\alpha}_0.$$

Assume (36) holds till some value of j. Then

$$\overline{\alpha}_j < \overline{\alpha}_{j+1} < \overline{\alpha}_j + \frac{\psi 6\alpha}{1-\psi 2\alpha}\widetilde{\alpha}_j, \tag{38}$$

and

$$\widetilde{\alpha}_{j+1} < \frac{2\psi 6\alpha}{1-\psi 2\alpha}\widetilde{\alpha}_j < \eta\widetilde{\alpha}_j. \tag{39}$$

The relation (39) shows the second part in (36) and, furthermore, it gives

$$\widetilde{\alpha}_m \le \eta^m 2\alpha \text{ for } m = 0, \ldots, j+1. \tag{40}$$

Going back to (38) we have

$$\overline{\alpha}_{j+1} < \overline{\alpha}_j + \frac{\eta}{2}\widetilde{\alpha}_j < \cdots < \overline{\alpha}_0 + \frac{\eta}{2}(\widetilde{\alpha}_0 + \cdots + \widetilde{\alpha}_j) < \alpha + \frac{\eta}{2}\sum_{m=0}^{j}\eta^m 2\alpha < \alpha\left(1 + \frac{\eta}{1-\eta}\right) < 2\alpha.$$

This ends the proof of the Claim.

For convenience we select $\eta = e^{-1} = \exp(-1)$. By going to (32) we see that one can take $\delta = \overline{K}\varepsilon^{c_3}$, where $c_3 = (1+\tau)^{-1}$, $\overline{K} = (2K(12e+2))^{c_3}$. As we want our final domain to contain $(D + \Delta/2) \times (\mathbb{T}^p + \Delta/2)$, it follows from (10) that the number of averaging steps is $N = N(\varepsilon) = E\left(\frac{\Delta/2}{\delta}\right) > \frac{\Delta/2}{\overline{K}\varepsilon^{c_3}} - 1$, where $E(\ )$ denotes the integer part. The final vector field, after these transformations, is $\varepsilon(\overline{f}_N + \widetilde{f}_N)$. The formulae (36) and (40) give $|\overline{f}_N|_{\Delta/2} < 2|f|_\Delta$ and $|\widetilde{f}_N|_{\Delta/2} < 2\alpha e^{-N} < c_1\exp(-c_2\varepsilon^{-c3})$, where $c_1 = 2\alpha\varepsilon$, $c_2 = \frac{\Delta/2}{\overline{K}}$. This proves the formula (3) and ends the proof of Theorem A.

3 An example: first step of averaging and existence of a quasiperiodic solution

As an illustration of the averaging theorem and to gain insight in the splitting under fast quasiperiodic forcing we consider the example (4). The change of the time variable $t = \tau/\varepsilon$ leads to the equation

$$\ddot{x} = \varepsilon^2 \sin(x + \sin t + \sin(\gamma t)), \quad \gamma = (1+\sqrt{5})/2. \tag{41}$$

Equivalently this can be written as

$$\dot{x} = \varepsilon y, \quad \dot{y} = \varepsilon \sin(x + \sin t + \sin(\gamma t)), \tag{42}$$

which is of the form (1) and, furthermore, is a Hamiltonian vector field with Hamiltonian function

$$H(x,y,t,\varepsilon)=\varepsilon\left(\frac{y^2}{2}+\cos(x+\sin t+\sin(\gamma t))\right). \tag{43}$$

As stated in Theorem B, the averaging process can be carried out, in this case, by using canonical transformations. It is useful, to discuss the behaviour of (41), to do first one step of averaging. To this end we split (43) in the averaged and the purely quasiperiodic part. Let us use the angles θ_1 and θ_2 to denote t and γt. Then (43) reads

$$\begin{aligned} H(x,y,\theta,\varepsilon) &= \varepsilon\left(\frac{y^2}{2}+\cos x\cos(\sin\theta_1+\sin\theta_2)-\sin x\sin(\sin\theta_1+\sin\theta_2)\right)= \\ &= \varepsilon(\overline{H}_0+\widetilde{H}_0), \end{aligned} \tag{44}$$

where $\overline{H}_0=\frac{y^2}{2}+\cos x\overline{\cos(\sin\theta_1+\sin\theta_2)}$. Note that $\widetilde{H}_0$ does not depend on y. This makes the first average easier. More explicitely (44) is of the form $\varepsilon(\frac{y^2}{2}+A\cos x+ m_1(t)\sin x+m_2(t)\cos x)$, where $\overline{m}_1=\overline{m}_2=0$.

We change from (x,y) to a new set of variables (ξ,η) by means of the canonical transformation $x=\frac{\partial S}{\partial y}$, $\eta=\frac{\partial S}{\partial \xi}$, where $S=S(\xi,y)$ is a generating function of the form $S=\xi y+\varepsilon S_1(\xi,\eta,t)$ and S_1 is determined by $\frac{\partial S_1}{\partial t}=\widetilde{H}_0$, $\overline{S_1}=0$. Hence $S_1= M_1(t)\sin\xi+M_2(t)\cos\xi$, where M_j, $j=1,2$, is the primitive of m_j with zero average.

Hence, the canonical change is given by

$$x=\xi,\quad \eta=y+\varepsilon(\cos\xi M_1(t)-\sin\xi M_2(t)),$$

leading to the Hamiltonian after the first average, which can be obtained in full explicit form

$$\begin{aligned} &\varepsilon\left(\frac{\eta^2}{2}+A\cos\xi\right)+\varepsilon^2\eta\left(-\cos\xi M_1(t)+\sin\xi M_2(t)\right)+ \\ &\quad+\varepsilon^3\left(-\sin(2\xi)\frac{1}{2}M_1M_2+\frac{1}{4}\cos(2\xi)(M_1^2-M_2^2)\right)=\varepsilon H_1+\varepsilon^2H_2+\varepsilon^3H_3. \end{aligned} \tag{45}$$

To determine the values of A, $m_1(t)$ and $m_2(t)$ we use the expansions (see Abramowitz and Stegun[1])

$$\begin{aligned} \cos(z\sin\alpha) &= J_0(z)+2\sum_{k\geq 1}J_{2k}(z)\cos(2k\alpha), \\ \sin(z\sin\alpha) &= 2\sum_{k\geq 0}J_{2k+1}(z)\sin(2(k+1)\alpha), \end{aligned} \tag{46}$$

where $J_n(z)$ stands for the Bessel function of first kind, order n and argument z. Therefore $A=(J_0(1))^2$ and M_1, M_2 can be easely obtained from (44), (45) and (46) as

$$-M_1(t) = 2J_0\sum_{k\geq 0}J_{2k+1}\left(\frac{\cos((2k+1)\theta_1)}{2k+1}+\frac{\cos((2k+1)\theta_2)}{(2k+1)\gamma}\right)+$$

$$
\begin{aligned}
&+2\sum_{k\geq 0}\sum_{l\geq 1} J_{2k+1}J_{2l}\left(\frac{\cos((2k+1)\theta_1+2l\theta_2)}{2k+1+2l\gamma}+\frac{\cos((2k+1)\theta_1-2l\theta_2)}{2k+1-2l\gamma}+\right.\\
&\left.+\frac{\cos(2l\theta_1+(2k+1)\theta_2)}{2l+(2k+1)\gamma}+\frac{\cos(-2l\theta_1+(2k+1)\theta_2)}{-2l+(2k+1)\gamma}\right),
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
M_2(t) \;=\; & 2J_0\sum_{k\geq 1} J_{2k}\left(\frac{\sin(2k\theta_1)}{2k}+\frac{\sin(2k\theta_2)}{2k\gamma}\right)+\\
&+2\sum_{k,l\geq 1} J_{2k}J_{2l}\left(\frac{\sin(2k\theta_1+2l\theta_2)}{2k+2l\gamma}+\frac{\sin(2k\theta_1-2l\theta_2)}{2k-2l\gamma}\right)+\\
&+2\sum_{k,l\geq 0} J_{2k+1}J_{2l+1}\left(\frac{\sin((2k+1)\theta_1+(2l+1)\theta_2)}{2k+1+(2l+1)\gamma}-\right.\\
&\qquad\left.-\frac{\sin((2k+1)\theta_1-(2l+1)\theta_2)}{2k+1-(2l+1)\gamma}\right),
\end{aligned}
$$

where all the Bessel functions are evaluated at 1.

At this point we want to do some remarks.

a) The example (4) can be generalized to

$$x''=\sin(x+\varepsilon^p(a\sin(\tau/\varepsilon)+b\sin(\gamma\tau/\varepsilon))),\quad a,b\in\mathbb{R},\quad p\geq 0. \tag{48}$$

Then H_1 is replaced by $\frac{\eta^2}{2}+J_0(a\varepsilon^p)J_0(b\varepsilon^p)\cos\xi$. If ε goes to zero and $p>0$ the numerical coefficient tends to 1. If $p=0$ we find $J_0(a)\cdot J_0(b)$. The character of the fixed points of H_1, $\xi=0 \pmod{\pi}$, $\eta=0$, depends on $J_0(a)\cdot J_0(b)$. If the product is positive then $(0,0)$ is hyperbolic and $(\pi,0)$ elliptic. They exchange the stability for negative values of the product.

b) A lot of papers in the last years deal with equations like

$$x''=\sin x+\varepsilon^q\sin(\tau/\varepsilon),\quad q\in\mathbb{R}, \tag{49}$$

(see Holmes et al.[8], Gelfreich[7], Delshams and M. Seara[4]). The difficulties seem to appear for $q\leq 0$. However, consider again (48). By introducing $z=x+\varepsilon^p(a\sin(\tau/\varepsilon)+b\sin(\gamma\tau/\varepsilon))$ we have

$$z''=\sin z-\varepsilon^{p-2}(a\sin(\tau/\varepsilon)+b\gamma^2\sin(\gamma\tau/\varepsilon)),) \tag{50}$$

which reduces to the form (49) if $q=p-2$, $a=-1$, $b=0$. But (50) has a Hamiltonian like (45) with arguments of the Bessel functions equal to $a\varepsilon^p$ or to $b\varepsilon^p$. Instead of (49) we have, introducing $z=x+\varepsilon^{q+2}\sin(\tau/\varepsilon)$, the equation $z''=\sin(z-\varepsilon^{q+2}\sin(\tau/\varepsilon))$. Proceeding as in the passage from (4) to (45) we get a Hamiltonian like (45) but now

$$
\begin{aligned}
&A=J_0(\varepsilon^{q+2}),\quad M_1(t)=-2\sum_{k\geq 0}J_{2k+1}(\varepsilon^{q+2})\frac{\cos((2k+1)t)}{2k+1},\\
&M_2(t)=2\sum_{k\geq 1}J_{2k}(\varepsilon^{q+2})\frac{\sin(2kt)}{2k}.
\end{aligned}
\tag{51}
$$

From (51) it follows that, for any $q \geq -2$, there is no difficulty in the study of (49), the Hamiltonian (45) being a relatively small perturbation of a Hamiltonian, εH_1, with H_1 autonomous and with well defined non degenerated limit when ε goes to zero. On the other hand, if $q < -2$ then $J_0(\varepsilon^{q+2})$ has infinitely many oscillations when ε goes to zero.

We return to the study of (45). The first thing is to prove the existence of an invariant two-dimensional torus which replaces the hyperbolic fixed point $(0,0)$ appearing in H_1. To this end we state the following result.

Theorem C. *Let us consider*

$$\dot{x} = \varepsilon[(A(\varepsilon) + \varepsilon Q(\theta, \varepsilon))x + \varepsilon g(\theta, \varepsilon) + h(x, \theta, \varepsilon)] \tag{52}$$

where $|\varepsilon| \leq \varepsilon_0$, x belongs to a closed ball of radius r around 0 in $\mathbb{R}^n$, $\theta \in \mathbb{T}^p + \Delta$, $\theta = \omega t$. Assume $A(\varepsilon)$ is a uniformly hyperbolic matrix (i.e. $|\mathrm{Re}\ \lambda| > a > 0$, $\forall \lambda \in \mathrm{Spec}\ A(\varepsilon)$, $|\varepsilon| \leq \varepsilon_0$) and any two eigenvalues of $A(\varepsilon)$, $\lambda_i(\varepsilon)$, $\lambda_j(\varepsilon)$, satisfy $\lambda_i(\varepsilon) \neq \lambda_j(\varepsilon)$ if $i \neq j$ $\forall |\varepsilon| \leq \varepsilon_0$. Furthermore assume ω satisfies a diophantine condition $|(k, \omega)| \geq \frac{c}{|k|^\tau}$ $\forall k \in \mathbb{Z}^p \setminus \{0\}$ for some $c, \tau > 0$ and A, Q, g, h are bounded with respect to ε, Q, g and h are analytic with respect to θ in $\mathbb{T}^p + \Delta$, h is analytic with respect to x and $h(0, \theta, \varepsilon) = 0$, $D_x h(0, \theta, \varepsilon) = 0$ $\forall \theta \in \mathbb{T}^p + \Delta$, $|\varepsilon| \leq \varepsilon_0$.

Then there exists $\varepsilon_1 \leq \varepsilon_0$ such that for any ε with $0 < |\varepsilon| < \varepsilon_1$, there exists an affine change of coordinates $y = R(\theta)x + v(\theta)$, depending analytically on θ for $\theta \in \mathbb{T}^p + \Delta/2$, transforming (52) to

$$\dot{y} = \varepsilon(A_\infty(\varepsilon)y + h_\infty(y, \theta, \varepsilon)), \tag{53}$$

where $h_\infty = O(|y|^2)$ uniformly in θ, ϵ. The matrix A_∞ is hyperbolic. Furthermore the equation (52) has a quasiperiodic solution, $x_\varepsilon(t)$, with basic frequencies $\omega_1, \ldots, \omega_p$ and $|x_\varepsilon| = O(\varepsilon^2)$ for $\varepsilon \to 0$.

The proof is close to the one of similar results that can be found in Bogoljubov et al.[2] or in Jorba and Simó.[9] The main idea is to do a sequence of transformations. After each one the equation (52) is of the form

$$\dot{x}_j = \varepsilon[(A_j(\varepsilon) + \varepsilon^{2^j} Q_j(\theta, \varepsilon))x_j + \varepsilon^{2^j} g_j(\theta, \varepsilon) + h_j(x_j, \theta, \varepsilon)]. \tag{54}$$

In (54) x_j is the current variable and the variables (x_j, θ) range on a somewhat smaller domain, $B_{r_j}(0) \times (\mathbb{T}^p + \Delta_j)$, where r_j and Δ_j have positive limits when j goes to ∞. In this case the difference $\Delta_j - \Delta_{j+1}$ can be chosen of the form $\frac{d}{j^2}$, for a suitable $d > 0$, for instance. Despite the norms of Q_j and g_j increase without bound for $j \to \infty$, the functions $\varepsilon^{2^j}|Q_j|$ and $\varepsilon^{2^j}|g_j|$ behave like η^{2^j} for some $0 < \eta < 1$ provided ε is small enough. The existence of the normally hyperbolic quasiperiodic solution follows from the fact that (53) has 0 as solution, by going back through the sequence of transformations. Full details on the proof and on the properties of the invariant manifold of the solution will appear elsewhere.[13]

Returning to our example (41), the quasiperiodic solution fills up densely a two-dimensional torus, $\mathbb{T}^2$. We shall talk loosely of $\mathbb{T}^2$ as the solution replacing the fixed point. The related manifolds, $W^{U,S}_{\mathbb{T}^2}$, are close to the separatrices of H_1 in (45). From the averaging theorem it follows that the splitting must be bounded by some expression like (3) (see Fontich and Simó[5,6] and references therein for the proximity argument and simpler cases). To check the behaviour of (41) we shall carry out some symbolic and numerical computations in the next section.

4 Approximations of the invariant torus and their invariant manifolds. Numerical results.

To obtain approximations of the invariant torus and their invariant manifolds we can proceed as explained before. That is, we can do some steps of the averaging process and then some ones along the lines of the proof of Theorem C. However we shall proceed, formally, in a simpler and more direct way. In contrast to the approach following the proofs of those Theorems, we do not claim the method proposed here is convergent.

We start directly with equation (41): $\ddot{x} = \varepsilon^2 \sin(x + \sin t + \sin(\gamma t))$. We expand it as

$$\ddot{x} = \varepsilon^2 \sum_{k\geq 0} x^k f_k(\overline{\theta}_1, \overline{\theta}_2), \tag{55}$$

where now $\overline{\theta}_1 = \exp(it)$, $\overline{\theta}_2 = \exp(i\gamma t)$, i.e. $\overline{\theta}_j = \exp(i\theta_j)$. The symbolic computation of the quasiperiodic solution is done by keeping $\overline{\theta}_1$, $\overline{\theta}_2$ as symbols and using complex numerical coefficients, for instance in FORTRAN double precision. If we keep terms up to 10^{-16}, as $\sin t + \sin(\gamma t)$ can reach any value close to 2 and $|t|$, $|\gamma t|$ can be reduced to something bounded by π, we use powers up to order 28. Let $C = \cos(\sin t + \sin(\gamma t))$, $S = \sin(\sin t + \sin(\gamma t))$ and $\sin t = \frac{1}{2i}(\overline{\theta}_1 - \overline{\theta}_1^{-1})$, $\sin(\gamma t) = \frac{1}{2i}(\overline{\theta}_2 - \overline{\theta}_2^{-1})$. Then $f_k = (-1)^{E(k/2)}\frac{1}{k!}\binom{C}{S}$, where C is used for k odd and S for k even.

We look for a formal solution expanded in even powers of ε:

$$x = \sum_{j\geq 1} \varepsilon^{2j} x_j(\overline{\theta}_1, \overline{\theta}_2). \tag{56}$$

By inserting (56) in (55) we get $\sum_{j\geq 1} \varepsilon^{2j}\ddot{x}_j = \varepsilon^2 \sum_{k\geq 0} f_k(\sum_{l\geq 1}\varepsilon^{2l}x_l)^k$ and, in particular, $\ddot{x}_1 = f_0$. When x_{n-1} has been obtained we compute the required products $x_{l_1}, x_{l_2}, \ldots, x_{l_s}$ with $l_1 + \cdots + l_s = n-1$, to obtain x_n by a double integration. If we have memory enough in the computer it is better to keep for x^k the terms of order $s = k, \ldots, n-1$ in ε^2 and actualize them to order n (computing also the terms of order n in ε^2 in the expansion of x^n) when x_n is available.

To obtain the invariant manifolds we split (55) as two first order equations

$$\dot{x} = \varepsilon y, \quad \dot{y} = \varepsilon \sin(x + \sin t + \sin(\gamma t)). \tag{57}$$

Then, from (56) we obtain $y = \varepsilon^{-1}\sum_{j\geq 1}\varepsilon^{2j}x_j$. Let

$$q^{(0)} = \sum_{j\geq 1}\varepsilon^{2j}x_j, \quad p^{(0)} = \sum_{j\geq 1}\varepsilon^{2j-1}\dot{x}_j, \tag{58}$$

the approximation of the quasiperiodic solution. We introduce

$$x = q^{(0)} + \xi, \quad y = p^{(0)} + \eta. \tag{59}$$

Then, as $q^{(0)}$, $p^{(0)}$, as defined by (58), satisfy the system (57), we get

$$\begin{aligned} \dot{\xi} &= \varepsilon\eta, \\ \dot{\eta} &= \varepsilon[\sin(q^{(0)}+\sin t+\sin(\gamma t)+\xi)-\sin(q^{(0)}+\sin t+\sin(\gamma t))] = \\ &= \varepsilon(\widehat{C}\xi-\widehat{S}\frac{\xi^2}{2!}-\widehat{C}\frac{\xi^3}{3!}+\ldots)=\varepsilon\sum_{k\geq 1}\xi^k m_k(\overline{\theta}_1,\overline{\theta}_2,\varepsilon), \end{aligned} \tag{60}$$

where

$$\widehat{C}=\cos(q^{(0)}+\sin t+\sin(\gamma t)), \quad \widehat{S}=\sin(q^{(0)}+\sin t+\sin(\gamma t)). \tag{61}$$

Those are available in powers (positive and negative, of course) of $\overline{\theta}_1$, $\overline{\theta}_2$ when (58) is available.

We search for the invariant manifold (either the unstable or the stable one) writting η as a power series in ε, ξ and coefficients sums of powers of $\overline{\theta}_1$, $\overline{\theta}_2$:

$$\eta=\sum_{k\geq 1}\xi^k p^{(k)}(\overline{\theta}_1,\overline{\theta}_2,\varepsilon)=\sum_{k\geq 1}\xi^k\sum_{l\geq 0}\varepsilon^l p_{k,l}(\overline{\theta}_1,\overline{\theta}_2)=\sum_{k\geq 1}\xi^k\sum_{l\geq 0}\varepsilon^l\sum_{r,s}p_{k,l,r,s}\overline{\theta}_1^r\overline{\theta}_2^s. \tag{62}$$

The invariance of (62) by the equation (60) gives

$$\begin{aligned} \varepsilon\sum_{k\geq 1}\xi^k\sum_{s\geq 0}\varepsilon^s m_{ks}(\overline{\theta}_1,\overline{\theta}_2) &= \sum_{k\geq 1}\xi^k\sum_{l\geq 0}\varepsilon^l i\left(\frac{\partial p_{kl}}{\partial\overline{\theta}_1}\overline{\theta}_1+\gamma\frac{\partial p_{kl}}{\partial\overline{\theta}_2}\overline{\theta}_2\right)+ \\ &+\sum_{k\geq 1}k\xi^{k-1}\sum_{l\geq 0}\varepsilon^l p_{kl}(\overline{\theta}_1,\overline{\theta}_2)\cdot\varepsilon\sum_{r\geq 1}\xi^r\sum_{m\geq 0}\varepsilon^m p_{rm}(\overline{\theta}_1,\overline{\theta}_2). \end{aligned} \tag{63}$$

If $\overline{p}_{k,l}$ denotes the average of $p_{k,l}$ then we write $p_{k,l}=\overline{p}_{k,l}+\widetilde{p}_{k,l}$. By looking at the terms in ε^0 in (63) we have $\widetilde{p}_{k,0}=0$. Equating the terms in ξ^u for $u\geq 1$ and among them the terms in ε^v, $v\geq 1$, in (63) we obtain, recurrently, the functions $\widetilde{p}_{u,v}$ and $\overline{p}_{u,v-1}$. The only algebraic condition to be satisfied to make the recurrence possible is $\overline{p}_{1,0}\neq 0$. For $u=v=1$ we obtain $\overline{p}_{1,0}^2=\overline{m}_{1,1}$. The value of $\overline{m}_{1,1}$ is the average of $\widehat{C}$ in (60) when $q^{(0)}$ is set equal to zero, that is, $(J_0(1))^2$. By taking $\overline{p}_{1,0}=J_0(1)$ we obtain the unstable manifold. If $\overline{p}_{1,0}=-J_0(1)$ we get the stable one. Then all the successive terms are uniquely determined. Furthermore, if $p_{k,l}^U$ and $p_{k,l}^S$ denote the functions $p_{k,l}$ appearing in the expansion of the unstable and stable manifolds, respectively, from the symmetries of the functions in (61) we have $p_{kl}^S=p_{kl}^U(-1)^{1+l}$. The function p_{kl}, as a function of $\overline{\theta}_1$ and $\overline{\theta}_2$, has the parity of $1+k+l$. Ax η must be real for ξ, ε and t real, it follows from (62) that $p_{k,l,-r,-s}$is the complex conjugate of $p_{k,l,r,s}$. This, together with the symmetries mentioned above save computer memory: It is enough to store terms with $r\geq 0$ and if $r=0$, with $s\geq 0$.

The previous algorithms have been implmented to obtain the coefficients $p_{k,l,r,s}$ with $1\leq k\leq 12$, $0\leq l\leq 24$, $|r|,|s|\leq 28$. In this way we obtain a formal representation of the invariant torus and the related manifolds.

To extend numerically the local manifolds obtained so far, given ε, ξ and t, we substitute in (56) and (62). As $(t,\gamma t)$ (mod 2π) fill $\mathbb{T}^2$ densely when t ranges in $\mathbb{R}$, for practical purposes we have taken a different approach. Initial values of the angles

(θ_1, θ_2), say (ψ_1, ψ_2), are taken in $\mathbb{T}^2$ and t is set to zero. Then the system to be integrated is

$$\dot{x} = \varepsilon y, \quad \dot{y} = \varepsilon \sin(x + \sin\theta_1 + \sin\theta_2), \quad \dot{\theta}_1 = 1, \quad \dot{\theta}_2 = \gamma, \tag{64}$$

with the initial conditions

$$t = 0, \quad \theta_1 = \psi_1, \quad \theta_2 = \psi_2, \quad x = q^{(0)}(\overline{\theta}_1, \overline{\theta}_2, \varepsilon) + \xi, \quad y = \sum_{k \geq 0} \xi^k p^{(k)}(\overline{\theta}_1, \overline{\theta}_2, \varepsilon), \tag{65}$$

as follows from (59) and (62). Before globalizing the manifolds the following tests have been performed:

a) If $\xi = 0$ in (64) we have the invariant torus. Given ε, ψ_1, ψ_2 we can compute x, y and also, by formal derivation, the values of $\dot{x}$, $\dot{y}$. These values have been checked against direct evaluation in (42). The agreement is uniformly very good in $\mathbb{T}^2$ until $\varepsilon = 0.07$ (say, errors less than 10^{-12} and much less if ε is very small) and acceptable till $\varepsilon = 0.1$ (errors less than 10^{-8}). No direct numerical integration of (64) with the initial conditions (65) has been done because of the hyperbolic character of $\mathbb{T}^2$.

b) If $\xi \neq 0$ is small (say $\xi = 0.02 \div 0.04$) and ε is fixed, we can integrate (64) with the conditions (65) for a moderate time interval (we have selected $t = t_f = 2\pi$ as final time). Then the final values x_f, y_f, $\theta_{1,f}$, $\theta_{2,f}$ have been obtained. From the third equation in (65) the current value of ξ, ξ_f, is obtained. Then the value, y_w, of y from the fourth equation in (65), evaluated at ε, ξ_f, $\theta_{1,f}$, $\theta_{2,f}$ is obtained. This value is compared with y_f. The agreement is similar to the one described for $\mathbb{T}^2$ or even better.

After the tests have been passed we have globalized the manifold $W^U_{\mathbb{T}^2}$ till x reaches the value π. Starting with values of ψ_1, ψ_2 in an equispaced lattice in $\mathbb{T}^2$ (typically 100×100 points are used) we obtain as final values $y = y^U$, $\theta_1 = \varphi_1$, $\theta_2 = \varphi_2$. Of course, the values of (φ_1, φ_2) are not equispaced in $W^U_{\mathbb{T}^2} \cap \{x = \pi\}$, because the travelling times are different. However, the map $(\psi_1, \psi_2) \to (\varphi_1, \varphi_2)$ (for a fixed initial value of ξ) is a diffeomorphism. It is not difficult to set up an interpolation procedure to obtain y^U as a function of (φ_1, φ_2) in $\mathbb{T}^2$ with enough accuracy: $y^U = m^U(\varphi_1, \varphi_2, \varepsilon)$. If we do the same process for $W^S_{\mathbb{T}^2}$, we obtain y^S in $W^S_{\mathbb{T}^2} \cap \{x = \pi\}$ (by backwards integration in time taking the correct "branch" of $W^S_{\mathbb{T}^2}$), given by $y^S = m^S(\varphi_1, \varphi_2, \varepsilon)$. By symmetry one has $m^S(\varphi_1, \varphi_2, \varepsilon) = m^U(2\pi - \varphi_1, 2\pi - \varphi_2, \varepsilon)$.

The splitting between the two manifolds is measured by the function

$$m(\varphi_1, \varphi_2, \varepsilon) = m^U(\varphi_1, \varphi_2, \varepsilon) - m^S(\varphi_1, \varphi_2, \varepsilon). \tag{66}$$

A measure of the "size" of the splitting can be the function

$$d(\varepsilon) = \max_{(\varphi_1, \varphi_2) \in \mathbb{T}^2} |m(\varphi_1, \varphi_2, \varepsilon)|.$$

Table I

ε	$10^6 \times d(\varepsilon)$	ε	$10^6 \times d(\varepsilon)$	ε	$10^6 \times d(\varepsilon)$
0.098	12309.4645	0.072	2148.8343	0.046	88.72111
0.096	11097.3677	0.070	1795.1467	0.044	57.03980
0.094	9963.8012	0.068	1486.7868	0.042	38.12663
0.092	8907.2905	0.066	1220.0508	0.040	24.57281
0.090	7926.2890	0.064	991.2397	0.038	15.19353
0.088	7019.1189	0.062	796.7054	0.036	8.97125
0.086	6183.9246	0.060	632.8989	0.034	5.03753
0.084	5418.6365	0.058	496.4097	0.032	2.67983
0.082	4720.9480	0.056	383.9985	0.030	1.34597
0.080	4088.3052	0.054	292.6163	0.028	0.63507
0.078	3517.9080	0.052	219.4063	0.026	0.27932
0.076	3006.7227	0.050	161.6829	0.024	0.11257
0.074	2551.5056	0.048	116.9086	0.022	0.04077

It is enough to let φ_1 range in $[0, 2\pi]$ and φ_2 in $[0, \pi]$. The Table I gives the values of $d(\varepsilon)$ obtained numerically for several values of ε. With the accuracy used on the computations we can only be confident on the results for the range of values of ε displayed here.

We note also that the values of φ_1, φ_2 such that $m(\varphi_1, \varphi_2, \varepsilon) = 0$ correspond to homoclinic orbits to the invariant $\mathbb{T}^2$. The figure 1 shows some plots of the sets $\mathcal{C}_h(\varepsilon) = \{(\varphi_1, \varphi_2) \in \mathbb{T}^2 | m(\varphi_1, \varphi_2, \varepsilon) = 0\}$ for different values of ε. We remark that the topology of $\mathcal{C}_h$ changes with ε. This is quite natural. In the limit when ε is very small the dynamics of (x, y) in (64) is quite slow. If a point belongs to $\mathcal{C}_h(\varepsilon)$ then the transport of that point for the irrational flow on the torus must be rather close to points belonging to $\mathcal{C}_h(\varepsilon)$. We shall return to this point in Section 6.

5 The first order splitting

In this section we shall compute formally, the "first order" terms of the splitting by using Melnikov's method after one step of averaging, i.e., using (45). So we obtain a theoretical value for the function m defined in (66). We do not claim this approximation is rigourously justified. However in figure 2 we plot the data of Table I against the predictions of the Melnikov's approach observing a good agreement. So, we will pursue in this direction, assuming that the Melnikov function gives a good description of the behaviour, and we shall derive several consequences. The ones concerning the bifurcation of the set of primary homoclinic points in $x = \pi$ are presented in Section 6.

From (45) the variation of H_1 due to the perturbations H_2 and H_3 is

$$\Delta H_1(\varphi_1, \varphi_2, \varepsilon) = \int_{\mathbb{R}} (\varepsilon^2 [H_1, H_2] + \varepsilon^3 [H_1, H_2])|_{\sigma(t, \varphi_1, \varphi_2, \varepsilon)}\, dt, \tag{67}$$

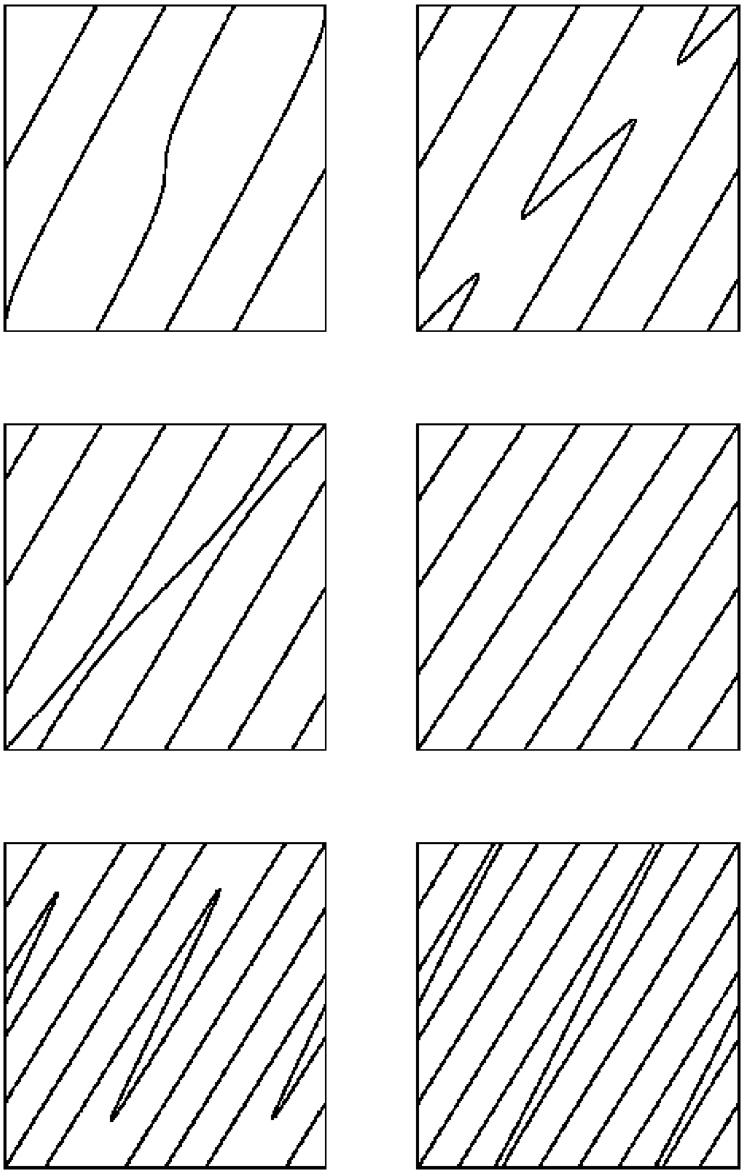

Figure 1: The set $\mathcal{C}_h(\varepsilon)$ of primary homoclinic points in the section $x = \pi$ for different values of ε. a) $\varepsilon = 0.098$; b) $\varepsilon = 0.08$; c) $\varepsilon = 0.078$; d) $\varepsilon = 0.05$; e) $\varepsilon = 0.0308$; f) $\varepsilon = 0.0304$

where the integration is carried out along the separatrix of H_1 and, in (67), the angles θ_1, θ_2 appearing in the functions M_1 and M_2 have phases φ_1, φ_2 at $\xi = \pi$. On the other hand the separatrix of H_1 is given by

$$\frac{\eta^2}{2} + J_0^2 \cos\xi = J_0^2. \tag{68}$$

By differentiation with respect to η at $\xi = \pi$ we have

$$m(\varphi_1, \varphi_2, \varepsilon) = \Delta\eta \simeq \frac{\Delta H_1}{\eta_0}, \tag{69}$$

where $\Delta H_1$is given by (67) and η_0 by $2J_0$. The result from (69) has to be compared with the values in (66). Here, and in what follows, the Bessel functions are evaluated at the point 1 unless explicitely stated.

The brackets in (67) are easely computed and the integrand is given by

$$\begin{aligned} &\varepsilon^2 J_0^2 (M_1(-2\sin\xi + 3\sin\xi\cos\xi) + M_2(-2\cos\xi + 2\cos^2\xi - \sin^2\xi)) + \\ &\qquad + \varepsilon^2 2J_0 \sin\frac{\xi}{2}\left(\cos 2\xi M_1 M_2 + \sin 2\xi \frac{M_1^2 - M_2^2)}{2}\right). \end{aligned} \tag{70}$$

The separatrix is parametrized by

$$\xi = 4\arctan(e^{\varepsilon J_0 t}), \tag{71}$$

and η is obtained from (68). By substitution of (71) in (70) the formula (67) is expressed as

$$\begin{aligned} &\int_{\mathbb{R}} \varepsilon^2 J_0^2 \left[M_1(t)\left(-\frac{\sinh(u)}{\cosh^2(u)} + \frac{12\sinh(u)}{\cosh^4(u)}\right) + M_2(t)\left(-\frac{8}{\cosh^2(u)} + \frac{12}{\cosh^4(u)}\right)\right] + \\ &\qquad +2J_0\varepsilon^3\left[M_1(t)M_2(t)\left(\frac{1}{\cosh(u)} - \frac{8}{\cos^3(u)} + \frac{8}{\cos^5(u)}\right) + \right. \\ &\qquad\qquad \left. +\frac{1}{2}(M_1(t)^2 - M_2(t)^2)\left(-\frac{4\sinh(u)}{\cosh^3(u)} + \frac{8\sinh(u)}{\cosh^5(u)}\right)\right], \end{aligned} \tag{72}$$

where $u = \varepsilon J_0 t$. The functions $M_1(t)$ and $M_2(t)$ in (72) are of the form

$$\begin{aligned} M_1(t) &= \sum_s \mathrm{coef}_s \cos(\beta_s t + p_s\varphi_1 + q_s\varphi_2) = \\ &= \sum_s \mathrm{coef}_s (\cos(\beta_s t)\cos(p_s\varphi_1 + q_s\varphi_2) - \sin(\beta_s t)\sin(p_s\varphi_1 + q_s\varphi_2)), \\ M_2(t) &= \sum_s \mathrm{coef}_s \sin(\beta_s t + p_s\varphi_1 + q_s\varphi_2) = \\ &= \sum_s \mathrm{coef}_s (\sin(\beta_s t)\cos(p_s\varphi_1 + q_s\varphi_2) - \cos(\beta_s t)\sin(p_s\varphi_1 + q_s\varphi_2)). \end{aligned} \tag{73}$$

Due to the parities of the terms in (72) and (73), the only terms giving a non zero contribution to (72) are the ones which appear in (73) multiplying $\sin(p_s\varphi_1 + q_s\varphi_2)$. The terms M_1, M_2 behave like M_2 and $M_1^2 - M_2^2$ like M_1. We summarize the values of the integrals appearing in (72):

$$
\begin{aligned}
\int_{\mathbb{R}} \sin(\beta t)\left(-\frac{\sinh(u)}{\cosh^2(u)} + \frac{12\sinh(u)}{\cosh^4(u)}\right) dt &= \frac{2\pi\beta^3}{\varepsilon^4 J_0^4 \cosh(\frac{\beta\pi}{2J_0\varepsilon})},\\
\int_{\mathbb{R}} \cos(\beta t)\left(-\frac{8}{\cosh^2(u)} + \frac{12}{\cosh^4(u)}\right) dt &= \frac{2\pi\beta^3}{\varepsilon^4 J_0^4 \sinh(\frac{\beta\pi}{2J_0\varepsilon})},\\
\int_{\mathbb{R}} \cos(\beta t)\left(\frac{1}{\cosh(u)} - \frac{8}{\cos^3(u)} + \frac{8}{\cos^5(u)}\right) dt &= \left(\frac{\beta^4}{3J_0^5\varepsilon^5} - \frac{2\beta^2}{3J_0^3\varepsilon^3}\right)\frac{\pi}{\cosh(\frac{\beta\pi}{2J_0\varepsilon})},\\
\int_{\mathbb{R}} \sin(\beta t)\left(-\frac{4\sinh(u)}{\cosh^3(u)} + \frac{8\sinh(u)}{\cosh^5(u)}\right) dt &= \left(\frac{\beta^4}{3J_0^5\varepsilon^5} - \frac{2\beta^2}{3J_0^3\varepsilon^3}\right)\frac{\pi}{\sinh(\frac{\beta\pi}{2J_0\varepsilon})},
\end{aligned}
\tag{74}
$$

where, as before, $u = \varepsilon J_0 t$. As we shall see later, for a fixed value of ε, the dominant terms in (67), and hence in (69), have values of β which are roughly of the orfder $\varepsilon^{1/2}$, while the ones coming from H_3 have a factor ε^0. However the coefficients in front of those terms (like coef$_s$ in (73)) having H_3 as source are larger, and they become much more important when ε decreases. So we have to keep both the contribution of H_2 and H_3 in (67).

Finally we obtain for ΔH_1 an expression of the form

$$
\sum_s c_s(p_s, q_s, \beta_s)\varepsilon^{r_s}\frac{1}{F(\frac{\beta_s\pi}{2J_0\varepsilon})}\sin(p_s\varphi_1 + q_s\varphi_2), \tag{75}
$$

where $p_s, q_s \in \mathbb{Z}$, $\beta_s = p_s + \gamma q_s$, the coefficients c_s do not depend on ε and F denotes one of the hyperbolic functions cosh or sinh. Looking at (75) we see that each one of the terms is of the form $\exp(-c/\varepsilon)$, where c is a positive constant. However, for a fixed ε, if β_s is big the contribution of this term is negligible. If β_s is very small, coming from a small divisor associated to suitable values of p_s and q_s, then $1/F$ is big, but the constant c_s turns out to be very small. The dominant terms in (75) shall be studied later.

The figure 2 shows the data of Table I against the maximum values of the function given by (69) by using (72), (74) and the explicit expressions given by (47). Despite the complexity of the formulae only a few of the harmonics present in (75) have relevant coefficients, specially when ε becomes small. Table II gives the dominant harmonics and the final coefficients in (75) for a few values of ε.

We see that the dominant contributions come from values (p_s, q_s) of the form $(F_j, -F_{j-1})$, where F_j denote the Fibonacci numbers. This is natural: this is the way to have β_s small, and then $1/F$ relatively big without having a coefficient c_s too small. To be explicit we define $F_1 = 1$, $F_2 = 2$, $F_j = F_{j-1} + F_{j-2}$. A short analysis shows that the terms coming from M_1M_2 and $M_1^2 - M_2^2$ are the most important. As those harmonics, of the form

$$
\sin(F_j\varphi_1 - F_{j-1}\varphi_2), \tag{76}
$$

Table II

ε	p	q	coef	ε	p	q	coef	ε	p	q	coef
0.03	2	-1	-.185166E-8	0.01	3	-2	.805440E-19	0.003	8	-5	-.107506E-30
	3	-2	.953812E-6		5	-3	-.205424E-13		13	-8	.728265E-29
	5	-3	-.103376E-5		8	-5	-.546092E-13	0.001	13	-8	.496686E-61
	8	-5	-.122753E-8		13	-8	.247705E-18		21	-13	.682913E-58

come from products of elements of M_1 and M_2, we have to examine the contributions of pairs (p_{s_1}, q_{s_1}), (p_{s_2}, q_{s_2}) such that

$$p_{s_1} + p_{s_2} = F_j, \quad q_{s_1} + q_{s_2} = -F_{j-1}. \tag{77}$$

Again an examination of the values of the coefficients $c_{s_i}(p_{s_i}, q_{s_i}, \beta_{s_i})$, $i = 1, 2$, with the constraints (77), shows that a large contribution comes from the choice $p_{s_1} = F_{j-1}$, $p_{s_2} = -q_{s_1} = F_{j-2}$, $-q_{s_2} = F_{j-3}$. In fact, there are several decompositions of (77) giving similar or even larger contributions. Using just this one, we underestimate, at most, by a factor of the form γ^{2j}, which turns out to be like ε^{-1}.

We recall here, for future use the explicit expression of F_j:

$$F_j = (\gamma^{j+1} + \gamma^j + (-1)^j \gamma^{-j})/(\gamma + 2), \tag{78}$$

and then $|F_j - F_{j-1}\gamma| = \gamma^{-j}$.

At this moment, to see the effect of powers of ε in the "perturbation" we consider a more general case. We shall see (48) with $a = b = 1$ instead of (4). This amounts to replace the argument 1 in the Bessel functions by ε^p. We also use the asymptotic value

$$J_n(z) \simeq \left(\frac{z}{2}\right)^n \frac{1}{n!}, \tag{79}$$

for z fixed and n large. Even for $z = 1$ this gives a relative error less than $\frac{1}{8(n+1)}$. Furthermore, for j large we can approximate (78) by

$$F_j \simeq \frac{\gamma + 1}{\gamma + 2} \gamma^j. \tag{80}$$

By using (79) and (80) and assuming β_s/ε not too small (to be checked later), the dominant part, ρ_j, of the coefficient of the harmonic (76) in (67) is given by

$$\rho_j = \frac{(\varepsilon^p)^{F_{j+1}}}{2^{F_{j+1}}(F_{j-1})!((F_{j-2})!)^2(F_{j-3})!} \frac{\gamma^{-2j-3}}{\exp(\frac{\pi\gamma^{-j}}{2\varepsilon J_0(\varepsilon^p)})}. \tag{81}$$

By taking logarithms and skipping some terms simply linear in j (where the contribution of the other decompositions in (77) is included), we have

$$\begin{aligned} \alpha_j &= \ln \rho_j \simeq \frac{\gamma+1}{\gamma+2}\left[\gamma\left(1 - \ln 2 - \ln\frac{\gamma+1}{\gamma+2}\right) + \left(1 + \frac{3}{\gamma}\right)\ln\gamma\right]\gamma^j - \\ &- \frac{\gamma+1}{\gamma+2}\gamma\ln\gamma \cdot \gamma^j \left(j - p\frac{\ln\varepsilon}{\ln\gamma}\right) - \frac{\pi\gamma^{-j}}{2\varepsilon J_0(\varepsilon^p)}, \end{aligned} \tag{82}$$

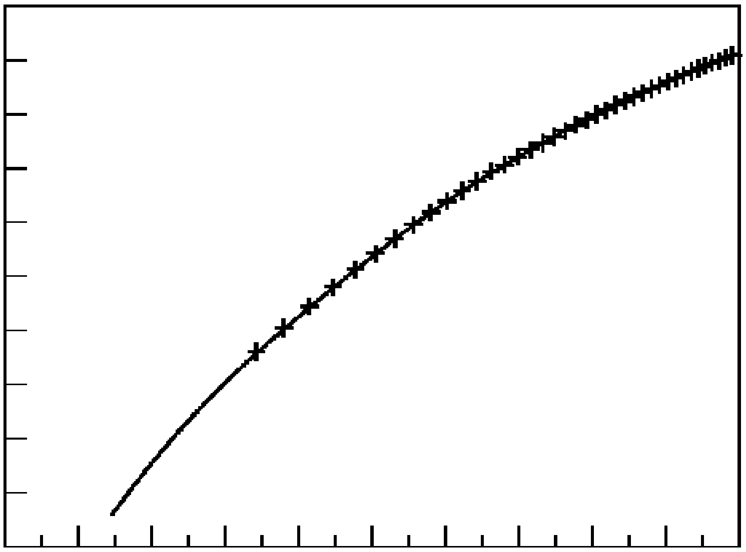

Figure 2: Comparison of the numerical values of the measure of the splitting, $d(\varepsilon)$, and the theoretical ones computed by using the Melnikov function. The figure displays $d(\varepsilon)$ versus ε, using $\log_{10}$ scales for both variables. The window in the horizontal variable is $[-2,-1]$, and in the vertical one is $[-11,-1]$.

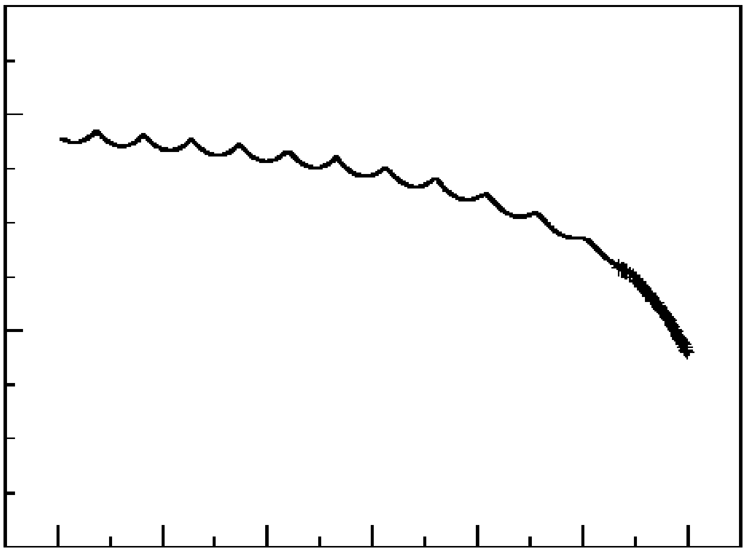

Figure 3: Behaviour of the function $c(\varepsilon)$. The function $c(\varepsilon)$ is plotted against ε. The $\log_{10}$ scale is used for ε. Window in the horizontal variable: $[-7.5,-0.5]$. In the vertical one $[0,2.5]$

where in (81) we have used Stirling's formula for the factorials and explicit use of (80) and the recurrence of the F_j has been made. Given ε we have to select j in (82) so that α_j is maximized. To this end we rewrite (82) as

$$R\gamma^j - S\gamma^j \left(j - p\frac{\ln\varepsilon}{\ln\gamma} \right) - T\gamma^{-j}\varepsilon^{-1}. \tag{83}$$

In (83) R and S are independent of ε and j. Also T can be considered as independent of ε. Indeed, if $p = 0$ then $T = \frac{\pi}{2J_0(1)}$, and if $p > 0$ T tends to $\frac{\pi}{2}$ when ε goes to zero. Let $\widetilde{S} = \frac{S}{T}$. Then the value of j maximizing (83) is

$$j = -\frac{\ln\varepsilon + \ln[\widetilde{S}(-\frac{1}{2} - p)\frac{\ln\varepsilon}{\ln\gamma}](1 + o(1))}{2\ln\gamma}, \text{ as } \varepsilon \to 0. \tag{84}$$

Of course, in (84) one has to take the closest integer to obtain the right value of j, and for some values of ε there are two consecutive values of j giving the same contribution.

By inserting the value of j as given by (84) in (82) we get the maximum value of ρ as a function of ε. This is given by

$$\rho_{\max} = -\left(\frac{|\ln\varepsilon|}{\varepsilon}\right)^{1/2} \left(\frac{2\gamma + 1}{\gamma + 2}\frac{\pi}{J_0}\right)^{1/2} (1 + 2p)^{1/2}. \tag{85}$$

As j must be taken integer there is a small difference in the correct value when j, as given by (84), is not an integer. Let us introduce the function $c(\varepsilon)$ defined by

$$c(\varepsilon) = -\ln d(\varepsilon) \left|\frac{\varepsilon}{\ln\varepsilon}\right|^{1/2}. \tag{86}$$

The minimum value of $c(\varepsilon)$ is obtained when the value of j given by (84) is integer. The maximum (and then two harmonics have equal contribution to $d(\varepsilon)$) when (84) produces (roughly) a value in $\mathbb{Z} + 1/2$. The behaviour of $c(\varepsilon)$ is shown in figure 3. The values corresponding to the data in Table I are also shown. When ε goes to zero we have

$$\begin{aligned} L &= \liminf_{\varepsilon\to 0} c(\varepsilon) = \left(\frac{2\gamma + 1}{\gamma + 2}\frac{\pi}{J_0}\right)^{1/2} (1 + 2p)^{1/2}, \\ U &= \limsup_{\varepsilon\to 0} c(\varepsilon) = \left(\frac{2\gamma + 5/4}{\gamma + 2}\frac{\pi}{J_0}\right)^{1/2} (1 + 2p)^{1/2}. \end{aligned} \tag{87}$$

The value of L in (87) follows immediately of (85) and the one of U is obtained by a small modification of the argument (asking for equal coefficients for two consecutive values of j). For completeness we give the numerical values of L and U:

$$\begin{aligned} L(p) &\simeq 1.917874016(1 + 2p)^{1/2}, \text{ if } p > 0, \;\; L(0) = 2.192467937, \\ U(p) &\simeq 1.973656366(1 + 2p)^{1/2}, \text{ if } p > 0, \;\; U(0) = 2.256236993. \end{aligned} \tag{88}$$

From (86) and (88), and always under the assumption of a good prediction of the splitting when using Melnikov's function, we have

$$d(\varepsilon) > \exp\left(-\frac{c|\ln\varepsilon|^{1/2}}{\varepsilon^{1/2}}\right) \tag{89}$$

for $\varepsilon \to 0$, where c is some positive constant. The relation (89) shows the optimality of the exponent in (3). We cannot change c_3 by anything else, at least in the current example. By comparing more carefully with (3) including the Remark 1, we notice in (89) an extra factor $|\ln\varepsilon|$. This is due to the special form of the equation (4) (or (48)). Indeed those are entire functions and in (47) the coefficient of the k-th harmonic decrases like $1/k!$. For analytic functions in some strip we would have a behaviour like $\exp(-\sigma|k|)$, with some positive σ, for the k-th harmonic. Furthermore, due to the special form of (47) in our case, in (77) only a reduced set of decompositions of $(F_j - F_{j-1})$ plays a significant role. A behaviour like $\exp(-\sigma|k|)$ produces that many terms contribute in a significant way. This implies that in the general case, even the estimates using Melnikov's approach give larger estimates, closer to the ones predicted by (3) and Remark 1.

As a final comment we point out that the averaging process leading first to (45), can be continuated for further steps. However it is not possible then to express it as a sum $\sum_j \varepsilon^j H_j$ with a finite number of terms. The second and successive averages introduce successive integrals of the functions M_1 and M_2 and their products. This gives rise to additional terms in the Melnikov function. The contribution of those terms is more relevant for smaller values of ε, because then some powers of small divisors play a significant role. This can explain the better agreement observed in figure 2 for larger rather than for small values of ε.

6 Bifurcations of the set of homoclinic orbits

From the discussion in the preceding section we have seen that for some values of ε, two different consecutive values of j give the same contribution. Let us consider in general the contribution to $m(\varphi_1, \varphi_2, \varepsilon)$ given by the two dominant harmonics of the form (76) with coefficients a and b:

$$m(\varphi_1, \varphi_2, \varepsilon) = a\sin(F_j\varphi_1 - F_{j-1}\varphi_2) + b\sin(F_{j-1}\varphi_1 - F_{j-2}\varphi_2), \quad a = a(\varepsilon), \quad b = b(\varepsilon). \tag{90}$$

Primary homoclinic orbits are given by $m(\varphi_1, \varphi_2, \varepsilon) = 0$. This gives a curve $\mathcal{C}_h(\varepsilon)$ in $\mathbb{T}^2$. If, say, $\left|\frac{b}{a}\right|$ is small, then $\mathcal{C}_h(\varepsilon)$ is close to the line in $\mathbb{T}^2$ with slope F_j/F_{j-1}. Changes in the topology of $\mathcal{C}_h(\varepsilon)$ are produced when m, $\frac{\partial m}{\partial\varphi_1}$ and $\frac{\partial m}{\partial\varphi_2}$ are simultaneously zero. From (90) and the properties of F_j it follows immediately that this happens if and only if $|a| = |b|$. Then the values of the arguments, $F_j\varphi_1 - F_{j-1}\varphi_2$, $F_{j-1}\varphi_1 - F_{j-2}\varphi_2$, must belong to $\{\pi/2, -\pi/2\}$. It follows also that the corresponding values of φ_1, φ_2 are multiples of $\pi/2$.

Going back to the numerical results, it is seen that a bifurcation is produced for some ε in $[0.078, 0.080]$, where the average slope of $\mathcal{C}_h(\varepsilon)$ changes from 2 to 3/2 and, more accurately, for some ε in $[0.03065, 0.03066]$, where it changes from $3/2$ to $5/3$ (see also figure 1). The predicted values of the bifurcations, looking only at the two dominant terms in (67) are close to 0.079199 and 0.030389, respectively.

From (84) it follows that to advance j in one unit, when ε goes to zero, one should decrease $\ln \varepsilon$ by $2 \ln \gamma$. Hence, if the successive bifurcations are produced for values of ε equal to $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ one has $\lim_{j\to\infty} \frac{\varepsilon_{j+1}}{\varepsilon_j} = \gamma^2$. At the bifurcation some of the orbits associated to $\mathcal{C}_h(\varepsilon)$ are no longer transversal. It is precisely at those orbits where the change of topology of $\mathcal{C}_h(\varepsilon)$ is produced. Of course, other bifurcations appear when the invariant manifolds are continuated and the successive folds give rise to homoclinic tangencies. But these are related to "secondary" homoclinic orbits, crossing several times the plane $\xi = \pi$.

Changes in the topology of the set of primary homoclinic orbits have been found in the context of the elliptic planar restricted three-body problem by Martínez and Pinyol[10] and in the general planar three-body problem by Martínez and Simó,[11] but the behaviour with respect to the parameters (for instance, the eccentricity of the orbit of the two dominant bodies) is different from the one displayed here. These bifurcations seem generically unavoidable in 3 degrees of freedom Hamiltonian systems depeding on parameters or in similar problems.

If a higher number of frequancies appears in the problem then the geometry of the bifurcations depends on the diophantine properties of the frequency vector and the current dominant harmonics for each value of ε.

7 Conclusions

A general Theorem on averaging under fast quasiperiodic forcing has been proved. Certainly this is not the most general case to be considered. Specially, to understand the dynamics in the case of Hamiltonian systems, it is necessary to take into account that the "forcing" frequencies are also varying. So the crossing of resonances is unavoidable. Even without having the frequencies locked at resonance, this problem presents major difficulties.

However the results obtained are useful when the forcing frequencies can be assumed constant and satisfying a diophantine condition. An example can be the study of the motion of asteroids not influencing the major planets, whose orbit can be considered as quasiperiodic for a moderate time interval.

In particular we have learned about the behaviour of the splitting of separatrices under quasiperiodic forcing. Furthermore, just one step of the averaging procedure can make easier the analysis of some problems.

A formal approach using the Melnikov function followed by a detailed analysis, seems to give some light on the amazing fact that each contribution to the splitting is of the type $\exp(-c/\varepsilon)$, but all together give much larger contributions, say $\exp(-c/\varepsilon^a)$, $a < 1$, as happens in the example presented here. The global behaviour is a kind of envelope of the splittings due to the individual harmonics associated to small divisors. As usual the small divisors become relevant when they are of the order of $\varepsilon^{1/2}$.

Finally a new kind of bifurcation of the set of primary homoclinic orbits shows up in a natural way and this fact makes a big difference with respect to the splitting under fast periodic forcing.

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