Variational computation of homoclinic orbits for twist maps

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Abstract

Variational methods have been used extensively in the study and computation of monotone twist maps of the annulus (dimension $d = 1$), and in their higher dimensional generalizations. In particular, periodic orbits can be characterized and computed as critical points of certain real-valued functions, constructed from the generating function of the map.

In this paper we propose a similar variational method for the computation of orbits *homoclinic* (or *heteroclinic* in some cases) to periodic orbits. This again consists in finding critical points of a certain function, constructed with knowledge of the generating function of the map and of the stable and unstable manifolds to the periodic orbits near the periodic points.

We also prove that under relatively simple conditions, when $d = 1$, there exist heteroclinic connections between fixed points.

1. Introduction

After the work of J.Mather and S.Aubry, it has become clear that variational principles are not only one the most powerful techniques for the theoretical study of twist maps, but also yield practical methods for numerical computations (see, for example, [9].)

Computation of homoclinic orbits is usually achieved either by some sort of shooting-Newton method, or by extending the stable and stable manifolds far enough away from the fixed point so as to have an intersection between them [4]. We propose here a method for computing them, for twist maps, that is based on the variational principle for the map. It is analogous to the method for finding periodic orbits as critical points of a 'periodic potential' (see [9], or next section). In contrast with the methods above mentioned, it requires only local knowledge of the stable and unstable manifolds (i.e., near the fixed point), and one can use minimization methods to find critical points, which, although slower, are in general better behaved than Newton methods.

In section 2 we detail the maps and variational principles we will be working with. Section 3 describes the function whose critical points are the orbits we are interested in, and gives an interpretation of it. The next section, 4, gives examples of computations for two of the most common twist maps. In section 5 we use this function to prove the existence of heteroclinic orbits between some fixed points of twist maps of the annulus.

2. Preliminaries

2.1. The Maps

Let $\mathrm{T}^{\mathrm{d}} = \mathrm{R}^{\mathrm{d}}/\mathrm{Z}^{\mathrm{d}}$ be the d-dimensional torus. We will study maps of the space $\mathrm{A}^{\mathrm{d}} = \mathrm{T}^{\mathrm{d}} \times \mathrm{R}^{\mathrm{d}}$, the d-dimensional annulus, into itself, that have a generating function. It is more convenient to work in the covering space $\mathrm{R}^{2\mathrm{d}}$ of A^{d}.

Since generating functions are central to the variational approach to twist maps, we start directly with them.

Definition 1. A map $S : \mathrm{R}^{\mathrm{d}} \times \mathrm{R}^{\mathrm{d}} \to \mathrm{R}$ is called a *generating function* if it satisfies

(a) S is a C^2 function,

(b) $S(x+m, x'+m) = S(x, x') \quad \forall m \in \mathrm{Z}^{\mathrm{d}},\ x, x' \in \mathrm{R}^{\mathrm{d}}$,

(c) the maps $x \to D_2S(x, x_0)$ and $x \to D_1S(x_0, x)$ are diffeomorphisms of R^{d} for any $x_0 \in \mathrm{R}^{\mathrm{d}}$.

A map $f : \mathrm{R}^{2\mathrm{d}} \to \mathrm{R}^{2\mathrm{d}}$ is said to be *generated* by such an S when

$$f(x, y) = (x', y') \quad \Longleftrightarrow \quad \left\{ \begin{array}{lcl} y & = & -D_1S(x, x') \\ y' & = & D_2S(x, x') \end{array} \right. \tag{1}$$

Some remarks:

To generate a function from a given S, we should able to solve the right hand side of 1 for (x', y'), given (x, y). This means that $y = -D_1S(x, x')$ should be solvable for x', for all y and x, so that some condition like 1(c) is necessary.

Equation 1(b) implies that a map f generated by S is a lift of a map $\bar{f} : \mathrm{A}^{\mathrm{d}} \to \mathrm{A}^{\mathrm{d}}$.

Actually,

Proposition 1. $\bar{f}$ is an *exact symplectic twist map*, i.e., it satisfies

(a) $\bar{f}$ is homotopic to the identity map,

(b) $\bar{f}^*(y\,dx) - y\,dx$ is an exact 1-form in A^{d}, and

(c) If $f(x, y) = (X, Y)$, then the map $y \to X(x_0, y)$ is a diffeomorphism of R^{d}, $\quad \forall x_0 \in \mathrm{R}^{\mathrm{d}}$.

Proposition 2. There is an homeomorphism between the set of lifts of C^1 exact symplectic twist maps of A^{d} and the set of generating functions on $\mathrm{R}^{2\mathrm{d}}$ satisfying $S(0, 0) = 0$.

Since S and $S +$ constant generate the same map, the normalizing condition $S(0, 0) = 0$ is necessary. For a proof of these two propositions we refer to [5] or [6].

Condition 1(c) in definition 1 of a generating function is sometimes replaced by

3'. There is a $\beta > 0$ such that $v^T . D_{12}S(x, x') . v \le -\beta\|v\|^2, \forall (x, x') \in R^{2d}, \forall v \in R^{2d}$.

This condition also implies that S generates an exact symplectic twist map as defined here. (Again, see [6]). We find it useful for the proof of proposition 5, so we adopt it from now on.

2.2. The Variational Principle

Having a generating function allows us to work with only half of the coordinates (the *base* coordinates, i.e., the x's); it gives the orbits as critical points of a real-valued function.

Definition 2. A *configuration* is an infinite sequence $(x_i)_{i=-\infty}^{\infty} \in (\mathrm{R}^{\mathrm{d}})^{\mathrm{Z}}$.

Proposition 3. The orbits under f are in one-to-one correspondence with the configurations that are critical points of the formal series (called the *action*)

$$W((x_i)_{i=-\infty}^{\infty}) = \sum_{i=-\infty}^{\infty} S(x_i, x_{i+1})$$

Note that although the series for W is not in general convergent, $\partial_{x_i} W$ involves only two terms of the series, and therefore ∇W *is* well defined. The orbit can be found from the critical configuration by using the relation /reff:S.

This function W can be used to find some special orbits of $\bar{f}$. For example,

Definition 3. A point $(x, y) \in \mathrm{R}^{2\mathrm{d}}$ is said to be a *p/q-periodic* point of $\bar{f}$ (where $q \in \mathrm{Z}$, and $p \in \mathrm{Z}^{\mathrm{d}}$) whenever $f^q(x, y) = (x + p, y)$.

A p/q-periodic point is then a point of period q that is translated by p in the base by the lift of the map. If $(x_i, y_i)_{i=-\infty}^{\infty}$ is a p/q-periodic orbit, we have $(x_{i+q}, y_{i+q}) = (x_i + p, y_i) \quad \forall i \in \mathrm{Z}$. Then $S(x_{i+q}, x_{i+q+1}) = S(x_i + p, x_{i+1} + p) = S(x_i, x_{i+1})$. Therefore, there are only q different terms in the series for W, and the information given by it is contained in a finite sum.

Definition 4. The *p/q-periodic action* is the function $W_{p/q} : (\mathrm{R}^{\mathrm{d}})^q \to \mathrm{R}$ given by

$$W_{p/q}(x_0, \ldots, x_{q-1}) = S(x_0, x_1) + S(x_1, x_2) + \cdots + S(x_{q-1}, x_0 + p).$$

Proposition 4. There is a one-to-one correspondence between p/q-periodic points and the critical points of $W_{p/q}$.

The proofs of these two last propositions are straightforward.

This result can be viewed in two different ways. On one hand, it has been used to prove existence of periodic points of all kind (i.e., for all p/q) for this kind of maps. For example, in [6] there is a rather simple proof that $W_{p/q}$ has a minimum, and therefore that $\bar{f}$ has a p/q-periodic orbit. (As an aside, note that working in the covering space is essential here. If we try to find minima of $S(x_0, x_1) + S(x_1, x_2) + \cdots + S(x_{q-1}, x_0)$, we would get the point $x_0 = \ldots = x_{q-1} = x$, a minimum of $S(x, x)$.) Actually, considering the topology of T^{d} and symmetries of $W_{p/q}$, it is shown there that there is a correct number (at least $d + 1$, and 2^d when they are all nondegenerate) of p/q-periodic orbits. On the other hand, this is not only a theoretical tool to prove the existence of orbits, but also a convenient way of finding them. Instead of trying to solve $f^q(x, y) - (x + p, y) = 0$, using Newton's method, for example, we can look for minima of $W_{p/q}$, a much more robust problem.

What we are interested in is a method that would do for homoclinic orbits what this one does for periodic ones.

3. The Method

3.1. The problem

A fixed point for $\bar{f}$ has as x-coordinates the critical points of $\phi(x) = W_{0/1}(x) = S(x, x)$. Let $\xi, \eta \in \mathrm{R}^{\mathrm{d}}$ be two such critical points, corresponding to fixed points

(ξ, y_ξ) and (η, y_η). Assume these to be hyperbolic. Since f is symplectic, $\det(Df(x,y)) = +1$, and there are as many eigenvalues with modulus smaller than one as there are with modulus larger than one. In other words, both the stable and unstable manifolds at these fixed points have dimension d.

We want to find an heteroclinic orbit between the fixed points, i.e. a configuration $(x_i)_{i=-\infty}^{\infty}$ that is a critical point of the action W, with $x_i \to \xi$ (as $i \to -\infty$), and $x_i \to \eta$ (as $i \to +\infty$).

3.2. On how to compute W

The problem we have now is how to compute W, or even ∇W. The first one may well be divergent, even if we consider only configurations going from ξ to η; as for the second one, while the gradient is well defined for any configuration, we need a way of restricting the search of critical points to a finite dimensional space. There is a relation between the generating function and the stable and unstable manifolds that does just that.

Let $(\bar{x}, \bar{y})$ be a hyperbolic fixed point of $\bar{f}$.

Proposition 5. There exists a neighborhood $\mathcal{U}$ of $\bar{x} \in \mathrm{R}^{\mathrm{d}}$ and functions $\Phi^u : \mathcal{U} \to \mathrm{R}$ and $\Phi^s : \mathcal{U} \to \mathrm{R}$ such that

$$\mathcal{W}^u_{loc}(\bar{x}, \bar{y}) = \mathrm{Graph}\,(\nabla\Phi^u) \;\text{ and }\; \mathcal{W}^s_{loc}(\bar{x}, \bar{y}) = \mathrm{Graph}\,(\nabla\Phi^s)$$

Proof: The manifold $\mathcal{W}^s_{loc}(\bar{x}, \bar{y})$ is Lagrangian, that is, the restriction of the symplectic form $\omega = d(y\,dx)$ to $T\mathcal{W}^s_{loc}(\bar{x}, \bar{y})$ is identically zero. To see this, take any two $u, v \in T_p\mathcal{W}^s_{loc}(\bar{x}, \bar{y})$. We know that $df^n(u) \to 0$ as $n \to \infty$, and the same is true for v, since they are both tangent to the stable manifold. As f preserves ω, we have

$$\omega(u, v) = (f^n)^*\omega(u, v) = \omega(df^n(u), df^n(v)) \to \omega(0, 0) = 0, \quad \text{as } n \to \infty.$$

The tangent plane to $\mathcal{W}^s_{loc}(\bar{x}, \bar{y})$ at $(\bar{x}, \bar{y})$ is given by the generalized eigenvectors of $M = Df(\bar{x}, \bar{y})$. For the manifold to be a graph of some function over the base near the fixed point it is enough to check that these tangent plane has no vertical component.

In terms of the generating function, the linear part of the map f at a fixed point is given by

$$M = Df(\bar{x}, \bar{y}) = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} = \begin{bmatrix} BA & B \\ CBA - B^{-T} & CB \end{bmatrix},$$

where $A = D_{11}S(\bar{x}, \bar{x}), B = -D_{12}S(\bar{x}, \bar{x})^{-1}$, and $C = D_{22}S(\bar{x}, \bar{x})$.

Now, suppose that the vertical vector $[0, w]^T$ is in the tangent space $T_p\mathcal{W}^s_{loc}(\bar{x}, \bar{y})$ to $\mathcal{W}^s_{loc}(\bar{x}, \bar{y})$ at the fixed point. This manifold is invariant, so we should also have $M\;[0, w]^T \in T_p\mathcal{W}^s_{loc}(\bar{x}, \bar{y})$.

Since $\mathcal{W}^s_{loc}(\bar{x},\bar{y})$ is a Lagrangian manifold, the usual symplectic product $<u,v>= u^T\ J\ v$
(where $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$, and I is the $d \times d$ identity matrix) is zero whenever u and v are in $T_p\mathcal{W}^s_{loc}(\bar{x},\bar{y})$. In particular, we have that

$$\begin{aligned} <M\ [0,w]^T\ ,\ [0,w]^T> &= [0,w]\ M^T \cdot J\ [0,w]^T = (B\ w)^T \cdot (-w) + \\ &+ (CB\ w)^T \cdot 0 = -w^T\ B^T\ w = 0, \end{aligned}$$

where we have used the form of M, as written above.

We see then, *assuming* $B = -D_{12}S(\bar{x},\bar{x})^{-1}$ *to be definite* (see condition *3'*) that $w = 0$; and that therefore the plane $T_p\mathcal{W}^s_{loc}(\bar{x},\bar{y})$ cannot contain a vertical direction.

We can assume then that $\mathcal{W}^s_{loc}(\bar{x},\bar{y})$ is the graph of a function $\varphi^s : \mathcal{U} \to \mathrm{R}^{\mathrm{d}}$. That the graph of φ^s is Lagrangian means that $\frac{\partial \varphi^s_j}{\partial x_i} = \frac{\partial \varphi^s_i}{\partial x_j}$, for all $i,j = 1,\ldots,d$, which is what we need for φ to be the gradient of a function Φ^s.

A similar argument applies to $T_p\mathcal{W}^u_{loc}(\bar{x},\bar{y})$.

□

We choose these functions so that $\Phi^s(\bar{x}) = \Phi^u(\bar{x}) = 0$, and denote $\varphi_s = \nabla\Phi^s$, $\varphi_u = \nabla\Phi^u$. These Φ^s and Φ^u have a nice interpretation, key to the method for computing the orbits.

Proposition 6. Given $x \in \mathcal{U}$, let $(x_i,y_i) = f^i(x,\varphi_s(x))$ for $i = 0,1,\ldots$, and $(x_i,y_i) = f^i(x,\varphi_u(x))$ for $i = 0,-1,\ldots$.

Then

$$\Phi^s(x) = -\sum_{i=0}^{\infty}(S(x_i,x_{i+1}) - S(\bar{x},\bar{x})), \quad \text{and} \tag{2}$$

$$\Phi^u(x) = \sum_{i=0}^{\infty}(S(x_{-i-1},x_{-i}) - S(\bar{x},\bar{x})). \tag{3}$$

For the proof we appeal to the following lemma, which appears in essentially the same form as part of a proof in Appendix 2 of [10], where it is credited to J.Mather:

Lemma 1. If an invariant manifold is the graph of a function ∇H, then we have

$$S(x,x') = H(x') - H(x) + \mathit{constant},$$

where x' is given by $f(x,\nabla H(x)) = (x',y')$.

Proof: (of the lemma). Since the manifold given by ∇H is invariant under f, we have $y' = \nabla H(x')$. Considering x' as a function of x, call $G(x) = S(x,x'(x))$ and $\overline{G}(x) = H(x'(x)) - H(x)$.

Then we have

$$\begin{aligned}\nabla G(x) &= D_1 S(x,x') + \frac{\mathrm{d}x'}{\mathrm{d}x} D_2 S(x,x') = -y + \frac{\mathrm{d}x'}{\mathrm{d}x} y' \\ &= -\nabla H(x) + \frac{\mathrm{d}x'}{\mathrm{d}x} \nabla H(x') = \nabla \overline{G}(x).\end{aligned}$$

Therefore, $G(x) - \overline{G}(x)$ is a constant. □

Using this lemma, the proof of the proposition is now very simple:

Proof: Consider Φ^s first. The manifold is the local stable manifold of $(\bar{x}, \bar{y})$. Using the lemma, we get that $S(x,x') = \Phi^s(x') - \Phi^s(x) + \text{constant}$. Since $f(\bar{x},\bar{y}) = (\bar{x},\bar{y}) \Rightarrow x'(\bar{x}) = \bar{x}$, evaluating at $\bar{x}$ gives that the value of the constant is $S(\bar{x},\bar{x})$. That is, $S(x,x') - S(\bar{x},\bar{x}) = \Phi^s(x') - \Phi^s(x) \quad \forall x \in \mathcal{U}$. If we now use this for each x_i (noting that $x'(x_i) = x_{i+1}$, and using $x_0 = x$), we get that

$$\sum_{i=0}^{N-1} (S(x_i, x_{i+1}) - S(\bar{x},\bar{x})) = \sum_{i=0}^{N-1} (\Phi^s(x_{i+1}) - \Phi^s(x_i)) = \Phi^s(x_N) - \Phi^s(x).$$

As $N \to \infty$, $\Phi^s(x_N) \to \Phi^s(\bar{x}) = 0$, and the series converges to $-\Phi^s(x)$.

The proof for Φ^u is completely analogous. □

Therefore, if $(x_i)_{i=-\infty}^{\infty}$ gives an heteroclinic orbit between (ξ, y_ξ) and (η, y_η), the tails of the series defining W are given by Φ^u and $-\Phi^s$, except for the substraction of $S(\xi,\xi)$ at the begining and of $S(\eta,\eta)$ at the end. Clearly we have to substract these two terms at the tails, or the series would not converge. But this does not change its 'gradient', so it has the same critical points as W, which is what we want.

3.3. On the finite dimensional problem

We can then compute W on an heteroclinic orbit if we know the manifolds near the fixed points, by using just a finite piece of the orbit. Moreover, we can set up a function from R^N to R that will have the same value as W on an heteroclinic orbit, and the right critical points:

Proposition 7. Let $\mathcal{U}_\xi, \mathcal{U}_\eta$ be neighborhoods of ξ, η, respectively, playing the same role as $\mathcal{U}$ before. Let $\Phi^u : \mathcal{U}_\xi \to \mathrm{R}$ be a function describing $\mathcal{W}^u_{loc}(\xi, y_\xi)$ and $\Phi^s : \mathcal{U}_\eta \to \mathrm{R}$ one describing $\mathcal{W}^s_{loc}(\eta, y_\eta)$, as above. For $N \in \mathrm{N}$, let

$$W_N(x_0, \ldots, x_N) = \Phi^u(x_0) + \sum_{i=0}^{N-1} S(x_i, x_{i+1}) - \Phi^s(x_N).$$

Then any critical point of W_N in $\mathcal{U}_\xi \times (\mathrm{R}^{\mathrm{d}})^{N-2} \times \mathcal{U}_\eta$ gives a piece of an heteroclinic orbit between (ξ, y_ξ) and (η, y_η), under the usual correspondence 1.

Proof: Let $(x_0, \ldots, x_N)$ be such a critical point. If

$$
\begin{aligned}
y_0 &= \nabla\Phi_u(x_0) = -D_1 S(x_0, x_1), && (4)\\
y_i &= -D_1 S(x_i, x_{i+1}) = D_2 S(x_{i-1}, x_i), \qquad \text{for } i = 1, \ldots, N-1; && (5)\\
y_N &= \nabla\Phi^s(x_N) = D_2 S(x_{N-1}, x_N), && (6)
\end{aligned}
$$

then it follows from the correspondence 1 between the map and its generating function and the definitions of Φ^s and Φ^u that $(x_0, y_0) \in \mathcal{W}^u_{loc}(\xi, y_\xi)$, $(x_N, y_N) \in \mathcal{W}^s_{loc}(\eta, y_\eta)$, and that $f(x_i, y_i) = (x_{i+1}, y_{i+1})$, for $i = 0, \ldots, N-1$. $\square$

This shows that instead of searching for critical points in an infinite dimensional space, $(\mathrm{R}^d)^{\mathrm{Z}}$, we can get away with using an N-dimensional one. We should have some care as to what value of N we use. If the neighborhoods of the fixed points are small, and if N is small, it is quite possible that N iterations of f would not be enough to go from one neighborhood to the other. The function W_N would not have any critical points; to find an heteroclinic orbit we would need a larger N. Of course, it is to one's advantage to take N not *too* large, since the task of finding critical points becomes harder the larger the space we are working in. Fortunately, heteroclinic orbits spend most of their time near the fixed points, so the necessary values of N don't get too large (between 10 and 20, typically).

A different problem is that we are assuming perfect knowledge of the manifolds $\mathcal{W}^u_{loc}(\xi, y_\xi)$ and $\mathcal{W}^s_{loc}(\eta, y_\eta)$, something rather unusual. But this method can still be used to find the heteroclinic orbits approximately, by using approximations to the stable and unstable manifolds. The simplest ones would be their tangent planes, i.e., the stable and unstable subspaces of the linearization of the map at the appropiate fixed points. (This is can be computed using linear algebra, or some numerical method). It is possible to use higher order approximations to the manifolds, but it seems easier just to increase the value of N, thus getting the endpoints of the approximation to the piece of the orbit closer to the fixed points.

To summarize, the steps in using this method would be:

- Find the fixed points to connect, and approximations to the stable and unstable manifolds.
- Compute approximations to Φ^u and to Φ^s by using the approximate manifolds.
- Pick a number $N \in \mathrm{N}$, and find an approximation to W_N using the approximations to Φ^u and to Φ^s.
- Use your favorite numerical method to compute critical values of this approximation. If none is found, try increasing N.
- For more precision, increase N.

In all of the examples throughout this work, we use the linear stable and unstable subspaces to approximate the manifolds, and we find minima (using a modified 'follow the gradient' method) as a way of finding critical points.

3.4. On periodic orbits

For simplicity, we have restricted the discussion in this section to fixed points, but the same method can be set up, and the same propositions proven, for orbits homoclinic to p/q-periodic orbits. (The sole exception is the part of the proof of proposition 5 where we show that the stable and unstable manifolds cannot be vertical. This would have to be shown by other means, or checked in each case.) We need only replace each single x-coordinate by a q-vector of x-coordinates, $S(x, x)$ by $W_{p/q}(x^0, \ldots, x^{q-1})$, and S by

$$\overline{S}((x^0, \ldots, x^{q-1}), (x'^0, \ldots, x'^{q-1})) = S(x^0, x^1) + \ldots + S(x^{q-1}, x'^0 + p).$$

For more details, see [12].

4. Examples

4.1. The Standard Map

Probably the most common example of the kind of map we are considering is *the* standard map.

A generating function of the form $S(x, x') = \frac{1}{2}\langle x - x', x - x'\rangle + V(x)$ generates a map

$$\begin{cases} x' &= x + y + \nabla V(x) \\ y' &= y + \nabla V(x) \end{cases}$$

ussually called a *standard map*. The whole class is the standard family. If $d = 1$ and $V(x) = -\frac{k}{(2\pi)^2}\cos(2\pi x)$, it is called *the* standard map. Here $\varphi = V$, which has a minimum at $x = 0$, corresponding to the hyperbolic fixed point $(0, 0)$.

Figure 1 gives the phase space portrait of this hyperbolic fixed point (the crosses at the borders $x = 0$ and $x = 1$) together with pieces of both $\mathcal{W}^u$ and $\mathcal{W}^s$, as well as the homoclinic connection computed by the method described before (the square boxes).

As is well known, and can be seen from the figure, between two consecutive points of the homoclinic orbit there is a point of another homoclinic orbit, where the manifolds cross with the opposite orientation. This other orbit does not give a minimum of W. This can be seen by remembering that the value of W on an orbit gives the area below the manifolds (see Proposition 6). More precisely, if $(\ldots, (x_0, y_0), \ldots)$ is an homoclinic orbit, we can form a curve by following first $\mathcal{W}^u$ from the fixed point until we reach (x_0, y_0), and the switching to $\mathcal{W}^s$, until reaching the fixed point again. The area below this curve is the value of W on the orbit.

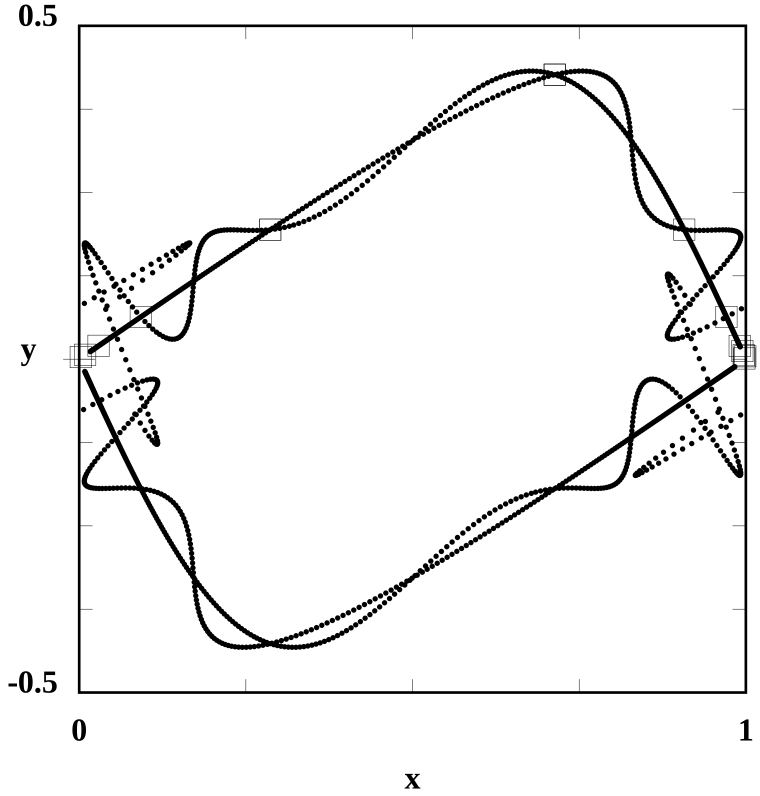

Figure 1. Homoclinic tangle of fixed point, with connection. ($k = 1.5$)

If we do that with the two orbits, the computed one and the other intersection, we see that the region below the second curve is completely above the first one, and therefore gives a larger value of W.

This other orbit is still a critical point of W, and can be found if a different method (other than finding minima) of finding critical points is used. Or, we can find it as a minimum, too, because of the symmetry of the potential for this particular map: it is a minimum of W in the invariant (for the ∇W flow) subspace

$$\mathbf{Symm} = \{(x_i)_{i=-\infty}^{\infty} : x_{-i} + x_i = 1, \quad \text{for all } i \in \mathrm{Z}\},$$

of all configurations symmetric with respect to the line $x = \frac{1}{2}$, with a point on it.

Another thing one can try is to connect the fixed point $(0, 0)$ not with the next representative $(1, 0)$, but with, say, $(2, 0)$. This is equivalent to using a map whose potential has a minimum of the same value in between the points we are trying to connect. This can cause problems.

If we try to find this minimum anyway, we get what we have in figure 2.

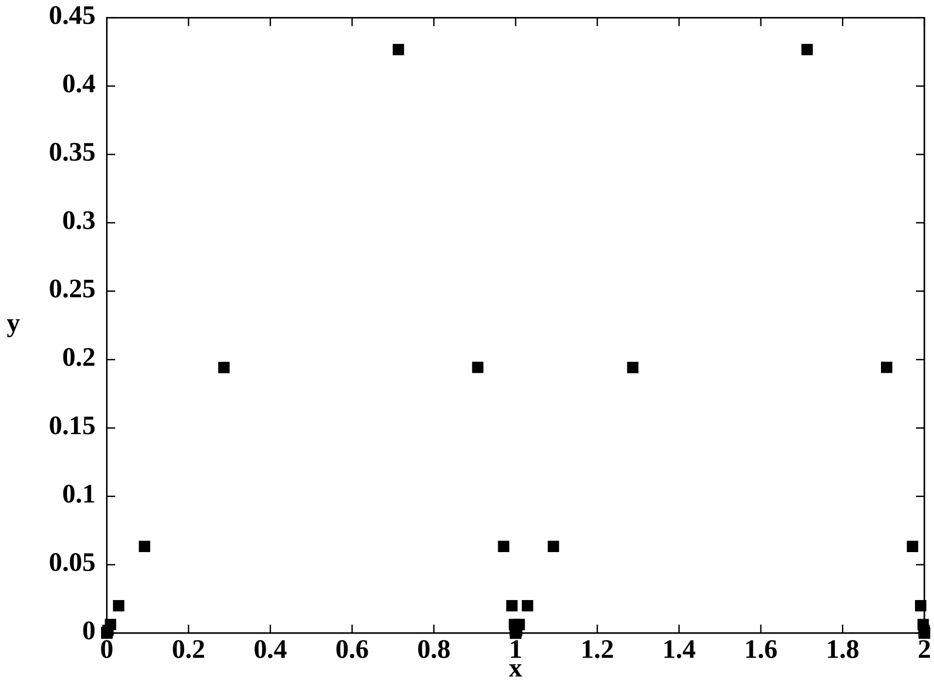

Figure 2. Trying to skip the next representative. ($k = 1.5$)

Points tend to cluster near the three representatives of the fixed point ($x = 0, 1$, or 2), where the potential achieves its minimum. Indeed, if we increase the number of points used in the computations, we get more and more points near these three points. It seems, then, that no minimum exists, but that the lowest possible value is that of the concatenation of two minimizing orbits, one from $(0, 0)$ to $(1, 0)$, the other one from $(1, 0)$ to $(2, 0)$.

As an illustration of computing orbits homoclinic to periodic ones, in figure 3 we have again the phase space of the standard map.

The crosses represent a hyperbolic $2/5$ periodic orbit, and the dots give the homoclinic connection computed by the described method. Actually, we have taken a point in the middle of the computed configuration (the one with x–coordinate 0.5), and iterated it 30 times forward and 30 backwards.

4.2. Froeschlé map

A common higher dimensional example ($d = 2$) is the Froeschlé map ([3],[8]), given by the generating function

$$S(x, x') = \frac{1}{2}(x - x').(x - x')^{\mathrm{T}} - V_{h,k}(x),$$

where

$$V_{h,k}(x) = \frac{k}{(2\pi)^2}\{\cos(2\pi x_0) + \cos(2\pi x_1)\} + \frac{h}{(2\pi)^2}\cos(2\pi(x_0 + x_1)).$$

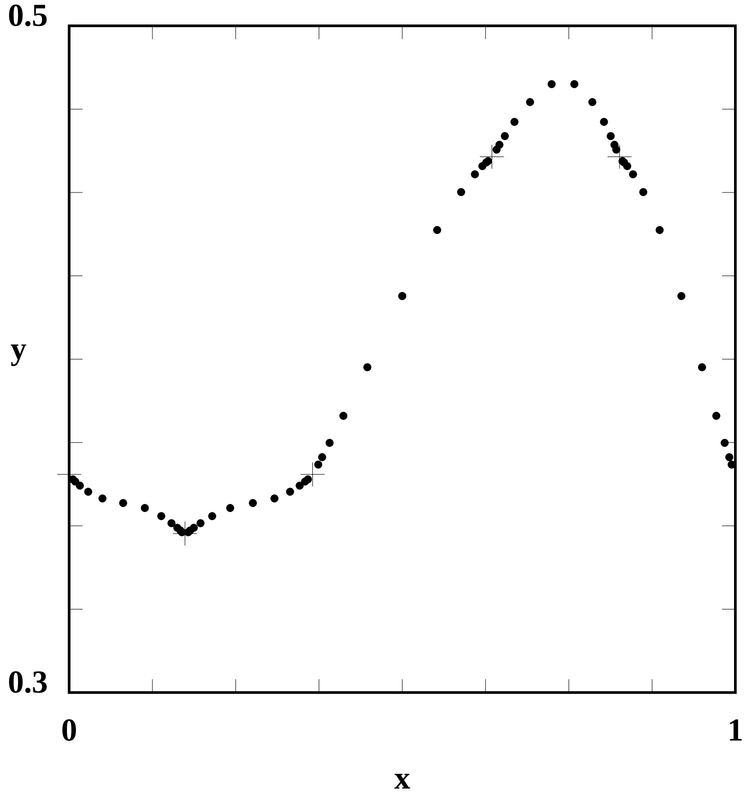

Figure 3. A 2/5 periodic orbit with the computed connection. $(k = 1.5)$

This map is a close relative of the standard map. As a matter of fact, for $h = 0$, we have the product of two uncoupled standard maps. It has an hyperbolic fixed point at $((0,0),(0,0))$, for $h, k > 0$.

If we try to find homoclinic connections for this point, immediately we find a problem that we did not have to face in $d = 1$: there is no obvious way of choosing to which other representative connect. We saw that (for $d = 1$) having points in between with the same value of the potential as the fixed points can bring trouble. Here there is not even a good way to make sense of what we mean by 'in between'.

We know there are orbits of the standard map connecting neighboring fixed points, so we get orbits of the Froeschlé map by taking, for example, the product of one of this orbits and the fixed points. In this way, we can get an orbit to $((1,0),(0,0))$ that has $x_1 = 0$ all along, and to $((0,1),(0,0))$ having $x_0 = 0$. If we perturb the map just a little bit, these orbits should remain there, and therefore there are orbits to the representatives $(1,0)$ and $(0,1)$ for small h. An orbit going to $(1,0)$, computed with the method we described before, is in figure 4.

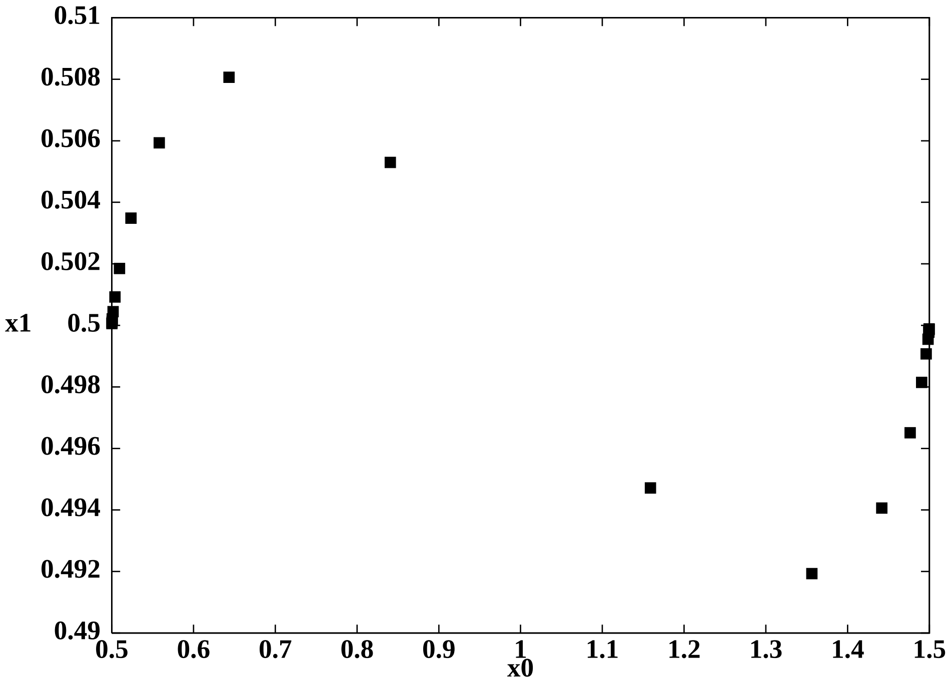

Figure 4. Homoclinic orbit of a 4D map. Only the two base coordinates are shown here ($k = 1, h = 0.1$)

If we now try to find connections to other representatives, $(2, 1)$ for example, we get what we have in figure 5, which resembles what we had for a similar situation with the standard map: a concatenation of orbits (three in this case), not just a single one.

5. A theorem

We turn now to showing the existence of connections. For this section, we consider only fixed points and $d = 1$; that is, we work with the classical twist maps of the annulus (cylinder, actually) $\mathrm{T}^1 \times \mathrm{R}$. Under these conditions, a fixed point that is a local minimum for $\varphi(x) = S(x, x)$ corresponds to a hyperbolic fixed point of the map f.

Aubry-Mather theory studies a special class of orbits, that of (globally) *minimizing orbits*. These are the ones that minimize W in a well defined sense: any finite segment of them minimizes the corresponding truncation of W, keeping the endpoints fixed. Given a number ω, there is a set $\mathcal{M}_\omega$—the *Mather set*—consisting of all the minimizing orbits with rotation number ω. If $\omega = p/q$ is rational, a global minimum of $W_{p/q}$ gives a minimizing periodic orbit. In this case, $\mathcal{M}_{p/q}$ consist of all such minima, and their connecting orbits. It can be proved (see [2], or [7]) that these homoclinic connections always exist between neighboring periodic orbits of $\mathcal{M}_{p/q}$. Under some conditions (roughly, that there be no invariant curves between them, see [11]) there are also heteroclinic orbits between two different $\mathcal{M}_\omega$.

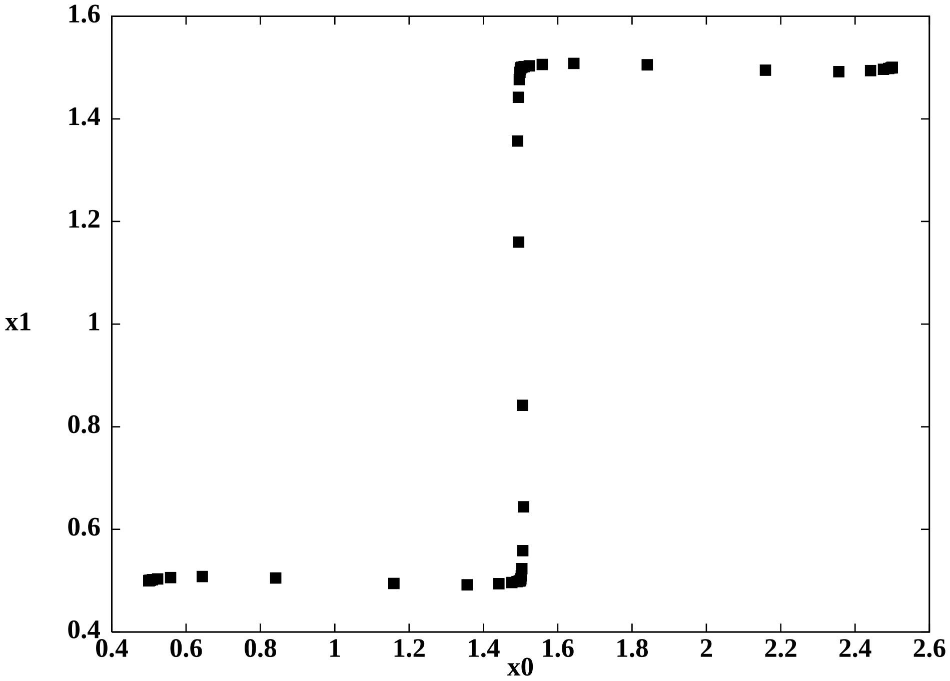

Figure 5. A concatenation of three orbits. Once again, the minimum seems to not be achieved by any single orbit ($k = 1, h = 0.1$)

The result we present below is only for fixed point, but it applies to some locally minimizing fixed points, points not in $\mathcal{M}_{0/1}$. Also, the proof is quite elementary, and, although we use the ordering of the real line that forms the base space, it seems possible that this method could be used to prove existence of heteroclinic orbits in higher dimensions, again much like in the case of periodic orbits.

5.1. An example

To see what conditions may be necessary for existence of connections, consider a standard map, with $d = 1$ and $V(x) = \frac{k}{2(2\pi)^2}\{\cos(4\pi x) + \sin(4\pi x)\}$. The potential φ is graphed in figure 6.

This shows that there are four hyperbolic fixed points, corresponding to the four minima of φ: $(0,0), (\frac{1}{4},0), (\frac{1}{2},0), (\frac{3}{4},0)$.

Figure 7 shows part of the phase space for this map: part of the unstable manifold of $(0,0)$, part of the stable manifolds of $(\frac{1}{2},0), (\frac{1}{4},0)$, and $(\frac{3}{4},0)$, and two invariant circles around the last two fixed points.

Since both $(\frac{1}{4},0)$ and $(\frac{3}{4},0)$ are surrounded by invariant circles, their stable and unstable manifolds are confined to stay inside, too; there cannot exist connections with any of the other fixed points.

The problem with $(0,0)$ and $(0,\frac{1}{4})$ is that, although φ has local minima at both, their values are not the same. We will ask that they be the same.

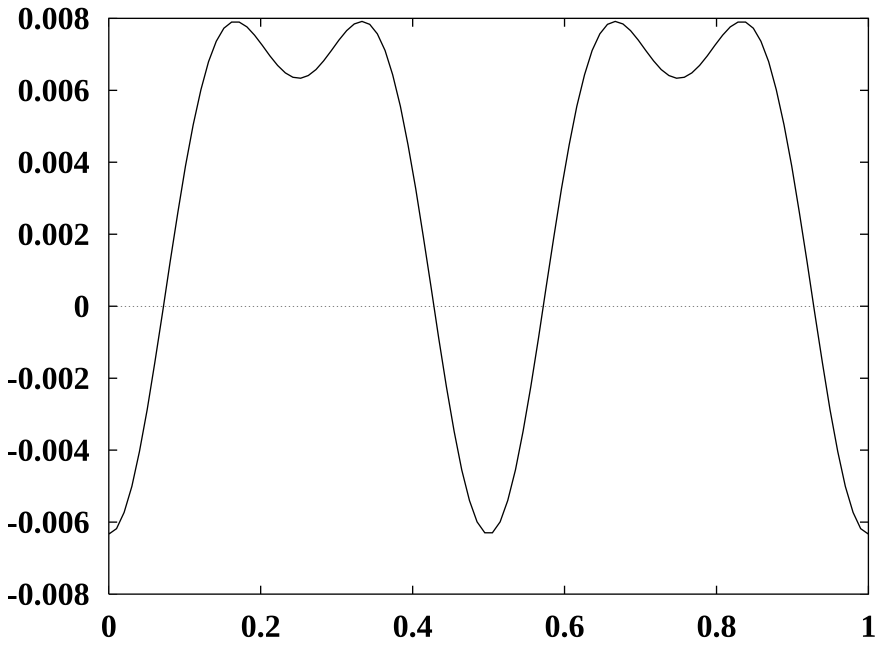

Figure 6. The potential of the map. It has four local minima, corresponding to the four hyperbolic fixed points

Although $(0, \frac{1}{4})$ and $(0, \frac{3}{4})$ give the same value of φ, there is still no connection. But in between them there is $(0, \frac{1}{2})$, which gives an even lower value. On the other hand, note that there is a connection going from $(0, 0)$ to $(0, \frac{1}{2})$, even though there is a local minima between them. This one, $(0, \frac{1}{4})$, gives a higher value of φ. The problem, then, is not the existence of a local minimum between the points, but whether the value of φ between fixed points is lower than at the fixed points themselves or not (see the standard map example). We will ask this not to happen, either.

Finally, we assume here the standard twist condition for maps of the annulus, condition $3'$ in the case $d = 1$: there exists a $\beta > 0$ such that $D_{12}S(x, x') \leq -\beta$, for any $(x, x') \in \mathrm{R}^2$.

These observations sum up the conditions we need.

5.2. The result

Theorem 1. Suppose that ξ and η are two isolated local minima of φ, with $\varphi(\xi) = \varphi(\eta) = 0$, and that $\varphi(x) > 0$ for any $x \in (\xi, \eta)$.

Then there is an heteroclinic connection between ξ and η.

Before getting to the proof, some remarks:

First, asking φ to be 0 at the fixed points is no extra restriction: changing S by a constant does not change the map it generates, so we might as well subtract the common value of φ at ξ and η.

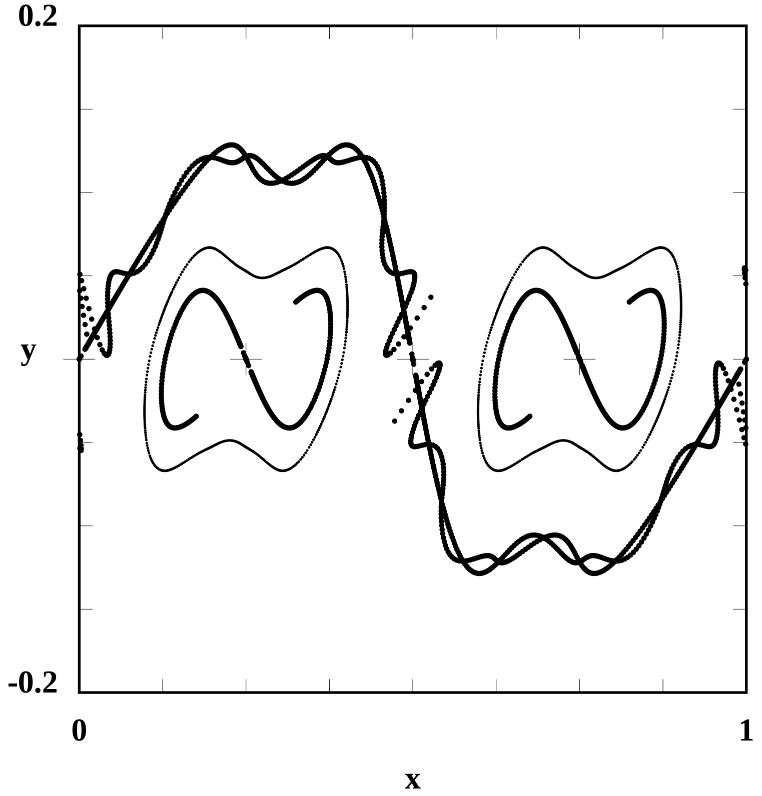

Figure 7. Phase space of the map

Second, if we choose η to be another representative of the point that ξ represents —if, in other words, $\xi - \eta \in \mathbb{Z}$—we would have *homoclinic* orbits. The conditions of the theorem are particularly well suited for homoclinic orbits, since $\varphi(\xi) = \varphi(\eta)$ whenever $\xi - \eta \in \mathbb{Z}$.

Third, and last, under the same conditions, we have a connection going in the opposite direction, from η to ξ: the function $S^*(x, x') := S(x', x)$ satisfies all the requirements of the theorem (note that $\varphi^*(x) := S^*(x, x) = \varphi(x)$), and generates a map f^* given by

$$f^*(x, y) = (x', y') \iff f(x', -y') = (x, -y).$$

If (x_i, y_i) is a connection for f^* from ξ to η, defining $x'_i = x_{-i}$ and $y'_i = -y_{-i}$ gives an orbit for f, going from η to ξ.

Proof: The idea is to show that W_N has a critical point, for some large N, by showing that it achieves its minimum value in a certain region at an interior point. For that we have to be more careful about the domain of definition of W_N. We know that near the fixed points $\mathcal{W}^u_{loc}(\xi, y_\xi)$ and $\mathcal{W}^s_{loc}(\eta, y_\eta)$ are graphs over the base of functions $\nabla\Phi^u$ and $\nabla\Phi^s$, respectively. Choose then

$\varepsilon > 0$ so small that this is true on $B_\varepsilon(\xi) = \{x : \xi \le x < \xi + \varepsilon\}$ and on $B_\varepsilon(\eta) = \{x : \eta - \varepsilon < x \le \eta\}$.

Near ξ we have $\mathcal{W}^u_{loc}(\xi, y_\xi) \approx T_\xi \mathcal{W}^u_{loc}(\xi, y_\xi)$, $f^{-1} \approx Df^{-1}(\xi)$, and $Df^{-1}(\xi)$ a contraction on $T_\xi \mathcal{W}^u_{loc}(\xi, y_\xi)$. If we define

$$\psi^u(x) = \pi_1 \circ f^{-1}(x, \nabla\Phi^u(x)), \tag{7}$$

(that is, ψ^u gives the effect of f^{-1} restricted to the unstable manifold on the x-coordinates), then for small enough ε we will have $\psi^u(x) < x$, for $\xi \le x < x + \varepsilon$. Similarly, $\psi^s(x) > x$, for $\eta - \varepsilon < x \le \eta$, where

$$\psi^s(x) = \pi_1 \circ f(x, \nabla\Phi^s(x)). \tag{8}$$

Pick an ε that makes all the above true. Then, if $B^*_\varepsilon(\xi) = \psi^u(B_\varepsilon(\xi))$ and $B^*_\varepsilon(\eta) = \psi^s(B_\varepsilon(\eta))$, we will have $\overline{B^*_\varepsilon(\xi)} \subset B_\varepsilon(\xi)$, and $\overline{B^*_\varepsilon(\eta)} \subset B_\varepsilon(\eta)$.

Let I be the interval $[\xi, \eta]$, and $\mathrm{J} = \mathrm{I} \setminus (B^*_\varepsilon(\xi) \cup B^*_\varepsilon(\eta))$.

The most natural space where to look for a critical point of W_N would be $X_N := \overline{B_\varepsilon(\xi)} \times \mathrm{I}^{N-1} \times \overline{B_\varepsilon(\eta)}$. It is more convenient (as is common in annulus maps) to consider only *ordered* configurations. We will work in $\mathcal{O}_N := \{\underline{x} = (x_0, \dots, x_N) \in \mathrm{X}_N : x_i \le x_{i+1}, \text{ for } i = 0, \dots, N-1\}$.

$\mathcal{O}_N$ is a compact space, and W_N a continous function, so it has a minimum in $\mathcal{O}_N$. As stated before, what we want is to show that the minimum value of W_N is achieved in the interior of $\mathcal{O}_N$, for some N. As we saw, the configuration $\underline{x}$ where this happens gives the heteroclinic orbit.

First let's check whether ∇W_N points outside $\mathcal{O}_N$ or not. We have

$$\underline{x} \in \partial\mathcal{O}_N \iff \begin{cases} x_0 = \xi \text{ or } \xi + \varepsilon, & \text{or} \\ x_N = \eta \text{ or } \eta - \varepsilon, & \text{or} \\ x_i = x_{i+1} \quad \text{for some } i = 0, \dots, N-1. \end{cases}$$

Except at $x_0 = \xi + \varepsilon$, or $x_N = \eta - \varepsilon$, ∇W_N points outside $\mathcal{O}_N$, regardless of N :

- $x_0 = \xi$: Then $\overline{n}$ (outer normal) is $(-1, 0, \dots, 0)$, and

  $$\nabla W_N \times \overline{n} = -\frac{\partial W_N}{\partial x_0} = -\nabla\Phi^u(\xi) - D_1 S(\xi, x_1).$$

  Since $D_{12}S < 0$, we have $D_1 S(\xi, \xi) \ge D_1 S(\xi, x_1)$, (with equality only when $x_1 = \xi$); and since $\nabla\Phi^u(\xi) = \{y\text{–coordinate of the fixed point corresponding to } \xi\} = D_2 S(\xi, \xi)$, we have

  $$\nabla W_N \times \overline{n} = -D_2 S(\xi, \xi) - D_1 S(\xi, x_1) \ge -D_2 S(\xi, \xi) - D_1 S(\xi, \xi) = 0,$$

  with equality only if $x_1 = x_0 = \xi$.

- $x_N = \eta$: $\overline{n} = (0, \dots, 0, 1)$ and, as above,

  $$\nabla W_N \times \overline{n} = \frac{\partial W_N}{\partial x_N} = D_2 S(x_{N-1}, \eta) - \nabla\Phi^s(\eta) \ge D_2 S(\eta, \eta) + D_1 S(\eta, \eta),$$

  with equality only if $x_N = x_{N-1} = \eta$.

- $x_i = x_{i+1}$ $(0 < i < N-1)$: $\quad \overline{n} = (0, \ldots, 0, \underbrace{1}_{i}, \underbrace{-1}_{i+1}, 0, \ldots, 0)$ and

$$\begin{aligned}\nabla W_N \times \overline{n} &= \frac{\partial W_N}{\partial x_i} - \frac{\partial W_N}{\partial x_{i+1}} \\ &= [D_2S(x_{i-1}, x_i) + D_1S(x_i, x_{i+1})] - [D_1S(x_{i+1}, x_{i+2}) + D_2S(x_i, x_{i+1})] \\ &= [D_2S(x_{i-1}, x_i) - D_2S(x_i, x_i)] + [D_1S(x_i, x_i) - D_1S(x_i, x_{i+2})] \geq 0,\end{aligned}$$

where the last inequality is again a consequence of $D_{12}S < 0$, and we can have equality only in the case when $x_{i-1} = x_i = x_{i+1} = x_{i+2}$.

- $x_0 = x_1$: $\quad \overline{n} = (1, -1, \ldots, 0)$ and

$$\begin{aligned}\nabla W_N \times \overline{n} &= \frac{\partial W_N}{\partial x_0} - \frac{\partial W_N}{\partial x_1} \\ &= [\nabla\Phi^u(x_0) + D_1S(x_0, x_1)] - [D_1S(x_1, x_2) + D_2S(x_0, x_1)] \\ &= [\nabla\Phi^u(x_0) - D_2S(x_0, x_0)] + [D_1S(x_0, x_0) - D_1S(x_0, x_2)]\end{aligned}$$

The last term of the sum is non-negative—once again because of $D_{12}S < 0$— and can be 0 only if $x_0 = x_1 = x_2$.

As for the first term, let $x_{-1} = \psi^u(x_0)$, that is, $f(x_{-1}, y_{-1}) = (x_0, \nabla\Phi^u(x_0))$. We chose ε so that $x_{-1} < x_0$, and therefore we have that $\nabla\Phi^u(x_0) = D_2S(x_{-1}, x_0) \leq D_2S(x_0, x_0)$, once again because of the twist condition, and once again with equality only when $x_{-1} = x_0$. The first term is non-negative then, and 0 only when $x_{-1} = x_0$. But this last means that $(x_0, \nabla\Phi^u(x_0))$ is a fixed point, and the only one near ξ is ξ itself. So, $\nabla W_N \times \overline{n} \geq 0$, with equality only when $x_0 = x_1 = x_2 = \xi$.

- $x_{N-1} = x_N$: As in the previous case, ∇W_N points outside this face of $\partial\mathcal{O}_N$ unless $x_{N-2} = x_{N-1} = x_N = \eta$, in which case it is tangent to it.

If W_N reaches its minimum on $\mathcal{O}_N$ at a point $\underline{x} = (x_i)_{i=0}^N$, either $\underline{x}$ is in the interior of $\mathcal{O}_N$ or at a point of $\partial\mathcal{O}_N$ where ∇W_N does not point outside of $\mathcal{O}_N$. Suppose that $x_i = x_{i+1}$. Then, as we just saw, we must have $x_{i-1} = x_i = x_{i+1} = x_{i+2}$, lest ∇W_N point outside the face $\{x_i = x_{i+1}\}$ of $\partial\mathcal{O}_N$. Repeating now this argument with the faces $\{x_{i+1} = x_{i+2}\}$ and $\{x_{i-1} = x_i\}$, etc.,etc., we get that we must have $\xi = x_0 = x_1 = \ldots = x_N = \eta$, clearly not true. We also saw that we cannot have $x_0 = \xi$ or $x_N = \eta$, and so we conclude that any point $\underline{x} \in \mathcal{O}_N$ where W_N reaches its minimum value must be *strictly ordered*, meaning that $\xi < x_0 < \ldots < x_N < \eta$.

The only possibility for $\underline{x}$ not being in the interior of $\mathcal{O}_N$ is that either $x_0 = \xi + \varepsilon$ or $x_N = \eta - \varepsilon$. In these last two cases, whether ∇W_N points outside $\mathcal{O}_N$ or not will depend on ε and N. But we'll see that if N is large enough, the minimum of W_N is achieved at a configuration $\underline{x}$ inside $\mathcal{O}_N$.

For this, we will exploit formulas in 6 to construct new configurations from given ones. Using ψ^u, we can rewrite 2 as

$$\Phi^u(x) = \sum_{i=0}^{\infty} S(\psi^u(x_{-i}), x_{-i})$$

(remember that $S(\xi,\xi)=0$), and therefore we have that

$$\Phi^u(x)=\Phi^u(\psi^u(x))+S(\psi^u(x),x). \tag{9}$$

Now, from the configuration $\underline{x}$, form a new one by dropping the $(j+1)^{th}$ component (for some $j=0,\dots,N-2$), and putting a new first component at $\psi^u(x_0)$. That is, let $\underline{x}'\in\mathcal{O}_N$ be given by

$$\begin{cases} x'_0=\psi^u(x_0) & , \\ x'_i=x_{i-1} & \text{for } i=1,\dots,j+1, \text{ and} \\ x'_i=x_i & \text{for } i=j+2,\dots N. \end{cases}$$

Then

$$\begin{aligned} W_N(\underline{x}') &= \Phi^u(x'_0)+S(x'_0,x'_1)+\sum_{i=1}^{j}S(x'_i,x'_{i+1})+S(x'_{j+1},x'_{j+2}) \\ &\quad+\sum_{i=j+2}^{N-1}S(x'_i,x'_{i+1})-\Phi^s(x'_N) \\ &= \Phi^u(\psi^u(x_0))+S(\psi^u(x_0),x_0)+\sum_{i=0}^{j-1}S(x_i,x_{i+1})+S(x_j,x_{j+2}) \\ &\quad+\sum_{i=j+2}^{N-1}S(x_i,x_{i+1})-\Phi^s(x_N) \\ &= W_N(\underline{x})+S(x_j,x_{j+2})-S(x_j,x_{j+1})-S(x_{j+1},x_{j+2}), \end{aligned}$$

and therefore

$$\Delta W_N:=W_N(\underline{x})-W_N(\underline{x}')=S(x_j,x_{j+1})+S(x_{j+1},x_{j+2})-S(x_j,x_{j+2}).$$

Now, we have that $\phi(x)=S(x,x)>m$, for some $m>0$, for all $x\in$ J. Since S is continous and J compact, there is a $\delta>0$ such that $S(x,x')>m$ whenever $\|x-x'\|<\delta$, and $x\in$ J. For a fixed $y\in$ I, $\|S(x',y)-S(x,y)\|<\frac{m}{2}$ whenever x' is close enough to x; therefore we can choose δ so that also $\|x-x'\|<\delta\Rightarrow\|S(x',y)-S(x,y)\|<\frac{m}{2}$, for any $y\in$ I and any $x\in$ J.

This implies that if $x_i\in$ J and $\|x_j-x_{j+1}\|<\delta$ we have $\Delta W>\frac{m}{2}>0$, and therefore that $\underline{x}$ does not give the minimum value of W_N in $\mathcal{O}_N$. Choosing $N>\frac{\text{length(I)}}{\delta}$ (so that any $\underline{x}\in\mathcal{O}_N$ must have coordinates closer than δ), we ensure that some coordinates of the $\underline{x}$ giving the minimum should be outside J. Since this coordinates are ordered, we must have either $x_0\in B^*_\varepsilon(\xi)$ or $x_N\in B^*_\varepsilon(\eta)$.

So far, we have shown that one of the endpoints of a minimizing configuration must be inside either $B_\varepsilon(\xi)$ or $B_\varepsilon(\eta)$. Instead of showing that the other end should also be inside, we again use formula 2 (or 3.2, depending on which end we are talking about) to produce a new minimal configuration with the desired property. Without losing any generality, suppose $\underline{x}$ minimizes W_N, and that $x_N \in B^*_\varepsilon(\eta)$.

Define $\underline{x}' \in \mathcal{O}_N$ by

$$\begin{cases} x'_0 = \psi^u(x_0), & \text{and} \\ x'_i = x_{i-1}, & \text{for } i = 1, \ldots, N. \end{cases}$$

Since $\underline{x}$ gives the minimum of W_N, and $x_N \in B^*_\varepsilon(\eta) \subset B_\varepsilon(\eta)$, we must have $x_N < \eta$, and therefore also

$$\frac{\partial W_N}{\partial x_N}(\underline{x}) = D_2S(x_{N-1}, x_N) - \nabla\Phi^s(x_N) = 0.$$

So, if we define $y_{N-1} = -D_1S(x_{N-1}, x_N)$, and $y_N = \nabla\Phi^s(x_N) = D_2S(x_{N-1}, x_N)$, we have that $f(x_{N-1}, y_{N-1}) = (x_N, y_N)$ (from the definition 1 of the map f), and that $(x_N, y_N) \in \mathcal{W}^s_{loc}(\eta, y_\eta)$. Therefore, by 8, $\psi^s(x_{N-1}) = x_N$. In the same manner we got formula 9, we can get that

$$\Phi^s(x) = \Phi^s(\psi^s(x)) - S(x, \psi^s(x)). \tag{10}$$

These two formulas show that

$$\begin{aligned} W_N(\underline{x}') &= \Phi^u(x'_0) + \sum_{i=0}^{N-1} S(x'_i, x'_{i+1}) - \Phi(x'_N) \\ &= \Phi^u(\psi^u(x_0)) + S(\psi^u(x_0), x_0) + \sum_{i=0}^{N-2} S(x_i, x_{i+1}) - \Phi(x_{N-1}) \\ &= \Phi^u(x_0) + \sum_{i=0}^{N-2} S(x_i, x_{i+1}) - \Phi^s(\psi^s(x_{N-1})) + S(x_{N-1}, \psi^s(x_{N-1})) \\ &= \Phi^u(x_0) + \sum_{i=0}^{N-2} S(x_i, x_{i+1}) + S(x_{N-1}, x_N) - \Phi^s(x_N) \\ &= W_N(\underline{x}). \end{aligned}$$

This shows that we have constructed a new minimizing configuration, namely $\underline{x}'$, and the claim is that this one is in the interior of $\mathcal{O}_N$. We already saw that $x'_N = x_{N-1} = (\psi^s)^{-1}(x_N)$. Together with the fact that $x_N \in B^*_\varepsilon(\eta) = \psi^s(B_\varepsilon(\eta))$, this gives that $x'_N \in B_\varepsilon(\eta)$.

At the other endpoint, $x'_0 = \psi^u(x_0)$, and since $x_0 \in \overline{B_\varepsilon(\eta)}$, we have that $x'_0 \in \overline{B^*_\varepsilon(\eta)} \subset B_\varepsilon(\eta)$.

This shows that the endpoints of the configuration $\underline{x}'$ are in the interior of the neighborhoods of the fixed points, and since it is a minimizing configuration, this is all we need to show that $\underline{x}' \in$ *interior* $\mathcal{O}_N$. □

Aknowledgments

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