Renormalization group approach to zero temperature Bose condensation.

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§*1 Introduction.*

In this lecture, I will present some recent results [B] about the renormalization group approach to the problem of Bose condensation at zero temperature for a three dimensional system of bosons, interacting with a repulsive short range potential.

Before starting, I have to acknowledge the contribution of many other people in the different stages of this work. In fact, I was introduced to this problem by G. Gallavotti and J. F. Perez two years ago, but at the beginning we were only able to understand the difficulties, without finding a solution, so that we gave up the project. I started again to work on the subject during a visit to the University of Sao Paulo, where I had many useful discussions with J. F. Perez; these discussions allowed to clarify the problem, that I could really understand only recently. In this last stage the comments and the suggestions of G. Gallavotti had again an important role, as well as the criticism of C. Castellani, C. Di Castro and M. Grilli, which allowed me to find and correct an important mistake in my calculations.

The ideas that I shall discuss in this lecture are not yet sufficient for a complete rigorous treatment of the Bose condensation problem, but they are formulated in a way, that makes reasonable the conjecture that such a treatment is possible. The aim of my talk will be to give convincing arguments that the usual picture of three dimensional Bose condensation (see, for example, [ADG], [P]) can be explained in terms of a renormalization group flow, showing asymptotic freedom and anomalous behaviour of the two-point correlation function $S(x-y)$, at least order by order in the running coupling constants.

As we shall see, this anomalous behaviour is related to the fact the Fourier transform of

$S(x)$ has a singularity of the type $(k_0^2 + v^2 \vec{k}^2)^{-1}$, to compare with the singularity $(-ik_0 + \vec{k}^2/2m)^{-1}$ of the free Bose gas. This anomaly explains, according to Landau's criterion [LaL], the superfluid properties of the system, whose spectrum is expressed, for small momenta, in terms of collective excitations with speed v.

The main missing point, in order to get a rigorous proof, is a suitable cluster expansion, similar to the one used to analyze the infrared ϕ_4^4 problem [GK]; this would solve the *large field problem*. However, I do not expect this is a trivial generalization of known techniques, since in the Bose gas problem one has to use complex gaussian measures, instead of positive gaussian measures, and this would introduce new technical problems.

There is of course a huge physical literature on the subject of Bose condensation at zero or small temperature. As far as I know, the more convincing results about the superfluid behaviour at zero temperature are contained in [NN] and [PS] (see also references therein), where the authors use arguments similar to the ones that I will explain (C. Castellani made me aware of this point). However, at my knowledge, there is no treatment of the problem explicitly based on renormalization group arguments.

In the next section I will give the relevant definitions and I will discuss the so called *Bogoliubov approximation* [Bo], which is the starting point of the usual picture of Bose condensation. In §3 I will introduce the renormalization group transformation and the associated effective potentials and running coupling constants. Finally, in §4 I will discuss the flow equations for the running coupling constants, truncated to second order, showing that they can solved; the solution implies that the theory is asymptotically free with a superfluid two-point correlation.

§*2 The Bogoliubov approximation.*

The problem is the following. Let:

$$H = \sum_{i=1}^{N} \left(-\frac{\Delta_{\vec{x}_i}}{2m} - \mu \right) + \lambda \sum_{i<j} v(\vec{x}_i - \vec{x}_j) \tag{2.1}$$

be the hamiltonian describing a system of N bosons in $\mathbf{R}^3$, enclosed in a periodic box of side size L, interacting with a pair potential $\lambda v(\vec{x} - \vec{y})$ which is supposed C^∞ and with short range p_0^{-1}. In fact we shall suppose that the interaction is repulsive in the sense that $v(0) > 0$, $v(\vec{x}) \geq 0$ and $\lambda \geq 0$.

Let $\varphi_{\vec{x}}^{\pm}$ be the creation and the annihilation operators for the bosons and

$$\varphi_x^\sigma = e^{Ht} \varphi_{\vec{x}}^\sigma e^{-Ht} \quad , \quad \sigma = \pm,\ x = (t, \vec{x}) \tag{2.2}$$

Define, for $\beta > t_i \geq 0$, $i = 1, \ldots, n$, with $t_i \neq t_j$ for $i \neq j$:

$$S_{\sigma_1 \cdots \sigma_n}(x_1, \ldots, x_n) = \lim_{\beta \to \infty} \lim_{L \to \infty} \frac{\text{Tr}\, \left[e^{-\beta H} \varphi_{x_{\pi(1)}}^{\sigma_{\pi(1)}} \cdots \varphi_{x_{\pi(n)}}^{\sigma_{\pi(n)}}\right]}{\text{Tr}\ e^{-\beta H}} \tag{2.3}$$

where $\sigma_i = \pm 1$ and π is the permutation of $(1, \ldots, n)$, such that $t_{\pi(1)} > t_{\pi(2)} > \ldots > t_{\pi(n)}$. In particular consider:

$$S(x) = S_{-+}(x, 0) \tag{2.4}$$

The functions (2.3), (2.4) describe the properties of the *ground state* of the above bosons system (essentially *by definition*) in the grand canonical ensemble with chemical potential μ.

The case $\lambda = 0$ is trivial and one finds that, if $\mu < 0$, the Schwinger functions at finite β and L (the r.h.s. of (2.3)) can be calculated by the Wick rule (see for example [NO]) with propagator:

$$S_{\beta,L}(x) = L^{-d} \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} e^{-\varepsilon(\vec{k})t} \Big(\frac{\vartheta(t > 0)}{1 - e^{-\beta\varepsilon(\vec{k})}} + \frac{\vartheta(t \leq 0) e^{-\beta\varepsilon(\vec{k})}}{1 - e^{-\beta\varepsilon(\vec{k})}} \Big) \tag{2.5}$$

where

$$\varepsilon(\vec{k}) = \frac{\vec{k}^2}{2m} - \mu \tag{2.6}$$

Note that it is not possible to take $\mu = 0$ in (2.5), since in this case the term in the sum with $k_0 = |\vec{k}| = 0$ involves a division by zero. However the interesting phenomena just occur when $\mu = 0$ and we shall therefore have to deal also with such case. For this reason we take μ as a function of β, L, which goes to 0 as $L, \beta \to \infty$, in such a way that the number of particles in the *condensate* (the state $\vec{k} = 0$) is fixed, that is:

$$e^{\beta\mu}(1 - e^{\beta\mu})^{-1} = L^d \rho \tag{2.7}$$

where ρ is the condensate density. One finds, if $x_0 = t$:

$$S(x) = S_\rho(x) \equiv \rho + \frac{1}{(2\pi)^3} \int d\vec{k} e^{-i\vec{k}\cdot\vec{x}} e^{-\frac{\vec{k}^2}{2m} x_0} \vartheta(x_0 > 0) \tag{2.8}$$

so that:

$$S_0(x) = \frac{1}{(2\pi)^4} \int dk\, \frac{e^{-ikx}}{-ik_0 + \frac{\vec{k}^2}{2m}} \tag{2.9}$$

where $k = (k_0, \vec{k})$.

The Schwinger functions generated by the above limiting procedure and by the Wick rule will describe the ground state of a system of non interacting bosons with density ρ. They describe a *Bose condensed state.*

Let us now consider the case $\lambda > 0$. As it is well known [NO], if β and L are finite, the Schwinger functions can be expressed as functional integrals in the following way:

$$S_{\sigma_1\ldots\sigma_n}(x_1\ldots x_n) = \int \varphi_{x_1}^{\sigma_1}\ldots\varphi_{x_n}^{\sigma_n}\frac{e^{-V(\varphi)}P(d\varphi)}{\int e^{-V(\varphi)}P(d\varphi)} \tag{2.10}$$

where $P(d\varphi)$ is a complex gaussian measure, such that the fields φ_x^-, $\varphi_x^+ = (\varphi_x^-)^*$ have covariance:

$$\begin{aligned} &\int \varphi_x^-\varphi_y^+ P(d\varphi) = S_{\beta,L}(x-y)\\ &\int \varphi_x^-\varphi_y^- P(d\varphi) = \int \varphi_x^+\varphi_y^+ P(d\varphi) = 0 \end{aligned} \tag{2.11}$$

and, if $x = (x_0, \vec{x})$, $y = (y_0, \vec{y})$, $\Lambda = [-\frac{1}{2}\beta, \frac{1}{2}\beta]\times[-\frac{1}{2}L, \frac{1}{2}L]^3$:

$$V(\varphi) = \lambda\int_\Lambda v(\vec{x}-\vec{y})\delta(x_0-y_0)\varphi_x^+\varphi_x^-\varphi_y^+\varphi_y^-\, dx\, dy \tag{2.12}$$

Note that the fields $\varphi_x^\pm$ satisfy periodic boundary conditions in Λ.

In order to study the limit $L, \beta\to\infty$, it is convenient to add to $V(\varphi)$ a term proportional to $\int\varphi_x^+\varphi_x^-\, dx$, by changing the definition of μ, and then fix μ so that (2.7) is satisfied. The limiting theory will be still described by the functional integrals (2.10), where $P(d\varphi)$ is the formal integration with propagator (2.8) and

$$V(\varphi) = \lambda\int_\Lambda v(\vec{x}-\vec{y})\delta(x_0-y_0)\varphi_x^+\varphi_x^-\varphi_y^+\varphi_y^-\, dx\, dy + \nu\int_\Lambda \varphi_x^+\varphi_x^-\, dx \tag{2.13}$$

ν has the role of a control parameter that should be fixed so that the limiting theory is meaningful as a perturbation of the free theory with propagator (2.8). If this program is successful, it is natural to say that there is Bose condensation at $T = 0$ with given density ρ and chemical potential $-\nu$. One could also fix the physical mass of the particles by adding to $V(\varphi)$ a term $\alpha\int\varphi_x^+(-\Delta_{\vec{x}}/2m)\varphi_x^-\, dx$, but we shall not do that, because we do not expect that the choice of α is critical.

The form of the covariance (2.8) shows that the field $\varphi_x^\pm$ can be represented as:

$$\varphi_x^\pm = \xi^\pm + \psi_x^\pm \tag{2.14}$$

with $\xi^\pm$ being variables independent from $\psi_x^\pm$ and with covariance $\langle\xi^-\xi^+\rangle = \rho$, while the fields $\psi_x^\pm$ have covariance $\langle\psi_x^-\psi_y^+\rangle = S_0(x-y)$.

The integration with respect to $\xi^\pm$ can be thought as a gaussian integral by writing $\xi^\pm = \xi_1 \pm i\xi_2$ and:

$$P(d\xi) = e^{-\frac{\xi_1^2+\xi_2^2}{\rho}}\frac{d\xi_1\, d\xi_2}{2\pi\rho} \tag{2.15}$$

Hence, if we define:

$$W^{(-\infty)}(\xi) = -\frac{1}{\Lambda} \log \int e^{-V(\xi+\psi)} P(d\psi) \tag{2.16}$$

we see that the computation of $\langle \xi^+ \xi^- \rangle$ in presence of interaction will lead to the integral:

$$\rho = \int \frac{d\xi_1 d\xi_2}{2\pi\rho} (\xi_1^2 + \xi_2^2)\, e^{-(\xi_1^2+\xi_2^2)/\rho}\, e^{-(W^{(-\infty)}(\xi) - W^{(-\infty)}(0))|\Lambda|} \tag{2.17}$$

and the equality to ρ of the above integral is just the requirement that the condensate density should be ρ. Therefore equality (2.17) can hold if and only if the function $W^\infty(\xi)$, which is a function of the product $\xi^+\xi^-$, by symmetry considerations, reaches its minimum at $\xi^+\xi^- = \rho$. And in this case $\xi^+\xi^-$ will be a sure random variable, provided the minimum is not degenerate.

This implies that, in order to get a condensate with density ρ, one has to choose ν so that the free energy (2.16) has a minimum in

$$\xi^+ = \xi^- = \sqrt{\rho} > 0 \tag{2.18}$$

These considerations are a heuristic justification of the so called *Bogoliubov approximation*, very usually used in the literature, consisting in replacing the fields $\xi^\pm$ by a real positive constant external field and by choosing its value so that the ground state energy is minimum.

At my knowledge, there is only one rigorous general result in this direction, found by Ginibre in 1967, [Gi]. Ginibre proved that, at any finite temperature and any fixed chemical potential, the finite volume pressure of the Bose system in the Bogoliubov approximation has a supremum (as a function of ξ), whose thermodynamical limit coincides with the thermodynamical limit of the real pressure. However, Ginibre was not able to prove a similar result for the correlation functions.

In any case, I will suppose that the Bogoliubov approximation is correct in the limit $\beta, L \to \infty$ and that the equation relating ν and ρ is invertible; this implies that, in order to study Bose condensation at $T = 0$, one has to consider the measure

$$\frac{1}{\mathcal{N}} e^{-V_\rho(\psi)} P(d\psi) \tag{2.19}$$

where $\mathcal{N}$ is a normalization constant, $V_\rho(\psi)$ is obtained from (2.13) trough the substitution (2.14), $\xi^\pm$ are positive constants satisfying (2.18) and ν has to be chosen so that the free energy is minimum for the fixed value of $\xi^\pm$.

The Schwinger functions of the field ψ are defined by an expression similar to (2.10) and we shall use the symbol $\tilde{S}$ to denote them. Their perturbation expansion is obtained in the usual way in terms of the propagator $S_0(x-y)$. Note that the measure (2.19) does not preserve the free measure property $\tilde{S}_{--}(x-y) = \tilde{S}_{++}(x-y) = 0$.

An old perturbative argument [HP] allows to show (see also [ADG]) that the free energy is minimum if the following formal equation is satisfied:

$$\Sigma_{-+}(0) = \Sigma_{++}(0) \tag{2.20}$$

where $\Sigma_{\sigma_1\sigma_2}(k)$ is the Fourier transform of the sum of all one particle irreducible graphs (connected graphs which can not become disconnected by cutting one leg) with two external lines $\psi_x^{\sigma_1}$, $\psi_y^{\sigma_2}$.

The formal proof of (2.20) is very simple. It starts from the observation that:

$$W^{(-\infty)}(\xi) = \sum_{s=0}^{\infty} w_s \rho^s \tag{2.21}$$

where w_s is the sum of all irreducible graphs with $2s$ zero momentum ξ external legs (s of type $+$ and s of type $-$), constructed by the interaction (2.13) and the propagator $S_{\beta,L} - \rho$ (whose limit for $\Lambda \to \infty$ is given by (2.9)); the irreducibility condition follows from the fact that the propagator is zero at $k = 0$. Hence the free energy is stationary (and it turns out that the stationary point is unique and a minimum, at least for λ small enough) if the following equation is satisfied:

$$\sum_{s=1}^{\infty} s w_s \rho^{s-1} = 0 \tag{2.22}$$

Let us now consider $\Sigma_{-+}(0)$. It is clear that, given a graph contributing to w_s, there are s^2 graphs contributing to $\Sigma_{-+}(0)$ and having the same value; they are obtained by substituting in all possible ways an external leg ξ^- with ψ^- and an external leg ξ^+ with ψ^+, so that:

$$\Sigma_{-+}(0) = \sum_{s=1}^{\infty} s^2 w_s \rho^{s-1} \tag{2.23}$$

A similar argument can be used for $\Sigma_{++}(0)$ or $\Sigma_{--}(0)$, which are equal by symmetry reasons. Given a graph contributing to w_s, there are $s(s-1)/2$ equal contributions to $\Sigma_{++}(0)$, obtained by substituting two external legs ξ^+ with two ψ^+. Moreover there is a factor 2, coming from the possibility of interchanging the two external ψ^+ fields in the calculation of $\Sigma_{++}(0)$; hence:

$$\Sigma_{++}(0) = \sum_{s=1}^{\infty} s(s-1) w_s \rho^{s-1} \tag{2.24}$$

The condition (2.20) immediately follows from (2.22), (2.23) and (2.24). Note also that we could repeat the previous arguments in presence of an ultraviolet cut off, that we shall indeed introduce in the following section.

The *renormalization condition* (2.20) has here the same role of the condition which determines the critical temperature in Statistical Mechanics; for example, in the case of the φ_4^4 infrared model, a similar condition allows to define the stable manifold of the trivial fixed point in the renormalization group flow [GK]. Therefore it is natural to use it, instead of the free energy minimum condition, as the condition fixing the chemical potential, in a renormalization group analysis of the measure (2.19).

§*3 The effective potentials.*

The problem we want to study is an infrared problem; therefore we shall consider a simplified model by substituting $S_0(x)$ with

$$g_{\leq 0}(x) = \frac{1}{(2\pi)^4} \int dk\, t_0(k) \frac{e^{-ikx}}{-ik_0 + \frac{\vec{k}^2}{2m}} \tag{3.1}$$

where $t_0(k)$ is a smooth function, which imposes an ultraviolet cutoff on scale p_0 (the scale of the potential). We shall choose $t_0(k)$ as a regularization of the characteristic function of the set $\{k_0^2 + \frac{\vec{k}^2}{2m}\frac{p_0^2}{2m} \leq (\frac{p_0^2}{2m})^2\}$, such that, if γ is a fixed positive number greater than 1 and $q_0 = p_0^2/2m$:

$$t_0(k) = \theta(k_0^2 + \frac{\vec{k}^2}{2m}\frac{p_0^2}{2m}) \quad , \quad \theta(\lambda) = \begin{cases} 1 & \text{if } |\lambda| \leq \frac{2\gamma^2}{\gamma^2+1} q_0^2 \\ 0 & \text{if } |\lambda| \geq \frac{2}{\gamma^2+1} q_0^2 \end{cases} \tag{3.2}$$

The particular choice of $\theta(\lambda)$ is not relevant; it is done only to simplify some calculations in the following.

Note that the assumed presence of the ultraviolet cut off on the scale of the interaction potential is reasonable only if $\rho p_0^{-3} \ll 1$, *i.e.* only if there is in mean much less than one particle in a cube with side equal to the range p_0^{-1} of the potential.

Hence we have to study the measure:

$$\frac{1}{\mathcal{N}} e^{-V_\rho(\psi)} P^{(\leq 0)}(d\psi) \tag{3.3}$$

where $P^{(\leq 0)}(d\psi)$ is the measure with covariance (3.1). We shall do that by a multiscale analysis, in the form presented in [G] and applied to a "similar" infrared problem, the one dimensional Fermi gas, in [BG], [BGPS].

At first, one could try to define a family of effective potentials based on a multiscale decomposition of the covariance (3.1). However, this approach does not work, as a consequence of the well known fact that the functions $\tilde{S}_{\sigma_1\sigma_2}(x-y)$ are expected to have a large distance behaviour very different from the behaviour of the *free propagator* (3.1). In fact, by simple (very formal) perturbative arguments, using the normalization condition (2.20), one can prove [ADG] that the Fourier transform of $\tilde{S}_{\sigma_1\sigma_2}(x-y)$ behaves for $k\to 0$ as

$$\frac{A}{k_0^2+c^2\vec{k}^2} \tag{3.4}$$

where A and c are suitable constants (c has the physical interpretation of sound waves velocity in the condensate). The behaviour (3.4) is considered typical of superfluids.

Hence we are faced with a model with an *anomalous* behaviour, as in the $d=1$ Fermi gas. The anomaly is now of a different nature, but we shall see that we can extend in a very natural way the renormalization group techniques of [BG] to cover also this case.

We start writing the potential V_ρ in the following way:

$$V_\rho = |\Lambda|(\nu\rho+\lambda\hat{v}(0)\rho^2)+\mathcal{L}V_\rho+\mathcal{R}V_\rho \tag{3.5}$$

where the *local part* $\mathcal{L}V_\rho$ is obtained by substituting $\psi_y^\pm$ with $\psi_x^\pm$ in the expression of $V_\rho(\psi)$, after extracting the constant term, so that:

$$\begin{aligned}\mathcal{L}V_\rho(\psi) = \lambda\hat{v}(0)\int_\Lambda(\psi_x^+\psi_x^-)^2\,dx + 2\lambda\hat{v}(0)\sqrt{\rho}\int_\Lambda \psi_x^+\psi_x^-\,(\psi_x^+ + \psi_x^-)\,dx+\\ +(2\lambda\hat{v}(0)\rho+\nu)\int_\Lambda \psi_x^+\psi_x^-\,dx+\lambda\hat{v}(0)\rho\int_\Lambda(\psi_x^+ + \psi_x^-)^2\,dx\end{aligned} \tag{3.6}$$

We now observe that, if the "remainder" $\mathcal{R}V_\rho$ and the local terms of order greater than two were not present in (3.6) (so obtaining the exactly soluble *Bogoliubov model*), then the condition (2.20) could be imposed, by choosing $\nu=-2\lambda\rho\hat{v}(0)$, that is by putting equal to zero the coefficient of $\psi^+\psi^-$ in (3.5). This observation implies that the "natural" way to study the measure (3.3) is to change the free measure, by absorbing in it the local terms of V_ρ, quadratic in the field, which remain after imposing the condition $\nu^0\equiv\nu+2\lambda\rho\hat{v}(0)=0$. We shall denote $P_B(d\psi)$ the new measure and we shall call it the *renormalized free measure*.

By using the results of Appendix 1 with $a=\lambda\rho\hat{v}(0)$, $b_0=b=c=0$, we see that the renormalized free measure has indeed the behaviour (3.4) for $k\to 0$, with $c^2=c_B^2\equiv 2\lambda\rho\hat{v}(0)$.

Therefore the measure (3.3) should be written:

$$\frac{1}{\mathcal{N}}e^{-\tilde{V}_0(\psi)}P_B(d\psi) \tag{3.7}$$

where

$$P_B(d\psi)\equiv P^{(\le 0)}(d\psi)e^{-\lambda\hat{v}(0)\rho\int_\Lambda(\psi_x^+ + \psi_x^-)^2\,dx} \tag{3.8}$$

and

$$
\begin{aligned}
\tilde{V}_0(\psi) =& \lambda\hat{v}(0)\int_\Lambda (\psi_x^+\psi_x^-)^2\, dx + \\
&+ 2\lambda\hat{v}(0)\sqrt{\rho}\int_\Lambda \psi_x^+\psi_x^-(\psi_x^+ + \psi_x^-)\, dx + \nu^0\int_\Lambda \psi_x^+\psi_x^-\, dx
\end{aligned}
\tag{3.9}
$$

and the parameter ν^0 has to be determined so that the condition (2.20) is satisfied.

It is convenient, before proceeding, to change the basic fields and to perform a rescaling (amounting at fixing $\rho = 1$); we set:

$$
\chi_x^\pm = \frac{1}{\sqrt{2\rho}}(\psi_x^+ \pm \psi_x^-), \qquad \psi^\pm = \sqrt{\frac{\rho}{2}}(\chi_x^+ \pm \chi_x^-)
\tag{3.10}
$$

$$
\varepsilon = \lambda\hat{v}(0)\rho\frac{2m}{p_0^2}
\tag{3.11}
$$

so that $\tilde{V}_0$ becomes a function of χ:

$$
\begin{aligned}
\tilde{V}_0(\chi) = \frac{p_0^2\rho}{2m}\Big(\frac{\varepsilon}{4}\int_\Lambda ((\chi_x^-)^4 - 2(\chi_x^-)^2(\chi_x^+)^2 + (\chi_x^+)^4) dx + \\
+ \varepsilon\sqrt{2}\int_\Lambda ((\chi_x^+)^3 - (\chi_x^-)^2\chi_x^+)\, dx + \frac{\nu^0}{2}\frac{2m}{p_0^2}\int_\Lambda ((\chi_x^+)^2 - (\chi_x^-)^2)\, dx\Big)
\end{aligned}
\tag{3.12}
$$

and the covariance of the fields $\chi^\pm$ in the distribution (3.8) is, in the limit $\Lambda \to \infty$:

$$
\tilde{C}^{\sigma_1\sigma_2}(x-y) = \int P_B(d\chi)\chi_x^{\sigma_1}\chi_y^{\sigma_2} = \frac{1}{(2\pi)^4}\int e^{-ik(x-y)}t_0(k)\tilde{G}_0^{-1}(k)_{\sigma_1\sigma_2}
\tag{3.13}
$$

where the matrix $\tilde{G}_0(k)$ is defined by (see appendix 1):

$$
\tilde{G}_0(k) = \rho\begin{pmatrix} \frac{\vec{k}^2}{2m} + 4\varepsilon\frac{p_0^2}{2m}t_0(k) & ik_0 \\ -ik_0 & -\frac{\vec{k}^2}{2m} \end{pmatrix}
\tag{3.14}
$$

where the first row and column correspond to $\sigma = +$ and the second row and column correspond to $\sigma = -$.

Note that $\tilde{C}^{--}(k)$ behaves, for $k \to 0$, as $1/k^2$ (like in φ_4^4 theory), while $\tilde{C}^{-+}(k)$ and $\tilde{C}^{++}(k)$ are less singular. In a sense, the field χ^+ behaves like the derivative of the field χ^-.

In order to study the measure (3.7), we make the scale decomposition

$$
\tilde{C}^{\sigma_1\sigma_2}(x) = \sum_{h=-\infty}^{0}\tilde{C}_h^{\sigma_1\sigma_2}(x)
\tag{3.15}
$$

where $\tilde C_h^{\sigma_1\sigma_2}(x)$ is obtained from (3.13) by substituting $t_0(k)$ with

$$\tilde T_h(k) = t_0(\gamma^{-h}k) - t_0(\gamma^{-h+1}k) \tag{3.16}$$

so that we have the scaling relations:

$$\tilde C_h^{\sigma_1\sigma_2}(x) = \gamma^{(3+\frac{\sigma_1+\sigma_2}{2})h}\tilde g_h^{\sigma_1\sigma_2}(\gamma^h x) \tag{3.17}$$

with

$$\tilde g_h^{\sigma_1\sigma_2}(x) = \frac{1}{(2\pi)^4}\int dk e^{-ikx}\tilde T_0(k)\tilde G_h^{-1}(k)_{\sigma_1\sigma_2} \tag{3.18}$$

and

$$\tilde G_h(k) = \rho\begin{pmatrix} \frac{\gamma^{2h}\vec k^2}{2m} + 4\varepsilon\frac{p_0^2}{2m}t_0(\gamma^h k) & ik_0 \\ -ik_0 & -\frac{\vec k^2}{2m}\end{pmatrix} \tag{3.19}$$

Note that $\tilde g_h^{\sigma_1\sigma_2}(x)$ are essentially independent of h, for $h\to-\infty$.

Let us now define the *effective potentials* $\tilde V_h(\chi)$ in the usual way [G]:

$$\begin{aligned} e^{-\tilde V_h(\chi)} &= \int \tilde P_B^{(h+1)}(d\chi^{(h+1)})\ldots\tilde P_B^{(0)}(d\chi^{(0)})e^{-\tilde V_0(\chi+\chi^{(h+1)}+\ldots+\chi^{(0)})} = \\ &= \int \tilde P_B^{(h+1)}(d\chi^{(h+1)})e^{-\tilde V_{h+1}(\chi+\chi^{(h+1)})} \end{aligned} \tag{3.20}$$

where $\tilde P_B^{(h)}(d\chi)$ denotes the measure with covariance $\tilde C_h$.

The scaling relations (3.17) imply that we can define dimensionless fields $\bar\chi^\pm$ by the relations:

$$\chi_x^- = \gamma^h\bar\chi^-_{\gamma^h x} \quad , \quad \chi_x^+ = \gamma^{2h}\bar\chi^+_{\gamma^h x} \tag{3.21}$$

and this allows us to analyze the relevant local part $\mathcal{L}\tilde V_h(\chi)$ of the effective potentials $\tilde V_h(\chi)$.

$\mathcal{L}\tilde V_h(\chi)$ is in principle a linear combination of all the local terms relevant or marginal. However only a few of them are different from zero, thanks to the symmetries of the original potential (2.13). Let us consider first the local terms not involving derivatives of the fields, that is terms of the form

$$F_{m_1m_2} = \int dx\, {\chi_x^-}^{m_1}{\chi_x^+}^{m_2} \tag{3.22}$$

By using (3.21), it is easy to show that $F_{m_1m_2}$ is not irrelevant only if $m_1+m_2-4\le 0$. The terms with $m_1+m_2=1$ are absent, since the fields $\chi^\pm$ have no $k=0$ Fourier component. If $m_1+m_2=2$, we have two relevant terms, F_{20} and F_{11}, and one marginal term, F_{02}. If $m_1+m_2\ge 3$, we have one relevant term, F_{30}, and two marginal terms, F_{40} and F_{31}. However, F_{11} and F_{30} are absent, as a consequence of the particular structure of the potential (2.13), which implies that the local monomials of order 2 and 3 in the fields $\psi^{\pm}$ *must* appear in the following combinations:

$$\begin{aligned}
(\psi_x^+)^3+(\psi_x^-)^3 &= 2\frac{\rho^{3/2}}{2^{3/2}}((\chi_x^+)^3+3\chi^+(\chi^-)^2)\\
\psi_x^+\psi_x^-(\psi_x^++\psi_x^-) &= 2\frac{\rho^{3/2}}{2^{3/2}}((\chi_x^+)^3-(\chi_x^+)(\chi_x^+)^2)\\
\psi_x^+\psi_x^- &= \frac{\rho}{2}((\chi_x^+)^2-(\chi_x^-)^2)\\
(\psi_x^+)^2+(\psi_x^-)^2 &= 2\frac{\rho}{2}((\chi_x^+)^2+(\chi_x^-)^2)
\end{aligned}\tag{3.23}$$

As regards the local terms involving derivatives of the field, we have three marginal terms, that is:

$$\begin{aligned}
D_{tt} &= -(\frac{2m}{p_0^2})^2\int_\Lambda(\partial_{x_0}\chi_x^-)^2\,dx \quad , \quad D_{ss}=-p_0^{-2}\int_\Lambda(\partial_{\vec{x}}\chi_x^-)^2\,dx\\
D_t &= -\frac{2m}{p_0^2}\int_\Lambda\chi_x^+\partial_{x_0}\chi_x^-\,dx
\end{aligned}\tag{3.24}$$

We did not include in the list the local term $-\int_\Lambda dx\chi_x^+\partial_{\vec{x}}\chi_x^- = (1/\Lambda)\sum_k\chi_k^+ i\vec{k}\,\chi_k^-$, since it is identically zero; in fact the fields χ_k^σ are even functions of $\vec{k}$. Finally, the local terms $\int_\Lambda dx\chi_x^-\partial\chi_x^-$ and $\int_\Lambda dx(\chi_x^-)^2\partial\chi_x^-$ are absent, since they are integrals of total derivatives and the fields satisfy periodic boundary conditions.

We can now define the relevant part of the effective potentials in terms of a *localization operator*, defined in the usual way [G]. For example, if (m,n) is equal to $(4,0)$, $(2,1)$ or $(0,2)$, we define:

$$\mathcal{L}\chi_{x_1}^-\ldots\chi_{x_m}^-\chi_{y_1}^+\ldots\chi_{y_n}^+=(\chi_{x_1}^-)^m(\chi_{x_1}^+)^n \tag{3.25}$$

and a similar definition is used for the other marginal terms. The monomials of positive dimension are localized trough a suitable Taylor expansion:

$$\begin{aligned}
\mathcal{L}\chi_x^-\chi_y^- &= (\chi_x^-)^2+\sum_{i=0}^3(y_i-x_i)\chi_x^-\partial_i\chi_x^-+\frac{1}{2}\sum_{i,j=0}^3(y_i-x_i)(y_j-x_j)\chi_x^-\partial_i\partial_j\chi_x^-\\
\mathcal{L}\chi_x^+\chi_y^- &= \chi_x^+\chi_x^-+\sum_{i=0}^3(y_i-x_i)\chi_x^+\partial_i\chi_x^-
\end{aligned}\tag{3.26}$$

Note that, according to the remark preceeding (3.23) and rotation invariance, many terms in the r.h.s. of (3.26) cancel out in the full effective potential.

The previous discussion implies that $\mathcal{L}\tilde{V}_h(\chi)$ is of the form

$$\begin{aligned}
\mathcal{L}\tilde{V}^{(h)}(\chi)=&\frac{p_0^2\rho}{2m}(\tilde{\lambda}_hF_{40}+\tilde{\mu}_hF_{21}+\gamma^{2h}\tilde{\nu}_h(F_{02}-F_{20})+\\
&+2\tilde{z}_hF_{02}+2\tilde{\zeta}_hD_{tt}+2\tilde{\alpha}_hD_{ss}+2\tilde{d}_hD_t)
\end{aligned}\tag{3.27}$$

where the factors 2 have no special meaning and the dimension fixing factor $\frac{p_0^2\rho}{2m}$ is introduced to keep track of the dimensions of the various quantities (its physical dimension is that of an action density, in space time).

As usual, we can hope to have a perturbative control of the model only if we can show that all the running constants appearing in (3.27) stay small for $h \to -\infty$, for a suitable choice of ν_0, such that:

$$|\tilde{\nu}_h| \leq \bar{\nu} \qquad , \qquad \text{for } h \to -\infty \tag{3.28}$$

This condition is equivalent to the renormalization condition (2.20). In fact it implies that, for $h \to -\infty$, the effective potential is of the second order in the fields $\psi^{\pm}$ and that they appear only in the combination $(\psi_x^+ + \psi_x^-)^2 = 2\rho F_{02}$, up to terms containing field derivatives, which do not influence (2.20). A simple calculation allows to prove that this structure of the effective potential implies the renormalization condition.

For $h = 0$ we have:

$$\tilde{\lambda}_0 = \frac{\varepsilon}{4}, \quad \tilde{\mu}_0 = -\varepsilon\sqrt{2}, \quad \tilde{\nu}_0 = \frac{\nu^0}{2}\frac{2m}{p_0^2} \tag{3.29}$$

and

$$\tilde{z}_0 = \tilde{\zeta}_0 = \tilde{\alpha}_0 = \tilde{d}_0 = 0 \tag{3.30}$$

However $\tilde{z}_h$, $\tilde{\zeta}_h$, $\tilde{d}_h$ and $\tilde{\alpha}_h$ will be different from zero for $h < 0$ and they could grow as $h \to -\infty$.

This problem can be solved by the same strategy used in passing from the representation (3.3) of the measure to the representation (3.7). We define iteratively a new family of effective potentials $V_h(\chi)$ in the following way.

Given $V_0(\chi) = \tilde{V}_0(\chi)$, we define $\tilde{V}_{-1}(\chi)$ as before, so that:

$$\begin{aligned} \frac{1}{\mathcal{N}}\int P_B(d\chi)e^{-V_0(\chi)} &= \frac{1}{\mathcal{N}}\int \tilde{P}_B^{(\leq -1)}(d\chi)\tilde{P}_B^{(0)}(d\chi^{(0)})e^{-V_0(\chi+\chi^{(0)})} = \\ &= \frac{1}{\mathcal{N}}\int \tilde{P}_B^{(\leq -1)}(d\chi)e^{-\tilde{V}_{-1}(\chi)} \end{aligned} \tag{3.31}$$

where $\tilde{P}_B^{(\leq -1)}(d\chi)$ and $\tilde{P}_B^{(0)}(d\chi)$ denote the measures with covariance $\sum_{h=-\infty}^{-1}\tilde{C}_h^{\sigma_1\sigma_2}$ and $\tilde{C}_0^{\sigma_1\sigma_2}$, respectively.

We then define $V_{-1}(\chi)$ by absorbing the terms proportional to $\tilde{z}_{-1}$, $\tilde{\zeta}_{-1}$, $\tilde{d}_{-1}$ and $\tilde{\alpha}_{-1}$ in the measure $\tilde{P}_B^{(\leq -1)}(d\chi)$, so that

$$\frac{1}{\mathcal{N}}\int \tilde{P}_B^{(\leq -1)}(d\chi)e^{-\tilde{V}_{-1}(\chi)} = \frac{1}{\mathcal{N}}\int P_B^{(\leq -1)}(d\chi)e^{-V_{-1}(\chi)} \tag{3.32}$$

We can iterate this procedure, so defining a family of measures $P_B^{(\leq h)}(d\chi)$ and a family of effective potentials $V_h(\chi)$, such that:

$$\mathcal{L}V_h(\chi) = \frac{p_0^2\rho}{2m}\left[\lambda_h F_{40} + \mu_h F_{21} + \gamma^{2h}\nu_h(F_{02} - F_{20})\right] \tag{3.33}$$

and the covariance $C^{\sigma_1\sigma_2}_{\leq h}$ of $P^{(\leq h)}_B(d\chi)$ is of the form (as follows from the calculations in appendix 1):

$$C^{\sigma_1\sigma_2}_{\leq h}(x-y) = \frac{1}{(2\pi)^4}\int dk e^{-ik(x-y)}\, t_0(\gamma^{-h}k)\, G^{-1}_{\leq h}(k)_{\sigma_1\sigma_2} \tag{3.34}$$

where

$$G_{\leq h}(k) = \rho \begin{pmatrix} \frac{\vec{k}^2}{2m} + 4\frac{p_0^2}{2m}\bar{Z}_h(k) & ik_0\bar{E}_h(k) \\ -ik_0\bar{E}_h(k) & -\left(\frac{\vec{k}^2}{2m} + 4\bar{B}_h(k)\frac{2m}{p_0^2}\,k_0^2 + 4\bar{A}_h(k)\frac{\vec{k}^2}{2m}\right) \end{pmatrix} \tag{3.35}$$

Note that $V_h(\chi)$ can be easily obtained from the potential $\bar{V}_h(\chi)$ defined by:

$$e^{-\bar{V}_{h-1}(\chi)} = \frac{1}{\mathcal{N}}\int \bar{P}^{(h)}_B(d\chi^{(0)})e^{-V_h(\chi+\chi^{(0)})} \tag{3.36}$$

where $\bar{P}^{(h)}_B(d\chi)$ is a single scale covariance obtained from (3.34) by substituting $t_0(\gamma^{-h}k)$ with

$$T_0(\gamma^{-h}k) = t_0(\gamma^{-h}k) - t_0(\gamma^{-h+1}k) \tag{3.37}$$

$V_h(\chi)$ is calculated by the equation:

$$V_h(\chi) = \bar{V}_h(\chi) - \frac{p_0^2\rho}{2m}\left[2z_hF_{02} + 2\zeta_hD_{tt} + 2\alpha_hD_{ss} + 2d_hD_t\right] \tag{3.38}$$

if the local part of $\bar{V}_h$ is written in the form

$$\begin{aligned}\mathcal{L}\bar{V}_h(\chi) = \frac{p_0^2\rho}{2m}[&\lambda_hF_{40} + \mu_hF_{21} + \gamma^{2h}\nu_h(F_{02}-F_{20}) + \\ &+ 2z_hF_{02} + 2\zeta_hD_{tt} + 2\alpha_hD_{ss} + 2d_hD_t]\end{aligned} \tag{3.39}$$

Note that the choice (3.2) of $t_0(k)$ implies that

$$T_h(k)T_{h'}(k) = 0 \qquad \text{if} |h-h'| > 1 \tag{3.40}$$

The functions $\bar{A}_h(k)$, $\bar{B}_h(k)$, $\bar{Z}_h(k)$ and $\bar{E}_h(k)$ satisfy, for $h \leq -1$, some recursion relations, which can be easily obtained using (3.38) and the relation ♣A1.11 of appendix 1, implying that

$$G_{\leq h}(k) = G_{\leq h+1} + 2\Delta_h(k)t_0(\gamma^{-h}k) \tag{3.41}$$

where:

$$\Delta_h(k) = \rho \begin{pmatrix} 2z_h\frac{p_0^2}{2m} & id_hk_0 \\ -id_hk_0 & -2\frac{2m}{p_0^2}k_0^2\zeta_h - 2\alpha_h\frac{\vec{k}^2}{2m} \end{pmatrix} \tag{3.42}$$

It follows that:

$$\begin{aligned}\bar{Z}_h(k) &= \bar{Z}_{h+1}(k) + z_h t_0(\gamma^{-h}k)\\ \bar{A}_h(k) &= \bar{A}_{h+1}(k) + \alpha_h t_0(\gamma^{-h}k)\\ \bar{B}_h(k) &= \bar{B}_{h+1}(k) + \zeta_h t_0(\gamma^{-h}k)\\ \bar{E}_h(k) &= \bar{E}_{h+1}(k) + 2d_h t_0(\gamma^{-h}k)\end{aligned} \tag{3.43}$$

with initial conditions:

$$\bar{Z}_0(k) = \varepsilon t_0(k) \quad , \quad \bar{A}_0(k) = \bar{B}_0(k) = 0 \quad , \quad \bar{E}_0(k) = 1 \tag{3.44}$$

The covariance $\bar{C}_h^{\sigma_1\sigma_2}(x)$ associated with the measure $\bar{P}_B^{(h)}(d\chi)$ satisfies the following scaling relations:

$$\bar{C}_h^{\sigma_1\sigma_2}(x) = \gamma^{(3+\frac{\sigma_1+\sigma_2}{2})h} g_h^{\sigma_1\sigma_2}(\gamma^h x) \tag{3.45}$$

where

$$g_h^{\sigma_1\sigma_2}(x) = \frac{1}{(2\pi)^4}\int dk \frac{e^{-ikx}\ T_0(k)\ p_h^{\sigma_1\sigma_2}(k)}{\rho\mathcal{D}_h(k)} \tag{3.46}$$

$$\mathcal{D}_h(k) = b_h(k)k_0^2 + a_h(k)\frac{v_0^2}{4}\vec{k}^2 + \gamma^{2h}\Big[\frac{\vec{k}^4}{4m^2}(1+4A_h(k)) + 4B_h(k)\frac{k_0^2\vec{k}^2}{p_0^2}\Big] \tag{3.47}$$

$$\begin{aligned}a_h(k) &= 4Z_h(k) + 16Z_h(k)A_h(k)\\ b_h(k) &= E_h(k)^2 + 16Z_h(k)B_h(k)\end{aligned} \tag{3.48}$$

$$\begin{aligned}&p_h^{-+}(k) = p_h^{+-}(-k) = -ik_0E_h(k) \quad , \quad p_h^{--}(k) = -\gamma^{2h}\frac{\vec{k}^2}{2m} - 4Z_h(k)\frac{p_0^2}{2m}\\ &p_h^{++}(k) = \frac{\vec{k}^2}{2m} + 4\Big[B_h(k)k_0^2\frac{2m}{p_0^2} + A_h(k)\frac{\vec{k}^2}{2m}\Big] \quad , \quad v_0 = \frac{p_0}{m}\end{aligned} \tag{3.49}$$

By using (3.43), (3.44) and (3.2), it is easy to see that, on the support of $T_0(k)$:

$$\begin{aligned}Z_h(k) &= \bar{Z}_h(\gamma^h k) = \sum_{h'=h}^{0} z_{h'}t_0(\gamma^{-(h'-h)}k) = \sum_{h'=h+1}^{0} z_{h'} + z_h t_0(k)\\ A_h(k) &= \bar{A}_h(\gamma^h k) = \sum_{h'=h}^{0} \alpha_{h'}t_0(\gamma^{-(h'-h)}k) = \sum_{h'=h+1}^{0} \alpha_{h'} + \alpha_h t_0(k)\\ B_h(k) &= \bar{B}_h(\gamma^h k) = \sum_{h'=h}^{0} \zeta_{h'}t_0(\gamma^{-(h'-h)}k) = \sum_{h'=h+1}^{0} \zeta_{h'} + \zeta_h t_0(k)\\ E_h(k) &= \bar{E}_h(\gamma^h k) = 1 + 2\sum_{h'=h}^{0} d_{h'}t_0(\gamma^{-(h'-h)}k) = 1 + 2\sum_{h'=h+1}^{0} d_{h'} + 2d_h t_0(k)\\ &z_0 = \varepsilon \quad , \quad \alpha_0 = \zeta_0 = d_0 = 0\end{aligned} \tag{3.50}$$

Note that there are some relations between λ_h, μ_h and $Z_h(0)$, which can be defined in another equivalent way by the following steps.

a) We first integrate in a single step the fluctuations up to scale $h+1$, without any free measure renormalization and by using the original representation (2.12) of the potential. The result of this operation can be written in the form:

$$\tilde{U}_h(\varphi) = \sum_{s=1}^{\infty} \int d\underline{x} d\underline{y} W_{h,s}(\underline{x},\underline{y}) \varphi^+_{x_1} \cdots \varphi^+_{x_s} \varphi^-_{y_1} \cdots \varphi^-_{y_s} \tag{3.51}$$

where $\underline{x} = (x_1, \ldots, x_s)$, $\underline{y} = (y_1, \ldots, y_s)$. $U_h(\varphi)$ satisfies the identity:

$$\int P(d\psi) e^{-V(\varphi)} = \frac{1}{\mathcal{N}} \int P_0^{(\le h)}(d\psi) e^{-\tilde{U}_h(\varphi)} \tag{3.52}$$

where $P_0^{(\le h)}(d\psi)$ is the measure with propagator:

$$C_0^{(\le h)}(x) = \frac{1}{(2\pi)^4} \int dk\, t_0(\gamma^{-h}k) \frac{e^{-ikx}}{-ik_0 + \vec{k}^2} \tag{3.53}$$

b) We insert the representation (2.14) of $\varphi^\pm$ in (3.51), we collect the terms containing the monomials $(\prod_{i=1}^{r_1} \psi^+_{x_i})(\prod_{j=1}^{r_2} \psi^-_{y_j})$, with $2 \le r_1 + r_2 \le 4$, and we localize them, by a Taylor expansion of the fields of order 0, if $r_1 + r_2 > 2$, and of order 2, if $r_1 + r_2 = 2$. We obtain various local terms; those which do not contain any derivative of a field are of the form

$$\lambda_h^{r_1,r_2} \int dx (\psi^+_x)^{r_1} (\psi^-_x)^{r_2} \tag{3.54}$$

with

$$\lambda_h^{r_1,r_2} = \sum_{s=1}^{\infty} \binom{s}{r_1}\binom{s}{r_2} \bar{W}_{h,s} \rho^{s-2} \quad , \quad \bar{W}_{h,s} = \lim_{\beta|\Lambda|\to\infty} \frac{1}{\beta\Lambda} \int d\underline{x} d\underline{y} W_{h,s}(\underline{x},\underline{y}) \tag{3.55}$$

c) We insert the representation (3.10) in (3.54) and we collect the terms proportional to F_{40}, F_{21}, $F_{02} - F_{20}$, F_{02}, D_{tt}, D_{ss} and D_t; let us call $\tilde{\lambda}_h$, $\tilde{\mu}_h$, $\tilde{\nu}_h$, $2\tilde{Z}_h$, $2\tilde{B}_h$, $2\tilde{A}_h$ and $(\tilde{E}_h - 1)$ their respective coefficients. By using (3.55), it is easy to prove that:

$$\begin{aligned}
\tilde{\lambda}_h &= \frac{1}{4} \sum_{s=2}^{\infty} \bar{W}_{h,s} \rho^s \sum_{r=0}^{4} \binom{s}{r}\binom{s}{4-r} \\
\tilde{\mu}_h &= -\frac{1}{\sqrt{2}} \sum_{s=2}^{\infty} s(s-1) \bar{W}_{h,s} \rho^s \\
\tilde{Z}_h &= \frac{1}{2} \sum_{s=2}^{\infty} s(s-1) \bar{W}_{h,s} \rho^s
\end{aligned} \tag{3.56}$$

Note that there is no contribution with $s=1$ in the equation for $\tilde Z_h$, because they could come only from the monomials $\psi^+_x\psi^-_y$, which only contribute to $F_{02}-F_{20}$, see (3.23).

d) We absorb in the free measure the local terms proportional to F_{02}, D_{tt}, D_{ss} and D_t. If we call $U_h(\chi)$ the remaining part of $\tilde U_h(\psi)$, thought as a function of χ, and $P^{(\le h)}_{0,B}(\chi)$ the renormalized free measure, we can write:

$$\int P^{(\le 0)}(d\psi)e^{-V(\varphi)}=\frac{1}{\mathcal N}\int P^{(\le h)}_{0,B}(d\chi)e^{-U_h(\chi)} \tag{3.57}$$

However, this identity and the property that the local terms proportional to F_{02}, D_{tt}, D_{ss} and D_t are equal to zero uniquely determine the effective potential $V_h(\chi)$ and the measure $P^{(\le h)}_B(d\chi)$. Hence $\tilde\lambda_h=(p_0^2\rho/2m)\lambda_h$, $\tilde\mu_h=(p_0^2\rho/2m)\mu_h$ and (by using the results of appendix 1) $\tilde Z_h t_0(\gamma^{-h}k)=(p_0^2\rho/2m)Z_h(k)$.

The last observation and (3.56) imply the *exact* identity:

$$Z_h(0)=-\frac{\mu_h}{\sqrt 2}\qquad,\qquad \forall h\le 0 \tag{3.58}$$

which will play a very important role in the discussion of the following section.

§4 *The beta function.*

We want to study perturbatively the flow of the *running couplings* λ_h, μ_h and ν_h (the *beta function* of our problem) and of the *renormalization functions* $Z_h(k)$, $A_h(k)$, $B_h(k)$ and $E_h(k)$, by keeping only the leading terms in the expansion of r_{h-1} in terms of $\{r_{h'}, h'\ge h\}$, if $r_h\equiv\{\lambda_h,\mu_h,\nu_h,Z_h,A_h,B_h,E_h\}$.

In order to understand which are the leading terms, we shall consider only the Feynman graphs calculated by using the single scale propagator (3.45); this corresponds to the approximation of neglecting the irrelevant part of $V_h(\chi)$ in calculating the relevant part of $V_{h-1}(\chi)$. In terms of the tree expansion described in [G], this approximation implies that we consider only trees with one tree vertex on scale h. The tree expansion allows to prove also that one obtains in this way the same bounds that one would obtain by summing over all trees with the same number of λ-vertices, μ-vertices and ν-vertices on different scales.

In order to perform this analysis, we need bounds on the rescaled propagators (3.46). If we define a scale h_0 so that:

$$v_0^2 z_0=\gamma^{2h_0}\frac{q_0^2}{4m^2} \tag{4.1}$$

then we have to distinguish the case $h \geq h_0$ from the case $h \leq h_0$. In the first case, if $T_0(k) \neq 0$, $\gamma^{2h}\vec{k}^4$ is dominating over $a_h\vec{k}^2$, at least if $|a_h\vec{k}^2| \leq cz_0$, that we shall prove is true for λ small enough. In the second case $a_h\vec{k}^2$ is dominating over $\gamma^{2h}\vec{k}^4$.

If $\gamma^{2h} \geq Z_h(0)$, which essentially means $h \geq h_0$, and

$$|A_h| \leq \eta \quad , \quad \gamma^{2h}|B_h| \leq \eta \quad , \quad |E_h - 1| \leq \eta \tag{4.2}$$

where η is some number sufficiently small, one can show that there exists a fast decaying function $f(x)$ such that:

$$|g_h^{\sigma_1\sigma_2}(x)| \leq \gamma^{-h\frac{\sigma_1+\sigma_2}{2}} f(\gamma^h x_0, \vec{x}) \tag{4.3}$$

Let us now consider a graph contributing to the flow equation for λ_h, μ_h or $Z_h(0)$ and denote n_σ the number of external lines of type χ^σ, n_λ the number of λ-vertices, n_μ the number of μ-vertices. In order to estimate the contribution of this graph, we observe that the bound (4.3) implies that each internal half-line of type χ^σ gives a contribution $\gamma^{-h\sigma/2}$, while each one of the $(n_\lambda + n_\mu - 1)$ integrations that one has to perform to evaluate the local part of the graph gives a contribution proportional to γ^{-h}. Hence, up to a constant, the graph can be bounded by:

$$\begin{aligned} &|\lambda_h|^{n_\lambda}|\mu_h|^{n_\mu}\gamma^{h[\frac{1}{2}(4n_\lambda+2n_\mu-n_-)-\frac{1}{2}(n_\mu-n_+)-(n_\lambda+n_\mu-1)]} = \\ &\quad = \lambda_h|^{n_\lambda}|\mu_h|^{n_\mu}\gamma^{h[n_\lambda-\frac{1}{2}n_\mu+\frac{1}{2}(n_+-n_-+2)]} \end{aligned} \tag{4.4}$$

If we denote N_L the number of independent loops of the graph (equal to the number of propagators minus $(n_\lambda + n_\mu - 1)$) and we use the identity (3.58), we can write the bound (4.4) in the form:

$$\lambda_h^{n_\lambda}(Z_h^2\gamma^{-2h})^{n_\mu/2}\gamma^{h(N_L+n_+)} \leq \lambda_h^{n_\lambda} Z_h^{n_\mu/2}\gamma^{h(N_L+n_+)} \tag{4.5}$$

Note that the contribution of a graph to the beta function is obtained by calculating its value at zero momentum of the external lines. Therefore no graph with $N_L = 0$ can contribute, because the single scale propagator vanishes at zero momentum. Moreover, (3.29) and (3.44) imply that $\lambda_0 = Z_0/(4)$, so that, by using (4.5), one can easily show that, if λ_0 is small enough,:

$$Z_h \leq 5\rho\lambda_h \tag{4.6}$$

In fact this inequality follows by induction from the remark that, if it is true for $h' \geq h$, then (4.5) implies that the leading contributions to the beta function are those with $N_L = 1$. Hence one expects that there exists a constant c such that:

$$\begin{aligned} &|\lambda_{h-1} - \lambda_h| \leq c\lambda_h^2\gamma^h \\ &|Z_{h-1} - Z_h| = \frac{|\mu_{h-1} - \mu_h|}{\sqrt{2}} \leq \frac{c}{\sqrt{2}}\lambda_h^{3/2}\gamma^{2h} \end{aligned} \tag{4.7}$$

which implies (4.6), together with

$$0 < Z_h \leq 2Z_0 \qquad , \qquad \text{if } 2h \geq h_0 \tag{4.8}$$

if λ_0 is small enough.

Note that all the previous bounds rest on the hypothesis (4.2); hence we have to check that (4.2) is consistent with (4.6) and (4.8). This can be easily done, by noticing that the graphs contributing to B_{h-1} and A_{h-1} are the graphs with $n_- = 2$ and $n_+ = 0$ and that their contribution is of the form $\int dx x_0^2 W_G(x)$ and $\int dx \vec{x}^2 W_G(x)$, respectively. Hence, we can use (4.5) for bounding $A_{h-1} - A_h$, while we have to multiply that bound by a factor γ^{-2h}, in the case of $B_{h-1} - B_h$; by using also (4.6), we find that:

$$|B_{h-1} - B_h| \leq c\lambda_h \gamma^{-h} \qquad , \qquad |A_{h-1} - A_h| \leq c\lambda_h \gamma^h \tag{4.9}$$

In a similar way one can prove that

$$|E_{h-1} - E_h| = 2|d_h| \leq c\lambda_h \gamma^h \tag{4.10}$$

Hence (4.2) are satisfied, if λ_0 is small enough.

Remark - In order to make rigorous the previous considerations, one can not really use a complete perturbative argument, because the formal series representing the beta function are at most asymptotic series. In fact, by applying the technique of reference [G], one can prove only $n!$ bounds on the sum of all graphs with n point vertices, like in the Fermi gas problem, see [BG].

The previous discussion shows that, in order to study the beta function, it is sufficient to consider carefully only the region $h < h_0$. The latter is the region where, if, for some constants c, c_1:

$$\frac{1}{2} \leq 1 + 4A_h \leq c, \qquad cZ_h \leq E_h^2 + 16B_h Z_h \leq c_1 Z_h, \quad 0 < Z_h < 2Z_0 \equiv 2\varepsilon \tag{4.11}$$

the rescaled propagators $g_h^{--}(x)$ and $g_h^{-+}(x)$ are essentially independent of h, that is they can be bounded by a rapidly decaying function $f(x)$, uniformly in A_h, B_h, Z_h, E_h verifying (4.11) (see appendix 2). On the contrary, the propagator $g_h^{++}(x)$ can be bounded by $f(x)/Z_h$.

The previous properties of the rescaled propagators and the identity (3.58) imply that, in the bound of a generic graph contributing to the beta function, two μ_h vertices are essentially equivalent to one λ_h vertex. A further simple analysis allows to prove that, if one wants to keep in each flow equation only the leading terms, then one has to consider only one loop graphs without ν_h vertices. Therefore, we shall study the flow equations, by keeping only these contributions; the properties of the corresponding solutions will be used to justify the approximation.

In this approximation ν_h does not appear in the flow equations of λ_h and μ_h; hence the growth of ν_h is controlled in a simple way by the right choice of ν_0, that is the right value of the chemical potential, if one can control the flow of λ_h and μ_h, which only depend on the renormalization functions.

It turns out that the leading terms of the beta function depend in a smooth way on the cutoff function $t_0(k)$, as one expects, because these terms determine the asymptotic behaviour of the model, which is independent of the choice of $t_0(k)$. Therefore we shall approximate the smooth cutoff function $t_0(k)$, whenever this is possible, by the characteristic function of the set $\{k_0^2 + \vec{k}^2 \le 1\}$.

This approximation simplifies the discussion of the beta function, because it implies that the supports of the Fourier transforms of the single scale covariances $\bar{C}_h^{\sigma_1\sigma_2}(x)$ are disjoint. This will guarantee that the leading terms in the beta function for the running couplings on scale $(h-1)$ will depend only on the running couplings on scale h; in fact we have to consider only graphs with up to four point vertices, one loop and all external momenta equal to 0, so that all the propagators belonging to the loop must have the same momentum. In terms of the tree expansion described in [G] this means that we must consider only trees with one tree vertex on scale h.

Note that this approximation is apparently not allowed everywhere; in fact the constants ζ_h and α_h (see (3.39)) depend on $t_0'(k)^2$, hence they are divergent when the regularization of $t_0(k)$ goes to zero. However there is a cancellation of these contributions, as well as of all contributions containing derivatives of $t_0(k)$, in the calculation of the leading terms, as explained in appendix 2. Therefore also in this case we can take the same approximation for the cutoff function.

There is another simplification following from the choice of $t_0(k)$ described before: $Z_h(k)$, $A_h(k)$ and $B_h(k)$ are independent of k, for $k^2 \le \gamma^h$, that is for the values of k in the support of the Fourier transform of $C_{\le h}^{\sigma_1\sigma_2}(x)$. This immediately follows from (3.43). Hence, also Z_h, A_h and B_h are constants.

There are very "few" graphs with one loop; therefore the calculation is straightforward: but quite long. Here we report the result, while the details are exposed in appendix 2. Neglecting terms of order γ^{2h} (which essentially come from the corrections to the scaling of the propagators, which become quickly scale independent), one finds after using (3.58) to eliminate μ_h from the row result (see appendix 2):

$$\begin{aligned}
\lambda_{h-1} &= \lambda_h - 36 q_0^3 \beta_{2,h} (4Z_h)^2 \big(\lambda_h - \frac{Z_h}{6}\big)^2 \\
Z_{h-1} &= Z_h - \frac{1}{4} q_0^3 \beta_{2,h} (4Z_h)^3 Z_h \\
E_{h-1} &= E_h - \frac{1}{4} q_0^3 \beta_{2,h} (4Z_h)^3 E_h \\
A_{h-1} &= A_h \\
B_{h-1} &= B_h + \frac{1}{16} q_0^3 \beta_{2,h} (4Z_h)^2 E_h^2
\end{aligned} \tag{4.12}$$

$$\nu_{h-1} = \gamma^2 \left(\nu_h + q_0\beta_{1,h}(24Z_h\lambda_h - 4Z_h^2)\right)$$

where:

$$\begin{aligned}\beta_{2,h} =& \frac{\log\gamma}{8\pi^2\sqrt{a_h^3 b_h}}\left(\frac{2}{v_0}\right)^3, \quad \beta_{1,h} = \frac{1-\gamma^{-2}}{8\pi^2(\sqrt{a_h b_h}+a_h)} q_0^2\left(\frac{2}{v_0}\right)^3, \quad v_0 = \frac{p_0}{m} \\ a_h =& 4Z_h(1+4A_h), \quad b_h = (E_h^2 + 16 Z_h B_h), \quad q_0 = \frac{p_0^2}{2m}\end{aligned} \tag{4.13}$$

It is very easy to analyze this flow, under the conditions (4.11), implying that $\beta_{2,h}$ is of order Z_h^{-2}. In fact, this observation is sufficient to prove that $Z_h = O(1/|h|)$ for $h\to-\infty$. But this property has to be true also for E_h, since, by (4.12):

$$\frac{E_{h-1}-E_h}{E_h} = \frac{Z_{h-1}-Z_h}{Z_h} \tag{4.14}$$

It is now very easy to check that

$$B_h \xrightarrow[h\to-\infty]{} B_{-\infty} > 0 \tag{4.15}$$

while A_h stays constant (indeed a small constant, roughly equal to its value on scale h_0).

Finally, if we define:

$$c_0 = \frac{p_0^3\log\gamma}{2\pi^2 v_0^3\sqrt{B_{-\infty}(1+4A_{-\infty})}} \tag{4.16}$$

we see that the first two equations in (4.12) can be written, for $h\to-\infty$, in the form:

$$\begin{aligned}\lambda_{h-1} =& \lambda_h - 36c_0Z_h^2\left(\frac{\lambda_h}{Z_h}-\frac{1}{6}\right)^2 \\ Z_{h-1} =& Z_h - c_0 Z_h^2\end{aligned} \tag{4.17}$$

The discussion of the above equations is elementary and, starting from initial data $Z_0=\varepsilon, \lambda_0 = Z_0/4$ (or any other close to them), the result is that, if ν_0 is chosen so that ν_h is bounded uniformly in h, then, asymptotically:

$$Z_h = c\,|h|^{-1}, \quad \lambda_h = \frac{1}{4}Z_h, \quad \nu_h = O(\lambda_h) \tag{4.18}$$

if c is a suitable constant (ε independent).

Note that these results are consistent with (4.11), which can be then proved inductively, together with (4.18).

At this point it is very easy to check that all the neglected terms in the beta function are at least of order $1/|h|^2$. Hence they can not change in a substantial way the asymptotic properties of the flow (up to convergence problems, see the remark above); only the values of $A_{-\infty}$, $B_{-\infty}$ and c depend on them, and $A_{-\infty}$ has to be a small number (of order ε). Note that this last observation is important, in order to be sure that the first condition in (4.11) is preserved, since we do not have a control on the sign of $A_{-\infty}$.

The main consequence of the previous discussion is that, for $k \to 0$ (that is for $h \to -\infty$), the model is gaussian (asymptotic freedom) and the pair Schwinger function of the fields $\psi^{\pm}$ behaves as:

$$\tilde{S}_{-+}(k) = -\tilde{S}_{--}(k) = -\tilde{S}_{++}(k) \simeq \frac{q_0}{8B_{-\infty}} \frac{1}{k_0^2 + c^2 \vec{k}^2} \tag{4.19}$$

where the sound speed c is given by:

$$c^2 = \frac{(1 + 4A_{-\infty}) v_0^2}{16 B_{-\infty}} \tag{4.20}$$

Appendix 1: Propagators for the Bose gas.

Let $P^{(t)}(d\psi)$ be the formal complex gaussian measure with covariance

$$g^{(t)}(x) = \frac{1}{(2\pi)^4} \int dk\, e^{-ikx} \frac{t(k)}{-ik_0 + \frac{\vec{k}^2}{2m}} \tag{A1.1}$$

where $x = (x_0, \vec{x})$, $k = (k_0, \vec{k})$ and $t(k)$ is a positive cutoff function.

We consider the fields $\chi_x^{\pm} = \frac{1}{\sqrt{2\rho}}(\psi_x^+ \pm \psi_x^-)$. Their propagator has the form:

$$\langle \chi_x^{\sigma} \chi_y^{\sigma'} \rangle = \frac{1}{(2\pi)^4} \int e^{-ikx} t(k) G^{-1}(k)_{\sigma\sigma'} \tag{A1.2}$$

where the matrix G, which we call the propagator matrix, is:

$$G = \rho \begin{pmatrix} \frac{\vec{k}^2}{2m} & ik_0 \\ -ik_0 & -\frac{\vec{k}^2}{2m} \end{pmatrix} \tag{A1.3}$$

Therefore the formal gaussian measure:

$$P^{(t)}(d\psi) e^{-\rho \sum_{\sigma\sigma'} \int \chi_{\sigma k}^{\sigma} \Delta_{\sigma\sigma'} \chi_{-\sigma' k}^{\sigma'} dk} \tag{A1.4}$$

has a propagator matrix $G' \equiv G + 2\rho\Delta t(k)$. In particular, if Δ is associated with the quadratic form:

$$\int dx[2a(\chi_x^+)^2 - 2\sum_{i=0}^{3} b_i(\partial_i\chi_x^-)^2 - 2c\chi_x^+\partial_0\chi_x^-] \qquad (A1.5)$$

with $a, b_0, b = b_1 = b_2 = b_3, c$ not negative real numbers, that is:

$$\Delta = \begin{pmatrix} 2a & ick_0 \\ -ick_0 & -2(b_0k_0^2 + b\vec{k}^2) \end{pmatrix} \qquad (A1.6)$$

then it is:

$$G' = \rho \begin{pmatrix} \frac{\vec{k}^2}{2m} + 4at(k) & ik_0[1 + 2ct(k)] \\ -ik_0[1 + 2ct(k)] & -\frac{\vec{k}^2}{2m} - 4(bk_0^2 + b\vec{k}^2)t(k) \end{pmatrix} \qquad (A1.7)$$

And the $\psi^\pm$ fields propagators with respect to the measure $P_{ab}^{(t)}(d\psi)$ with propagator matrix G' can be immediately checked to be:

$$\int P_{a,b}^{(t)}(d\psi)\psi_x^-\psi_y^+ = \frac{1}{(2\pi)^4}\int dk\, e^{-ik(x-y)}.$$
$$\cdot \frac{ik_0[1 + 2ct(k)] + \frac{\vec{k}^2}{2m} + 2d^+(k)t(k)}{[1 + 2ct(k)]^2k_0^2 + \frac{\vec{k}^4}{4m^2} + 4d^+(k)\frac{\vec{k}^2}{2m}t(k) + 16a(\sum_{i=0}^3 b_ik_i^2)t(k)^2}\, t(k) \qquad (A1.8)$$

and:

$$\int P_{a,b}^{(t)}(d\psi)\psi_x^+\psi_y^+ = \int P_{a,b}^{(t)}(d\psi)\psi_x^-\psi_y^- = \frac{1}{(2\pi)^4}\int dk\, e^{-ik(x-y)}.$$
$$\cdot \frac{-2d^-(k)t(k)}{[1 + 2ct(k)]^2k_0^2 + \frac{\vec{k}^4}{4m^2} + 4d^+(k)\frac{\vec{k}^2}{2m}t(k) + 16a(\sum_{i=0}^3 b_ik_i^2)t(k)^2}\, t(k) \qquad (A1.9)$$

where $d^\pm(k) = a \pm (b_0k_0^2 + b\vec{k}^2)$. Note that (A1.9) can be derived also by performing a Bogoliubov transformation.

The previous calculation can be immediately generalized, in the sense that, if the measure $P_G^{(t)}(d\psi)$ has propagator matrix G and cutoff function $t(k)$, then the measure

$$P_G^{(t)}(d\psi)e^{-\rho\sum_{\sigma\sigma'}\int \chi_{\sigma k}^\sigma \Delta_{\sigma\sigma'}\chi_{-\sigma' k}^{\sigma'} dk} \qquad (A1.10)$$

has propagator matrix

$$G' = G + 2\rho t(k)\Delta \qquad (A1.11)$$

Appendix 2: The second order beta function.

In this appendix we want to prove (4.12). We first note that all the terms in the r.h.s. of (4.12) are obtained by applying the localization operator to:

$$V_h^{\leq 4}\equiv\mathcal{E}_h(V_h)-\frac{1}{2!}\mathcal{E}_h^T(V_h^2)+\frac{1}{3!}\mathcal{E}_h^T(V_h^3)-\frac{1}{4!}\mathcal{E}_h^T(V_h^4) \tag{A2.1}$$

where $\mathcal{E}_h$ and $\mathcal{E}_h^T$ denote, respectively, the expectation and the truncated expectation with respect to the measure describing the fluctuations on scale h, whose propagator is given by (3.45)-(3.49).

Some other remarks are important.

1) In order to calculate the beta function one has to evaluate some Feynman graphs at zero momentum of the external lines. Therefore only terms without internal lines or terms with at least one loop can contribute, since the single scale propagator vanishes at zero momentum.

2) The previous remark also implies that, in the graphs with only one loop, all internal lines must carry the same momentum. Hence, if we suitably choose the cutoff function $t_0(k)$, the internal lines of the loop may only have propagators of scale h or $h+1$; in fact at least one propagator must be of scale h and the supports of the Fourier transforms of the propagators of scale h and $h'\geq h$ are disjoint if $h'>h+1$. This implies that the trees (see [G] for the definitions), that one has to consider in evaluating the contribution of such graphs, are the trees with only one vertex of scale h and at most three endpoints and the trees with one vertex on scale h and one non trivial vertex on scale $h+1$, corresponding to some irrelevant contribution.

3) In the calculation of the graphs with only one loop, the subgraph associated with the tree vertex of scale $h+1$ has no loop. Therefore in this tree vertex the $\mathcal{R}$ operation coincides with the identity.

4) Since we are interested only in the leading orders, we can neglect in the rescaled propagators (3.46) the terms proportional to γ^{2h} and the dependence on k of Z_h, A_h, B_h and E_h (see (3.50)). For the same reason we can approximate Z_{h+1}, A_{h+1}, B_{h+1} and E_{h+1} by Z_h, A_h, B_h and E_h in the expression of $g_{h+1}^{\sigma_1\sigma_2}(x)$. Finally we can neglect the difference between λ_{h+1}, μ_{h+1} and λ_h, μ_h in the endpoints of the trees involving a tree vertex on scale $h+1$.

The previous remarks imply that the leading terms in the beta function can be obtained by the following steps:

a) Evaluate the graphs with one loop and propagator given by the sum of the single scale propagators of scale h and $h+1$, approximated as explained in remark 4), that is:

$$\gamma^{(3+\frac{1}{2}(\sigma_1+\sigma_2))h} g_{1,h}^{\sigma_1\sigma_2}(\gamma^h x) \tag{A2.2}$$

where

$$g_{1,h}^{\sigma_1\sigma_2}(x) = \frac{1}{(2\pi)^4}\int dk\; e^{-ikx}\frac{T_1(k)\,\bar{p}_h^{\sigma_1\sigma_2}(k)}{\rho\mathcal{D}_h(k)} \tag{A2.3}$$

Here $\mathcal{D}_h(k)$ and $\bar{p}_h^{\sigma_1\sigma_2}$ denote the expressions (3.47) and (3.49), modified as explained in remark 4) above, that is:

$$\begin{aligned}&\mathcal{D}_h(k) = b_h k_0^2 + a_h\frac{v_0^2}{4}\vec{k}^2\\ &a_h = 4Z_h(1+4A_h) \qquad , \qquad b_h = E_h^2 + 16 Z_h B_h\end{aligned} \tag{A2.4}$$

$$\begin{aligned}&\bar{p}_h^{--}(k) = -4q_0 Z_h \qquad , \qquad \bar{p}_h^{-+}(k) = \bar{p}_h^{+-}(-k) = -ik_0 E_h\\ &\bar{p}_h^{++}(k) = \frac{16 Z_h B_h k_0^2 + a_h \vec{k}^2\frac{v_0^2}{4}}{4q_0 Z_h} \qquad , \qquad q_0 = \frac{p_0^2}{2m}, \qquad v_0 = \frac{p_0}{m}\end{aligned} \tag{A2.5}$$

and

$$T_1(k) = T_0(k) + T_0(\gamma^{-1}k) = t_0(\gamma^{-1}k) - t_0(\gamma k) \tag{A2.6}$$

b) Evaluate the same graphs with propagator of scale $h+1$, again approximated as in remark 4). This propagator is obtained from (A2.3) by substituting $T_1(k)$ with

$$T_2(k) = T_0(\gamma^{-1}k) = t_0(\gamma^{-1}k) - t_0(k) \tag{A2.7}$$

We shall denote $g_{2,h}^{\sigma_1\sigma_2}$ the corresponding rescaled propagator.

c) Subtract the values found in b) from the values found in a) and add the trivial graphs without any internal line.

d) Approximate in the result the cutoff function $t_0(k)$ by the characteristic function of the set $\{k_0^2 + \frac{\vec{k}^2}{2m}\frac{p_0^2}{2m} \le (\frac{p_0^2}{2m})^2\}$. Note that this approximation is everywhere equivalent to calculating the graphs with all propagators on the single scale h, except in the case of the beta function for B_{h-1}, A_{h-1} and E_{h-1}, which involve derivatives with respect to the loop momentum. Hence, except in this case, we have to calculate the graphs by using the propagator obtained from (A2.3) by substituting $T_1(k)$ with $T_0(k)$; we shall denote again $g_h^{\sigma_1\sigma_2}$ this rescaled propagator.

The trivial graphs give the linear terms in the r.h.s. of (4.12), except the term linear in λ_h appearing in the equation for ν_{h-1}. This term is obtained by contracting two χ^- fields in the λ_h vertex; one gets:

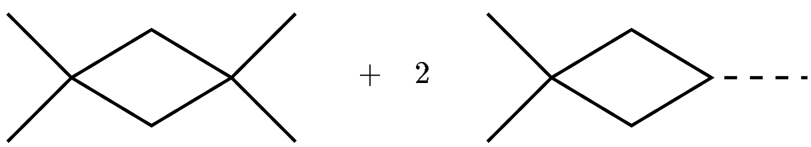
$$-\binom{4}{2}\frac{p_0^2}{2m}\lambda_h\gamma^{2h}4q_0 Z_h\beta_{1,h}F_{20} \tag{A2.8}$$

where F_{20} is defined as in (3.22) and

$$\beta_{1,h} = \int \frac{dk}{(2\pi)^4} \frac{T_0(k)}{\mathcal{D}_h(k)} = \frac{1-\gamma^{-2}}{8\pi^2(\sqrt{a_h b_h} + a_h)} q_0^2 \left(\frac{2}{v_0}\right)^3 \tag{A2.9}$$

The quadratic terms in the equations for λ_h and μ_h are associated with the graphs drawn in Fig. ♣A2.10 , where the heavy lines represent the χ^- fields and the dotted ones represent χ^+.

Note that the coefficients in front of the different graphs, here and in the following figures, indicate how many times the graph appears, if one expands the powers of the potential in the r.h.s. of (A2.1) in terms of the different monomials of the field, whose sum gives the potential, and consider the different possibilities of connecting different point vertices, giving rise to the same graph. In order to get the right contribution to the beta function, one has to consider also the coefficients of the expectations in (A2.1) and the combinatorial factors which count the different possibilities of choosing the external lines in the different vertices of the graph and the different possibilities of contracting the internal lines.



(A2.10)

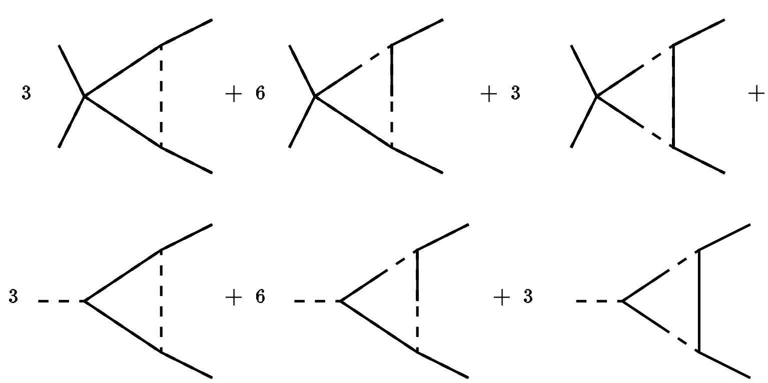
The contribution to $\mathcal{L}V_{h-1}(\chi)$ coming from the graphs in (A2.10) is:

$$-\frac{1}{2}\left[\binom{4}{2}^2 2\lambda_h^2 (4q_0 Z_h)^2 \beta_{2,h} F_{40} + 2\binom{4}{2} 2\lambda_h \mu_h (4q_0 Z_h)^2 \beta_{2,h} F_{21}\right] \left(\frac{p_0^2}{2m}\right)^2 \tag{A2.11}$$

where

$$\beta_{2,h} = \int \frac{dk}{(2\pi)^4} \frac{T_0(k)}{\mathcal{D}_h(k)^2} = \frac{\log\gamma}{8\pi^2\sqrt{a_h^3 b_h}} \left(\frac{2}{v_0}\right)^3 \tag{A2.12}$$

The cubic terms in the equations for λ_h and μ_h are associated with the graphs drawn in ♣A2.13 .



(A2.13)

The contribution to $\mathcal{L}V_{h-1}(\chi)$ coming from the graphs in (A2.13) is:

$$\frac{1}{6}\, 3\cdot 8\left[\binom{4}{2}\lambda_h \mu_h^2 F_{40} + \mu_h^3 F_{21}\right]\left(\frac{p_0^2}{2m}\right)^3 \beta_{3,h} \tag{A2.14}$$

where:

$$\begin{aligned}\beta_{3,h} &= \rho^3 \int \frac{dk}{(2\pi)^4}\Big[g_h^{--}(k)^2 g_h^{++}(k) + 2 g_h^{--}(k) g_h^{-+}(k)^2 + \\ &+ g_h^{--}(k) g_h^{-+}(k) g_h^{+-}(k)\Big] = \\ &= \rho^3 \int \frac{dk}{(2\pi)^4}\Big[g_h^{--}(k)^2 g_h^{++}(k) + g_h^{--}(k) g_h^{-+}(k)^2\Big] = 4 q_0 Z_h \beta_{2,h}\end{aligned} \tag{A2.15}$$

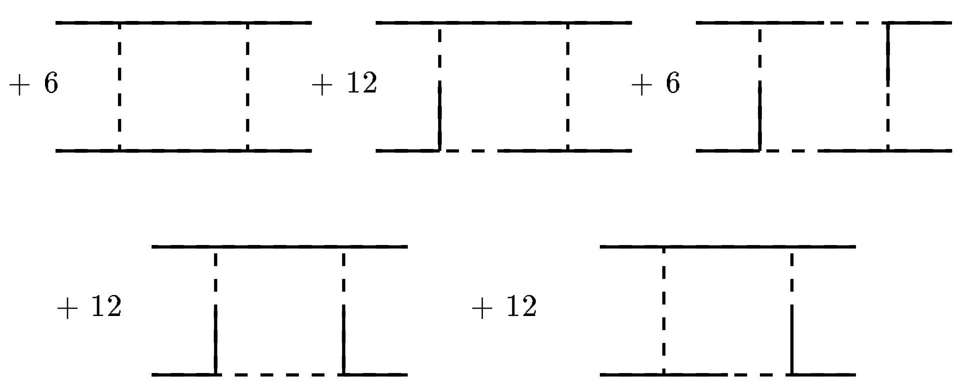
The last equality follows from the identity:

$$\rho^2 [g_h^{--}(k) g_h^{++}(k) + g_h^{-+}(k)^2] = -\frac{T_0(k)^2}{\mathcal{D}_h(k)} \tag{A2.16}$$

and from the observation that $T_0(k)^2 = T_0(k)$, in the approximation of item d) above. We also used the fact that $g_h^{-+}(k) = -g_h^{+-}(k) = -g_h^{-+}(-k)$. Hence (A2.14) can be written as:

$$4\left[\binom{4}{2}\lambda_h \mu_h^2 F_{40} + \mu_h^3 F_{21}\right] 4 q_0 Z_h \left(\frac{p_0^2}{2m}\right)^3 \beta_{2,h} \tag{A2.17}$$

The quartic term in the equation for λ_h is associated with the graphs drawn in ♣A2.18 .



(A2.18)

The contribution to $\mathcal{L}V_{h-1}(\chi)$ coming from the graphs in (A2.18) is:

$$-\frac{1}{4!}\, 2^4 \cdot 6 \left(\frac{p_0^2}{2m}\right)^4 \mu_h^4 \beta_{4,h} F_{40} \tag{A2.19}$$

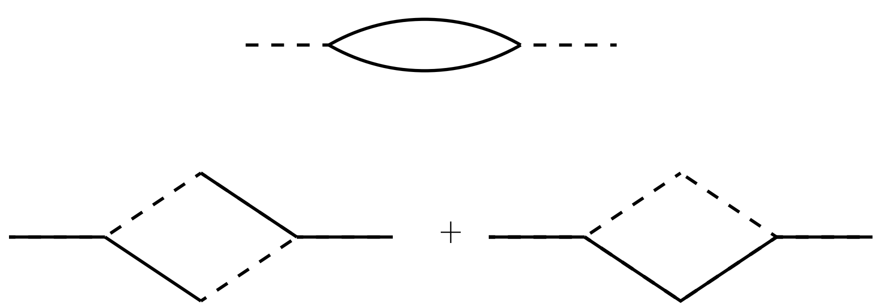
where:

$$
\begin{aligned}
\beta_{4,h} = \rho^4 \int \frac{dk}{(2\pi)^4} \Big[& g_h^{--}(k)^2 g_h^{++}(k)^2 + 2 g_h^{--}(k) g_h^{-+}(k)^2 g_h^{++}(k) + \\
& + g_h^{-+}(k)^4 - 2 g_h^{--}(k) g_h^{-+}(k)^2 g_h^{++}(k) + 2 g_h^{--}(k) g_h^{-+}(k)^2 g_h^{++}(k) \Big] = \\
= \rho^4 \int \frac{dk}{(2\pi)^4} & \Big[g_h^{--}(k) g_h^{++}(k) + g_h^{-+}(k)^2 \Big]^2 = \beta_{2,h}
\end{aligned}
\tag{A2.20}
$$

where we used again the identity (A2.16).

The leading contribution in the flow equation for Z_h is the quadratic term associated with the graph in the first line of ♣A2.22 , whose contribution to the local part is:

$$
-\frac{1}{2}\, 2 \left(\frac{p_0^2}{2m} \right)^2 \mu_h^2 (4 q_0 Z_h)^2 \beta_{2,h} F_{02}
\tag{A2.21}
$$



(A2.22)

and no operators involving field derivatives arise because this is a marginal operator.

The contribution to the local part of the last two graphs of (A2.22) is:

$$
\begin{aligned}
& -\frac{1}{2} 2^2 \left(\frac{p_0^2 \rho}{2m} \right)^2 \mu_h^2 \Big\{ \gamma^{2h} F_{20} \int \frac{dk}{(2\pi)^4} \Big[g_h^{--}(k) g_h^{++}(k) + g_h^{-+}(k)^2 \Big] - \\
& - \frac{1}{2} [\beta_{t,h}^{(1)} - \beta_{t,h}^{(2)}] q_0^2 D_{tt} - \frac{1}{6} [\beta_{s,h}^{(1)} - \beta_{s,h}^{(2)}] 4 m^2 q_0^2 D_{ss} \Big\}
\end{aligned}
\tag{A2.23}
$$

where D_{tt} and D_{ss} are defined as in (3.24) and

$$
\begin{aligned}
\beta_{t,h}^{(i)} &= \int dx\ x_0^2 \Big[g_{i,h}^{-+}(x)^2 - g_{i,h}^{--}(x) g_{i,h}^{++}(x) \Big] \\
\beta_{s,h}^{(i)} &= \int dx\ \vec{x}^2 \Big[g_{i,h}^{-+}(x)^2 - g_{i,h}^{--}(x) g_{i,h}^{++}(x) \Big]
\end{aligned}
\tag{A2.24}
$$

$$
g_{i,h}^{\sigma_1 \sigma_2}(x) = \frac{1}{(2\pi)^4} \int dk \frac{e^{-ikx}\, T_i(k)\, \bar{p}_h^{\sigma_1 \sigma_2}(k)}{\rho \mathcal{D}_h(k)}
\tag{A2.25}
$$

We have:

$$\rho^2 \beta^{(i)}_{t,h} = \int \frac{dk}{(2\pi)^4} \Big\{ E_h^2 \Big[\frac{\partial}{\partial k_0} \frac{k_0 T_i}{\mathcal{D}_h} \Big]^2 + \Big[\frac{\partial}{\partial k_0} \frac{T_i}{\mathcal{D}_h} \Big] \frac{\partial}{\partial k_0} \Big[\Big(1 - \frac{E_h^2 k_0^2}{\mathcal{D}_h} \Big) T_i \Big] \Big\}$$
$$= \int \frac{dk}{(2\pi)^4} \Big\{ E_h^2 \frac{T_i^2}{\mathcal{D}_h^2} + \Big(\frac{\partial}{\partial k_0} \frac{T_i}{\mathcal{D}_h} \Big) \frac{\partial T_i}{\partial k_0} \Big\} \tag{A2.26}$$

It is easy to see that the terms containing the derivatives of T_i give the same contribution to $\beta^{(1)}_{t,h}$ and $\beta^{(2)}_{t,h}$, so that, by doing in the remaining term the approximation of item d) above, we get:

$$\rho^2 [\beta^{(1)}_{t,h} - \beta^{(2)}_{t,h}] = \beta_{2,h} E_h^2 \tag{A2.27}$$

We have also:

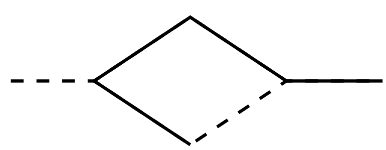
$$\rho^2 \beta^{(i)}_{s,h} = \sum_{j=1}^{3} \int \frac{dk}{(2\pi)^4} \Big\{ E_h^2 \Big[\frac{\partial}{\partial k_j} \frac{k_0 T_i}{\mathcal{D}_h} \Big]^2 + \Big[\frac{\partial}{\partial k_j} \frac{T_i}{\mathcal{D}_h} \Big] \frac{\partial}{\partial k_j} \Big[\Big(1 - \frac{E_h^2 k_0^2}{\mathcal{D}_h} \Big) T_i \Big] \Big\} =$$
$$= \sum_{j=1}^{3} \int \frac{dk}{(2\pi)^4} \Big(\frac{\partial}{\partial k_j} \frac{T_i}{\mathcal{D}_h} \Big) \frac{\partial T_i}{\partial k_j} \tag{A2.28}$$

and one can see again that the terms containing the derivative of T_i give the same contribution to $\beta^{(1)}_{s,h}$ and $\beta^{(2)}_{s,h}$, so that:

$$\beta^{(1)}_{s,h} - \beta^{(2)}_{s,h} = 0 \tag{A2.29}$$

By summarizing, the contribution to the local part of the last two graphs of (A2.22) is:

$$2\beta_{1,h} \left(\frac{p_0^2}{2m} \right)^2 \mu_h^2 \gamma^{2h} F_{20} + \beta_{2,h} E_h^2 \left(\frac{p_0^2}{2m} \right)^2 \mu_h^2 q_0^2 D_{tt} \tag{A2.30}$$



(A2.31)

The leading contribution in the flow equation for E_h is associated with the graph drawn in (A2.31), whose local part is calculated by taking the first order Taylor expansion of the external χ^- field. The term of order zero, which would contribute to the relevant term F_{11}, cancels out for parity reasons, as well as the term of order one containing the spatial derivatives, so that the local part is equal to:

$$-\frac{1}{2}\ 2\ 2\ 2 \left(\frac{p_0^2}{2m} \right)^2 \mu_h^2 (\beta^{(1)}_{5,h} - \beta^{(2)}_{5,h}) q_0 (-D_t) \tag{A2.32}$$

where D_t is defined as in (3.24) and, if we use also the definition (A2.25):

$$\beta_{5,h}^{(i)} = \rho^2 \int dx\ x_0 g_{i,h}^{--}(x) g_{i,h}^{+-}(x) \tag{A2.33}$$

We have:

$$\beta_{5,h}^{(i)} = 4q_0 Z_h E_h \int \frac{dk}{(2\pi)^4} \frac{k_0 T_i}{\mathcal{D}_h} \left[\frac{\partial}{\partial k_0} \frac{T_i}{\mathcal{D}_h} \right] \tag{A2.34}$$

and we can see, as before, that the terms containing the derivatives of T_i give the same contribution to $\beta_{5,h}^{(1)}$ and $\beta_{5,h}^{(2)}$, so that, in the usual approximation:

$$\beta_{5,h}^{(1)} - \beta_{5,h}^{(2)} = -8q_0 Z_h E_h b_h \int \frac{dk}{(2\pi)^4} \frac{k_0^2 T_0(k)}{\mathcal{D}_h^3} = -2q_0 Z_h E_h \beta_{2,h} \tag{A2.35}$$

where the last equality follows from an explicit calculation and $\beta_{2,h}$ is exactly the same function of a_h and b_h defined in (A2.12). Hence the local part of the graph drawn in (A2.31) can be written as:

$$-8 \left(\frac{p_0^2}{2m} \right)^2 q_0^2 \mu_h^2 Z_h E_h \beta_{2,h} D_t \tag{A2.36}$$

The flow equations immediately follow from (A2.8), (A2.11), (A2.17), (A2.19), (A2.21), (A2.30) and (A2.36):

$$\begin{aligned}
\lambda_{h-1} &= \lambda_h - 36 q_0^3 (4Z_h)^2 \beta_{2,h} \left[\lambda_h - \frac{\mu_h^2}{12 Z_h} \right]^2 \\
\mu_{h-1} &= \mu_h - 12 q_0^3 (4Z_h)^2 \beta_{2,h} \mu_h \left[\lambda_h - \frac{\mu_h^2}{12 Z_h} \right] \\
Z_{h-1} &= Z_h - 8 q_0^3 Z_h^2 \beta_{2,h} \mu_h^2 \\
A_{h-1} &= A_h \\
B_{h-1} &= B_h + \frac{1}{2} q_0^3 \beta_{2,h} E_h^2 \mu_h^2 \\
E_{h-1} &= E_h - 8 q_0^3 Z_h E_h \beta_{2,h} \mu_h^2 \\
\nu_{h-1} &= \gamma^2 \left[\nu_h + 24 q_0 Z_h \beta_{1,h} \lambda_h - 2 q_0 \beta_{1,h} \mu_h^2 \right]
\end{aligned} \tag{A2.37}$$

where:

$$\beta_{2,h} = \frac{\log\gamma}{8\pi^2 \sqrt{a_h^3 b_h}} \left(\frac{2}{v_0} \right)^3 \quad , \quad \beta_{1,h} = \frac{1-\gamma^{-2}}{8\pi^2 (\sqrt{a_h b_h} + a_h)} q_0^2 \left(\frac{2}{v_0} \right)^3 \tag{A2.38}$$

with:

$$a_h = 4Z_h(1 + 4A_h) \qquad , \qquad b_h = E_h^2 + 16 Z_h B_h \tag{A2.39}$$

and the (4.12) follow immediately from the above by replacing μ_h via (3.58).

One may wonder whether it might be that the equation for μ_h is compatible with the others. Of course, if (3.58) is valid, it must be; this can also be seen directly (and in fact it is this remark that would make one conjecture the exact relation (3.58), if one did not know it). However the compatibility is true only to the order of the calculation that we are performing, *i.e.* within the neglected corrections: if we did not know *a priori* the validity of (3.58) we could not guarantee that the corrections and the incertitude on the initial data would not spoil the relation (3.58) and turn the flow equation into an unmanageable relation of little use.

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