A RIGOROUS RENORMALIZATION GROUP METHOD FOR INTERFACES IN RANDOM MEDIA

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Abstract: We prove the existence Gibbs states describing rigid interfaces in a disordered solid-on-solid (SOS) for low temperatures and for weak disorder in dimension $D \geq 4$. This extends earlier results for hierarchical models to the more realistic models and proves a long-standing conjecture. The proof is based on the renormalization group method of Bricmont and Kupiainen originally developed for the analysis of low-temperature phases of the random field Ising model. In a broader context, we generalize this method to a class of systems with non-compact single-site state space.

Key Words: Disordered systems, interfaces, SOS-model, renormalization group, contour models

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I. Introduction

In 1988 a remarkable article by Bricmont and Kupiainen [BK1] settled the long-standing dispute on the lower critical dimension of the random field Ising model through a rigorous mathematical proof of the existence of at least two phases at low temperatures in dimension three and above (the less disputed absence of a phase transition in dimension two was later proven by Aizenman and Wehr [AW]). Their proof was based on a renormalization group (RG) analysis that clearly should provide a valuable tool for the investigation of the low temperature phase of disordered systems in general. Unfortunately, the technical complexity of this approach has so far prevented more wide-spread applications, a notable exception being the proof by the same authors [BK2] of the diffusive behaviour of random walks in asymmetric random environments in dimensions greater than two.

Another problem of considerable interest that invites an application of this method is that of the stability of interfaces in random media; one may think in particular of domain walls in random bond or random field Ising models. In a series of articles [BoP,BoK1,BoK2] a hierarchical approximation of such interface models has been investigated; the purpose of the present paper is to go beyond this hierarchical approximation and to analyse the physically more realistic solid-on-solid (SOS) model. We emphasize that the analysis of the hierarchical models shed considerable light on some aspects of this problem, in particular the more probabilistic ones, and has helped us in finding our way through the full model. We recommend reading of in particular Ref. [BoK1] as a warm-up before entering the technical parts of the present work. This reference also contains a fairly detailed introduction into the physical background and heuristic arguments which we prefer not to repeat here again in order to keep down the size of the present paper. For even more physical background on interfaces in random systems, we recommend the review by Forgacs et al. in Domb and Lebowitz Vol. 14 [FLN].

As is to be expected, the analysis of the interface model is in several respects considerable more complicated than that of the random field model; however, sometimes it is the case that added complications entail more clarity: it is our hope to convince the reader of the enormous virtues of this approach and of its conceptual clarity – and even simplicity – and in particular of its wide applicability and flexibility. From this point of view, we would like see the present work in a broader context as a generalization of the RG method for the analysis of the low-temperature phase of disordered systems to models with possibly non-compact single site state space. With this in mind, we have tried to give a fairly detailed and, hopefully, somewhat pedagogical exposition, emphasizing the conceptual ideas and presenting the method in more detail than has been done in [BK].

In presenting our approach we have chosen to stick to a concrete model and show how the RG method can be used to solve it rather than to aim directly at more generally valid results. Overall, we have tried to stress the physically relevant ideas and keep the level of mathematical abstraction as low as compatible with rigour. This is clearly to some extent a matter of taste, but we hope that this choice will make our work more accessible to a wider audience.

We would like to mention that another approach to the low temperature phase of disordered systems has recently been announced by Zahradnik [Za1]. This approach is based on the Pirogov-Sinai theory and aims at dealing with systems with finite spin space but possibly asymmetric ground states (like q-state Potts models). Although full details of this method have not yet been published, it is our believe that the two techniques are not incompatible and that an 'ultimate theory' of the low temperature disordered systems may be obtained by melting together these methods.

Let us now describe the model we want to analyse. A SOS-surface is described by a family of heights, $\{h_x\}_{x\in\mathbb{Z}^d}$, where h_x takes values in $\mathbb{Z}$. The Hamiltonian, that describes the energy difference between the 'flat' surface ($h_x \equiv 0$) and an arbitrary one is formally given by

$$H(h) = \sum_{x,y\in\mathbb{Z}^d:|x-y|=1} |h_x - h_y| + \sum_{x\in\mathbb{Z}^d} J_x(h_x) \tag{1.1}$$

where $J_x(h)$ are random variables that describe the disorder in the system. We will generally assume that for $x \neq x'$, $\{J_x(h)\}_{h\in\mathbb{Z}}$ and $\{J_{x'}(h)\}_{h\in\mathbb{Z}}$ are independent stochastic sequences with identical distributions The properties of the stochastic sequences $\{J_x(h)\}_{h\in\mathbb{Z}}$ themselves depend on the particular physical system under consideration. Two particular examples were highlighted in our previous work [BoP,BoK1,BoK2]:

(i) (Random bond model) The distribution of the sequence $\{J_x(h)\}_{h\in\mathbb{Z}}$ is stationary with respect to translations $h \to h+k$, $k \in \mathbb{Z}$. The marginal distributions satisfy gaussian bounds of the form

$$\mathbb{P}\left(|J_x(h)| > \epsilon\right) \leq e^{-\frac{\epsilon^2}{2\sigma^2}} \tag{1.2}$$

and the $J_x(h)$ are centered, i.e.

$$\mathbb{E}\, J_x(h) = 0 \tag{1.3}$$

In fact, one may think of the $J_x(h)$ as sequences of i.i.d. random variables. However, it turns out in the proofs that independence is unessential and impossible to maintain in the course of renormalization, while stationarity is an important invariant property.

(ii) (Random field model) Here, a priori the $J_x(h)$ should be thought of as sums of i.i.d. random variables. But again, this is not a property that is maintained under renormalization and is replaced by a weaker condition: Let $D_x(h,h') \equiv J_x(h) - J_x(h')$. Then the distribution of the stochastic array $\{D_x(h,h')\}_{h,h'\in\mathbb{Z}}$ is invariant under the diagonal translations $(h,h') \to$

$(h+k, h'+k)$, $k \in \mathbb{Z}$,

$$\mathbb{E} D_x(h,h') = 0 \tag{1.4}$$

and the marginals satisfy gaussian bounds of the form

$$\mathbb{P}\left(D_x(h,h') > \epsilon\right) \leq e^{-\frac{\epsilon^2}{2\sigma^2 |h-h'|}} \tag{1.5}$$

For further physical motivations of these choices we refer to our previous articles. Let us remark that the hamiltonian (1.1) differs from the one of the d-dimensional random field Ising model essentially only in that the variables h take values in $\mathbb{Z}$ rather than $\{-1,1\}$ (this has been observed in [BFG]) which fact suggests the application of the techniques of [BK1]. In the present paper we will actually consider only the case (i); the details in the case (ii) may be found in [K].

Our aim is to prove that, for $d \geq 3$, at low temperature and for small σ, there exist infinite volume Gibbs states corresponding to the Hamiltonian (1.1) describing surfaces with everywhere finite heights, for almost all realizations of the disorder. To be more precise, let us denote by $\Omega \equiv \mathbb{Z}^{\mathbb{Z}^d}$ the configuration space and Σ the Borel sigma-algebra of Ω. For any finite subset $\Lambda \subset \mathbb{Z}^d$, we set $\Omega_\Lambda \equiv \mathbb{Z}^\Lambda$ and denote by Σ_Λ the sigma algebra generated by the functions $h_x, x \in \Lambda$. For any configuration $h \in \Omega$ we write h_Λ, h_{Λ^c} for the restrictions of the function h to Λ and Λ^c, respectively. For two configurations h and η we write $(h_\Lambda, \eta_{\Lambda^c})$ for the element of Ω for which

$$(h_\Lambda, \eta_{\Lambda^c})_x = \begin{cases} h_x & \text{, if } x \in \Lambda \\ \eta_x & \text{, if } x \notin \Lambda \end{cases} \tag{1.6}$$

We set, for any finite volume Λ

$$H_{J,\Lambda}(h_\Lambda, \eta_{\Lambda^c}) \equiv \sum_{x,y \in \Lambda : |x-y|=1} |h_x - h_y| + \sum_{\substack{x \in \Lambda, y \in \Lambda^c \\ |x-y|=1}} |h_x - \eta_y| + \sum_{x \in \Lambda} J_x(h_x) \tag{1.7}$$

This is of course always a finite sum. The *local specifications* (or *finite volume Gibbs measures*) are probability kernels on Ω such that for any Σ-measurable function f,

$$\mu^\eta_{\Lambda,\beta,J}(f) \equiv \frac{1}{Z^\eta_{\Lambda,\beta,J}} \int_{\Omega_\Lambda} f(h_\Lambda, \eta_{\Lambda^c}) e^{-\beta H_{J,\Lambda}(h_\Lambda, \eta_{\Lambda^c})} dh_\Lambda \tag{1.8}$$

where dh_Λ denotes the counting measure on Ω_Λ. The constant $Z^\eta_{\Lambda,\beta,J}$ is a normalization constant chosen such that $\mu^0_{\Lambda,\beta,J}(1) = 1$, usually called the *partition function.* Measures $\mu^\eta_{\beta,J}$ on (Ω, Σ) are in fact called *Gibbs measures* for η, if for all finite Λ, this measure conditioned on η_{Λ^c} coincides with $\mu^\eta_{\Lambda,\beta,J}$ (these are the so-called DLR-equations (see [Ge]). More important for us is the fact that (at least) the *extremal* Gibbs measures can be constructed as weak limit points of sequences $\mu^\eta_{\Lambda_n,\beta,J}$, for sequences Λ_n that increase to $\mathbb{Z}^d$ [Ge]. The problem of statistical mechanics is then to investigate the structure of the set of these limit points. Here, however, our ambitions are somewhat more modest: we want to show that for constant configurations $\eta_x \equiv k$, $k \in \mathbb{Z}$, and suitable sequences of volumes Λ_n, the sequences of measures $\mu^k_{\Lambda_n,\beta,J}$ converge to a limiting measure, for almost all J. It should be noted that for our models not even the existence of a limit point is a non-trivial question, since a priori a sequence of probability measures on Ω need not converge to a measure, due to the non-compactness of the space $\mathbb{Z}$! (As an example for such a situation, take the sequence of probability measures ρ_n on $\mathbb{Z}$, which assign mass $1/n$ to the atoms $\{1,\dots,n\}$ and mass zero to all others. Clearly this sequence has no limit point in the space of measures (cf. [CT] chap. 1.5, ex.6)).

Finally we must mention that all the objects introduced above are of course random variables on some underlying probability space $(\Theta, \mathcal{F}, \mathbb{P})$ on which the $J_x(h)$ are defined. It should be noted in particular that due to the definition of $H_{J,\Lambda}$, the local specifications $\mu^\eta_{\Lambda,\beta,J}$ are measurable w.r.t. the sigma algebras $\mathcal{F}_\Lambda$ (the sub-sigma algebras generated by the functions $\{J_x(h)\}_{x\in\Lambda}$). Care should be taken that then limits are taken, neither Λ_n nor η should depend on J. It is frequently possible to produce pathological results by choosing random boundary conditions[1]. The central result of this paper is then the following

THEOREM 1: *Let $d \geq 3$ and assume that the random variables $J_x(h)$ satisfy the conditions detailed under (i). Then there exists $\beta_0 < \infty$, $\sigma_0 > 0$, such that for all $\beta \geq \beta_0$ and $\sigma \leq \sigma_0$ there exists an increasing sequence of cubes $\Lambda_n \uparrow \mathbb{Z}^d$, centered at the origin, such that the sequence of measures $\mu^k_{\Lambda_n,\beta,J}$ converges to a Gibbs measure $\mu^k_{\beta,J}$, for $\mathbb{P}$-almost all J. For $k \neq k'$, $\mu^k_{\beta,J}$ and $\mu^{k'}_{\beta,J}$ are disjoint.*

Remark: The condition that the sequence of volumes be a sequence of cubes is only made to simplify some technical aspects of the proof. It should be possible, with more work, to prove the theorem for far more general (non-random) sequences of increasing and absorbing volumes. The measures constructed in Theorem 1 are presumably the only extremal Gibbs measures corresponding to 'translational invariant' boundary conditions. To analyse the full structure of the set of Gibbs measures remains an interesting, but difficult question.

Before entering the details of the proof of this theorem, we would like to explain some of the main ideas and features of the RG approach. As always in statistical mechanics, the principal idea is to find a way of arranging the summations involved in the expression (1.8) for the local specifications in a suitable way as a convergent sum. In the low temperature phase, the usual way of doing this is by first finding the ground states (minima of H_Λ) and then representing all other configurations as (local) deformations of these ground states (often called 'contours' or 'Peierls

[1] Newman and Stein [NS] have recently investigated interesting phenomena of this type in the context of spin glass models.

contours'). Under favourable circumstances, one may arrange the sum over all these deformations as a convergent expansion ('low temperature expansion'). As opposed to many 'ordered' systems, the first (and in some sense *main*) difficulty in most disordered systems is that the ground state configuration depends in general on the particular realization of the disorder, and, worse, may in principle depend strongly on the shape and size of the finite volume Λ! In our particular model this means that a ground state for the infinite system may not even exist! This latter situation is actually expected to occur in dimensions $d \leq 2$.[2] In dimension $d \geq 3$, we expect, on the contrary, that a ground state in the infinite volume exists and moreover that this ground state itself may be seen as a 'small' deformation of the ground state of the ordered system. This property must, however, be proven in the course of the computation.

The crucial observation that forms the ideological basis for the renormalization group approach is that while for large volumes Λ we have no a priori control on the ground state, for sufficiently small volumes we can give conditions on the random variables J that are fulfilled with large probability under which the ground state in this volume is actually the same as the one without randomness. Moreover, the size of the regions for which this holds true will depend on the variance of the r.v.'s and increases to infinity as the latter decreases. This allows to find 'conditioned' ground states, where the conditioning is on some property of the configuration on this scale (e.g. mean height over a certain region), except in some small region of space. Re-summing then over the fluctuations about these conditioned ground states one obtains a new effective model for the conditions (the coarse grained variables) with effective random variables that (hopefully!!) have smaller variance than the previous ones. In this case, this procedure may be iterated, as now conditioned ground states on a larger scale can be found. This is the basic idea of the renormalization group.

To implement these ideas one has to overcome two major difficulties. The first is that one needs to find a formulation of the model, i.e. a representation of the degrees of freedom and of the interactions that is sufficiently general that its form remains invariant under the renormalization group transformation. There has been an extensive discussion recently in the literature (see [EFS]) on some 'pathological' aspects of the RG that indicates that a 'spin system' like formulation (like (1.1)) will in general be inadequate. We will see that a adequate solution of this problem can be given through a class of contour models. The second, and really the most fundamental difficulty is that the re-summation procedure as indicated above can only be performed outside a small, random region of space, called the 'bad region'. Now while in the first step this may look like no big problem, in the process of renormalization even a very thin region will 'infect' a larger and larger portion of space, if nothing is done. Moreover, in each step some more bad regions are created from

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[2] It is expected that the methods of Aizenman and Wehr [AW] used to prove the uniqueness of the Gibbs state in the two-dimensional random field Ising model can be used to prove such a result.

regions in which the new effective random variables have bad properties. This requires to get some control also in the bad regions and to get a precise notion of how regions with a certain degree of badness can be regarded as 'harmless' and be removed on the next scale. For the method to succeed we must then find ourselves in a situation where the bad regions 'die out' over the scales much faster than new ones are produced. This will generally depend on the geometry of the system and in particular on the dimension.

The remainder of this paper is organized in three stages. In the next section we give a more detailed and more specific outline of the renormalization group method. This will serve to expose the conceptual framework and to introduce most of the notation for later use. It should give the reader who may not be bothered with the hard technical work a fairly good idea of what we are doing. Then, in Section III, these ideas are set to work for the analysis of the 'ground states' (i.e. the case of zero temperature) and to prove the corresponding special case of Theorem 1. Here again we have two purposes in mind: First, this case is still considerably less complicated than the case of finite temperature while already exhibiting most of the interesting features. Second, all of the estimates used here are also needed in the more general case and separating those pertinent to the ground states from those related to expansions about them may make things only more transparent. This section also contains all the probabilistic estimates that then apply unaltered in the finite temperature case. Section IV finally contains the analysis of the finite temperature Gibbs states and the proof of Theorem 1. In Section V we conclude with some remarks on possible future developments. An appendix contains the proofs of some estimates of geometric nature.

II. The renormalization group and contour models

This section is intended to serve two purposes. First, we want to describe the principal ideas behind the renormalization group approach for disordered systems in the low-temperature regime. We hope to give the reader an outline of what he is to expect before exposing him to the, admittedly, somewhat complicated technical details. Second, we want to present the particular types of contour models on which the renormalization group will act. In this sense the present section introduces the notation for the later chapters. Most of the basic ideas outlined here are contained explicitly or implicitly in [BK].

II.1. The renormalization group for measure spaces

Let us recall first what is generally understood by a renormalization group transformation in a statistical mechanics system. We consider a statistical mechanics system to be given by a probability space (Ω, Σ, μ), where μ is an (infinite volume) Gibbs measure. One may think for the moment of Ω as the 'spin'-state over the lattice $\mathbb{Z}^d$, but we shall need more general spaces later. What we shall, however, assume is that Ω is associated with the lattice $\mathbb{Z}^d$ in such a way that for any finite subset $\Lambda \subset \mathbb{Z}^d$ there exists a subset $\Omega_\Lambda \subset \Omega$ and sub-sigma algebras, Σ_Λ, relative to Ω_Λ that satisfy $\Sigma_\Lambda \subset \Sigma_{\Lambda'}$, if and only if $\Lambda \subset \Lambda'$. Note that in this case any increasing and absorbing sequence of finite volumes, $\{\Lambda_n\}_{n\in\mathbb{Z}_+}$, induces a filtration $\{\Sigma_n \equiv \Sigma_{\Lambda_n}\}_{n\in\mathbb{Z}_+}$ of Σ. It should always be kept in mind that in the situations we are interested in we have, a priori, no explicit knowledge of the measures μ, but only of their local specifications for finite volumes, i.e. the expectations of μ conditioned on Σ_{Λ^c} (finite volume Gibbs measures with 'boundary conditions'). The other important notion that should be kept in mind is that the measures μ are, by Kolmogorov's theorem [Ge], uniquely determined by their values on all cylinder functions on all finite volumes Λ ('local observables').

Ideally, a *renormalization group transformation* is a measurable map, $\mathcal{R}$, that maps $\mathbb{Z}^d \to \mathbb{Z}^d$ and $(\Omega, \Sigma) \to (\Omega, \Sigma)$ in such a way that for any $\Lambda \subset \mathbb{Z}^d$,

(i) $\mathcal{R}(\Lambda) \subset \Lambda$, and moreover $\exists_{n<\infty} : \mathcal{R}^n(\Lambda) = \{0\}$, where n of course may depend on Λ.

(ii) $\mathcal{R}(\Omega_\Lambda) = \Omega_{\mathcal{R}(\Lambda)}$

We will see later that these conditions are slightly too restrictive in general, but for the moment we will stick to them. Note that the use of the same name, $\mathcal{R}$, for the action of renormalization on the lattice and on the space Ω which should not create confusion. The action of $\mathcal{R}$ on space will generally be blocking[3], e.g. $\mathcal{R}(x) = \mathcal{L}^{-1}x \equiv \text{ int}(x/L)$. The action on Ω has to be compatible with this blocking but needs to be defined carefully. We should like to stress that in different situations

[3] We call the blocking operator $\mathcal{L}^{-1}$ for historical reasons.

it may be appropriate to use RG maps that are not based on the simple blocking operation $\mathcal{L}^{-1}$.

Having the action of $\mathcal{R}$ on the measurable space (Ω, Σ) we may of course lift it to the measures on (Ω, Σ) via

$$(\mathcal{R}\mu)(\mathcal{A}') = \mu(\mathcal{R}^{-1}(\mathcal{A}')) \tag{2.1}$$

for any Borel-set $\mathcal{A} \in \mathcal{R}(\Sigma)$. The fundamental relation of the renormalization group allows to decompose the measure μ into a conditioned expectation and the renormalized measure on the condition, i.e. for any Borel-set $\mathcal{A} \in \Sigma$ we have

$$\mu(\mathcal{A}) = (\mathcal{R}\mu)(\mu^{\mathcal{R}=\cdot}(\mathcal{A})) \equiv \int_{\mathcal{R}(\Omega)} \mu\left(\mathcal{A}|\mathcal{R} = \omega'\right)(\mathcal{R}\mu)(d\omega') \tag{2.2}$$

A priori, all of the above are just trivialities. They are useful only if the map $\mathcal{R}$ can be chosen in such a manner that equation (2.2) and its iterates are useful in computing expectations of interest. This requires that the measure $\mathcal{R}\mu$ is 'simpler' in an appropriate sense than the measure μ itself, and that the conditioned expectations are more easy to control at least on a subspace that has large measure w.r.t. $\mathcal{R}\mu$. This has to be verified in explicit computations. An example in which this can be easily carried through in full detail is given for instance by the hierarchical model treated in [BoK2].

So far, we have not made reference to the specific situation in random systems. In a random system, the measure μ is itself random, i.e. is a measure valued random variable on some underlying probability space $(\Theta, \mathcal{F}, I\!\!P)$ that describes the randomness of the system. In such a situation it may – and will – turn out that the specific choice of the renormalization group transformation has to be adapted to the particular realization of the disorder, i.e. will itself have to be a – complicated – random function. In particular, in such a case the renormalization group transformation cannot be simply iterated since after each step the properties of the new measure have to be taken into account when specifying the new map. As a matter of fact we will even go one step further and allow the underlying spaces Ω to be random and to change under the application of the renormalization group map (although this point is to some extend a question of taste and convenience).

A final aspect that should be kept in mind is that of course the renormalized measures (or even their local specifications) can only *in principle* be computed exactly. In practice we must restrict our knowledge to certain bounds, and it is only on these that the renormalization maps may depend.

II.2. Contour models

The concept of *"contours"* has been fundamental in the analysis of low-temperature phases of spin-systems since its introduction by Robert Peierls in 1936 [P]. It formed the basis for the first proof of the existence of a phase transition in the Ising model, and as such, in any model of statistical mechanics. They play the fundamental role in the most powerful modern method to analyse phase transitions, the Pirogov-Sinai theory (see e.g. [Za2]). The basic idea behind contours is to make explicit the region in space where a spin-configuration deviates from a ground state configuration and to use the fact that these regions carry energy proportional to their volume and therefore are suppressed sufficiently to counter their entropy, if the temperature is sufficiently small. It is thus natural that contour models should constitute the proper context for our analysis as well.

As we have said before, the main new problem that arises in disordered systems is that the ground states are very difficult to come by, which seems to make the implementation of the contour idea impossible. However, in sufficiently simple situations – like the one we are studying –, one may guess that for sufficiently weak disorder the ground state should look *almost* like that of the ordered system. It is thus natural to build the contour model on the basis of the 'ideal' ground states and to let the contours themselves keep track of the deviations of the true ground state from these ideal ones. Section III will pinpoint this idea by dealing exclusively with the ground state problem while omitting the added complications of the thermal fluctuations. The basic notion here is that of the *"bad regions"* introduced in [BK1]. They are those regions in space where the randomness *locally* is sufficiently strong to potentially influence the ground state configuration. As long as these regions have a small density, they will be treated in a sense like deviations from ground states and kept track of with the contours.

We will now become more specific and give the precise definitions of contours in our situation.

DEFINITION 2.1: *A contour, Γ, is a pair $(\underline{\Gamma}, h)$, where $\underline{\Gamma}$ is a subset of $\mathbb{Z}^d$, called the support of Γ, and $h \equiv h(\Gamma) : \mathbb{Z}^d \to \mathbb{Z}$ is a map that is constant on connected components of $\underline{\Gamma}^c$.*

Note that h alone does not specify the contour as we do not require that $\underline{\Gamma}$ be restricted to the region where h is non-constant. We follow the usage of [BK1] in calling $\underline{\Gamma}$ the *support*, although this may be misleading. Of course, when mapping our SOS-model to a contour model, we must give a one-to-one map from heights to contours, but this information will always be assumed to be contained in the measures (in other words, we start with a model where only certain contours have non-zero measure). In the sequel, Ω shall denote the space of all contours. Also, Ω_Λ will denote the space of contours in the finite volume Λ. [4]

[4] Let us remark that the space of contours can also be described as a more general spin (h, σ), where σ_x takes the values zero if $x \in \underline{\Gamma}$ and zero otherwise.

We will also need to consider spaces of contours satisfying some further constraints. To explain this, we must introduce some notation. Let D denote some subset of $\mathbb{Z}^d \times \mathbb{Z}$. Such sets will later arise as the so-called 'bad regions'. Let us note already here that the fact that bad regions are subsets of $\mathbb{Z}^d \times \mathbb{Z}$, that is the basic lattice times the space of the 'heights' is a main difference to the random field case treated in [BK1], where bad regions are subsets of the base space $\mathbb{Z}^d$ only. It is the non-compactness of our 'spin-space' that makes this modification necessary. Given D, we denote by $D(h)$ the slices of D at height h, i.e.

$$D(h) \equiv \{(x,h') \in D | h' = h\} \tag{2.3}$$

Of course $D(h)$ can be identified in a natural way with a subset of $\mathbb{Z}^d$ by simply projecting. We must also define, for a given contour Γ the set

$$D(\Gamma) \equiv \{x \in \mathbb{Z}^d | (x, h_x(\Gamma) \in D\} \tag{2.4}$$

Given a set D, we will denote by $\Omega(D)$ all those contours whose support contains $D(\Gamma)$, i.e.

$$\Omega(D) \equiv \{\Gamma \in \Omega | D(\Gamma) \subset \underline{\Gamma}\} \tag{2.5}$$

Note that this definition is consistent since on the one hand $D(\Gamma)$ really only depends on the height part of Γ and the condition in the definition of (2.5) only affects the support of Γ. As a matter of fact, given any contour in $\Gamma' \in \Omega$ we can easily associate to it a contour in $\Omega(D)$ by first computing $D(\Gamma')$, and then setting $\underline{\Gamma} \equiv \underline{\Gamma}' \cup D(\Gamma')$. Then $\Gamma = (\underline{\Gamma}, h(\Gamma')$ is a contour in $\Omega(D)$.

As we have indicated above, a renormalization group transformation may depend on the starting measure. In particular, the transformations we will use depend on the set D (which of course will be chosen in accordance with the measure μ, or more precisely some of the random parameters μ depends on). The sets D will necessarily also be affected by the renormalization, so that we will have to construct maps $\mathcal{R}_D$, depending on D (as well as other parameters) that map the spaces $\Omega(D)$ into $\Omega(D')$ for a suitably computed D'. The resulting structure will then be a measurable map $\mathcal{R}_D : (\Omega(D), \Sigma(D)) \to (\Omega(D'), \Sigma(D'))$ that can be lifted to the measure μ s.t. for any $\mathcal{A} \in \Sigma(D')$,

$$(\mathcal{R}_D \mu)(\mathcal{A}) = \mu(\mathcal{R}_D^{-1}(\mathcal{A})) \tag{2.6}$$

Of course we want to iterate this procedure. At least here it becomes clear that it is necessary to find a parameterization of the measures we are dealing with that remains invariant under the RG transformations. As a first step, let us rewrite the original SOS model as a contour model.

II.3. The SOS model as a contour model

One of the basic ideas in the reformulation of our model in terms of contours is that the form in which the contour weights are written should make manifest how the energy is associated to the supports of the contours, and more specifically, to the connected components of the supports. To do this, we first need to introduce some more notation. First, we will always use the metric $d(x,y) = \max_{i=1}^{d} |x_i - y_i|$ for points in $\mathbb{Z}^d$. We will call a set $A \subset \mathbb{Z}^d$ connected, iff for all $x \in A$, $d(x, A \backslash \{x\}) \leq 1$. A maximal connected subset of a set A will be called a connected component of A. We will also use the notation $\overline{A}$ for the set of points whose distance from A is not bigger than 1, and we will write $\partial A \equiv \overline{A} \backslash A$ and call ∂A the boundary of A. A further important notion is that of the *interior* of a set A, $\mathrm{int}A$. It is defined as follows: For any set $A \subset \mathbb{Z}^d$, let $\hat{A} \subset \mathbb{R}^d$ denote the set in $\mathbb{R}^d$ obtained by embedding A into $\mathbb{R}^d$ and surrounding each of its points by the unit cube in d dimensions. Then the complement of $\hat{A}$ may have finite connected components. Their union with $\hat{A}$ is called $\mathrm{int}\hat{A}$, and the intersection of this set with $\mathbb{Z}^d$ is defined to be $\mathrm{int}A$.

Important operations will consist of the cutting and glueing of contours. First, for any contour Γ we may decompose its support, $\underline{\Gamma}$, into connected components, $\underline{\gamma}_i$ in the sense described above. Note that a contour is uniquely described by specifying its support $\underline{\Gamma}$, the values of h on the support and the values h takes on each of the connected components of the boundary of the support.This makes it possible to associate with each connected component $\underline{\gamma}_i$ of the support a contour, γ_i, by furnishing the additional information of the heights on $\underline{\gamma}_i$ and on the connected components of $\partial\underline{\gamma}_i$. We will call a contour with connected support a *connected contour.* In the same spirit we call a connected contour γ_i obtained from a connected component of the support of a contour Γ a *connected component* of Γ. A collection, $\{\gamma_1, \ldots, \gamma_n\}$ of connected contours is called *compatible*, if there exists a contour Γ, such that $\gamma_i, \ldots, \gamma_n$ are the connected components of Γ. This contour will also be called $(\gamma_1, \ldots, \gamma_n)$. It is clear that the notion of compatibility has a simple expression in terms of the heights on the connected components of the boundaries of the $\underline{\gamma}_i$.

Notice that the connected components of a contour are not entirely independent of each other, since a connected component may lie within the interior of another and thus has to adjust its exterior height to the corresponding height of this outer contour. Therefore, we will have two use a second notion of connectedness that we call *weak connectedness.* We say that a set $\mathcal{A} \subset \mathbb{Z}^d$ is *weakly connected*, if int A is connected. All the notions of the previous paragraph then find their weak analogs.

Finally, let us define the *level sets* $V_h(\Gamma)$ of a contour by

$$V_h(\Gamma) \equiv \{x \in \mathbb{Z}^d | h_x(\Gamma) = h\} \tag{2.8}$$

For any function $F : \mathbb{Z}^d \times \mathbb{Z} \to I\!R$, we introduce the notion

$$(F, V(\Gamma)) \equiv \sum_{h\in\mathbb{Z}} \sum_{x\in V_h(\Gamma)} F_x(h) \tag{2.9}$$

This notation will receive a considerable generalization shortly.

We now come to the representation of the SOS-model as a contour model. Defining

$$E_s(\Gamma) = \sum_{\substack{x,y\in\underline{\Gamma} \\ |x-y|=1}} |h_x(\Gamma) - h_y(\Gamma)| \tag{2.9}$$

we could write

$$H(h) = E_s(\Gamma) + (J, V(\Gamma)) \tag{2.10}$$

with $\underline{\Gamma}$ defined for a given function h as the set of x that possess a nearest neighbor y for which $h_y \neq h_x$. Then the term $E_s(\Gamma)$ could be written as a sum over connected components, $E_s(\Gamma) = \sum_i E_s(\gamma_i)$; this would produce the contour activities and the term $(J, V(\Gamma))$ would play the role of a 'field'. This would be reasonable, if the configuration $h \equiv 0$ were indeed the true ground state. However, the fields terms here may and will alter the ground state configuration, although only locally in rare places. As indicated above, we want to take this into account by adopting the definition of contours more closely to the real situation. Of course, due to our limited control over the random terms, this can be achieved only on a given finite length scale. To implement this, we allow only $J_x(h)$ that are small enough to remain in the field term. For a fixed $\delta > 0$ that will be chosen appropriately later, we set

$$S_x(h) \equiv J_x(h) \mathbb{I}_{|J_x(h)|<\delta} \tag{2.11}$$

Here and everywhere in the sequel $\mathbb{I}_X$ denotes the indicator function of the event X (i.e. $\mathbb{I}_X = 1$ if X holds and $\mathbb{I}_X = 0$ otherwise). For fields that are not small in this sense, we introduce a new field that only keeps roughly track of their size. It is defined as

$$N_x(h) \equiv \delta^{-1} \mathbb{I}_{|J_x(h)|\geq\delta} |J_x(h)| \tag{2.12}$$

Here we chose the prefactor δ^{-1} so that the control fields have minimal size one. This is not particularly significant but makes some of our later estimates more easily comparable with those in [BK]. The region D, the *bad region* is then defined as

$$D \equiv \left\{ (x,h) | \sup_{k\in\mathbb{Z}} \left(N_x(h+k) - \frac{c}{L}|k| \right) > 0 \right\} \tag{2.13}$$

where L is a positive integer (it will be the blocking scale of the RG transformation) and $c > 0$ is a suitably determined positive constant. Now we define the mass of a contour Γ as

$$\mu(\Gamma) = \begin{cases} \rho(\Gamma) e^{-\beta(S,V(\Gamma))} & \text{, if } \underline{\Gamma} = \{x | \exists_{y:|x-y|=1} : h_y(\Gamma) \neq h_x(\Gamma)\} \cup D(\Gamma) \\ 0 & \text{, otherwise} \end{cases} \tag{2.14}$$

where

$$\rho(\Gamma) = e^{-\beta(E_s(\Gamma)+(J,V(\Gamma)\cap D(\Gamma)\cap\underline{\Gamma})} \tag{2.15}$$

The important fact here is that $\rho(\Gamma)$ factors over the connected components of Γ, i.e. if $\Gamma = (\gamma_1, \ldots, \gamma_n)$, then

$$\rho(\Gamma) = \prod_{i=1}^{n} \rho(\gamma_i) \tag{2.16}$$

(Note that it is to have (2.16) that we wrote $D(\Gamma) \cup \underline{\Gamma}$ rather than simply $D(\Gamma)$. We must note here that the connected components of a contour will, of course, not in general satisfy the constraint to contain the bad region of that contour!).

Note that (2.14) gives a one-to-one relation between height-configurations and contours with non-zero weight. All quantities of interest in the SOS-model can thus be computed in the contour model defined above.

In an ideal situation, we would hope that the form of the measures on the contours would remain in this form under renormalization, i.e. activities factorizing over connected components plus a 'small-field' contribution. Unfortunately, the truth will not be that simple, except in the case of zero temperature, as will be shown in Section III. In general, the renormalization will introduce non-local interactions between connected components of supports as well as a non-local 'small random field', $\{S_C\}$ indexed by the connected subsets C of $\mathbb{Z}^d$. The final structure of the contour measures will be the following:

$$\mu(\Gamma) = \frac{1}{Z} e^{-\beta(S,V(\Gamma))} \sum_{\mathbb{Z}^d \supset G \supset \underline{\Gamma}} \rho(\Gamma, G) \tag{2.17}$$

where $\rho(\Gamma, G)$ are activities that factor over the connected components of G. Z is as usual the normalization constant that turns μ into a probability measure. For non-local fields the notion $(S, V(\Gamma))$ is extended to abbreviate

$$(S, V(\Gamma)) \equiv \sum_{h\in\mathbb{Z}} \sum_{C\subset V_h(\Gamma)} S_C(h) \tag{2.18}$$

the sum over C being here and always a sum over connected sets. The functions S, the activities ρ and the fields N will be the parameters on which the action of the renormalization group will finally be controlled. Of course these quantities will have to satisfy certain bounds that will be specified later, and it will be these bounds that eventually have to be controlled in the process of renormalization.

II.4. Renormalization of the contours

As the last point of this section we will now define the action of the renormalization group on the contours themselves. We should remark that this cannot yet be done completely, since, as indicated above, the renormalization group map will depend on the bad regions and even to some extend on the starting measure μ itself (basically through the fields $N_x(h)$). Thus at present we give the general outline while the details must be filled in the appropriate places of Sections III and IV.

The renormalization group transformation we shall construct consists of three distinct steps:

(i) Summation of small connected components of contours

(ii) Blocking of the remaining large contours

(iii) Dressing of the supports by the new bad region

Note that step (iii) is to some extent cosmetic and requires already the knowledge of the renormalized bad regions. We note that this causes no problem, as the bad regions may in practice already be computed after step (i).

Let us now give a brief description of the individual steps.

STEP 1: In principle we would like to sum in this step over all those classes of contours for which we can get a convergent expansion in spite of the random fields. In practice, we restrict ourselves to a much smaller, but sufficiently large, class of contours. Namely we define a connected component as 'small', if it is geometrically small (in the sense that $d(\underline{\gamma_i}) < L$) and if its support does not intersect the bad region, with the exception of a suitably defined 'harmless' subset of the bad region. This latter point is important since it will allow us to forget about this harmless part in the next stage of the iteration and this will assure that the successive bad regions become sparser and sparser. Precise definitions (although certainly not the optimal ones) are given in Section III.

A contour which contains no small connected component is called large, and we denote by $\Omega^l(D)$ the subspace of large contours. The first step of RG transformation is then nothing but the canonical projection form $\Omega(D)$ to $\Omega^l(D)$, i.e. to any contour in Ω we associated the large contour composed of only the large components of Γ.

STEP 2: In this step the large contours are mapped to a coarse-grained lattice. We choose the simplest action of $\mathcal{R}$ on $\mathbb{Z}^d$, namely $(\mathcal{R}x)_i = \mathcal{L}^{-1} \equiv \text{ int}\,(x_i/L)$. We will denote by $\mathcal{L}x$ the set of all points y s.t. $\mathcal{L}^{-1}y = x$. Now the action of $\mathcal{L}^{-1}$ on height configurations of large contours is defined as averaging, i.e.

$$(\mathcal{L}^{-1}h)_y = \frac{1}{L^d}\sum_{x\in\mathcal{L}y} h_x \tag{2.19}$$

with this definition we have the action of $\mathcal{L}^{-1}$ on large contours as

$$\mathcal{L}^{-1}\Gamma \equiv (\mathcal{L}^{-1}\underline{\Gamma}, \mathcal{L}^{-1}h) \tag{2.20}$$

STEP 3: The action of $\mathcal{R}$ given by (2.19) does not yet give a contour in $\Omega(D')$. Thus, the last step in the RG transformation consists of enlarging the supports of the contours by the newly created bad regions, which of course requires first to compute those. This will in fact be the most subtle and important part of the entire renormalization program and will be explained later. Given a new region D', the effect on the contours is just to replace $\underline{\mathcal{L}^{-1}\Gamma}$ by $\underline{\mathcal{L}^{-1}\Gamma} \cup D'(\mathcal{L}^{-1}\Gamma)$, so that finally the full RG transformation on the contours can be written as

$$\mathcal{R}_D(\Gamma) \equiv (D'(\mathcal{L}^{-1}\Gamma^l(\Gamma)) \cup \mathcal{L}^{-1}\underline{\Gamma}^l(\Gamma), \mathcal{L}^{-1}h(\Gamma^l(\Gamma))) \tag{2.21}$$

We have now set up the basic formalism for our RG program. The remaining task is now to analyze the induced action on measures of the type described above and to show that this technique has the computational power to prove the theorem announced in the introduction.

III. The ground states

As we have indicated in the introduction, the crucial new feature in the analysis of the low temperature phase of disordered systems as opposed to that of ordered ones lies in the fact that even the analysis of the properties of the ground state becomes highly non-trivial and the result crucial for the structure of the low-temperature phases. But while the essential conceptual features are already present, on the technical level this is still much simpler than the situation at finite temperature, mainly due to the fact that we need not perform any cluster expansions. We think it is helpful for the understanding of this method to separate difficulties of different origin and therefore we devote this section entirely to this particular case. The results obtained here will then prove useful in the general case that we will treat in Section IV.

III.1 Formalism and set-up

We will try to make this section as self-contained as possible, but refer to notations introduced in Section II. Let Γ denote a contour as defined in Definition 2.1. and let Ω be the space of all contours. For a given energy function $H : \Omega \rightarrow I\!R$, we must define the proper notion of a ground state contour; in particular, we are interested in ground states corresponding to 'boundary conditions zero at infinity'. In the sequel, let Λ always denote a *finite* subset of $Z\!\!Z^d$. We need to define restrictions H_Λ of H to finite volumes that are finite functions from Ω into $I\!R$. The precise definition of these restrictions for contour models will be given in a moment. Now define the sets $\mathcal{G}_\Lambda^{(\Gamma)}$ to be the contours of lowest energy in Λ for given external configuration Γ_{Λ^c}, i.e.

$$\mathcal{G}_\Lambda^{(\Gamma)} \equiv \left\{ \Gamma^* \in \Omega \middle| \Gamma^*_{\Lambda^c} = \Gamma_{\Lambda^c} \wedge H_\Lambda(\Gamma^*) = \inf_{\Gamma' : \Gamma'_{\Lambda^c} = \Gamma_{\Lambda^c}} H_\Lambda(\Gamma') \right\} \tag{3.1}$$

Here Γ_{Λ^c} denotes the restriction of Γ to Λ^c. The set of all infinite volume ground states is usually defined (see [AL]) as

$$\mathcal{G}_\infty \equiv \left\{ \Gamma \in \Omega \,\middle|\, \forall_{\Lambda \subset Z\!\!Z^d} \Gamma \in \mathcal{G}_\Lambda^{(\Gamma)} \right\} \tag{3.2}$$

Remark: Under some weak smoothness assumptions on the measure $I\!P$ the sets $\mathcal{G}_\Lambda^{(\Gamma)}$ consist $I\!P$-a.s. of single elements.

The problem of determining the entire set of ground states is, at least in our situation, far too ambitious and we will content ourselves with proving the existence of ground states corresponding to roughly flat interfaces with a given typical height. More precisely, we will proceed as follows. Let Λ_n denote the cube centered at the origin of side length L^n. Then define the cylinder set

$$\hat{\mathcal{G}}^{(h)}_{\Lambda_n,\Lambda} \equiv \left\{ \Gamma \in \Omega \,\middle|\, \exists_{\Gamma^* \in \mathcal{G}_{\Lambda_n}^{(\emptyset, h_x \equiv h)}} : \Gamma_\Lambda = \Gamma^*_\Lambda \right\} \tag{3.3}$$

(this is the set of all contours that within Λ look like a ground state for the finite volume Λ_n with boundary condition $(\emptyset, h_x \equiv h)$. Now set

$$\mathcal{G}_\infty^{(h)} \equiv \bigcap_{\Lambda\subset\mathbb{Z}^d} \bigcup_{n_0=0}^{\infty} \bigcap_{n\geq n_0} \hat{\mathcal{G}}_{\Lambda_n,\Lambda}^{(h)} \tag{3.4}$$

It is easy to convince oneself that

$$\mathcal{G}_\infty^{(h)} \subset \mathcal{G}_\infty \tag{3.5}$$

(this is in fact analogous to the observation that weak limit points of local specifications yield infinite volume Gibbs states). The purpose of the present section is to prove that $\mathcal{G}_\infty^{(h)}$ is non-empty if $d \geq 3$ and the disorder weak enough. It will also follow that these sets are disjoint for different values of h. By stationarity, it will in fact be enough to consider the case $h = 0$ which we will do from now on.

To construct an element of $\mathcal{G}_\infty^{(h)}$ we have to study elements Γ^* of the sets $\mathcal{G}_{\Lambda_n}^{(0)}$ and show that for any fixed finite volume Λ the restriction Γ^*_Λ becomes independent of n for n sufficiently large. Let us introduce the abbreviation $\Omega_n \equiv \Omega_{\Lambda_n}^{(0)}$. The analysis of ground states via the renormalization group method then consists of the following inductive procedure. Let $\mathcal{R}$ be a map $\mathcal{R} : \Omega_n \to \Omega_{n-1}$. Then clearly

$$\inf_{\Gamma\in\Omega_n} H_{\Lambda_n} = \inf_{\tilde\Gamma\in\Omega_{n-1}} \left(\inf_{\Gamma\in\mathcal{R}^{-1}\tilde\Gamma} H_{\Lambda_n}(\Gamma)\right) \tag{3.6}$$

which suggests to define

$$(\mathcal{R}H_{\Lambda_{n-1}})(\tilde\Gamma) \equiv \inf_{\Gamma\in\mathcal{R}^{-1}\tilde\Gamma} H_{\Lambda_n}(\Gamma) \tag{3.7}$$

Since $\mathcal{R}$ is in general not invertible, $\mathcal{R}^{-1}\tilde\Gamma$ denotes the set pre-images of $\tilde\Gamma$ in Ω_n. Then, defining $\mathcal{R}\mathcal{G}_{\Lambda_{n-1}}^{(0)}$ to be the set of ground states with respect to the energy function $\mathcal{R}H$. Then we have that

$$\mathcal{G}_{\Lambda_n}^{(0)} = \left\{ \Gamma^* \,\middle|\, H_{\Lambda_n}(\Gamma^*) = \inf_{\Gamma\in\mathcal{R}^{-1}(\mathcal{R}\mathcal{G}_{\Lambda_{n-1}}^{(0)})} H_{\Lambda_n}(\Gamma) \right\} \tag{3.8}$$

that is, if we can determine the ground states with respect to $\mathcal{R}H$ in the smaller volume Λ_{n-1}, then we have to search for the true ground state only within the inverse image of this set. The proper setting up of a RG scheme consists of finding maps $\mathcal{R}$ such that both tasks become simpler than the original one.

We will now give a precise description of the class of admissible energy functions. The original energy function describing the SOS-model was already introduced in the introduction and adopted to the contour formulation in Section II. In that section the relation between the original random fields $J_x(h)$ and the 'small field' $S_x(h)$ and the auxiliary field $N_x(h)$ and the 'bad region' D was explained. To describe the general class of models that will appear in the RG process, we begin with the auxiliary or 'control' fields N. Thus we let $\{N_x(h)\}_{x\in\Lambda_n}^{h\in\mathbb{Z}}$ be a family of non-negative real numbers. In fact, they will later be assumed to be a random field satisfying certain specific probabilistic assumption. Given such N, we may now define the 'bad region' corresponding to it, namely

DEFINITION 3.1: *Given a control field N, the set*

$$D \equiv D(N) \equiv \left\{ (x,h) \in \Lambda_n \times \mathbb{Z} \Big| \sup_{h' \in \mathbb{Z}} \left(N_x(h+h') - \frac{c}{L}|h'| \right) > 0 \right\} \tag{3.9}$$

is called the 'bad region'. Here c is a constant that may be chosen e.g. $c = 1/8$. Given a contour $\Gamma \in \Omega$, we denote by $D(\Gamma)$ the set

$$D(\Gamma) \equiv \{ x \in \Lambda_n | (x, h_x(\Gamma)) \in D \} \tag{3.10}$$

and we denote by $\Omega_n(D) \subset \Omega_n$ the space

$$\Omega_n(D) \equiv \{ \Gamma \in \Omega_n | D(\Gamma) \subset \underline{\Gamma} \} \tag{3.11}$$

DEFINITION 3.2: *An N-bounded contour energy ϵ of level k is a map $\epsilon : \Omega_n(D) \to \mathbb{R}$, s.t.*

(1) If $\gamma_1, \dots, \gamma_m$ are the connected components of Γ, then

$$\epsilon(\Gamma) = \sum_{i=1}^{m} \epsilon(\gamma_i) \tag{3.12}$$

(ii) If γ is a connected contour in $\Omega_n(D)$ then

$$\epsilon(\gamma) \geq E_s(\gamma) + L^{-(d-2)k} |\underline{\gamma} \backslash \overline{D}(\gamma)| - (N, V(\gamma) \cap \underline{\gamma}) \tag{3.13}$$

where $E_s(\gamma)$ is the strictly deterministic surface energy[5] *defined in (2.9).*

(iii) Let $C \subset D(h)$ be connected and $\gamma = (C, h_x \equiv h)$ be the connected component of a contour $\Gamma \subset \Omega_n(D)$. Then

$$\epsilon(\gamma) \leq \sum_{x \in C} N_x(h) \tag{3.14}$$

An N-bounded energy function of level k is a map $H_{\Lambda_n} : \Omega_n \to \mathbb{R}$ of the form

$$H_{\Lambda_n}(\Gamma) = \epsilon(\Gamma) + (S, V(\Gamma)) \tag{3.15}$$

[5] Observe that due to our boundary conditions, $E(\gamma)$ contains a term $\sum_{x \in \underline{\overline{\gamma}} \cap \partial(\Lambda_n^c)} |h_x(\gamma)|$. This is important for treating boundary effects.

where $S_x(h)$ are bounded random fields (see e.g. (2.11)) and ϵ is a N-bounded contour energy of level k.

Remark: The 'level' k in the definition refers to the fact that some properties of energy function change under the application of the RG transformation. A energy function for the SOS-model will be a energy function of level k after k iterations of the RG.

Remark: The appearance of the dimension and k dependent constant in the lower bound (3.14) is due to the fact that in the RG process no uniform constant suppressing supports of contours outside the bad region is maintained. The specific form of this constant is somewhat technical.

Remark: Restrictions of energy functions to more general finite volumes Λ are defined as

$$H_\Lambda(\Gamma) = \sum_{i:\gamma_i \cap \Lambda \neq \emptyset} \epsilon(\gamma_i) + (S, V_h(\Gamma) \cap \Lambda) \tag{3.16}$$

We are now ready to define what we mean specifically be a RG transformation

DEFINITION 3.3: *For a given control field N, a proper renormalization group transformation, $\mathcal{R}^{(N)}$, is a map from $\Omega_n(D(N))$ into $\Omega_{n-1}(D(N'))$, such that if H_{Λ_n} is of the form (3.16) with ϵ a N-bounded contour energy of level k, then $H'_{\Lambda_{n-1}} \equiv \mathcal{R}^{(N)} H_{\Lambda_n}$ is of the form*

$$H'_{\Lambda_{n-1}}(\Gamma) = \epsilon'(\Gamma) + (S', V(\Gamma)) \tag{3.17}$$

where ϵ' is a N'-bounded contour energy of level $k+1$, and S' is a new bounded random field and N' is a new control field.

From the above definition it is obvious that in order to make use of a RG transformation, it is crucial to be able to compute N' and S', i.e. to study the action of the RG on the random and control fields. As both are random fields, this control will be probabilistic, i.e. consist of statements an the effective probability distributions. We must therefore specify more precisely the corresponding assumptions.

Recall that the energy functions H are random functions on a probability space $(\Theta, \mathcal{F}, I\!P)$ and that H_{Λ_n} is assumed to be $\mathcal{F}_{\Lambda_n}$-measurable (this is evident e.g. in the original SOS-model, where H_{Λ_n} is a function of the stochastic sequences $J_x(h)$ with $x \in \Lambda_n$ only, and $\mathcal{F}_{\Lambda_n}$ is the sigma-algebra generated by these sequences). Of course, the renormalized energy functions are still random variables on this same probability space. It is useful to consider an action of the RG map on the sigma-algebras and to introduce $\mathcal{F}^{(k)} = \mathcal{R}^k \mathcal{F}$, where in particular $\mathcal{R}\mathcal{F}^{(k)}_\Lambda \subset \mathcal{F}^{(k-1)}_{\mathcal{L}\Lambda}$, such that after k iterations of the RG the resulting energy function is $\mathcal{F}^{(k)}$-measurable. Naturally, $\mathcal{F}^{(k)}$ is endowed with a filtration with respect to the renormalized lattice. In the general step we will drop the reference to the level in the specification of this sigma-algebra and write simply $\mathcal{F}$. We need to maintain certain such locality properties that we state as follows:

(i) The stochastic sequences $\{N_x(h)\}_{h\in\mathbb{Z}}$ and $\{S_x(h)\}_{h\in\mathbb{Z}}$ are measurable w.r.t. the sigma-algebras $\mathcal{F}_{\overline{x}}$.

(ii) For any connected contour $\gamma \in \Omega_n(D)$, $\epsilon(\gamma)$ is measurable w.r.t. $\mathcal{F}_{\overline{\underline{\gamma}}}$

A further important assumption on the random quantities is that of stationarity w.r.t. shift in the height variables:

(iii) The stochastic sequences $\{N_x(h)\}_{x\in\tilde{\Lambda}_n, h\in\mathbb{Z}}$ and $\{S_x(h)\}_{x\in\tilde{\Lambda}_n, h\in\mathbb{Z}}$ and the contour energies $\{\epsilon(\gamma)\}_{\underline{\gamma}\subset\tilde{\Lambda}_n}$ are simultaneously stationary w.r.t. the shifts $h \to h+h'$, $(\underline{\gamma}, \{h_x\}_{x\in\Lambda_n}) \to (\underline{\gamma}, \{h_x + h'\}_{x\in\Lambda_n})$, $h' \in \mathbb{Z}$. Here $\tilde{\Lambda}_n \equiv \{x \in \Lambda_n \,|\, d(x, \Lambda_N^c > 1\}$ [5]

$h' \in \mathbb{Z}$. (Note that for components touching the boundary of Λ_n, stationarity holds only if the apply the shift also to the external height).

Finally, we need assumptions on the smallness of the disorder. Here the S-fields are centered and bounded, i.e.

(iv) $\mathbb{E} S_x(h) = 0$

(v) $|S_x(h)| \leq \delta$, with δ some suitable small constant (for instance $\delta = \frac{1}{8L}$ will work).

The distribution of the the S satisfies the bounds

(vii)

$$\begin{aligned} &\mathbb{P}\left[S_x(h) \geq \epsilon\right] \leq \exp\left(-\frac{\epsilon^2}{2\sigma_k^2}\right) \quad \text{and} \\ &\mathbb{P}\left[S_x(h) \leq -\epsilon\right] \leq \exp\left(-\frac{\epsilon^2}{2\sigma_k^2}\right) \end{aligned} \tag{3.18}$$

Here the constants σ_k are parameters that will change in the course of renormalization (we will prove later that $\sigma_k^2 = L^{d-2-\eta}\sigma_0^2$) and those flow will have to be controlled. The control fields $N_x(h)$ should also satisfies bounds like (3.18), but actually the situation there is quite more complicated. Notice that in the original model the N-fields as defined in (2.12) satisfy bounds $\mathbb{P}(N_x(h) > z) \leq 2\exp\left(-\frac{z^2}{2\sigma^2}\right)$, and moreover the smallest non-zero value they take is δ. This latter fact is crucial in that it ensures that $D(N)$ is a fairly sparse set! It will be important to maintain such a property in the course of the RG process. As the exact form of these constraints is fairly complicated and difficult to motivate a priori, we postpone the precise formulation to Section III.5.

We have now a sufficient description of the general class of models on which the RG is to be performed. The RG transformation is now performed in three steps, as indicated in Section II.

[5] Stationarity is of course broken through the boundary conditions. This is the reason why we demand it only for objects 'well inside' Λ_n.

III.2 Absorbtion of small contours

In Section II we explained that the first part of the RG map consists of the re-summing of so-called 'small contours'. These can be defined as connected components of small size (on scale L) with support outside the bad regions. Now the definition of the bad regions is such that they exclude in fact the existence of such small components in a (conditioned) ground state contour. Actually, there is even a large portion of the bad region that may be removed if we are willing to allow for the appearance of 'flat' small contours, i.e contours with non-empty supports but constant height even on their support. It will be crucial to take advantage of this fact. The following definition describes this 'harmless' part of the bad region.

DEFINITION 3.4: *Let $D_i(h)$ denote the $L^{1/2}$-connected*[6] *components of $D(h)$. Such a connected component is called* small, *on level k, if*

(i) $|D_i(h)| < L^{(1-\alpha)/2}$

(ii) $d(D_i(h)) \leq L/4$

(iii) $\sum_{y\in D_i(h)} \sup_{h'\in\mathbb{Z}}(N_y(h+h') - \frac{1}{8L}|h'|) < LL^{-(d-2)k}\sigma^2$

Here $\alpha > 0$ is a constants that will be fixed later and $\sigma^2 \equiv \sigma_0^2$ refers to the variance of the original random fields, not to those at level k. Define now

$$\begin{aligned} \mathcal{D}(h) &\equiv \bigcup_{D_i(h)\,\text{small}} D_i(h) \quad \text{and} \\ \mathcal{D} &\equiv \bigcup_{h\in\mathbb{Z}} (\mathcal{D}(h) \times \{h\}) \end{aligned} \tag{3.19}$$

Remark: The definition of the slices $\mathcal{D}(h)$ follows closely that used in [BK], allowing us to carry over many of the geometric estimates used there. It is certainly not optimal. An important aspect of the definition of $\mathcal{D}$ is that it is 'local' in the following sense: If we consider a fixed point x and a set $E \subset \Lambda_n$ containing x, then the event $\{E$ is a component of $\mathcal{D}(h)\}$ can only depend on $N_{x'}$-fields within the region $d(x,x') \leq L/3$, i.e. the sigma-algebra generated by such events is independent of $\{N_{x'}(h)\}_{x':x'\in A, h\in\mathbb{Z}}$ if $d(x,A) > L/3$.

In the light of this definition, we may now define the 'small contours':

DEFINITION 3.5: *A connected contour $\gamma \in \Omega_n(D)$ is called* small (given N), *iff*

(i) $d(\underline{\gamma}) < L$, *and*

[6] It should be clear what is meant by $L^{1/2}$-connectedness: A set A is said to be $L^{1/2}$-connected, if there exists a path in A with steps of length less that or equal to $L^{1/2}$ joining each point in A.

(ii) $(D(h_\gamma)\backslash\mathcal{D}(h_\gamma))\cap\ \mathrm{int}\underline{\gamma}=\emptyset$

Here h_γ is defined as the height $h(\gamma)$ on the boundary of $\mathrm{int}\underline{\gamma}$. A contour Γ is called small iff the maximal connected component of each weakly connected component is small. A contour that is not small is called large. We denote by $\Omega_n^s(D)$ the set of small contours and by $\Omega_n^l(D)$ the set of large contours.

Remark: Notice that $\Omega_n^l(D)\subset\Omega_n(D\backslash\mathcal{D})$, but in general it is *not* a subset of $\Omega_n(D)$! Note that by our definition a small contour may have connected components that are not small, *provided* their support is contained in the interior of the support of a connected component that is small.

DEFINITION 3.6: *The first step in the RG transformation is a map T_1 that is nothing but the canonical projection from $\Omega_n(D)$ onto $\Omega_n^l(D)$, i.e. if $\Gamma=(\gamma_1,\dots,\gamma_r,\gamma_{r+1},\dots,\gamma_q)$ with γ_i large for $i=1,\dots,t$ and small for $i=r+1,\dots,q$, then*

$$T_1(\Gamma)\equiv\Gamma^l\equiv(\gamma_1,\dots,\gamma_r) \tag{3.20}$$

To give a precise description the conditioned ground states under the projection T_1, we need to define the following sets. First let $\overline{\mathcal{D}_i(h)}$ denote the *ordinary* connected components of $\overline{\mathcal{D}(h)}$ (in contrast to the definition of $D_i(h)$!). Given a contour $\Gamma^l\in\Omega_n^l(D)$ we write $\mathcal{B}_i(\Gamma^l,h)\equiv\overline{\mathcal{D}_i(h)}\backslash\underline{\Gamma^l}$ for all those components such that $\overline{\mathcal{D}_i(h)}\subset V_h(\Gamma^l)\backslash\underline{\Gamma^l}$. Let $\mathcal{B}(\Gamma^l)\equiv\bigcup_{i,h}\mathcal{B}_i(\Gamma^l,h)=\overline{\mathcal{D}}(\Gamma^l)\backslash\underline{\Gamma^l}$. Finally we set $\mathcal{D}_i(h)=\overline{\mathcal{D}_i(h)}\cap\mathcal{D}(h)$. Note that these $\mathcal{D}_i(h)$ need not be connected.

Let us denote by $\mathcal{G}_{\Gamma^l,1}$ the set of contours in $\Omega_n(D)$ that minimize H_n under the condition that $T_1\Gamma=\Gamma^l$. We have the following characterization of this set:

LEMMA 3.1: *Let $\Gamma^l\in\Omega_\Lambda^l(D)$ Then, for any $\Gamma\in\mathcal{G}_{1,\Gamma^l}$*

(i) $\underline{\Gamma}\backslash\underline{\Gamma^l}\subset\mathcal{B}(\Gamma^l)$, *and*

(ii) For all x, $h_x(\Gamma)\equiv h_x(\Gamma^l)$.

Remark: This Lemma is the crucial result of the first step of the RG transformation. It makes manifest that fluctuations on length scale L can only arise due to 'large fields in the bad regions'. Since this statement will hold in each iteration of the RG, it shows that any fluctuations of the surface are localized in the bad regions. We will come back to this more specifically later.

Before giving the proof of this Lemma, the following Lemma gives a formula for the renormalized energy function under T_1. We set

$$\epsilon^h(\mathcal{B}_i(\Gamma^l,h))\equiv\inf_{\gamma:\mathcal{D}_i(h)\subset\underline{\gamma}\subset\mathcal{B}_i(\Gamma^l,h),\gamma=(\underline{\gamma},h_x\equiv h)}\epsilon(\gamma) \tag{3.21}$$

(Note that γ here is not necessarily connected).

LEMMA 3.2: *Let for any $\Gamma^l \in \Omega_n^l(D)$ denote*

$$(T_1 H_n)(\Gamma^l) \equiv \inf_{\Gamma \in \Omega_n(D) : T_1(\Gamma)=\Gamma^l} H_n(\Gamma) \tag{3.22}$$

Then

$$(T_1 H_n)(\Gamma^l) - H_n(\Gamma^l) = \sum_{i,h} \epsilon^h(\mathcal{B}_i(\Gamma^l, h)) \tag{3.23}$$

Note that in the expression $H_n(\Gamma^l)$, we view Γ^l as a contour in $\Omega_n(D \backslash \mathcal{D})$; that is, the contributions to the energy in the regions $\mathcal{D} \backslash \underline{\Gamma^l}$ is ignored.

Proof: (Of Lemmas 3.1 and 3.2) We first prove Lemma 3.1. We will show that for any Γ s.t. $T_1(\Gamma) = \Gamma^l$ the quantity

$$E(\Gamma) - \left(E(\Gamma^l) + \sum_{j,h} \epsilon^h(\mathcal{B}_j(\Gamma^l, h)) \right) > 0 \tag{3.24}$$

unless Γ obeys the conditions stated in Lemma 3.1. Let $\gamma_1, \ldots, \gamma_r$ denote the small *weakly* connected components of Γ. Clearly, the difference $H_n(\Gamma) - H_n(\Gamma^l)$ can be written as a sum over these weakly connected components, namely

$$H_n(\Gamma) - H_n(\Gamma^l) = \sum_{i=1}^{r} \left(\epsilon(\gamma_i) + (S, V(\Gamma) \cap \ \mathrm{int}\, \underline{\gamma_i}) - (S, V(\Gamma^l) \cap \ \mathrm{int}\, \underline{\gamma_i}) \right) \tag{3.25}$$

Similarly, we may split the sum over the $\mathcal{B}_j(\Gamma^l, h)$ into sums over those contained in a given $\mathrm{int}\, \underline{\gamma_i}$ (Note that all $\mathcal{B}_j(\Gamma^l, h)$ must be contained in some such connected component, as $\underline{\Gamma}$ is constrained to contain all of D, and thus the supports of the small connected components of it must contain $\mathcal{D} \backslash \underline{\Gamma^l}$). Thus we are left to show that for any small weakly connected component γ of Γ, the quantities

$$\Delta E(\gamma) \equiv \epsilon(\gamma) + (S, V(\Gamma) \cap \ \mathrm{int}\, \underline{\gamma}) - (S, V(\Gamma^l) \cap \ \mathrm{int}\, \underline{\gamma}) - \sum_{j,h : \mathcal{B}_j(\Gamma^l, h) \subset \mathrm{int}\, \underline{\gamma}} \epsilon^h(\mathcal{B}_j(\Gamma^l, h)) \tag{3.26}$$

are strictly positive, unless γ has constant height and support contained in $\overline{\mathcal{D}}$. To show this, we insert the lower bounds (3.14) on ϵ for $\epsilon(\gamma)$, and an upper bound on $\epsilon^h(\mathcal{B}_j(\Gamma^l, h))$ obtained by bounding the infimum in (3.21) by the value obtained with the flat contour whose support is $\mathcal{D}_j(\Gamma^l, h)$ and bounding the result through the upper bound (3.15). Using moreover that $|S_x(h)| \leq \delta$, this yields

$$\begin{aligned} \Delta E(\gamma) \geq E_s(\gamma) + L^{-(d-2)k} |\underline{\gamma} \backslash \overline{D}(\gamma)| - (N, V(\gamma) \cap \underline{\gamma}) - \sum_{j,h: \mathcal{B}_j(\Gamma^l, h) \subset \mathrm{int}\, (\underline{\gamma})} \quad \sum_{x \in \mathcal{D}_j(\Gamma^l, h)} N_x^{(k)}(h) \\ - \delta \sum_{x \in \mathrm{int}\, \underline{\gamma}} \mathbb{I}_{h_x(\Gamma) \neq h_x(\Gamma^l)} \end{aligned} \tag{3.27}$$

To continue, we need the following geometrical Lemma:

LEMMA 3.3: (local lower bound on the surface energy) *Let γ be a weakly connected contour s.t. $d(\ int\underline{\gamma}) \leq L$. Let h_γ denote the height of γ on $\partial\ int\underline{\gamma}$. Then*

$$E_s(\gamma) \geq \frac{2d}{L} \sum_{x \in \operatorname{int}(\underline{\gamma})} |h_x(\gamma) - h_\gamma| \tag{3.28}$$

The proof of this Lemma will be given in appendix G, where all geometrical estimates of this type are collected and proven. Using this bound, we get

$$\begin{aligned}\Delta E(\gamma) \geq \sum_{x \in \operatorname{int}\underline{\gamma}} \Big(& \frac{2d}{L} |h_x(\gamma) - h_x(\Gamma^l)| - 2\delta \mathbb{I}_{h_x(\gamma) \neq h_x(\Gamma^l)} \\ & - N_x(h(\gamma)) - N_x(h(\Gamma^l)) \Big) + L^{-(d-2)k} |\underline{\gamma} \backslash \overline{D}(\gamma)|\end{aligned} \tag{3.29}$$

Now using condition (iii) from Definition 3.4 of $\mathcal{D}$, we see that

$$\begin{aligned}&\sum_{x \in \operatorname{int}\underline{\gamma}} \Big(\frac{d+2\delta L}{L} |h_x(\gamma) - h_x(\Gamma^l)| - 2\delta \mathbb{I}_{h_x(\gamma) \neq h_x(\Gamma^l)} - N_x(h(\gamma)) - N_x(h(\Gamma^l)) \Big) + \frac{1}{2} L^{-(d-2)k} |\underline{\gamma} \backslash \overline{D}(\gamma)| \\ &\quad \geq -2LL^{-(d-2)k}\sigma^2\end{aligned} \tag{3.30}$$

Note that the lower bound corresponds to such γ those support contains a single component $D_i(h)$; for flat contours containing several such components must have support with volume of the order of $L^{1/2}$ outside of $\overline{\mathcal{D}(\gamma)}$, due to the fact that the $D_i(h)$ are the $L^{1/2}$-connected components of $D(h)$. Thus

$$\Delta E(\gamma) \geq \sum_{x \in \operatorname{int}\underline{\gamma}} \frac{d - 2L\delta}{L} |h_x(\gamma) - h_x(\Gamma^l)| + \frac{1}{2} L^{-(d-2)k} |\underline{\gamma} \backslash \overline{D}(\gamma)| - 2LL^{-(d-2)k}\sigma^2 \tag{3.31}$$

Now if $\frac{d-L\delta}{L} > 2LL^{-(d-2)k}\sigma^2$ and $2L\sigma^2 < 1$, the lower bound in (3.31) is strictly positive unless $h_x(\Gamma) \equiv h_x(\Gamma^l)$ and $\underline{\gamma} \backslash \overline{D}(\gamma) = \emptyset$, which proves Lemma 3.1. Lemma 3.2 now follows immediately. $\diamondsuit$

Remark: Note that the prove imposes a smallness condition on σ^2 w.r.t. L and provides a reason for the choice of the constant $1/8L$ in (iii) of Definition 3.4.

From the previous Lemmas, and the Definition 3.4, we finally obtain the following uniform bounds on the ϵ^h.

LEMMA 3.4: *For any Γ^l, and any component $\mathcal{B}_i(\Gamma^l, h)$*

$$|\epsilon^h(\mathcal{B}_i(\Gamma^l, h)| \leq LL^{-(d-2)k}\sigma^2 \tag{3.32}$$

Here we see an additional rational for the definition of the harmless part of the large field region, namely that the ground state contours supported in then only introduce an extremely small correction to the energy which can, as we will see in the next step, be absorbed locally in the small fields.

III.3 The blocking

We now come to the crucial step in the RG transformation, that is the mapping of the configuration space Ω_n to Ω_{n-1}. The corresponding operator, T_2, will be chosen as $T_2 \equiv \mathcal{L}^{-1}$, with $\mathcal{L}^{-1}$ defined in Section II (c.f. eq. (2.19 ,20)). We will generally use the name $\mathcal{L}^{-1}$ when referring to the purely geometric action of T_2. Notice that $\mathcal{L}^{-1}$ is naturally a map from $\Omega_n(D\backslash\mathcal{D})$ into $\Omega_{n-1}(\mathcal{L}^{-1}(D\backslash\mathcal{D}))$, where $\mathcal{L}^{-1}(D\backslash\mathcal{D})$ is naturally defined as the union of the sets $\mathcal{L}^{-1}(D(h)\backslash\mathcal{D}(h))$. We must now construct the induced action of this map on the energy functions and on the random fields S and N. We consider first the small fields. Recall that we wanted to absorb the contributions of the small contours into the renormalized small fields. This would be trivial, if there were no interaction between the small contours and the supports of the large ones, i.e. if the $\mathcal{B}_i(\Gamma^l, h)$ did not depend on Γ^l. To take this effect into account, we proceed by writing

$$\epsilon^h(\mathcal{B}_i(\Gamma^l, h)) = \epsilon^h(\overline{\mathcal{D}_i(h)}) + \left(\epsilon^h(\mathcal{B}_i(\Gamma^l, h)) - \epsilon^h(\overline{\mathcal{D}_i(h)})\right) \tag{3.33}$$

and adding the first term to the small fields while the second is non-zero only for $\mathcal{D}_i(h)$ that touch the contours of Γ^l and will later be absorbed in the new contour energies. Thus we define the (preliminary) new small fields by

$$\tilde{S}'_y(h) \equiv L^{-(d-1-\alpha)} \left(\sum_{x \in \mathcal{L}y} S_x(h) + \sum_{i: \overline{\mathcal{D}_i(h)} \cap \mathcal{L}y \neq \emptyset} \frac{\epsilon^h(\overline{\mathcal{D}_i(h)})}{|\mathcal{L}^{-1}\overline{\mathcal{D}_i(h)}|} \right) \tag{3.34}$$

The pre-factor in this definition anticipates the scaling factor of the surface energy term under blocking. Note here that the $\tilde{S}'_y$ satisfy the locality conditions (i): $\tilde{S}'_y$ and $\tilde{S}'_{y'}$ are independent stochastic sequences if $|y - y'| > 1$, since the $\overline{\mathcal{D}_i(h)}$ cannot extend over distances larger than L.

The (preliminary) new control field is defined as

$$\tilde{N}'_y(h) \equiv L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}y \backslash \mathcal{D}(h)} \sup_{h' \in \mathbb{Z}} \left(N_x(h + h') - \frac{1}{8L}|h'| \right) \tag{3.35}$$

Note here that the summation over x excludes the regions $\mathcal{D}$, as the contributions there are dealt with elsewhere. This is crucial, as otherwise the regions with positive $\tilde{N}'$ would grow rather than shrink in the RG process. Note that the constant $\frac{1}{8L}$ in the definition is to some extend arbitrary.

We now define the induced energy function $T_2 T_1 H_n$ on $\Omega_{n-1}(\mathcal{L}^{-1}(D\backslash\mathcal{D}))$ by

$$(T_2 T_1 H_n)(\Gamma') \equiv \inf_{\Gamma^l : \mathcal{L}^{-1}\Gamma^l = \Gamma'} (T_1 H_n)(\Gamma^l) \tag{3.36}$$

The following Lemma states that this energy function is essentially of the same form as H_n:

LEMMA 3.5: *For any $\Gamma' \in \Omega_{n-1}(\mathcal{L}^{-1}(D\backslash\mathcal{D}))$ we have*

$$(T_2 T_1 H_n)(\Gamma') = L^{d-1-\alpha} \left(\sum_{i=1}^{q} \bar{\epsilon}(\gamma_i') + (\tilde{S}', V(\Gamma')) \right) \tag{3.37}$$

where the γ_i' are the connected components of Γ', and $\bar{\epsilon}$ satisfies the lower bound

$$\bar{\epsilon}(\gamma') \geq c_1 L^{\alpha} E_s(\gamma') + c_2 L^{\alpha} L L^{-(d-2)(k+1)} |\underline{\gamma'} \backslash \overline{\tilde{D}'}(\gamma')| - (\tilde{N}', V(\gamma') \cap \underline{\gamma'}) \tag{3.38}$$

where $\tilde{D}' \equiv D(\tilde{N}')$ is the preliminary bad field region. Moreover, for flat contours of the form $\gamma' = (C, H_y \equiv h)$ with $C \subset \tilde{D}'(h)$ connected, we have the upper bound

$$\bar{\epsilon}(\gamma') \leq (\tilde{N}', V(\gamma') \cap C) \tag{3.39}$$

Proof: Let us write the energy of a large contour in the form

$$\begin{aligned} &(T_1 E(\Gamma^l) - (S, V(\Gamma^l)) = \\ &\sum_i \epsilon(\gamma_i) + \sum_{\substack{i,h : \overline{\mathcal{D}_i(h)} \cap \overline{\underline{\Gamma^l}} \neq \emptyset \\ \overline{\mathcal{D}_i(h)} \subset V_h(\Gamma) \backslash \underline{\Gamma^l}}} \left(\epsilon^h(\mathcal{B}_i(\Gamma^l, h)) - \epsilon^h(\overline{\mathcal{D}_i(h)}) \right) + \sum_{i,h : \overline{\mathcal{D}_i(h)} \subset V_h(\Gamma^l) \backslash \underline{\Gamma^l}} \epsilon^h(\overline{\mathcal{D}_i(h)}) \end{aligned} \tag{3.40}$$

Note that here the condition that $\overline{\mathcal{D}_i(h)} \cap \overline{\underline{\Gamma^l}} \neq \emptyset$ in the second sum makes explicit the fact that the summand vanishes otherwise. Let us introduce the notation

$$S_C(h) \equiv \sum_i \epsilon^h(\overline{\mathcal{D}_i(h)}) \mathbb{1}_{C = \overline{\mathcal{D}_i(h)}} \tag{3.41}$$

for any finite subset $C \subset \Lambda_n$. Let us note that while here these object have only a transitory significance, in the finite temperature case they will acquire the meaning of non-local random fields. Notice that here $S_C(h)$ is strictly zero whenever $d(C) \geq \frac{L}{4}$.

Using this notation, we may write that

$$\begin{aligned} &(T_2 T_1 H_n)(\Gamma') - L^{d-1-\alpha}(\tilde{S}', V(\Gamma')) \\ &= \inf_{\Gamma^l : \mathcal{R}\Gamma^l = \Gamma'} \left(\sum_i \epsilon(\gamma_i) + \sum_{\substack{i,h : \overline{\mathcal{D}_i(h)} \cap \overline{\underline{\Gamma^l}} \neq \emptyset \\ \overline{\mathcal{D}_i(h)} \subset V_h(\Gamma^l) \backslash \underline{\Gamma^l}}} \left(\epsilon^h(\mathcal{B}_i(\Gamma^l, h)) - \epsilon^h(\overline{\mathcal{D}_i(h)}) \right) \right. \\ &\left. + \sum_{x \in \Lambda_n} \left(S_x(h_x(\Gamma^l)) - S_x(h_{\mathcal{L}^{-1}x}(\Gamma')) \right) + \sum_{\substack{h \in \mathbb{Z}, C \subset \Lambda_n \\ |C| > 1}} S_C(h) \left(\mathbb{1}_{C \subset V_h(\Gamma) \backslash \underline{\Gamma}} - \sum_{y \in \mathcal{L}^{-1}C} \frac{\mathbb{1}_{h_y(\Gamma') = h}}{|\mathcal{L}^{-1}C|} \right) \right) \end{aligned} \tag{3.42}$$

Notice that the first term in the last line gives a non-vanishing contribution only from x s.t. $\mathcal{L}^{-1}x \in \underline{\Gamma}'$ and the second one only from such C that intersect $\overline{\underline{\Gamma}^l} \cup \mathcal{L}\underline{\Gamma}'$. Therefore, the expression in the infimum can be unambiguously split into a sum over the connected components of Γ' and moreover, the infimum may be taken separately in all the terms of the sum. Thus, if Γ' can be decomposed in connected components as $\Gamma' = (\gamma_1', \ldots, \gamma_q')$, we get that

$$(T_2 T_1 H_n)(\Gamma') - L^{d-1-\alpha}(\tilde{S}', V(\Gamma')) = \sum_{i=1}^{q} L^{d-1-\alpha} \overline{\epsilon}(\gamma') \tag{3.43}$$

where

$$\begin{aligned} L^{d-1-\alpha}\overline{\epsilon}(\gamma') = \inf_{\Gamma^l : \mathcal{L}^{-1}\Gamma^l = \gamma'} \Bigg(& \epsilon(\Gamma^l) + \sum_{\substack{i,h : \overline{\mathcal{D}_i(h)} \cap \overline{\underline{\Gamma}^l} \neq \emptyset \\ \overline{\mathcal{D}_i(h)} \subset V_h(\Gamma^l) \setminus \underline{\Gamma}^l}} \left(\epsilon^h(\mathcal{B}_i(\Gamma^l, h)) - \epsilon^h(\overline{\mathcal{D}_i(h)}) \right) \\ & + \sum_{x \in \Lambda_n} \left(S_x(h_x(\Gamma^l)) - S_x(h_{\mathcal{L}^{-1}x}(\Gamma')) \right) + \sum_{h \in \mathbb{Z}, C \subset \Lambda_n} S_C(h) \left(\mathbb{I}_{C \subset V_h(\Gamma) \setminus \underline{\Gamma}} - \sum_{y \in \mathcal{L}^{-1}C} \frac{\mathbb{I}_{h_y(\Gamma')=h}}{|\mathcal{L}^{-1}C|} \right) \Bigg) \end{aligned} \tag{3.44}$$

Notice that the locality condition for $\overline{\epsilon}$ is obvious from the above remarks, i.e. $\overline{\epsilon}(\gamma')$ is measurable w.r.t. the sigma-algebra $\mathcal{F}_{\mathcal{L}\overline{\underline{\gamma'}}}$.

Let us now prove the lower bound on $\overline{\epsilon}$. Inserting the lower bound on $\epsilon(\Gamma^l)$ into (3.44) and noting that trivially $\epsilon^h(\mathcal{B}_i(\Gamma^l, h)) \geq \epsilon^h(\overline{\mathcal{D}_i(h)})$, we get

$$\begin{aligned} L^{d-1-\alpha}\overline{\epsilon}(\gamma') \geq \inf_{\Gamma^l : \mathcal{L}^{-1}\Gamma^l = \gamma'} \Bigg(& E_s(\Gamma^l) + L^{-(d-2)k} |\underline{\Gamma}^l \setminus \overline{D}(\Gamma^l)| - (N, V(\Gamma^l) \cap \underline{\Gamma}^l) \\ & + \sum_{x \in \Lambda_n} \left(S_x(h_x(\Gamma^l)) - S_x(h_{\mathcal{L}^{-1}x}(\Gamma')) \right) + \sum_{h \in \mathbb{Z}, C \subset \Lambda_n} S_C(h) \left(\mathbb{I}_{C \subset V_h(\Gamma) \setminus \underline{\Gamma}^l} - \sum_{y \in \mathcal{L}^{-1}C} \frac{\mathbb{I}_{h_y(\Gamma')=h}}{|\mathcal{L}^{-1}C|} \right) \Bigg) \end{aligned} \tag{3.45}$$

Now the terms in the second line are all small and will be bounded uniformly against some fraction of the surface energy and volume energy of the support of the Γ^l, while the remaining surface and volume energies together with the N-term will give the effective renormalized bounds. To see this, we rearrange the terms in (3.45) in the following form:

$$\begin{aligned} L^{d-1-\alpha}\overline{\epsilon}(\gamma') \geq \inf_{\Gamma^l : \mathcal{L}^{-1}\Gamma^l = \gamma'} \Bigg(& \frac{1}{4} E_s(\Gamma^l) \\ & + \frac{1}{4} E_s(\Gamma^l) + L^{-(d-2)k} |\underline{\Gamma}^l \setminus \overline{D}(\Gamma^l)| - \sum_{h \in \mathbb{Z}, C \subset V_h(\Gamma^l) : C \cap \underline{\Gamma}^l \neq \emptyset} S_C(h) \\ & - \left((N, V(\Gamma^l) \cap \underline{\Gamma}^l) - \frac{1}{4} E_s(\Gamma^l) \right) \\ & + \frac{1}{4} E_s(\Gamma^l) + \sum_{x \in \Lambda_n} \left(S_x(h_x(\Gamma^l)) - S_x(h_{\mathcal{L}^{-1}x}(\gamma')) \right) + \sum_{h \in \mathbb{Z}, C \subset \Lambda_n} S_C(h) \left(\mathbb{I}_{C \subset V_h(\Gamma^l)} - \mathbb{I}_{C \in \mathcal{R} V_h(\gamma')} \right) \Bigg) \end{aligned} \tag{3.46}$$

Note that we have split the $S_C(h)$ terms in such a way that the term appearing in the last line vanishes if Γ^l is a flat contour.

To treat (3.46) further, we need the following geometrical Lemmas, those proof will again be given the appendix.

LEMMA 3.6: *Let* $h' = \mathrm{Rnd}\left(\overline{h}\right)$ *where* $\overline{h} = L^{-d}\sum_{x\in\mathcal{L}0} h_x$. *Then*

$$\sum_{<x,y>:x,y\in\mathcal{L}0} |h_x - h_y| \geq \frac{1}{L}\sum_{x\in\mathcal{L}0}|h_x - h'| \tag{3.47}$$

From this Lemma and the boundedness of the small fields S we see immediately that e.g.

$$\begin{aligned}&\frac{1}{8}E_s(\Gamma^l) + \sum_{x\in\Lambda_n}\left(S_x(h_x(\Gamma^l)) - S_x(h_{\mathcal{L}^{-1}x}(\Gamma'))\right)\\ &\geq \sum_{x\in\mathcal{R}\gamma'}\left(S_x(h_x(\Gamma^l)) - (S_x(h_{\mathcal{L}^{-1}x}(\gamma')) - \frac{1}{8L}|h_x(\Gamma^l)) - h_{\mathcal{L}^{-1}x}(\gamma')|\right) \geq 0\end{aligned} \tag{3.48}$$

provided only $\delta \leq \frac{1}{16L}$. In much the same way we can deal with the remaining terms in the last line of (3.46). Just notice that only such C in that sum give a non-zero contribution that contain at least one site x for which $h_x(\Gamma^l) \neq h_{\mathcal{L}^{-1}x}(\gamma')$ and for each such site only the C that equals the $\overline{D_i(h)}$ that contains x gives a contribution, which in turn is bounded by $LL^{-(d-2)k}\sigma^2$. Therefore, provided this quantity is smaller than $\frac{1}{16L}$, the remainder of the last line of (3.46) is also non-negative. Note that the last condition holds true for $k \geq 1$ if $\sigma^2 \leq \frac{1}{8L}$; if $k = 0$, i.e. in the first RG step these terms even do not exist since $\mathcal{D}$ is empty in this case.

Thus we have shown that the last line in (3.46) is uniformly non-negative.

Let us now consider the third line in (3.46). We split the N-term as

$$(N, V(\Gamma^l)\cap\underline{\Gamma^l}) = \sum_{x\in\underline{\Gamma^l}:\, x\notin\mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))} N_x(h_x(\Gamma^l)) + \sum_{x\in\underline{\Gamma^l}:\, x\in\mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))} N_x(h_x(\Gamma^l)) \tag{3.49}$$

With the help of Lemma 3.6, the first term from (3.49) together with a piece of the surface energy can be bounded by the new large fields:

$$\begin{aligned}&\sum_{x\in\underline{\Gamma^l}:\, x\notin\mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))} N_x(h_x(\Gamma^l)) - \frac{1}{8}E_s(\Gamma^l)\\ &\leq \sum_{x\in\underline{\Gamma^l}:\, x\notin\mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))}\left(N_x(h_x(\Gamma^l)) - \frac{1}{8L}|h_x(\Gamma^l) - h_{\mathcal{L}^{-1}x}(\gamma')|\right)\\ &\leq \sum_{x\in\mathcal{R}\underline{\gamma'}:\, x\notin\mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))}\sup_{h'\in\mathbb{Z}}\left(N_x(h') - \frac{1}{8L}|h' - h_{\mathcal{L}^{-1}x}(\gamma')|\right)\\ &= L^{d-1-\alpha}(\tilde{N}', V(\gamma')\cap\underline{\gamma'})\end{aligned} \tag{3.50}$$

The second term in (3.49) gives for flat contours Γ^l a small contribution proportional to $|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}|$; taking advantage of the remaining $\frac{1}{8}E_s(\Gamma^l)$ we get the same bound for an arbitrary contour

$$
\begin{aligned}
&\sum_{x\in\underline{\Gamma^l}:\, x\in\mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))} N_x(h_x(\Gamma^l)) - \frac{1}{8}E_s(\Gamma) \\
&\le \sum_{x\in\underline{\Gamma^l}:\, x\in\mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))} \left(N_x(h_x(\Gamma^l)) - \frac{1}{16L}|h_x(\Gamma) - h_{\mathcal{L}^{-1}x}(\gamma')| - \frac{1}{16}E_s(\Gamma^l)\right) \\
&\le \sum_{x\in\underline{\Gamma^l}:\, x\in\mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))} \left(LL^{-(d-2)k}\sigma^2 - \frac{1}{16L}|h_x(\Gamma) - h_{\mathcal{L}^{-1}x}(\gamma')|\right) \\
&\le LL^{-(d-2)k}\sigma^2|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}|
\end{aligned}
\tag{3.51}
$$

where the last inequality was obtained by noting that in the previous sum only such sites x for which $h_x(\Gamma^l) = h_{\mathcal{L}^{-1}x}(\gamma')$ give a positive contribution and that for such sites $x \in \mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))$ implies $x \in \mathcal{D}(\Gamma^l)$. The final bound in (3.51) is of course just a tiny fraction of the volume term in the second line and will be absorbed in it.

The S_C term in the second line is in fact dominated by the same bound, i.e.

$$
\sum_{h\in\mathbb{Z},C\subset V_h(\Gamma^l):C\cap\underline{\Gamma^l}\neq\emptyset} |S_C(h)| \le LL^{-(d-2)k}\sigma^2|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}|
\tag{3.52}
$$

where the volume term arises as a trivial upper bound on the number of connected components of $\mathcal{D}(\Gamma^l)$ that may intersect $\underline{\Gamma^l}$. Putting theses results together, we have arrived at

$$
L^{d-1-\alpha}\overline{\epsilon}(\gamma') \ge \inf_{\Gamma^l\in\mathcal{L}\gamma'} \left(\frac{1}{4}E_s(\Gamma^l) + \frac{1}{4}E_s(\Gamma^l) + L^{-(d-2)k}|\underline{\Gamma^l}\backslash\overline{\mathcal{D}}(\Gamma^l)| - 2LL^{-(d-2)k}\sigma^2|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}|\right)
\tag{3.53}
$$

The first quarter of surface energy here yields the new surface energy by the following Lemma (whose proof may again be found in the appendix):

LEMMA 3.7: *Let $\Gamma\in\mathcal{L}^{-1}\gamma'$. Then*

$$
E_s(\Gamma) \ge \frac{L^{d-1}}{d+1}E_s(\gamma')
\tag{3.54}
$$

Note that in order to use this bound, we must choose α such that $L^\alpha \ge 4(d+1)$.

To bound the remaining terms, let us set $\tilde{D}' \equiv D(\tilde{N}')$. Clearly we want to bound the remaining surface and volume energies by a term proportional to $|\underline{\gamma'}\backslash\overline{\tilde{D}'}(\gamma')|$. The tricky part here is that the original estimate is only in the volume of $\underline{\Gamma^l}$ *outside* the bad region, while the new estimate involves the new bad region, which is smaller than the image of the bad region under $\mathcal{L}^{-1}$ since the harmless part, $\mathcal{D}$, has been excluded in the definition of the $\tilde{N}'$. In fact, the geometric constraints in the definition of $\mathcal{D}$ were essentially made in order to get the desired estimate nonetheless, as we will see from the following two Lemmas, that are again proven in the appendix.

LEMMA 3.8: *Let γ be connected and large. Then*

$$L^{(1-\alpha)/2}E_s(\gamma)+|\underline{\gamma}\backslash\overline{\mathcal{D}}(\gamma)|\geq\frac{1}{2}|\underline{\gamma}\cap\overline{\mathcal{D}}(\gamma)| \tag{3.55}$$

LEMMA 3.9: *Let $\Gamma\in\mathcal{L}^{-1}\gamma'$. Then there exists a constant $c_6>0$ s.t.*

$$LE_s(\Gamma)+|\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma))|\geq c_6L|\underline{\gamma'}\backslash\overline{\tilde{D}'}(\gamma')| \tag{3.56}$$

where $\tilde{D}'\equiv D(\tilde{N}')$.

To use these Lemmas, we just have to rearrange terms slightly: Setting $q\equiv\frac{1}{5}$, we may write

$$\begin{aligned}&\frac{1}{4}E_s(\Gamma^l)+L^{-(d-2)k}|\underline{\Gamma^l}\backslash\overline{D}(\Gamma^l)|-2LL^{-(d-2)k}\sigma^2|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}|\\&\geq\frac{1}{8}E_s(\Gamma^l)+\frac{1}{8}E_s(\Gamma^l)+(1-q)L^{-(d-2)k}|\underline{\Gamma^l}\backslash\overline{D}(\Gamma^l)|+qL^{-(d-2)k}|\underline{\Gamma^L}\backslash(\overline{D}(\Gamma^l)\backslash\overline{\mathcal{D}}(\Gamma^l))|\\&\quad-qL^{-(d-2)k}|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}|-2LL^{-(d-2)k}\sigma^2|\overline{\mathcal{D}}(\Gamma)\cap\underline{\Gamma^l}|\\&\geq\frac{1}{8}E_s(\Gamma^l)+qL^{-(d-2)k}|\underline{\Gamma^l}\backslash(\overline{D}(\Gamma^l)\backslash\overline{\mathcal{D}}(\Gamma^l))|\\&\quad+\left(L^{-(d-2)k}(\frac{1}{2}(1-q)-q)-2LL^{-(d-2)k}\sigma^2\right)|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}|\\&\geq c_6qL^{-(d-2)k}|\underline{\gamma'}\backslash\overline{\tilde{D}'}(\gamma')|\end{aligned} \tag{3.57}$$

Here the last inequality was obtained assuming that σ^2 is sufficiently small, so that the term proportional to $|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}|$ in the one but last line is positive. We have also assumed that $\frac{1}{8}L^{-(1-\alpha)/2}\geq L^{-(d-2)k}(1-q)$.

Collecting everything, we get the desired lower bound (3.38).

To conclude the proof of the Lemma, it remains to prove the upper bound (3.39) for flat contours γ'. Thus let $C'\subset\tilde{D}'(h)$ be connected and set $\gamma'\equiv(C',h_x\equiv h)$. Then we bound the infimum over Γ^l from above by the term for which $\Gamma=((D(h)\backslash\mathcal{D}(h))\cap\mathcal{L}C',h_x\equiv h)$. It is easy to check that for this contour, only the term $\epsilon(\Gamma^l)$ in (3.44) gives a non-zero contribution. It is then a trivial matter to conclude from the upper bound on $\epsilon(\Gamma^l)$ and the definition of $\tilde{N}'$, that

$$L^{d-1-\alpha}\bar{\epsilon}(\gamma')\leq\sum_{x\in\mathcal{L}C'}N_x^{(k)}(h)\leq L^{d-1-\alpha}\sum_{y\in C'}\tilde{N}'_y(h) \tag{3.58}$$

This concludes the proof of Lemma 3.5. $\diamondsuit$

III.4 Final shape-up

The hard part of the RG transformation is now done. However, not all of the properties of the original model are yet shared by the renormalized quantities; in particular, the renormalized weak field $\tilde{S}'$ is not centered and it may have become too large. Both defaults are, however, easily rectified. We define

$$S'_y(h) \equiv \tilde{S}'_y(h) 1_{|\tilde{S}'_y(h)|<\delta} - I\!E\left(\tilde{S}'_y(h) 1_{|\tilde{S}'_y(h)|<\delta}\right) \tag{3.59}$$

It is important here that due to the stationarity assumptions, the expectation in (3.59) does not depend on h, i.e $I\!E\left(\tilde{S}'_y(h) 1_{|\tilde{S}'_y(h)|<\delta}\right) = I\!E\left(\tilde{S}'_y(0) 1_{|\tilde{S}'_y(0)|<\delta}\right)$ except if y is next to the boundary of Λ_{n-1}. What is left, i.e. the large part of the small field is taken account of through the redefined control field. In fact, the definition of the renormalized control field is somewhat subtle and maybe not immediately intuitive. We set first

$$\mathcal{N}_x(h) \equiv \begin{cases} \sup_{h'\in\mathbb{Z}}\left(N_x(h+h') - \frac{1}{32L}|h'|\right), & \text{if } \sup_{h'\in\mathbb{Z}}\left(N_x(h+h') - \frac{c}{2L}|h'|\right) \geq \frac{1}{32L} \\ N_x(h), & \text{otherwise} \end{cases} \tag{3.60}$$

and define the final renormalized control field by

$$N'_y(h) \equiv\equiv L^{-(d-1-\alpha)} \sum_{x\in\mathcal{L}y\setminus\mathcal{D}(h)} \mathcal{N}_x(h) + |\tilde{S}'_y(h)| 1_{|\tilde{S}'_y(h)|>\delta} \tag{3.61}$$

Remark: It would appear natural at this point to define N' simply as $\tilde{N}'_y(h) + |\tilde{S}'_y(h)| 1_{|\tilde{S}'_y(h)|>\delta}$. The reader will check easily that our N' is larger than this quantity. On the other hand due to this definition, we will be able to show later that it has the property to be either zero or greater than some strictly positive, k-dependent quantity. This will turn out to be convenient for the probabilistic estimates. More importantly, this property will be needed in the case of finite temperatures and we find it convenient (and hopefully less confusing) to work with the same definitions in both situations.

Given N', we may now define $D' \equiv D(N')$ as in Definition 3.1. Then let T_3 (given N') be the map from Ω_{n-1} to $\Omega_{n-i}(D')$ defined through

$$T_3(\Gamma) = (h(\Gamma), \underline{\Gamma} \cup D'(\Gamma)) \tag{3.62}$$

We define the contour energies

$$\begin{aligned}\epsilon'(\gamma'') \equiv & \inf_{\substack{\gamma':h(\gamma')=h(\gamma'')\\ \underline{\gamma'}\subset\underline{\gamma''}}} \overline{\epsilon}(\gamma') + \sum_{y\in\underline{\gamma''}} \tilde{S}'_y\left(h_y(\gamma'')\right) \mathbb{1}_{\left|\tilde{S}'_y(h_y(\gamma''))\right|\geq\frac{1}{16L}} \\ & \sum_{\substack{y\in\underline{\gamma''}\\ d(y,\Lambda^c_{n-1})=1}} \left(I\!E\left[S'_y\left(h_y(\gamma'')\right)\mathbb{1}_{\left|\tilde{S}'_y(h_y(\gamma''))\right|<\frac{1}{16L}}\right] - I\!E\left[S'_y(0)\,\mathbb{1}_{\left|\tilde{S}'_y(0)\right|<\frac{1}{16L}}\right]\right)\end{aligned} \tag{3.63}$$

Notice that the terms in the second line of (3.63) are a boundary term that is due to the fact that that the renormalized quantities are stationary with respect to shifts in h only if they are at least at a distance 2 from the boundary. The importance of the stationarity assumptions does lie in fact that it implies that all corresponding terms of this form with $d(y,\Lambda_{n-1}^c)>1$ are identically zero.

The final form of the renormalization group map is then given through the following

LEMMA 3.10: *For any $\Gamma'\in\Omega_{\Lambda'}(N')$ we have*

$$T_3T_2T_1E(\Gamma') = L^{d-1-\alpha}\left(\epsilon'(\Gamma')+(S',V(\Gamma'))+\sum_{y\in\Lambda'}I\!E(\tilde S'_y(0)1_{|\tilde S'_y(0)|>\delta'})\right) \tag{3.64}$$

where ϵ' is a N'-bounded contour energy of level $k+1$.

Proof: Notice that for all $\Gamma\in T_3^{-1}\Gamma'$ $(S',V(\Gamma))=(S',V(\Gamma'))$. Moreover, $|\underline{\Gamma}\backslash\widetilde{D'}(\gamma')|\geq|\underline{\gamma'}\backslash D'(\gamma')|$ and, as remarked above, for any $\Gamma\in T_3^{-1}\gamma'$,

$$(\tilde N',V(\Gamma)\cap\underline{\Gamma})+(|\tilde S'|1_{|S'|>\delta'},V(\gamma')\cap\underline{\gamma'})\leq(N',V(\gamma')\cap\underline{\gamma'}) \tag{3.65}$$

Then we get immediately from Lemma 3.5 that

$$\begin{aligned}\epsilon'(\gamma')\geq{}& c_1L^\alpha E_s(\gamma')+c_2L^\alpha L^{-(d-2)(k+1)}|\underline{g'}\backslash D'(\gamma')|-(N',V(\gamma')\cap\underline{\gamma'})\\ &\sum_{\substack{y\in\underline{\gamma'}\\ d(y,\Lambda_{n-1}^c)=1}}\left(I\!E\left[S'_y\left(h_y(\gamma')\right)\mathbb{1}_{|\tilde S'_y(h_y(\gamma'))|<\frac{1}{16L}}\right]-I\!E\left[S'_y\left(0\right)\mathbb{1}_{|\tilde S'_y(0)|<\frac{1}{16L}}\right]\right)\end{aligned} \tag{3.66}$$

This is the desired form except for the boundary term in the second line. However, as it exists only for contours touching the boundary it can easily be estimated away against a piece of the surface energy. To do so, we use that under the assumptions on the distributions of the small fields, $I\!E\left|S'_y(h)\right|\leq C_1\sigma_k$ and thus

$$\begin{aligned}&\sum_{\substack{y\in\underline{\gamma'}\\ d(y,\Lambda_{n-1}^c)=1}}\left|I\!E\left[S'_y\left(h_y(\gamma')\right)\mathbb{1}_{|\tilde S'_y(h_y(\gamma'))|<\frac{1}{16L}}\right]-I\!E\left[S'_y\left(0\right)\mathbb{1}_{|\tilde S'_y(0)|<\frac{1}{16L}}\right]\right|\\ &2C_1\sigma_k\sum_{\substack{y\in\underline{\gamma'}\\ d(y,\Lambda_{n-1}^c)=1}}|h_y(\gamma')|\leq 2C_1\sigma_kE_s(\gamma')\end{aligned} \tag{3.67}$$

(see footnote to Definition 3.2).

The upper bound on ϵ' follows immediately from Lemma 3.5. The locality condition on ϵ' being trivially verified, one also sees immediately that all stationarity conditions for the renormalized quantities follow directly from the corresponding assumptions and the construction of these quantities. This concludes the proof of Lemma 3.10. $\diamondsuit$

This concludes the construction of the entire RG transformation. We may summarize the results of the previous three subsections in the following

PROPOSITION 3.1: *Let $\mathcal{R}^{(N)} \equiv T_3 T_2 T_1 : \Omega_n(D(N)) \to \Omega_{n-1}(D(N'))$ with T_1, T_2 and T_3 defined above; let N' and S' and ϵ' be defined as above and define $H'_{n-1} \equiv L^{-(d-1-\alpha)}(\mathcal{R}^{(N)} H_n)$ through*

$$H'_{n-1}(\Gamma) = \epsilon'(\Gamma) + (S', V(\Gamma)) \tag{3.68}$$

Then, if H_n is a N-bounded energy function of level k, then H'_{n-1} is a N'-bounded energy function of level $k+1$.

This proposition allows us to control the flow of the RG transformation on the energies through its action on the random fields S and N. What is now left to do is to study the evolution of the probability distributions of these random fields under the RG map. This will be done in the next subsection.

III.5 Probabilistic estimates

Our task is now to control the action of the RG transformation on the random fields S and N, i.e. given the probability distribution of these random fields, we would like to compute the distribution of the renormalized random fields S' and N' as defined through eqs.. (3.34), (3.35) and (3.59), (3.60), (3.61). Of course, rather than the precise probability distributions themselves we only compute certain bounds on these distributions.

Let us begin with the simpler small fields. In the k-th level of iteration, the distributions of the random fields are governed by a parameter σ_k^2 (essentially the variance of $S_x(h)$), that decreases exponentially fast to zero with k. We will set

$$\sigma_k^2 \equiv L^{-(d-2-\eta)} \sigma^2 \tag{3.69}$$

where η may be chosen as $\eta \equiv 3\alpha$. We denote by $S^{(k)}$ the small random field obtained from S after k iterations of the RG map $\mathcal{R}$. (Where the action of $\mathcal{R}$ on S is defined through (3.59) and (3.34)). We then have the following

PROPOSITION 3.2: *Let $d \geq 3$. Assume that the initial S satisfies assumptions (i), (iii) and (iv)-(vi) with σ^2 sufficiently small. Then, for all $k \in I\!N$ and for all $\epsilon \geq 0$,*

$$\begin{aligned} I\!P\left[S_y^{(k)}(h) \geq \epsilon\right] &\leq \exp\left(-\frac{\epsilon^2}{2\sigma_k^2}\right) \quad \textit{and} \\ I\!P\left[S_y^{(k)}(h) \leq -\epsilon\right] &\leq \exp\left(-\frac{\epsilon^2}{2\sigma_k^2}\right) \end{aligned} \tag{3.70}$$

with σ_k defined through (3.69), and $S^{(k)}$ satisfy assumptions (i),(iii), (iv), and (v).

Remark: This Lemma is essentially equivalent to Proposition 1 of [BK]. We give a slightly different proof for the convenience of the reader.

Proof: The proof of this proposition relies on a general probabilistic result on convolutions of random variables satisfying Gaussian estimates. It reads:

LEMMA 3.11: *Let $\{X_j\}_{j=1,\dots,N}$ be a family of random variables and define $X \equiv \sum_{j=1}^N X_j$. Assume that there is a decomposition $\{1,\dots,N\} = \bigcup_{i=1}^m V_i$ s.t.*

(i) For each i, $\{X_j\}_{j\in V_i}$ is a family of independent random variables

(ii) $\mathbb{E}X_j = 0$ for all j

(iii) For all $\epsilon \geq 1$, $\mathbb{P}[X_j > \epsilon] \leq e^{-\frac{\epsilon^2}{2}}$ and $\mathbb{P}[X_j < -\epsilon] \leq e^{-\frac{\epsilon^2}{2}}$.

Then there exists a constant $C > 1$ independent of N and m, s.t. for all $\epsilon > 0$, $\mathbb{P}[X > \epsilon] \leq e^{-\frac{\epsilon^2}{2CmN}}$ and $\mathbb{P}[X < -\epsilon] \leq e^{-\frac{\epsilon^2}{2CmN}}$

Proof: Notice first that the information on $\mathbb{P}$ in the assumptions of the Lemma are completely symmetric, so that it suffices to prove the bound on $\mathbb{P}[X > \epsilon]$. To do this, we will first prove a bound on the Laplace transform $\mathbb{E}e^{tX}$ for $t \geq 0$ from which the desired estimate will follow by the exponential Markov inequality (see e.g. [CT]). To do this, we need a bound on $\mathbb{E}e^{tX_j}$ first. This will be derived from the assumptions as in [BoK]: We distinguish the cases $t \geq 1$ and $t < 1$. For $t \geq 1$ we have

$$\begin{aligned} \mathbb{E}e^{tX_j} &= t\int_{-\infty}^{\infty} e^{tx}\mathbb{P}[X_j \geq x]dx \\ &\leq t\int_{-\infty}^{1} e^{tx}dx + t\int_{1}^{\infty} e^{tx}\mathbb{P}[X \geq x]dx \\ &\leq e^t + t\int_1^\infty e^{tx}e^{-\frac{x^2}{2}}dx \leq e^{\frac{C_1^2t^2}{2}} \end{aligned} \tag{3.71}$$

with some constant C_1^2. For $t < 1$ we use

$$\begin{aligned} \mathbb{E}e^{tX_j} &\leq 1 + \frac{t^2}{2}\left(\mathbb{E}[X_j^2 \mathbb{1}_{X_j<0}] + \mathbb{E}[X_j^2 e^{tX_j}\mathbb{1}_{X_j\geq 0}]\right) \\ &\leq \exp\left[\frac{t^2}{2}\left(\mathbb{E}[X_j^2\mathbb{1}_{X_j<0}] + \mathbb{E}[X_j^2 e^{X_j}\mathbb{1}_{X_j\geq 0}]\right)\right] \end{aligned} \tag{3.72}$$

where in the last line we have estimated the second term in the argument of the exponential by its value for $t = 1$. Using the bounds (iii), it is easy to see that the expectations in the second line of (3.72) are bounded by universal numerical constants, so that we see that there exists a universal constant C s.t.

$$\mathbb{E}e^{tX_j} \leq e^{\frac{C^2t^2}{2}} \quad \text{for all} \quad t \geq 0 \tag{3.73}$$

Using this, we can now bound the Laplace Transform of X

$$\begin{aligned} I\!E\left[e^{tX}\right] &= I\!E\left[e^{\sum_{i=1}^{m}\sum_{j\in V_i} tX_j}\right] \leq \prod_{i=1}^{m}\left(I\!E\left[e^{m\sum_{j\in V_i} tX_j}\right]\right)^{\frac{1}{m}} \\ &= \prod_{j=1}^{N}\left(I\!E\left[e^{mtX_j}\right]\right)^{\frac{1}{m}} \leq e^{\frac{mNCt^2}{2}} \end{aligned} \tag{3.74}$$

where the first inequality is an application of the Hölder inequality. The bound on $I\!P[X > \epsilon]$ is now an immediate consequence of (3.74) by the exponential Markov inequality. $\diamondsuit$

To prove the proposition, recall that

$$S_y^{(k+1)}(h) \equiv \tilde{S}_y^{(k+1)}(h)1\!\!1_{|\tilde{S}_y^{(k+1)}(h)|<\delta} - I\!E\left(\tilde{S}_y^{(k+1)}(h)1\!\!1_{|\tilde{S}_y^{(k+1)}(h)|<\delta}\right) \tag{3.75}$$

where

$$\tilde{S}_y^{(k+1)}(h) \equiv L^{-(d-1-\alpha)}\left(\sum_{x\in\mathcal{L}y} S_x^{(k)}(h) + \sum_{i:\overline{\mathcal{D}_i(h)}\cap\mathcal{L}y\neq\emptyset}\frac{\epsilon^h(\overline{\mathcal{D}_i(h)})}{|\mathcal{L}^{-1}\overline{\mathcal{D}_i(h)}|}\right) \tag{3.76}$$

We will first prove bounds for $\tilde{S}^{(k+1)}$. Note that the second term in (3.76) is uniformly bounded by

$$L^{-(d-1-\alpha)}\sum_{i:\overline{\mathcal{D}_i(h)}\cap\mathcal{L}y\neq\emptyset}\frac{\epsilon^h(\overline{\mathcal{D}_i(h)})}{|\mathcal{L}^{-1}\overline{\mathcal{D}_i(h)}|} \leq L^{1+\alpha-d}C_1LL^{-(d-2)k}\left(L^{\frac{1}{2}}\right)^d\sigma^2 \leq L^{-(d-4-2\alpha)/2-k\eta}C_1\sigma_k^2 \tag{3.77}$$

with some constant $C_1 > 0$. Now notice that, for $k \geq 1$,

$$\begin{aligned} &L^{-(d-4-2\alpha)/2}C_1L^{-k\eta}\sigma_k^2 = \sigma_{k+1}L^{1-\eta/2-k\eta}C_1\sigma_k \\ &\leq \sigma_{k+1}L^{-\eta+\alpha+(3-d)/2}C_1L^{1/2}\sigma \leq \frac{1}{2}\sigma_{k+1} \end{aligned} \tag{3.78}$$

if $d > 3-\eta+\alpha$, $L^{1/2}\sigma \leq 1$ (which has been assumed before) and η and L are large enough s.t. $L^{-\eta+(3-d)/2}C_1 \leq 1$. Under these assumptions we have that for all $\epsilon \geq \sigma_{k+1}$

$$I\!P\left[\tilde{S}_y^{(k+1)}(h) \geq \epsilon\right] \leq I\!P\left[L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}y}S_x^{(k)}(h) \geq \frac{\epsilon}{2}\right] \tag{3.79}$$

Now, since the S_x are independent for x's that are separated by distances greater than one, one may easily group the L^d variables in the sum into 2^d sets of mutually independent ones. This allows us to apply Lemma 3.11 which yields

$$I\!P\left[L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}y}S_x^{(k)}(h) \geq \frac{\epsilon}{2}\right] \leq e^{-\frac{\epsilon^2}{2^d\cdot 8CL^{-(\eta-2\alpha)}\sigma_{k+1}^2}} \tag{3.80}$$

With this bound for $\tilde{S}^{(k+1)}$, it is now easy to derive the desired bound on $S^{(k+1)}$. Note first that

$$\mathbb{E}\left(\tilde{S}_y^{(k+1)}(h)\mathbb{1}_{|\tilde{S}_y^{(k+1)}(h)|<\delta}\right) \leq \left(\mathbb{E}\left[\tilde{S}_y^{(k+1)}(h)\right]^2\right)^{1/2} \leq 2\sigma_{k+1} \tag{3.81}$$

where the last bound uses (3.80) and assumes $8dCL^{-(\eta-2\alpha)} \leq 1$. Using this bound, we see easily that for $\epsilon > 3\sigma_{k+1}$

$$\mathbb{P}\left[S_y^{(k+1)}(h) > \epsilon\right] \leq P\left[\tilde{S}_y^{(k+1)}(h) > \epsilon - 2\sigma_{k+1}\right] \leq e^{-\frac{(\epsilon-2\sigma_{k+1})^2}{2^d 8CL^{-(\eta-2\alpha)}\sigma_{k+1}^2}} \leq e^{-\frac{\epsilon^2}{2^d\cdot 9\cdot 8CL^{-(\eta-2\alpha)}\sigma_{k+1}^2}} \tag{3.82}$$

Using once more Lemma 3.11, we get from this that *for all* $\epsilon > 0$,

$$\mathbb{P}\left[S_y^{(k+1)}(h) > \epsilon\right] \leq e^{-\frac{\epsilon^2}{2^d\cdot 9\cdot 8C^2L^{-(\eta-2\alpha)}\sigma_{k+1}^2}} \tag{3.83}$$

and assuming that L and η are large enough s.t. $2^d 36C^2L^{-(\eta-2\alpha)} \leq 1$, this gives the desired bound. The bound on $\mathbb{P}\left[S_y^{(k+1)}(h) < -\epsilon\right]$ follows in exactly the same way. The centeredness of the S is true by definition, the locality and stationarity properties have already been established in the course of the previous sections. Thus Proposition 3.2 is proven. $\diamondsuit$

Now we come to the central estimate on the distribution of the control fields. In the same spirit as above, we denote by $N_x^{(k)}(h)$ the fields obtained after k iterations of the RG transformation from a starting field $N^{(0)}$, where the iterative steps are defined by equations (3.35) and (3.60). In the same spirit we will denote by $D^{(k)}$ and $\mathcal{D}^{(k)}$ the bad regions and harmless bad regions in the k-th RG step. What we need to prove for these control fields are two types of results: First of all, they must be large only with very small probabilities; second, and more important, they must be equal to zero with large and larger probability, as k increases. This second fact is crucial for the 'bad regions' to become smaller and smaller in each iteration of the RG group (remember that we have good control over the ground states only outside these bad regions!). The proof of this second fact must take into account the absorbtion of parts of the bad regions, the $\mathcal{D}$, in each step. Morally, what is happening is that once a large field has been scaled down sufficiently, it will actually drop to zero, since it finds itself in the region $\mathcal{D}$. Actually, due to the complications arising from interactions between neighboring blocks, this is not quite true, as the field really drops to zero only if the fields at neighboring sites, too, are small. This has been taken into account in [BK] by considering an upper bound on the control field that is essentially the sum of the original N over small blocks. We follow their procedure for simplicity by defining

$$\begin{aligned}
\bar{N}_y^{(0)}(h) &= N_y^{(0)}(h)\\
\bar{\mathcal{N}}_x^{(k)}(h) &\equiv \begin{cases} \sup_{h'\in\mathbb{Z}}\left(N_x^{(k)}(h+h') - \frac{1}{16L}|h'|\right), & \text{if } \sup_{h'\in\mathbb{Z}}\left(N_x^{(k)}(h+h') - \frac{c}{2L}|h'|\right) \geq \frac{1}{16L}\\ \bar{N}_x^{(k)}(h), & \text{otherwise}\end{cases}\\
\bar{N}_y^{(k+1)}(h) &= L^{-(d-1-\alpha)} \sum_{x\in\mathcal{L}\{y\}\cap D^{(k+1)}(h)\setminus\mathcal{D}^{(k)}(h)} \overline{\mathcal{N}}_x^{(k)}(h) + \left|\tilde{S}_y^{(k+1)}(h)\right| \mathbb{1}_{|\tilde{S}_y^{(k+1)}(h)|>\delta}
\end{aligned} \tag{3.84}$$

The fields $\bar{N}$ bound the original N from above, but also, in an appropriate sense, from below, namely

LEMMA 3.12: *The fields $\bar{N}^{(k)}$ defined in (3.84) satisfy*

$$\bar{N}_x^{(k)}(h) \geq N_x^{(k)}(h) \tag{3.85}$$

Moreover, if M is an arbitrary subset of $\mathbb{Z}^d$ and if $K \subset \mathbb{Z}^d$ denotes the union of the connected components of $D^{(k)}(h)$ that intersect M, then

$$\sum_{x \in M} \bar{N}_x^{(k)}(h) \leq C_1^k \sum_{x \in \overline{\overline{M}} \cap K} N_x^{(k)}(h) \tag{3.86}$$

Proof: The lower bound (3.85) is obvious. The upper bound can be proven by induction. Thus we assume (3.86) for k. To show that then it also holds for $k+1$, all we need to show is that

$$\begin{aligned} \sum_{y \in M} \sum_{x \in \mathcal{L}\overline{\{y\} \cap D^{(k+1)}(h)} \setminus \mathcal{D}^{(k)}(h)} \bar{\mathcal{N}}_x^{(k)}(h) &\leq C_1^k \sum_{y \in \overline{\overline{M}} \cap K} \sum_{x \in \mathcal{L}y \setminus \mathcal{D}^{(k)}(h)} \mathcal{N}_x^{(k)} \\ &= C_1^{k+1} \sum_{x \in \mathcal{L}\left(\overline{\overline{M}} \cap K\right) \setminus \mathcal{D}^{(k)}(h)} \mathcal{N}_x^{(k)}(h) \end{aligned} \tag{3.87}$$

Now, quite obviously,

$$\sum_{y \in M} \sum_{x \in \mathcal{L}\overline{\{y\} \cap D^{(k+1)}(h)} \setminus \mathcal{D}^{(k)}(h)} \bar{\mathcal{N}}_x^{(k)}(h) \leq \left|\overline{\{0\}}\right| \sum_{x \in \mathcal{L}\left(\overline{M} \cup K\right) \setminus \mathcal{D}^{(k)}(h)} \bar{\mathcal{N}}_x^{(k)}(h) \tag{3.88}$$

where $\left|\overline{\{0\}}\right| = 3^d$ takes into account the maximal possible overcounting due to the double sum over y and x and the restriction of the x summation to the image of K is justified since for other x must either lie in $\mathcal{D}^{(k)}(h)$ give a zero contribution. Using the induction hypothesis and the definition of $\bar{\mathcal{N}}^{(k)}$, a simple calculation shows now that

$$\sum_{x \in \mathcal{L}\left(\overline{M} \cup K\right) \setminus \mathcal{D}^{(k)}(h)} \bar{\mathcal{N}}_x^{(k)}(h) \leq (1 + C_1^k) \sum_{x \in \mathcal{L}\left(\overline{M} \cup K\right) \setminus \mathcal{D}^{(k)}(h)} \mathcal{N}_x^{(k)}(h) \tag{3.89}$$

from which we get (3.87) if C_1 is chosen for instance $2 \cdot 3^d$, which proves the lemma. $\diamondsuit$

Remark: The bound (3.86) will become relevant in the estimates for the finite temperature case only.

The main properties of the control fields are now given by the following

PROPOSITION 3.3: *Let* $f_d(z) \equiv z^2 \mathbb{I}_{z\geq 1} + z^{\frac{d-2}{d-1}} \mathbb{I}_{u<1}$ *and let* $\gamma_k \equiv L^{k(1-\eta)/2}\sigma^{-2}$. *Then*

$$\mathbb{P}\left[L^{-(d-3/2)k}\sigma > \bar{N}_y^{(k)}(h) > 0\right] = 0 \tag{3.90}$$

and if $z \geq L^{-(d-3/2)k}\sigma$

$$\mathbb{P}\left[\bar{N}_y^{(k)}(h) \geq z\right] \leq e^{-f_d(z)\frac{\delta^2}{\sigma_k^2}}, \quad \textit{if } z \geq L^{-(d-3/2)k}\sigma \tag{3.91}$$

Proof: The proof of this proposition will be by induction over k. Note that it is trivially verified for $k = 0$. Thus we assume (3.90) and (3.91) for k.

Let us first show that (3.90) holds for $k+1$. Notice first that the event under consideration cannot occur if $|\tilde{S}_y^{(k+1)}(h)| > \delta$. Therefore, unless $\bar{N}_y^{(k+1)}(h) = 0$, the site y must lie within $\tilde{D}^{(k+1)}(h)$. But this implies that

$$\mathcal{L}y \cap \left(D^{(k)}(h)\backslash\mathcal{D}^{(k)}(h)\right) \neq \emptyset \tag{3.92}$$

and hence there must exist a $L^{\frac{1}{2}}$-connected component $D_i(h) \subset D^{(k)}(h)$ intersecting $\mathcal{L}y$ that violates one of the conditions of 'smallness' from Definition 3.4. Assume first that *only* condition (iii) is violated. In this case, $D_i(h)$ is so small that it is contained in $\mathcal{L}\bar{y}$ and therefore contributes a term larger than $L^{-(d-2)(k+1)+\alpha}\sigma^2$ to $\bar{N}_y(h)$, and since $\sigma^2 \sim 1/L$, this already exceeds $L^{-(d-3/2)(k+1)}\sigma$. Thus, either condition (i) or (ii) must be violated. In both cases, this implies that the number of sites in $D_i(h)$ exceeds $L^{(1-\alpha)/2}$. The crucial remark here is that if $\sup_{h'\in\mathbb{Z}}\left(\bar{N}_x^{(k)}(h+h') - \frac{1}{8L}|h'|\right) > 0$, then either $\sup_{h'\in\mathbb{Z}}\left(\bar{N}_x^{(k)}(h+h') - \frac{1}{16L}|h'|\right) > \frac{1}{16L}$ or $N_x^{(k)}(h) > 0$. Thus, any site in $D_i(h)$ contributes either at least $\frac{1}{16L}$ or the minimal nonzero value of $N_x^{(k)}(h)$ which by inductive assumption is $L^{-(d-3/2)k}\sigma$. Therefore

$$\bar{N}_y^{(k+1)}(h) \geq L^{-(d-1-\alpha)} \sum_{x\in D_i(h)\cap\mathcal{L}y} L^{-(d-3/2)k}\sigma \geq L^{(1-\alpha)/2}L^{-(d-1-\alpha)}L^{-(d-3/2)k}\sigma = L^{-(d-3/2)(k+1)} \tag{3.93}$$

But this proves (3.90).

To prove (3.91), we show first that $\bar{\mathcal{N}}_x^{(k)}(h)$ satisfies essentially the same bound as $\bar{N}_x^{(k)}(h)$ itself, namely

$$\mathbb{P}\left[\bar{\mathcal{N}}_x^{(k)}(h) \geq z\right] \leq e^{-f_d(z)\frac{\delta^2}{b\sigma_k^2}}, \quad \text{if } z \geq L^{-(d-3/2)k}\sigma \tag{3.94}$$

To see this, notice that

$$\begin{aligned}
I\!P\left[\bar{N}_x^{(k)}(h) \geq z\right] &\leq I\!P\left[\sup_{h'}(\bar{N}_x^{(k)}(h+h') - \frac{1}{16L}|h'|) \geq z\right] \\
&\leq I\!P\left[N_x^{(k)}(h) \geq z\right] + \sum_{|h'|\geq 1} I\!P\left[\bar{N}_x^{(k)}(h+h') \geq z + \frac{1}{16L}|h'|\right] \\
&\leq e^{-f_d(z)\frac{\delta^2}{\sigma_k^2}} + \sum_{|h'|\geq 1} e^{-f_d(z+\frac{1}{16L}|h'|)\frac{\delta^2}{\sigma_k^2}}
\end{aligned} \tag{3.95}$$

Here the second sum converges and is in fact much smaller than the first term which, together with the assumption on the minimal size of z allows to get (3.94) with b slightly bigger than 1.

To complete the proof of (3.91) we need some property of the function f_d. Before stating it, let us point out that it is crucial to have the function $f_d(z)$ rather than simply z^2; namely, our goal is to show that $\bar{N}_x^{(k)}(h)$ is non-zero with very small probability which is true if $f_d(L^{-(d-3/2)k}\sigma)\frac{\delta^2}{\sigma_k^2}$ is large and grows with k. Apparently, this is true if f_d, for small values of its argument, is strictly smaller than linear! The way f_d can be chosen is governed by the following Lemma, as we will see shortly.

LEMMA 3.13: *The function f_d defined in proposition 3.3 satisfies*

$$\sum_{x\in\mathcal{L}\bar{y}} f_d(\bar{N}_x^{(k)}(h)) \geq L^{d-2-\eta} f_d\left(L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}} \bar{N}_x^{(k)}(h)\right) \tag{3.96}$$

Proof: We distinguish the cases $L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}} \bar{N}_x^{(k)}(h) \leq 1$ and $L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}} \bar{N}_x^{(k)}(h) \geq 1$. In the first case, just notice that

$$\begin{aligned}
\sum_{x\in\mathcal{L}\bar{y}} f_d(\bar{N}_x^{(k)}(h)) \geq \sum_{x\in\mathcal{L}\bar{y}} \left(\bar{N}_x^{(k)}(h)\right)^{\frac{d-2}{d-1}} &\geq \left(\sum_{x\in\mathcal{L}\bar{y}} \bar{N}_x^{(k)}(h)\right)^{\frac{d-2}{d-1}} \\
&= L^{(d-1-\alpha)\frac{d-2}{d-1}}\left(L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}} \bar{N}_x^{(k)}(h)\right)^{\frac{d-2}{d-1}} \\
&= L^{d-2-\alpha\frac{d-2}{d-1}} f_d\left(L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}} \bar{N}_x^{(k)}(h)\right)
\end{aligned} \tag{3.97}$$

In the second case, use the Schwarz inequality to get

$$
\begin{aligned}
\sum_{x\in\mathcal{L}\bar{y}} f_d(\bar{N}_x^{(k)}(h)) &\geq \sum_{x\in\mathcal{L}\bar{y}} \left(\bar{N}_x^{(k)}(h)\right)^2 \geq \frac{\left(\sum_{x\in\mathcal{L}\bar{y}} \bar{N}_x^{(k)}(h)\right)^2}{(3L)^d} \\
&= \frac{L^{2(d-1-\alpha)}}{(3L)^d} \left(L^{-(d-1-\alpha)} \sum_{x\in\mathcal{L}\bar{y}} \bar{N}_x^{(k)}(h)\right)^2 \\
&= L^{d-2-2\alpha-\frac{d\ln 3}{\ln L}} f_d \left(L^{-(d-1-\alpha)} \sum_{x\in\mathcal{L}\bar{y}} \bar{N}_x^{(k)}(h)\right)
\end{aligned}
\tag{3.98}
$$

from which the Lemma follows, for $\alpha \geq \frac{d\ln 3}{\ln L}$. $\diamondsuit$

We are now ready to proof (3.91) for $k+1$. Obviously,

$$
\begin{aligned}
I\!P\left[\bar{N}_y^{(k+1)}(h) \geq z\right] &\leq I\!P\left[L^{-(d-1-\alpha)} \sum_{x\in\mathcal{L}\bar{y}\setminus\mathcal{D}(h)} \bar{\mathcal{N}}_x^{(k)}(h) \geq z\right] \\
+ I\!P&\left[L^{-(d-1-\alpha)} \sum_{x\in\mathcal{L}\bar{y}\setminus\mathcal{D}(h)} \bar{\mathcal{N}}_x^{(k)}(h) \geq z-\delta \ \wedge\ |\tilde{S}_y^{(k)}(h)| > \delta\right]
\end{aligned}
\tag{3.99}
$$

Let us consider the first term in (3.99). By Lemma 3.13,

$$
\begin{aligned}
&P\left[L^{-(d-1-\alpha)} \sum_{x\in\mathcal{L}\bar{y}\setminus\mathcal{D}(h)} \bar{\mathcal{N}}_x^{(k)}(h) \geq z\right] = I\!P\left[f_d\left(L^{-(d-1-\alpha)} \sum_{x\in\mathcal{L}\bar{y}\setminus\mathcal{D}(h)} \bar{\mathcal{N}}_x^{(k)}(h)\right) \geq f_d(z)\right] \\
&\leq I\!P\left[\sum_{x\in\mathcal{L}\bar{y}} f_d\left(\bar{\mathcal{N}}_x^{(k)}(h)\right) \geq L^{d-2-\eta} f_d(z)\right]
\end{aligned}
\tag{3.100}
$$

Now the variables $f_d(N_x^{(k)}(h))$ are essentially exponentially distributed in their tails. Moreover, we can bound their Laplace transform by

$$
\begin{aligned}
&I\!E\left(e^{tf_d(N_x^{(k)}(h))}\right) \leq I\!P\left[N_x^{(k)}(h) = 0\right] + e^{-\gamma_k} e^{tf_0} + t\int_{f_0}^{\infty} e^{tf} e^{-f\alpha_k} df \\
&\leq 1 + e^{tf_0-\gamma_k} + t\frac{e^{(t-\alpha_k)f_0}}{\alpha_k - t}
\end{aligned}
\tag{3.101}
$$

where we have set $f_0 \equiv f_d(L^{-(d-3/2)k}\sigma)$ and $\alpha_k \equiv \frac{\delta^2}{\sigma^2}$. We will now bound the Laplace transform *uniformly* for all $t \leq t^* \equiv (1-\epsilon)\alpha_k$, for some small $\epsilon > 0$. Noticing that $\gamma_k \gg (1-\epsilon)\alpha_k f_0$ (check!), we get in this range

$$
I\!E\left(e^{tf_d(N_x^{(k)}(h))}\right) \leq 1 + \frac{1-\epsilon}{\epsilon} e^{-\epsilon f_0 \alpha_k}
\tag{3.102}
$$

Now using the independence of well-separated $\bar{N}_x^{(k)}$, We find

$$
\begin{aligned}
&I\!P\left[\sum_{x\in\mathcal{L}\bar{y}} f_d\left(\bar{\mathcal{N}}_x^{(k)}(h)\right) \geq L^{d-2-\eta} f_d(z)\right] \leq e^{-L^{d-2-\eta} f_d(z)\frac{t^*}{5^d}} I\!E\left(e^{\frac{t^*}{5^d}\sum_{x\in\mathcal{L}\bar{y}} f_d\left(\bar{\mathcal{N}}_x^{(k)}(h)\right)}\right) \\
&\leq e^{-L^{d-2-\eta} f_d(z)\frac{\alpha_k(1-\epsilon)}{5^d}} \left[I\!E\left(e^{t^* f_d\left(\bar{\mathcal{N}}_x^{(k)}(h)\right)}\right)\right]^{\frac{L^d}{5^d}} \\
&\leq e^{-L^{d-2-\eta} f_d(z)\frac{\alpha_k(1-\epsilon)}{5^d}} \left[1+\frac{1-\epsilon}{\epsilon}e^{-\epsilon f_0\alpha_k}\right]^{\frac{L^d}{5^d}}
\end{aligned}
\tag{3.103}
$$

The last factor in (3.103) is in fact close to one and may be absorbed in a constant in the exponent, since we want a bound only for $z \geq L^{-(d-3/2)(k+1)}\sigma$. This gives a bound of the desired form for the first term in (3.99). The second term in (3.99) is simply bounded by the minimum of the probabilities of the two events, making use of Proposition 3.2. This leads to a bound of the same type. We leave it to the readers to check the details themselves or to consult [BK]. The proof of Proposition 3.3 is now finished. $\diamondsuit$

III.6 Construction of the ground states

With the probabilistic estimates on the random fields obtained in the last subsection, we are now ready to make use of the RG transformations to analyse the infinite volume ground states. The first and quite immediate consequence of proposition 3.3 is that for any given point $x \in \mathbb{Z}^d$ it is quite unlikely to be contained in $D(0)$ in any iteration of the RG. Namely, we get from proposition 3.3 the

COROLLARY 3.4 *Let $d \geq 3$, σ^2 small enough. Then, there exists a constant c' (of order unity) such that for any $x \in \mathbb{Z}^d$*

$$
I\!P\left[\exists_{k\geq 0}\ :\ \sup_{h'\in\mathbb{Z}}\left(N_{\mathcal{L}^{-k}x}^{(k)}(h+h')-\frac{1}{8L}|h'|\right)\neq 0\right] \leq \exp\left(-\frac{\delta^2}{c'\sigma^{2-\frac{d-2}{d-1}}}\right) \tag{3.104}
$$

Proof: The supremum in (3.104) is clearly bounded from above by $\bar{\mathcal{N}}_x^{(k)}(h)$. Moreover, $\bar{\mathcal{N}}_x^{(k)}(h)$ is either zero or larger than $L^{-(d-3/2)k}\sigma$. Therefore

$$
\begin{aligned}
I\!P\left[\exists_{k\geq 0}\ :\ \sup_{h'\in\mathbb{Z}}\left(N_{\mathcal{L}^{-k}x}^{(k)}(h+h')-\frac{1}{8L}|h'|\right)\neq 0\right] &\leq I\!P\left[\exists_{k\geq 0}\bar{\mathcal{N}}_{\mathcal{L}^{-k}x}^{(k)}(h)\neq 0\right] \\
&\leq \sum_{k=0}^{\infty}\exp\left(-L^{\left(\frac{d-2}{2(d-1)}-\eta\right)k}\frac{\delta^2}{a\sigma^{2-\frac{d-2}{d-1}}}\right)
\end{aligned}
\tag{3.105}
$$

which gives (3.104) for a suitable constant c'. $\diamondsuit$

Let us denote by $D^{(k)}$, $\mathcal{D}^{(k)}$ the bad regions and 'harmless' bad regions in the k-th level. Set further

$$\Delta^{(k)}(h) \equiv \bigcup_{i=0}^{k} \mathcal{L}^i \overline{\text{ int}\, \mathcal{D}^{(i)}(h)} \tag{3.106}$$

One may keep in mind that the sets $D^{(k)}$ depend in principle on the finite volume in which we are working; however, this dependence is quite weak and only occurs near the boundary. We therefore suppress this dependence in our notations.

In this terminology corollary 3.4 states that even $\Delta^{(\infty)}(h)$ is a very sparse set, for any h. But this statement has an immediate implication for the ground states, via the following

PROPOSITION 3.5: *Let $\Lambda_n \equiv \mathcal{L}^n 0$, and let $\mathcal{G}_{\Lambda_n}^{(0)}$ be defined through (3.1). Then for any $\Gamma^* \in \mathcal{G}_{\Lambda_n}^{(0)}$,*

$$\underline{\Gamma^*} \subset \Delta^{(n-1)}(0) \cup \mathcal{L}^n D^{(n)}(0) \tag{3.107}$$

Proof: Let γ_i^* denote the maximal weakly connected components of Γ^*. It is clear that for all these components $h_{\partial \text{ int} \underline{\gamma_i^*}} = 0$. Let $\tilde{\gamma}_i^*$ denote the 'outer' connected component of γ_i^*, i.e. the connected component of γ_i^* s.t. the supports of all its other connected components are contained in the interior of its support (by the definition of weak connectivity, such a component must exist). If $\tilde{\gamma}_i^*$ is 'small' (in the sense of definition 3.5), since it occurs in a ground state, by Lemma 3.1, it is 'flat' (i.e. $h_x(\tilde{\gamma}_i^*) \equiv 0$) and its support is contained in $\overline{\mathcal{D}(0)}$ (in the first step this set is even empty). Then all the other connected components of γ_i^* are also small, so that γ_i^* is flat and its support is contained in $\overline{\mathcal{D}(0)}$. Thus $\underline{\Gamma^*} \subset \text{ int}\, \underline{\Gamma^{*,l}} \cup \overline{\mathcal{D}^{(0)}(0)}$. On the other hand, $\underline{\Gamma^{*,l}} \subset \mathcal{L}(\underline{\mathcal{R}\Gamma^*})$; again the support of the small components of $\mathcal{R}\Gamma^*$ will be contained, by the same argument in the closure of the small parts of the new bad regions, and so $\underline{\mathcal{R}\Gamma^{*,s}} \subset \overline{\mathcal{D}^{(1)}(0)}$, while $\underline{\mathcal{R}\Gamma^{*,l}} \subset \mathcal{L} \text{ int}\, (\underline{\mathcal{R}^2\Gamma^{*,l}})$. This may be iterated as long as the renormalized contours still have non-empty supports; in the worst case, after n steps, we are left with $\mathcal{R}^n\Gamma^*$, whose support, if not empty, may only consist of the single point 0, and this only if 0 is in the n-th level bad set $D^{(n)}(0)$. But this proves the proposition. $\diamondsuit$

This proposition, together with Corollary 3.4 already gives a very nice information about the ground states, namely that for any site x, the value of the height is zero with probability that is exponentially close to one. From this it is almost evident that a 'flat' ground state in the sense of (3.6) will exist. What is left to show is that the deviations from height zero are locally determined by the randomness and get independent of the size of the volume. The key observation is the

LEMMA 3.14: *Fix $k \geq 0$. If for all $n \geq k+1$*

$$\mathcal{L}^k \overline{\{0\}} \cap \left(\mathcal{L}^n D^{(n)} \bigcup_{i=0}^{k} \mathcal{L}^i \overline{\ int \mathcal{D}^{(i)}(h)} \right) = \emptyset \tag{3.108}$$

then the sets $\mathcal{G}^{(0)}_{\Lambda_n,\Lambda_k}$ are independent of n for all $n \geq k+1$.

Proof: Let $n \geq k+1$ be fixed and Let $\Gamma^* \in \mathcal{G}^{(0)}_{\Lambda_n}$. Loosely speaking the lemma states that if (3.108) holds, then the restriction of Γ^* is n-independent. But indeed the condition (3.108) implies that $\mathrm{int}\, \underline{\mathcal{R}^k\Gamma^*} \cap \overline{0} = \emptyset$ and therefore also $h_x(\mathcal{R}^k\Gamma^*) = 0$ for $x \in \overline{0}$. Therefore all connected components of $I\!R^{k-1}\Gamma^*$ whose support intersects $\mathcal{L}\overline{0}$ are *small* and thus local functions of the disorder, whence independent of n. The same is then obviously true for the components contour Γ^* itself whose support intersect $\mathcal{L}^k 0$. This proves the lemma. $\diamondsuit$

We are now ready to proof the main theorem of this section.

THEOREM 3.1: *Let $d \geq 3$ and σ^2 small enough. Then, for any integer $h \in Z\!\!\!Z$, the set $\mathcal{G}^{(0)}_\infty \neq \emptyset$, $I\!P$-a.s.*

Proof: Let us denote by $\mathcal{E}_{k,n}$ the event (3.107). By Lemma 3.14 it is clear that on the event

$$\mathcal{E} \equiv \bigcup_{n_0\geq 0} \bigcap_{n\geq n_0} \bigcup_{k\geq 0} \mathcal{E}_{k,n} \tag{3.109}$$

$\mathcal{G}^{(0)}_\infty \neq \emptyset$. Thus we just have to show that $\mathcal{E}$ occurs with probability one under the hypothesis of Theorem 3.1. To show this, we consider the complement of the event $\mathcal{E}$. Note first that we can decompose

$$\mathcal{E}^c_{k,n} = \mathcal{B}_{k,n} \cup \bigcup_{i=k+1}^{n-1} \tilde{\mathcal{B}}_{k,i} \tag{3.110}$$

where $\mathcal{B}_{k,n} \equiv \left\{\mathcal{L}^k\overline{0} \cap \mathcal{L}^n D^{(n)}(0) \neq \emptyset\right\}$ and $\tilde{\mathcal{B}}_{k,i} \equiv \left\{\mathcal{L}^k\overline{0} \cap \mathcal{L}^i \overline{\int \mathcal{D}^{(i)}(0)} \neq \emptyset\right\}$. Note that the latter event is strictly independent of n, i.e. on the finite volume, for $i < n$. We have to estimate

$$\begin{aligned} I\!P\left[\mathcal{E}^c\right] &= I\!P\left[\cap_{n_0} \cup_{n\geq n_0} \cup_{k<n} \mathcal{E}_{k,n}\right] \\ &\leq I\!P\left[\cup_k \cup_{n>k} \mathcal{B}_{k,n}\right] + I\!P\left[\cap_k \cup_{i=k+1}^\infty \tilde{\mathcal{B}}_{k,i}\right] \\ &\leq \limsup_{k\uparrow\infty} \sum_{i=k+1}^{\infty} \left(I\!P\left[\mathcal{B}_{k,i}\right] + I\!P\left[\tilde{\mathcal{B}}_{k,i}\right]\right) \end{aligned} \tag{3.111}$$

where in the second line we have used the n-independence of $\tilde{\mathcal{B}}_{k,i}$. It is now straightforward to show that the right hand side of (3.111) equals zero. Just note that

$$\tilde{\mathcal{B}}_{k,i} \subset \left\{\exists_{y\in\mathcal{L}0} \mid \sup_{h'\in Z\!\!\!Z} \left(N^{(i)}_y(h+h') - \frac{1}{8L}|h'|\right) \neq 0\right\} \tag{3.112}$$

The probability of this last event is bounded $L^d \exp\left(-\frac{\delta^2}{c'\sigma_i^{2-\frac{d-2}{d-1}}}\right)$, proceeding just as in the proof of Corollary 3.1 to estimate the probability of this latter event by. The probability of the events $\mathcal{B}_{k,i}$ is estimated in the same way. From this the theorem follows immediately. $\diamondsuit\diamondsuit$

The main result for the $T=0$ case is now proven: There exist 'flat' infinite volume ground states in dimension $d \geq 3$ if the disorder is sufficiently weak. It is, by the way, also clear that this flat ground state is unique under some weak assumptions on the distribution of the J that excludes local degeneracies; continuity, for instance is sufficient (but not necessary). Also, we already know how the ground state looks outside a very sparse region, but so far we have not said anything about how it may look like within that region. To get a precise knowledge of about the ground states in the bad regions would require a more careful analysis of the RG map, keeping track on more parameters in the renormalized contours within the bad region than we do (for example, if in the blocking procedure the height on a block has large fluctuations about its (maybe) small average, we simply forget about them, although they must be accompanied by large excess surface energy. One might thus for instance carry through the estimates an extra parameter keeping track of maximal and minimal heights that occurred in the history of a block. We will not go into such a detailed analysis here). However, even without doing this, the results we already have can give rough estimates on the probability distribution of the height, say at 0 of the ground state contour and we will present them as a last result of this section.

PROPOSITION 3.6: *Let Γ^* be an element of $\mathcal{G}_\infty^{(0)}$. Then,*

$$\mathbb{P}\left[|h_0(\Gamma^*)| \geq h\right] \leq \exp\left(-\frac{h^{\gamma_d}}{\sigma^{2-\mu_d}}\right) \tag{3.113}$$

where γ_d, μ_d are positive, d dependent constants for $d \geq 3$.

Proof: Let us denote by Γ_n^* an element of $\mathcal{G}_{\Lambda_n}^{(0)}$, and let A_n be the event that $h_0(\Gamma^*) = h_0(\Gamma_n^*)$. The point is that $h_0(\Gamma_n^*)$ can be estimated in probability a priori, for n not too large (depending on h), while it is quite unlikely that A_n occurs only for very large n. Thus let $B_n \equiv \cap_{k\geq n}^\infty A_n$. Then, for any n we have that

$$\begin{aligned}\mathbb{P}\left[|h_0(\Gamma^*)| \geq h\right] =&\mathbb{P}\left[|h_0(\Gamma^*)| \geq h \wedge (B_n \vee B_n^c)\right]\\ &\leq \mathbb{P}\left[|h_0(\Gamma_n^*)| \geq h \wedge B_n\right] + \mathbb{P}\left[|h_0(\Gamma^*)| \geq h \wedge B_n^c\right]\\ &\leq \mathbb{P}\left[|h_0(\Gamma_n^*)| \geq h\right] + \mathbb{P}\left[B_n^c\right]\end{aligned} \tag{3.114}$$

The desired bound will be obtained by choosing n in dependence on h such as to minimize the right-hand side of (3.114). Now B_n^c occurs only if the site zero is contained in the interior of the support of a connected component of Γ^* that depasses Λ_n. By proposition 3.5 and the estimates used in the proof of the theorem, it is clear that the probability of this to happen is bounded by

$$\mathbb{P}\left[B_n^c\right] \leq \exp\left(-L^{\left(\frac{d-2}{2(d-1)}-\eta'\right)n}\frac{\delta^2}{\sigma^{2-\frac{d-2}{d-1}}}\right) \tag{3.115}$$

To bound the other contribution in (3.114), notice that for any possible height-function that vanishes outside Λ_n, by Lemma 3.3,

$$E_s(h(\Gamma_n^*)) \geq \frac{|h_0(\Gamma_n^*)|}{2}\frac{d}{L^n}\sum_{x\in\mathcal{L}^n 0}|h_x(\Gamma_n^*)| \tag{3.116}$$

Therefore,

$$\mathbb{P}\left[|h_0(\Gamma_n^*)| \geq h\right] \leq \mathbb{P}\left[\sum_{x\in\mathcal{L}^n 0} \inf_{h_x\in\mathbb{Z}} \left(\frac{d}{L^n}|h_x| + J_x(h_x) - J_x(0)\right) < -\frac{h}{2}\right] \tag{3.117}$$

Let us set

$$I_x^{(n)} \equiv \inf_{h\in\mathbb{Z}} \left(\frac{d}{L^n}|h| + J_x(h) - J_x(0)\right) \tag{3.118}$$

Given the bounds on J, it is easy to show that

$$\mathbb{P}\left[I_x^{(n)} \leq -z\right] \leq \frac{\pi\sigma L^n}{d} \exp\left(-\frac{z^2}{4\sigma^2}\right) \tag{3.119}$$

which in turn is bounded by $\exp\left(-\frac{z^2}{8\sigma^2}\right)$, for $z \geq z_0 \equiv \sigma\sqrt{8\ln(\frac{2\pi\omega L^n}{d})}$. From that a simple computation shows that

$$0 \geq \mathbb{E}(I_x^{(n)}) \geq -\sigma\sqrt{8\ln\frac{2\pi\omega L^n}{d}} \tag{3.120}$$

and with $\tilde{I}_x^{(n)} \equiv I_x^{(n)} - \mathbb{E}(I_x^{(n)})$,

$$\begin{aligned} &\mathbb{P}\left[\tilde{I}_x^{(n)} \leq -z\right] \leq \exp\left(-\frac{z^2}{8\sigma^2}\right) \\ &\mathbb{P}\left[\tilde{I}_x^{(n)} \geq z\right] = 0 \leq \exp\left(-\frac{z^2}{8\sigma^2}\right) \end{aligned} \tag{3.121}$$

for $z \geq z_0$. Using essentially Lemma 3.11, some rescaling and some rather rough overestimation, we get from this that

$$\mathbb{P}\left[\sum_{x\in\mathcal{L}^n 0} \tilde{I}_x^{(n)} \leq -z\right] \leq \begin{cases} \exp\left(-\frac{z^2}{8\sigma^2 L^{nd}}\right), & \text{if } z_0 \leq 1 \\ \exp\left(-\frac{z^2}{8\sigma^2 L^{nd} z_0^2}\right), & \text{if } z_0 \geq 1 \end{cases} \tag{3.122}$$

and so

$$\mathbb{P}\left[\sum_{x\in\mathcal{L}^n 0} I_x^{(n)} \leq -\frac{h}{2}\right] \leq \begin{cases} \exp\left(-\frac{(h/2-L^{dn}z_0)^2}{8\sigma^2 L^{nd}}\right), & \text{if } z_0 \leq 1 \\ \exp\left(-\frac{(h/2-L^{nd}z_0)^2}{8\sigma^2 L^{nd} z_0^2}\right), & \text{if } z_0 \geq 1 \end{cases} \tag{3.123}$$

We see that to make use of this, we must choose n small enough, s.t. $h/2 \geq L^{dn}z_0$. The optimal choice of n is now in principal found by equating the bound from (3.123) with the probability of B_n^c; a rough estimate of the yields the solution

$$\mathbb{P}\left[|h_0(\Gamma^*)| \geq h\right] \leq \exp\left(-\frac{h^{\rho/d}\delta^2}{\sigma^{1+\rho/d-\gamma}\left(\ln(\frac{2\pi\sigma L^n}{d})\right)^{\rho/d}}\right) \tag{3.124}$$

with $\rho = \frac{d-2}{2(d-1)} - \eta'$ and $\gamma = \frac{d-2}{d-1}$ which proves the proposition. $\diamond$

With this bound on the height we conclude our analysis of the ground state. It should be clear that further and more detailed information can in principle be extracted from the RG and our estimates. The task of the next section will be to carry over these results to the finite temperature case and the Gibbs measures.

IV. The Gibbs states at finite temperature

In this section we repeat the construction and analysis of the renormalization maps from Section 3 for the finite temperature Gibbs measures. The steps will follow closely those of the previous section and we will be able to make use of many of the results obtained there. In fact, no new 'serious' problems will have to be dealt with here, and in particular the probabilistic analysis of Section 3.5 will mostly carry over. The difficulties here lie mostly in the technicalities of the various expansions that we will have to use. Our main emphasis will be on the proper set-up of a RG invariant parameterization of the contour measures.

IV.1 Set-up and inductive assumptions

Just as in Section 3 an object of crucial importance will be the control field $N_x(h)$. Given such a field, the bad region $D \equiv D(N)$ is defined exactly as in Definition 3.1.

Analogously to Definition 3.2 we now define an N-bounded contour measure:

DEFINITION 4.1: *A N-bounded contour measure is a probability measure on $\Omega_n(D)$ of the form*

$$\mu(\Gamma) = \frac{1}{Z} e^{-\beta(S,V(\Gamma))} \sum_{\Lambda_n \supset G \supset \underline{\Gamma}} \rho(\Gamma, G) \tag{4.1}$$

where

(i) S is a non-local small random field, that is a map that assign to each connected (non-empty) set $C \subset \Lambda_n$ and each height h a real number $S_C(h)$ that satisfy

$$|S_C(h)| \leq e^{-\tilde{\beta}|C|}, \quad \textit{if} \quad |C| > 1 \tag{4.2}$$

and for sets made of a single point x,

$$|S_x(h)| \leq \delta \tag{4.3}$$

The notion $(S, V(\Gamma))$ is shorthand for

$$(S, V(\Gamma)) \equiv \sum_{h \in \mathbb{Z}} \sum_{C \subset V_h(\Gamma)} S_C(h) \tag{4.4}$$

(ii) $\rho(\Gamma, G)$ are positive activities factorizing over connected components of G, i.e. if $(G_1, \ldots, G_l)$ are the connected components of G and if Γ_i denotes the contour made from those connected components of Γ those supports are contained in G_i, then

$$\rho(\Gamma, G) = \prod_{i=1}^{l} \rho(\Gamma_i, G_i) \tag{4.5}$$

where it is understood that $\rho(\Gamma, G) = 0$ *if* $\underline{\Gamma} = \emptyset$. *They satisfy the upper bound*

$$0 \leq \rho(\Gamma, G) \leq e^{-\beta E_s(\Gamma) - \tilde{\beta}|G \setminus \overline{D(\Gamma)}| + \beta B(N, V(\Gamma) \cap \underline{\Gamma}) + A|G \cap D(\Gamma)|} \tag{4.6}$$

Let $C \subset D(h)$ *be connected and* $\gamma = (C, h_x(\Gamma) \equiv h)$ *be a connected component of a contour* $\Gamma \subset \Omega_n(D)$. *Then*

$$\rho(\gamma, C) \geq e^{-\beta(N, V(\gamma) \cap C)} \tag{4.7}$$

Z is of course the partition function that turns μ *into a probability measure.*

Here β and $\tilde{\beta}$ are parameters ('temperatures') that will be renormalized in the course of the iterations. In the k-th level, they will be shown to behave as $\beta^{(k)} = L^{(d-1-\alpha)k}$ and $\tilde{\beta}^{(k)} = L^{(1-\alpha)k}$. B and A are further k-dependent constants. B will actually be chosen close to 1, i.e. with $B = 1$ in level $k = 0$ we can show that in all levels $1 \leq B \leq 2$. A is close to zero, in fact $A \sim e^{-\tilde{\beta}^{(k)}}$. These constants are in fact quite irrelevant, but cannot be completely avoided for technical reasons.

Note that we have not adorned the μ and ρ's with all these parameters as indices, nor with the finite volumes Λ_n, although of course they depend on these parameters as well as on others, in order to keep notations as streamlined as possible.

We must remark on some differences between our assumptions and those used in [BK]. Loosely speaking, the sets G are what [BK] call the 'outer supports'; however, in their method, a renormalization of the normal supports is not maintained. They are, in fact, forgotten after each RG step and the outer supports become the new inner supports, while a new outer support is created. This allows to perform the RG really only on spin configurations but not on contours. We felt that a formulation that allows to renormalize contour models more appealing, particularly in view of the analysis of the ground states. In fact, in the limit as $T \downarrow 0$, our contours tend to the ground state contours, while the sets G completely disappear. Also, [BK] keep track of an extra non-local interaction, called $W(\Gamma)$. It turns out this is unnecessary and disturbing.

The probabilistic assumptions on stationarity and locality of the quantities appearing here are completely analogous to those in Section III and we will not restate them; all quantities depending on sets C are of course supposed to be measurable w.r.t. $\mathcal{B}_{\overline{C}}$, etc.

The definition of a proper RG transformation will now be adopted to this set-up.

DEFINITION 4.2: *For a given control field* N, *a proper renormalization group transformation,* $\mathcal{R}^{(N)}$, *is a map from* $\Omega_n(D(N))$ *into* $\Omega_{n-1}(D(N'))$, *such that if* μ *is a* N*-bounded contour measure on* $\Omega_n(D(N))$ *with 'temperatures'* β *and* $\tilde{\beta}$ *and small field* S *(of level* k*), then* $\mu'_{\Lambda_{n-1}} \equiv \mathcal{R}^{(N)} \mu_{\Lambda_n}$ *is a* N'*-bounded contour measure on* $\Omega_{n-1}(D(N')$ *for some control field* N', *with temperatures* β' *and* $\tilde{\beta}'$ *and small field* S' *(of level* $k+1$*).*

IV.2 Absorption of small contours

The construction of the map T_1 on the level of contours proceeds now exactly as before, i.e. Definition 3.4 still defines the harmless large field region, Definition 3.5 the 'small' contours and Definition 3.6 the map T_1. What we have to do is to control the induced action of T_1 on the contour measures. Let us for convenience denote by $\hat{\mu} \equiv Z\mu$ the non-normalized measures; this only simplifies notations since T_1 leave the partition functions invariant (i.e. $T_1\mu = \frac{1}{Z}T_1\hat{\mu}$).

Of course we have for any $\Gamma^l \in \Omega_N^l(D)$

$$
\begin{aligned}
(T_1\hat{\mu})(\Gamma^l) &\equiv \sum_{\Gamma:T_1(\Gamma)=\Gamma^l} \hat{\mu}(\Gamma) \\
&= \sum_{\Gamma:T_1(\Gamma)=\Gamma^l} e^{-\beta(S,V(\Gamma))} \sum_{G\supset\underline{\Gamma}} \rho(\Gamma, G)
\end{aligned}
\tag{4.8}
$$

Now we write

$$
(S, V(\Gamma)) = (S, V(\Gamma^l)) + \left[(S, V(\Gamma)) - (S, V(\Gamma^l))\right] \tag{4.9}
$$

Here the first term is of course what we would like to have; the second reads explicitly

$$
\begin{aligned}
\left[(S, V(\Gamma)) - (S, V(\Gamma^l))\right] &= \sum_{h\in\mathbb{Z}} \left[\sum_{x\in V_h(\Gamma)\cap \operatorname{int}\underline{\Gamma^s}} S_x(h) - \sum_{x\in V_h(\Gamma^l)\cap \operatorname{int}\underline{\Gamma^s}} S_x(h)\right] \\
&+ \sum_{h\in\mathbb{Z}} \left[\sum_{\substack{C\subset V_h(\Gamma)\\ C\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset}} S_C(h) - \sum_{\substack{C\subset V_h(\Gamma^l)\\ C\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset}} S_C(h)\right] \\
&\equiv \delta S_{loc}(\Gamma, \Gamma^l) + \delta S_{nl}(\Gamma, \Gamma^l)
\end{aligned}
\tag{4.10}
$$

where we used the suggestive notation $\underline{\Gamma^s} \equiv \underline{\Gamma}\backslash\underline{\Gamma^l}$. Note also that all sets C are assumed to have volume at least 2 and are assumed to be connected. The conditions on C (resp. x) to intersect $\underline{\Gamma^s}$ just make manifest that otherwise the two contributions cancel. Thus all these unwanted terms are attached to the supports of the 'small' components of Γ. The local piece, δS_{loc} thus poses no particular problem. The non-local piece, however, may join up 'small' and 'large' components, which spoils the factorization properties of ρ. To overcome this difficulty, we apply a cluster-expansion, a trick that will be used again later. It is useful to introduce the notation

$$
\tilde{\sigma}_{\Gamma,\Gamma^l}(C) \equiv \sum_{h\in\mathbb{Z}} S_C(h)\left(\mathbb{1}_{C\subset V_h(\Gamma)} - \mathbb{1}_{C\subset V_h(\Gamma^l)}\right) \tag{4.11}
$$

so that

$$
\delta S_{nl}(\Gamma, \Gamma^l) = \sum_{C\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset} \tilde{\sigma}_{\Gamma,\Gamma^l}(C) \tag{4.12}
$$

Unfortunately the $\tilde{\sigma}_{\Gamma,\Gamma^l}(C)$ have arbitrary signs. Therefore expanding $\exp(-\beta\delta S_{nl})$ directly would produce a polymer systems with possibly negative activities (see below). However, by assumption,

$$|\tilde{\sigma}_{\Gamma,\Gamma^l}(C)| \leq 2 \sup_{h\in\mathbb{Z}} |S_C(h)| \leq 2e^{-\tilde{\beta}|C|} \equiv f(C) \tag{4.13}$$

Therefore, $\tilde{\sigma}_{\Gamma,\Gamma^l}(C) - f(C) \leq 0$ and setting

$$F(\text{ int }\underline{\Gamma^s}) \equiv \sum_{C\cap \text{int }\underline{\Gamma^s}\neq\emptyset} f(C) \tag{4.14}$$

we get

$$e^{-\beta\delta S_{nl}(\Gamma,\Gamma^l)} = e^{-\beta F(\text{int }\underline{\Gamma^s})} e^{\beta\sum_{C\cap \text{int }\underline{\Gamma^s}\neq\emptyset}(f(C)-\tilde{\sigma}_{\Gamma,\Gamma^l}(C))} \tag{4.15}$$

where the second exponential could be expanded in a sum over positive activities. The first exponential is not yet quite what we would like, since it does not factor over connected components. However, it is dominated by such a term, and the remainder may be added to the σ-terms. This will follow from the next Lemma.

LEMMA 4.1: *Let $A \subset \mathbb{Z}^d$ and let $(A_1,\dots,A_l)$ be its connected components. Let $F(A)$ be as defined in (4.15) and set*

$$\delta F(A) \equiv F(A) - \sum_{i=1}^{l} F(A_i) \tag{4.16}$$

Then

$$\delta F(A) = -\sum_{C\cap A\neq\emptyset} k(A,C)f(C) \tag{4.17}$$

where

$$0 \leq k(A,C)f(C) \leq e^{-\tilde{\beta}(1-\kappa)|C|} \tag{4.18}$$

for $\kappa = \tilde{\beta}^{-1}$

Proof: The proof of this lemma is very simple. Obviously, $\sum_{i=1}^{l} F(A_i)$ counts all C that intersect k connected components of A exactly k times, whereas in $F(A)$ such a C appears only once. Thus (4.17) holds with $k(A,C) = \#\{A_i : A_i\cap C \neq \emptyset\} - 1$. Furthermore, if C intersects k components, then certainly $|C| \geq k$, from which the upper bound in (4.18) follows. $\diamondsuit$

Now we can bring the non-local terms in their final form:

LEMMA 4.2: *Let $\delta S_{nl}(\Gamma,\Gamma^l)$ be defined in (4.10). Then*

$$\begin{aligned} e^{-\beta\delta S_{nl}(\Gamma,\Gamma^l)} &= r(\underline{\Gamma^s}) \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\substack{C_1,\dots,C_l \\ C_i\cap \text{int }\underline{\Gamma^s}\neq\emptyset \\ C_i\neq C_j}} \prod_{i=1}^{l} \phi_{\Gamma,\Gamma^l}(C_i) \\ &\equiv r(\underline{\Gamma^s}) \sum_{\mathcal{C}:\mathcal{C}\cap \text{int }\underline{\Gamma^s}\neq\emptyset} \phi_{\Gamma,\Gamma^l}(\mathcal{C}) \end{aligned} \tag{4.19}$$

where $\phi_{\Gamma,\Gamma^l}(C)$ *satisfies*

$$0 \leq \phi_{\Gamma,\Gamma^l}(C) \leq e^{-\tilde{\beta}|C|/2} \tag{4.20}$$

$r(\underline{\Gamma^s})$ *is a non-random positive activity factoring over connected components of* int$\underline{\Gamma^s}$*; for a weakly connected component* γ^s,

$$1 \geq r(\underline{\gamma^s}) \equiv e^{-\beta F(\operatorname{int}\underline{\gamma^s})} \geq e^{-\beta|\operatorname{int}\underline{\gamma^s}|e^{-a\tilde{\beta}}} \tag{4.21}$$

with some constant $0 < a < 1$.

Proof: Define for $|C| \geq 2$

$$\sigma_{\Gamma,\Gamma^l}(C) \equiv \tilde{\sigma}_{\Gamma,\Gamma^l}(C) - f(C)(k(\operatorname{int}\underline{\Gamma^s}, C) + 1) \tag{4.22}$$

Then we may write

$$\begin{aligned} e^{-\beta \sum_{C\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset} \sigma_{\Gamma,\Gamma^l}(C)} &= \prod_{C\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset} \left(e^{-\beta\sigma_{\Gamma,\Gamma^l}(C)} - 1 + 1\right) \\ &= \sum_{l=0}^{\infty} \sum_{\substack{C_1,\ldots,C_l \\ C_i\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset \\ C_i\neq C_j}} \prod_{i=1}^{l} \left(e^{-\beta\sigma_{\Gamma,\Gamma^l}(C_i)} - 1\right) \end{aligned} \tag{4.23}$$

which gives (4.19). But since $|\sigma_{\Gamma,\Gamma^l}(C)| \leq 2e^{-\tilde{\beta}(1-\kappa)|C|}$ by (4.18) and the assumption on $S_C(h)$, (4.20) follows if only $2\beta \leq e^{\tilde{\beta}(1-2\kappa)/2}$. Let us remark that given the behaviour of β and $\tilde{\beta}$ as given in the remark after Definition 4.2, this relation holds if it holds initially. The initial choice will be $\tilde{\beta} = \beta/L$, and with this relation we must only choose β large enough, e.g. $\beta \geq L(\ln L)^2$ will do.

The properties of $r(\underline{\Gamma^s})$ follow from Lemma 4.1. Note that these activities depend only on the geometry of the support of Γ^s and are otherwise non-random. $\diamondsuit$

We can now write

$$\begin{aligned} (T_1\hat{\mu})(\Gamma^l) &= e^{-\beta(S,V(\Gamma^l))} \sum_{\Gamma:T_1(\Gamma)=\Gamma^l} r(\underline{\Gamma^s}) \sum_{G\supset\underline{\Gamma}} \rho(\Gamma,G) e^{-\beta\delta S_{loc}(\Gamma,\Gamma^l)} \sum_{\mathcal{C}:\mathcal{C}\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset} \phi_{\Gamma,\Gamma^l}(\mathcal{C}) \\ &= e^{-\beta(S,V(\Gamma^l))} \sum_{\Gamma:T_1(\Gamma)=\Gamma^l} \sum_{K\supset\underline{\Gamma}} \sum_{\underline{\Gamma}\subset G\subset K} \sum_{\substack{\mathcal{C}\subset K \\ \mathcal{C}\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset \\ \mathcal{C}\cup G=K}} \\ &\quad \times r(\underline{\Gamma^s})\rho(\Gamma,G)e^{-\beta\delta S_{loc}(\Gamma,\Gamma^l)}\phi_{\Gamma,\Gamma^l}(\mathcal{C}) \end{aligned} \tag{4.24}$$

Now we may decompose the set K into its connected components and call K_1 the union of those components that contain components of $\underline{\Gamma^l}$. Naturally we call $K_2 = K\backslash K_1$. Note that everything factorizes over these two sets, including the sum over Γ (the possible small contours that can be inserted into Γ^l being independent from each other in these sets). We can make this explicit by writing

$$
\begin{aligned}
(T_1\hat{\mu})(\Gamma^l) = e^{-\beta(S,V(\Gamma^l))} & \sum_{K_1 \supset \underline{\Gamma^l}} \sum_{\Gamma_1 : T_1(\Gamma_1)=\Gamma^l} \sum_{\underline{\Gamma_1} \subset G_1 \subset K_1} \sum_{\substack{\mathcal{C}_1 \subset K_1 \\ \mathcal{C}_1 \cap \operatorname{int} \underline{\Gamma_1^s} \neq \emptyset \\ \mathcal{C}_1 \cup G_1 = \overline{K_1}}} \\
& r(\underline{\Gamma_1^s})\rho(\Gamma_1, G_1) e^{-\beta\delta S_{loc}(\Gamma_1,\Gamma^l)} \phi_{\Gamma_1,\Gamma^l}(\mathcal{C}_1) \\
& \times \sum_{K_2 : K_2 \cap \overline{K_1} = \emptyset} \sum_{\Gamma_2 : T_1(\Gamma_2)=\Gamma^l} \sum_{\underline{\Gamma_2} \subset G_2 \subset K_2} \sum_{\substack{\mathcal{C}_2 \subset K_2 \\ \mathcal{C}_2 \cap \operatorname{int} \underline{\Gamma_2^s} \neq \emptyset \\ \mathcal{C}_2 \cup G_2 = \overline{K_2}}} \\
& r(\underline{\Gamma_2^s})\rho(\Gamma_2, G_2) e^{-\beta\delta S_{loc}(\Gamma_2,\Gamma^l)} \phi_{\Gamma_2,\Gamma^l}(\mathcal{C}_2) \\
& \equiv e^{-\beta(S,V(\Gamma^l))} \sum_{K_1 \supset \underline{\Gamma^l}} \hat{\rho}(\Gamma^l, K_1) \sum_{K_2 : K_2 \cap \overline{K_1} = \emptyset} \tilde{\rho}(\Gamma^l, K_2)
\end{aligned}
\tag{4.25}
$$

Here, of course, the contours Γ_1 and Γ_2 are understood to have small components with supports only within the sets K_1 and K_2, respectively. Also, of course, the set K_2 must contain $D(\Gamma^l) \cap K_1^c$. Now the final form of (4.25) is almost the original one, except for the sum over K_2. This latter will give rise to an additional (non-local) field term, as we will explain now.[6]

Notice that the sum over K_2 can be factored over the connected components of K_1^c. In these components, $\tilde{\rho}$ depends on Γ^l only through the (constant) height $h(\Gamma^l)$ in this component. Let Y denote such a connected component and let h be the corresponding height. We have

LEMMA 4.3: *Let $\tilde{\rho}$ be defined in (4.25). Then*

$$
\sum_{\mathcal{D}(h) \cap Y \subset K \subset Y} \tilde{\rho}(\Gamma^l, K) = \left(\prod_i \tilde{\rho}'(h, \mathcal{B}_i^Y(h)) \right) e^{-\beta \sum_{C \subset Y} \bar{\psi}_C(h) - \beta \sum_{C \subset Y, \overline{C} \cap Y^c \neq \emptyset} \psi_C^s(Y,h)}
\tag{4.26}
$$

Here the $\mathcal{B}_i^Y(h)$ denote the connected components of the set $\mathcal{B}^Y(h) \equiv \overline{\mathcal{D}(h)} \cap Y = \mathcal{B}(\Gamma^l) \cap Y$ in Y. The sum over C is over connected sets such that $C \backslash \overline{D(h)} \neq \emptyset$.

The fields $\bar{\psi}_C(h)$ are independent of Y and Γ^l, as the geometry-dependent boundary contributions are made explicit in the contributions $\psi_C^s(Y,h)$. Moreover, there exists a strictly positive constant $1 > g > 0$, such that

$$
\begin{aligned}
\left|\bar{\psi}_C(h)\right| &\leq e^{-g\tilde{\beta}|C \backslash \overline{D(h)}|} \quad \text{and} \\
|\psi_C^s(Y,h)| &\leq e^{-g\tilde{\beta}|C \backslash \overline{D(h)}|}
\end{aligned}
\tag{4.27}
$$

and a constant $C_{19} > 0$ such that

$$
\left| \frac{1}{\beta} \ln \left(\tilde{\rho}'(h, \mathcal{B}_i^Y(h)) \right) \right| \leq B \sum_{x \in \mathcal{D}_i(h)} N_x(h) + \frac{C_{19}}{\beta} |\mathcal{B}_i^Y(h)|
\tag{4.28}
$$

---

[6] The fact that a non-local field is produced here even then initially no such field is present is of course the reason to include such fields in the inductive assumptions

Proof: Naturally, the form (4.26) will be obtained through a Mayer-expansion. That is as usual the connected components of K will be considered as polymers subjected to a hard-core interaction. However, an extra complication arises in the present situation due to the fact that these polymers are further constrained by the condition that their union must contain the set $\mathcal{D}(h) \cap Y$. Thus we define the set $\mathcal{G}(Y,h)$ of permissible polymers through

$$\mathcal{G}(Y,h) = \{K \subset Y, \text{conn.}, K \cap \mathcal{B}(\Gamma^l) = \cup_{\mathcal{B}_i^Y(h) \cap K \neq \emptyset} \mathcal{B}_i^Y(h)\} \tag{4.29}$$

That is to say, any polymer in this set will *contain* all the connected components of $\overline{\mathcal{D}(h)} \cap Y$ it intersects. For such polymers we define the activities

$$\begin{aligned} \tilde{\rho}'(h,K) = & \sum_{\substack{\tilde{K}: K \cap \mathcal{D}(h) \subset \tilde{K} \\ \tilde{K} \setminus \mathcal{B}^Y(h) = K \setminus \mathcal{B}^Y(h)}} \sum_{\Gamma: T_1(\Gamma) = (\emptyset, h)} \\ & \sum_{\underline{\Gamma} \subset G \subset \tilde{K}} \sum_{\substack{\mathcal{C} \subset \tilde{K} \\ \mathcal{C} \cap \operatorname{int} \underline{\Gamma}^s \neq \emptyset \vee \mathcal{C} = \emptyset \\ \mathcal{C} \cup G = \tilde{K}}} e^{-\beta \sum_{x \in \underline{\Gamma}} [S_x(h_x(\Gamma)) - S_x(h)]} r(\underline{\Gamma}) \rho(\Gamma, G) \phi_{\Gamma,h}(\mathcal{C}) \end{aligned} \tag{4.30}$$

Note that by summing over $\tilde{K}$ we collect all polymers that differ only within $\mathcal{B}^Y(h)$. Thus we get

$$\begin{aligned} & \sum_{\mathcal{D}(h) \cap Y \subset K \subset Y} \tilde{\rho}(\Gamma^l, K) \\ & = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\substack{K_1,\dots,K_N : K_i \in \mathcal{G}(Y,h) \\ \bigcup_{i=1}^N K_i \supset \mathcal{B}(\Gamma^l)}} \prod_{i=1}^N \tilde{\rho}'(K_i) \prod_{1 \le i < j \le N} \mathbb{1}_{\overline{K_i} \cap K_j = \emptyset} \end{aligned} \tag{3.31}$$

Next we have to extract the contributions of those polymers that can occur in the ground states. We set

$$\bar{\rho}(h,K) = \frac{\tilde{\rho}'(h,K)}{\prod_{\mathcal{B}_i^Y(h) \subset K} \tilde{\rho}'(h, \mathcal{B}_i^Y(h))} \tag{4.32}$$

Then $\bar{\rho}(h, \mathcal{B}_i^Y(h)) = 1$, i.e. the $\mathcal{B}_i^Y(h)$ play the role of the empty polymer. This procedure allows us to remove the restriction $\bigcup_{i=1}^N K_i \supset \mathcal{B}(\Gamma^l)$ in the following way. Set

$$\mathcal{G}'(Y,h) = \mathcal{G}(Y,h) \setminus \{\mathcal{B}_i^Y(h), i, h \in \mathbb{Z}\} \tag{4.33}$$

Then each polymer K_i is either in $\mathcal{G}'(Y,h)$ or is one of the $\mathcal{B}_i^Y(h)$. Moreover, once all the $K_i \in \mathcal{G}'(Y,h)$ are chosen, the hard-core interaction plus the constraint $\bigcup_{i=1}^N K_i \supset \mathcal{B}(\Gamma^l)$ fix the remaining K_i uniquely up to permutations. Since their activities $\bar{\rho}$ are equal to one, the entire sum over these polymers outside $\mathcal{G}'(Y,h)$ just contributes a factor 1. Therefore we get finally

$$\begin{aligned} & \sum_{\mathcal{D}(h) \cap Y \subset K \subset Y} \tilde{\rho}(\Gamma^l, K) \\ & = \left(\prod_{\mathcal{B}_i^Y(h) \subset Y} \tilde{\rho}'(h, \mathcal{B}_i^Y(h)) \right) \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\substack{K_1,\dots,K_N \\ K_i \in \mathcal{G}'(Y,h)}} \prod_{i=1}^N \bar{\rho}(h, K_i) \prod_{1 \le i < j \le N} \mathbb{1}_{\overline{K_i} \cap K_j = \emptyset} \end{aligned} \tag{4.34}$$

This is now (up to the prefactor) the standard form of a polymer partition function with hard-core interaction. It can be exponentiated and yields the estimates of the lemma provided we get the bound

$$\bar{\rho}(h,K) \leq e^{-c_2\tilde{\beta}|K\setminus\overline{\mathcal{D}(h)}|} \tag{4.35}$$

on the activities (we will not present the details of the Mayer expansion which from now on are entirely standard and can be found e.g. in [Br]).

To prove the bounds (4.35) note first that

$$\tilde{\rho}'(h,\mathcal{B}_i^Y(h)) \geq r(\mathcal{D}_i(h))\rho((\mathcal{D}_i(h),h_x\equiv h),\mathcal{D}_i(h)) \geq e^{-\beta|\mathcal{D}_i(h)|e^{-a\tilde{\beta}}-\beta B\sum_{x\in\mathcal{D}_i(h)}N_x(h)} \tag{4.36}$$

Note that this already 'half' of the bound (4.28) (using that $e^{-a\tilde{\beta}} \leq \frac{C_{19}}{\beta}$) but we also use it to bound the denominators in the activities $\bar{\rho}$. This yields

$$\begin{aligned}\bar{\rho}(h,K) \leq e^{\beta|\mathcal{D}_i(h)\cap K|e^{-a\tilde{\beta}}+\beta B\sum_{x\in\mathcal{D}_i(h)\cap K}N_x(h)} &\sum_{\substack{\tilde{K}:K\cap\mathcal{D}(h)\subset\tilde{K}\\ \tilde{K}\setminus\mathcal{B}^Y(h)=K\setminus\mathcal{B}^Y(h)}}\ \sum_{\Gamma:T_1(\Gamma)=(\emptyset,h)}\\ \times\sum_{\underline{\Gamma}\subset G\subset\tilde{K}}\ \sum_{\substack{\mathcal{C}\subset\tilde{K}\\ \mathcal{C}\cap\operatorname{int}\underline{\Gamma}^s\neq\emptyset\vee\mathcal{C}=\emptyset\\ \mathcal{C}\cup G=\tilde{K}}} &e^{-\beta\sum_{x\in\underline{\Gamma}}[S_x(h_x(\Gamma))-S_x(h)]}r(\underline{\Gamma})\rho(\Gamma,G)\phi_{\Gamma,h}(\mathcal{C})\end{aligned} \tag{4.37}$$

We will first estimate the sum over $\mathcal{C}$. Such an estimate will be needed again later and we state it in the form of

LEMMA 4.4: *For $\tilde{\beta}$ large enough, there exists a finite constant $0<g<1$,*

$$\sum_{\substack{\mathcal{C}\subset K,\mathcal{C}\cup G=K\\ \mathcal{C}\cap\operatorname{int}\underline{\Gamma}\neq\emptyset}}\phi_{\Gamma,h}(\mathcal{C}) \leq e^{|\operatorname{int}\underline{\Gamma}|e^{-g\tilde{\beta}}}e^{-g\tilde{\beta}|K\setminus G|} \tag{4.38}$$

Proof: Set $K\setminus G = M$. We will only use the fact that in this case $\mathcal{C}$ must have at least volume $|M|$. This gives

$$\begin{aligned}\left|\sum_{\substack{\mathcal{C}\subset K,\mathcal{C}\cup G=K\\ \mathcal{C}\cap\operatorname{int}\underline{\Gamma}^s\neq\emptyset\vee\mathcal{C}=\emptyset}}\phi_{\Gamma,h}(\mathcal{C})\right| &\leq \sum_{l=0}^{\infty}\frac{1}{l!}\sum_{\substack{C_1,\dots,C_l,|C_i|\geq 2\\ C_i\cap\operatorname{int}\underline{\Gamma}^s\neq\emptyset,C_i\subset K\\ \cup_{i=1}^l C_i\cup G=K}}\prod_{i=1}^{l}e^{-\tilde{\beta}|C_i|/2}\\ &\leq \sum_{l=0}^{\infty}\frac{1}{l!}|\operatorname{int}\underline{\Gamma}^s|^l\sum_{\substack{v_1,\dots,v_l\geq 2\\ \sum_i v_i\geq|M|}}\prod_{i=1}^{l}b^{v_i}e^{-\tilde{\beta}v_i/2}\\ &\leq \inf_{t\geq 0}e^{-t|M|}\sum_{l=0}^{\infty}\frac{1}{l!}|\operatorname{int}\underline{\Gamma}^s|^l\sum_{v_1,\dots,v_l\geq 2}\prod_{i=1}^{l}b^{v_i}e^{-\tilde{\beta}v_i/2}e^{tv_i}\\ &= \inf_{t\geq 0}e^{-t|M|}\exp\left(|\operatorname{int}\underline{\Gamma}^s|\frac{\left(be^{-\tilde{\beta}/2}e^t\right)}{1-be^{-\tilde{\beta}/2+t}}\right)\end{aligned} \tag{4.39}$$

Here b is some dimension-dependent constant such that b^v bounds the number of connected subsets of volume v that contain a specified point. The idea to introduce $e^{-t|M|+t\sum_i v_i} \geq 1$ into the constrained sum and to then remove the constraint is of course nothing but an exponential Markov inequality. From the last line of (4.39) one gets (4.38) easily, e.g. by bounding the infimum over t by the value of the function for $t = \tilde{\beta}/4$. $\diamondsuit$

We are now ready to estimate the sum over all the non-local terms in the definition of $\tilde{\rho}$. Remembering the assumption (4.8), we get that

$$
\begin{aligned}
&\sum_{\underline{\Gamma} \subset G \subset \tilde{K}} \sum_{\substack{\mathcal{C} \subset \tilde{K} \\ \mathcal{C} \cap \operatorname{int} \underline{\Gamma}^s \neq \emptyset \\ \mathcal{C} \cup G = \tilde{K}}} r(\underline{\Gamma}) \rho(\Gamma, G) \phi_{\Gamma,h}(\mathcal{C}) \\
&\leq \sum_{\underline{\Gamma} \subset G \subset \tilde{K}} e^{-\beta E_s(\Gamma) + \beta B(N, V(\Gamma) \cap \underline{\Gamma}) - \tilde{\beta}|G \setminus \overline{\mathcal{D}(\Gamma)}| + A|G \cap \mathcal{D}(\Gamma)|} e^{|\operatorname{int} \underline{\Gamma}| e^{-g\tilde{\beta}}} e^{-g\tilde{\beta}|\tilde{K} \setminus G|} \\
&\leq e^{|\operatorname{int} \underline{\Gamma}|(e^{-g\tilde{\beta}} + A)} e^{-\beta E_s(\Gamma) + \beta B(N, V(\Gamma) \cap \underline{\Gamma}) - g\tilde{\beta}|K \setminus \underline{\Gamma}| - \tilde{\beta}|\underline{\Gamma} \setminus \overline{\mathcal{D}(\Gamma)}|} \sum_{\underline{\Gamma} \subset G \subset \tilde{K}} e^{-(\tilde{\beta} - g\tilde{\beta})|G \setminus \underline{\Gamma}|} \\
&\leq e^{|\operatorname{int} \underline{\Gamma}|(e^{-g\tilde{\beta}} + A)} e^{-\beta E_s(\Gamma) + \beta B(N, V(\Gamma) \cap \underline{\Gamma}) - \tilde{\beta}|\Gamma \setminus \overline{\mathcal{D}(\Gamma)}| - g\tilde{\beta}|\tilde{K} \setminus \underline{\Gamma}|} e^{|\tilde{K} \setminus \underline{\Gamma}| e^{-(\tilde{\beta} - g\tilde{\beta})}} \\
&\leq e^{|\operatorname{int} \underline{\Gamma}|(e^{-g\tilde{\beta}} + A)} e^{-g'\tilde{\beta}|K \setminus \overline{\mathcal{D}(h)}|} e^{-\beta E_s(\Gamma) + \beta(N, V(\Gamma) \cap \underline{\Gamma}) - \tilde{\beta}|\underline{\Gamma} \setminus \overline{\mathcal{D}(\Gamma)}| + g'\tilde{\beta}|\underline{\Gamma} \setminus \overline{\mathcal{D}(h)}|}
\end{aligned}
\tag{4.40}
$$

where $g'\tilde{\beta} \equiv g\tilde{\beta} - e^{-(\tilde{\beta} - g\tilde{\beta})}$. Note that with A of the order of $e^{-\tilde{\beta}}$ we can replace $e^{-g\tilde{\beta}} + A$ by $e^{-g''\tilde{\beta}}$. Next we have to estimate the sum over the small contours within G. Combining (4.37) with (4.40) we can now write, for any $K \subset \mathcal{G}'(Y, h)$ that

$$
\begin{aligned}
\bar{\rho}(\Gamma^l, K) \leq\; & e^{-\frac{g'\tilde{\beta}}{2}|K \setminus \overline{\mathcal{D}(h)}| + |\operatorname{int} \underline{\Gamma}|(e^{-g\tilde{\beta}} + A)} \sum_{\substack{\tilde{K} : K \cap \mathcal{D}(h) \subset \tilde{K} \\ \tilde{K} \setminus \mathcal{B}^Y(h) = K \setminus \mathcal{B}^Y(h)}} \sum_{\substack{\Gamma \\ \mathcal{D}(h) \cap K \subset \operatorname{int} \underline{\Gamma} \subset \tilde{K}}} e^{-\frac{\beta}{2} E_s(\Gamma)} \\
&\times e^{-\tilde{\beta}|\underline{\Gamma} \setminus \overline{\mathcal{D}(\Gamma)}| + g'\tilde{\beta}|\underline{\Gamma} \setminus \overline{\mathcal{D}(h)}| - \frac{\beta}{4} E_s(\Gamma)} \\
&\times e^{-\frac{\beta}{4} E_s(\Gamma) - \frac{g\tilde{\beta}}{2}|K \setminus \overline{\mathcal{D}(h)}| + \beta B(N, V(\Gamma) \cap \underline{\Gamma}) + \beta B \sum_{x \in \operatorname{int} \underline{\Gamma}} N_x(h) - \beta \sum_{x \in \operatorname{int} \underline{\Gamma}} [S_x(h_x(\Gamma)) - S_x(h)]}
\end{aligned}
\tag{4.41}
$$

Here we have split the terms in such a way that the last two lines in (4.41) can now be estimated *uniformly* in Γ, following the same reasoning that was used in the proofs of Lemmas 3.1 and 3.2. Recalling these estimates, one finds easily that the one-but-last line is bounded by 1 while the last line is bounded by $e^{2\beta B \sum_{x \in \operatorname{int} \underline{\Gamma}} N_x(h)} \leq e^{2\beta B L L^{-(d-2)k} \sigma^2 |\mathcal{D}(h) \cap K|}$.

Estimating the remaining sum over Γ is now a rather easy matter, in particular since we can be fairly generous. Using Lemma 3.3, we get that

$$
\begin{aligned}
&\sum_{\substack{\Gamma \\ \mathcal{D}(h) \cap K \subset \operatorname{int} \underline{\Gamma} \subset \tilde{K}}} e^{-\frac{\beta}{2} E_s(\Gamma)} \leq \sum_{\substack{\underline{\Gamma} \\ \operatorname{int} \underline{\Gamma} \subset K}} \sum_{h_x,\, x \in \operatorname{int} \underline{\Gamma}} e^{-\frac{\beta}{2L} \sum_{x \in \operatorname{int} \underline{\Gamma}} |h_x - h|} \\
&\leq \sum_{\substack{\underline{\Gamma} \\ \operatorname{int} \underline{\Gamma} \subset K}} e^{|\operatorname{int} \underline{\Gamma}| e^{-\frac{\beta}{3L}}} \leq \sum_{\substack{G \\ G \subset K}} 2^{|G|} e^{|G| e^{-\frac{\beta}{3L}}} \leq \left(1 + 2e^{e^{-\frac{\beta}{3L}}}\right)^{|K|} \leq 4^{|K|}
\end{aligned}
\tag{4.42}
$$

In the same spirit we estimate the sum over $\tilde{K}$ by $2^{|K|}$ we get thus

$$\bar{\rho}(\Gamma^l, K) \leq e^{-\frac{g'\tilde{\beta}}{2}|K\backslash\overline{\mathcal{D}(h)}|}9^{|K|}e^{2\beta LL^{-(d-2)k}\sigma^2|\mathcal{D}(h)\cap K|} \tag{4.43}$$

where the $9^{|K|}$ absorbs also the factors $e^{|\mathrm{int}\,\underline{\Gamma}|e^{-g'\tilde{\beta}}}$. The reason why we may be so generous with the factors $C^{|K|}$ is that the conditions (i) and (ii) in the Definition 3.4 of the regions $\mathcal{D}$ assure that $\mathcal{D}(h)$ is sufficiently rare in K such that $|\mathcal{D}(h) \cup K| \leq const|K\backslash\overline{\mathcal{D}(h)}|$ where the constant depends on L. Since also $|K| \leq |K\backslash\overline{\mathcal{D}(h)}| + const'|\mathcal{D}(h) \cap K|$ for some other constant and since $\beta LL^{-(d-2)k}\sigma^2 = \frac{\beta^{(0)}}{\tilde{\beta}^{(0)}}\sigma^2 L\tilde{\beta} \ll \tilde{\beta}$, we get finally that, say,

$$\bar{\rho}(\Gamma^l, K) \leq e^{-\frac{g'\tilde{\beta}}{8}|K\backslash\overline{\mathcal{D}(h)}|} \tag{4.44}$$

as desired. This proves the bound (4.35) and thus (4.26) and (4.27). To complete the proof of Lemma 4.3 we need to derive (4.28). The corresponding lower bound on $\tilde{\rho}'(h, \mathcal{B}_i^Y(h)$ was already given by (4.36). For the upper bound, we recall that

$$\begin{aligned}&\tilde{\rho}'(h, \mathcal{B}_i^Y(h)) \\ &= \sum_{\Gamma: T_1(\Gamma)=(\emptyset,h)} \sum_{\underline{\Gamma}\subset G\subset\mathcal{B}_i^Y(h)} \sum_{\substack{\mathcal{C}\subset\mathcal{B}_i^Y(h) \\ \mathcal{C}\cap\,\mathrm{int}\,\underline{\Gamma^s}\neq\emptyset \,\vee\, \mathcal{C}=\emptyset \\ \mathcal{D}_i(h)\subset\mathcal{C}\cup G\subset\mathcal{B}_i^Y(h)}} e^{-\beta\sum_{x\in\underline{\Gamma}}[S_x(h_x(\Gamma))-S_x(h)]} r(\underline{\Gamma})\rho(\Gamma, G)\phi_{\Gamma,h}(\mathcal{C})\end{aligned} \tag{4.45}$$

We may proceed essentially in the same way as in the derivation of (4.35) and being rather generous obtain the bound

$$\tilde{\rho}'(h, \mathcal{B}_i^Y(h)) \leq 48^{|\mathcal{B}_i^Y(h)|}e^{\beta B\sum_{x\in\mathcal{D}_i(h)}N_x(h)} \leq e^{C_{19}|\mathcal{B}_i^Y(h)|+\beta B\sum_{x\in\mathcal{D}_i(h)}N_x(h)} \tag{4.46}$$

with some constant C_19 which gives in fact (4.28) and concludes the proof of the lemma. $\diamondsuit$

Next we need to control the activities $\hat{\rho}(\Gamma^l, K)$. Our aim here is to show that it satisfies essentially the same bounds as the original ρ. With the work to prove Lemma 4.3 already done, this will not be too difficult. In fact we perform a redefinition just as in (4.32) and set

$$\hat{\rho}'(\Gamma^l, K) \equiv \frac{\hat{\rho}(\Gamma^l, K)}{\prod_{h,i:\,\overline{\mathcal{D}_i(h)}\subset V_h(\Gamma^l)\cap\overline{K}} \tilde{\rho}'(h, \overline{\mathcal{D}_i(h)})} \tag{4.47}$$

This allows us to write the expression for $T_1\hat{\mu}$ in the following rather pleasant form:

$$\begin{aligned}(T_1\hat{\mu})(\Gamma^l) &= e^{-\beta(S, V(\Gamma^l))} \sum_{K\supset\underline{\Gamma^l}} \hat{\rho}'(\Gamma^l, K) \prod_{h,i:\,\overline{\mathcal{D}_i(h)}\subset V_h(\Gamma^l)\cap\overline{K}} \tilde{\rho}'(h, \overline{\mathcal{D}_i(h)}) \\ &\times \prod_{h,i:\,\mathcal{D}_i(h)\subset V_h(\Gamma^l)\cap\overline{K}^c} \tilde{\rho}'(h, \overline{\mathcal{D}_i(h)} \cap \overline{K}^c) \exp\left(-\beta(\bar{\psi}, V(\Gamma^l)\cap\overline{K}^c) - \beta \sum_{\substack{h,\mathcal{C}:\mathcal{C}\subset V_h(\Gamma^l)\cap\overline{K}^c \\ \overline{\mathcal{C}}\cap\overline{K}\neq\emptyset}} \psi_{\mathcal{C}}^s(K, h)\right)\end{aligned} \tag{4.48}$$

Here the $\tilde{\rho}'(h, \overline{\mathcal{D}_i(h)})$ are independent of the contour and K and can be exponentiated to yield a nonlocal field. For the activities we have the following bounds.

LEMMA 4.5:

$$0 \leq \hat{\rho}'(\Gamma^l, K) \leq e^{-\beta E_s(\Gamma^l) - \frac{g\beta}{2}|K \setminus \overline{D(\Gamma^l)}| + \beta B(N, V(\Gamma^l) \cap \underline{\Gamma^l})} 25^{|K|} \tag{4.49}$$

For contours $\Gamma^l = (C, h_x \equiv h)$, with $C \subset D(h) \backslash \mathcal{D}(h)$ connected we have moreover

$$\hat{\rho}'(\Gamma^l, \underline{\Gamma^l}) \geq e^{-\beta B(N, V(\Gamma^l) \cap \underline{\Gamma^l})} \tag{4.50}$$

Remark: The constant 25 in (4.49) is of course quite arbitrary and could be improved.

Proof: Notice first that $\hat{\rho}'(\Gamma^l, \underline{\Gamma^l}) = \hat{\rho}(\Gamma^l, \underline{\Gamma^l}) = \rho(\Gamma^l, \underline{\Gamma^l})$ so that (4.50) is trivial from the assumptions on ρ. The upper bound (4.49) is proven in exactly the same way as the upper bound on $\tilde{\rho}$, since small contours can be summed over in each connected component of the complement of $\underline{\Gamma^l}$ in K. We do not repeat the the details of the estimations. $\diamondsuit$

This concludes the summation over small contours.

IV.3 The blocking

We now turn to the main step of the RG transformation, the blocking. As before, nothing changes as far as the action of $\mathcal{R}$ on contours is concerned and all we have to do is to study the effect on the contour measures.

First we exponentiate all terms in (4.48) that give rise to the new random fields. We set

$$z_C(h) \equiv \sum_i \mathbb{I}_{C = \overline{\mathcal{D}_i(h)}} \left(-\frac{1}{\beta} \ln \left(\tilde{\rho}'(h, \overline{\mathcal{D}_i(h)}) \right) \right) \tag{4.51}$$

Setting now

$$\tilde{S}_C(h) \equiv S_C(h) + z_C(h) + \bar{\psi}_C(h) \tag{4.52}$$

and noticing that

$$(\bar{\psi}, V(\Gamma^l) \cap \overline{K}^c) = (\bar{\psi}, V(\Gamma^l)) - \sum_{\substack{h, C \subset V_h(\Gamma^l) \\ C \cap \overline{K} \neq \emptyset}} \bar{\psi}_C(h) \tag{4.53}$$

We have that

$$(T_1\hat{\mu})(\Gamma^l) = e^{-\beta(\tilde{S},V(\Gamma^l))} \sum_{K\supset\underline{\Gamma^l}} \hat{\rho}'(\Gamma^l, K) \left(\prod_{\substack{\mathcal{D}_i(h)\subset V_h(\Gamma^l)\cap\overline{K}^c \\ \overline{\mathcal{D}_i(h)}\not\subset\overline{K}^c}} \frac{\tilde{\rho}'(h, \overline{\mathcal{D}_i(h)}\cap\overline{K}^c)}{\tilde{\rho}'(h, \overline{\mathcal{D}_i(h)})} \right)$$
$$\times \exp\left(\beta \sum_{\substack{h, C\subset V_h(\Gamma^l) \\ C\cap\overline{K}\neq\emptyset}} \bar{\psi}_C(h) - \beta \sum_{\substack{h,C: C\subset V_h(\Gamma^l)\cap\overline{K}^c \\ \overline{C}\cap\overline{K}\neq\emptyset}} \psi^s_C(K,h) \right) \tag{4.53}$$

where now the random field and the activity-like contributions are almost well separated. We first prepare the field term for blocking. For given $\Gamma'\subset\Omega_{n-1}(\mathcal{L}^{-1}D)$, we can split the term into three parts

$$(\tilde{S}, V_h(\Gamma^l)) = L^{d-1-\alpha}(\tilde{S}', V_h(\Gamma')) + \delta\tilde{S}_{loc}(\Gamma^l,\Gamma') + \delta\tilde{S}_{nl}(\Gamma^l,\Gamma') \tag{4.54}$$

where for single points y

$$\tilde{S}'_y(h) \equiv L^{-(d-1-\alpha)} \left(\sum_{x\in\mathcal{L}y} \tilde{S}_x(h_x(\Gamma')) + \sum_{\substack{C\subset V_h(\Gamma'):\, C\cap\mathcal{L}y\neq\emptyset \\ d(C)<L/4\vee C\subset\mathcal{L}y}} \frac{\tilde{S}_C(h)}{|\mathcal{L}^{-1}(C)|} \right) \tag{4.55}$$

and for $|C'|>1$,

$$\tilde{S}'_{C'}(h) \equiv L^{-(d-1-\alpha)} \sum_{\substack{C:\mathcal{L}^{-1}(C)=C' \\ d(C)\geq L/4}} \tilde{S}_C(h) \tag{4.56}$$

Eqs. (4.55) and (4.56) are the analogues of (3.34) and almost the final definitions of the renormalized 'small random fields'. Furthermore

$$\delta\tilde{S}_{loc}(\Gamma^l,\Gamma') \equiv \sum_{y\in\Lambda_{n-1}} \left[\sum_{x\in\mathcal{L}y} \left(\tilde{S}_x(h_x(\Gamma^l)) - \tilde{S}_x(h_{\mathcal{L}^{-1}x}(\Gamma')) \right) \right.$$
$$\left. + \sum_{h\in\mathbb{Z}} \sum_{\substack{C:\, C\cap\mathcal{L}y\neq\emptyset \\ d(C)<L/4\vee C\subset\mathcal{L}y}} \tilde{S}_C(h) \left[\mathbb{I}_{C\subset V_h(\Gamma^l)} - \frac{\mathbb{I}_{h_y(\Gamma')=h}}{|\mathcal{L}^{-1}C|} \right] \right] \tag{4.57}$$

and

$$\delta\tilde{S}_{nl}(\Gamma^l,\Gamma') \equiv \sum_{h\in\mathbb{Z}} \sum_{\substack{C:\, C\subset\Lambda_{n-1} \\ d(C)\geq L/4\wedge|\mathcal{L}^{-1}C|\geq 2}} \tilde{S}_C(h) \left[\mathbb{I}_{C\subset V_h(\Gamma^l)} - \mathbb{I}_{\mathcal{L}^{-1}C\subset V_h(\Gamma^l)} \right]$$
$$\equiv \sum_{\substack{C:\, C\subset\Lambda_n \\ d(C)\geq L/4\wedge|\mathcal{L}^{-1}C|\geq 2}} \tilde{s}_{\Gamma^l,\Gamma'}(C) \tag{4.58}$$

The point here is that the contributions from $\delta\tilde{S}_{loc}$ will factor over the connected components of the blocked K, while the non-local $\delta\tilde{S}_{nl}$ can be expanded and gives only very small contributions, due to the minimal size condition on the C occurring in it.

In a similar way we decompose the exponent on the last line of (4.53). Here it is convenient to slightly enlarge the supports of the ψ and to define

$$\tilde{\psi}_{\Gamma^l,K}(\tilde{C}) \equiv - \sum_{\substack{h,C\subset V_h(\Gamma^l),\overline{C}=\tilde{C}\\ C\cap\overline{K}\neq\emptyset}} \bar{\psi}_C(h) + \sum_{\substack{h,C:C\subset V_h(\Gamma^l)\cap\overline{K}^c,\overline{\overline{C}}=\tilde{C}\\ \overline{C}\cap\overline{K}\neq\emptyset}} \psi^s_C(K,h) \tag{4.59}$$

This has the advantage that now $\tilde{\psi}_{\Gamma^l,K}(C)=0$, if $C\cap K=\emptyset$. We then decompose

$$\begin{aligned} &- \sum_{\substack{h,C\subset V_h(\Gamma^l)\\ C\cap\overline{K}\neq\emptyset}} \bar{\psi}_C(h) + \sum_{\substack{h,C:C\subset V_h(\Gamma^l)\cap\overline{K}^c\\ \overline{C}\cap\overline{K}\neq\emptyset}} \psi^s_C(K,h) = \sum_{C:C\cap K\neq\emptyset} \tilde{\psi}_{\Gamma^l,K}(C)\\ &= \sum_{y\in\Lambda_{n-1}} \sum_{\substack{C\cap\mathcal{L}y\neq\emptyset\\ C\cap K\neq\emptyset\\ d(C)<L/4\vee C\subset\mathcal{L}y}} \frac{\tilde{\psi}_{\Gamma^l,K}(C)}{|\mathcal{L}^{-1}C|} + \sum_{\substack{C\cap K\neq\emptyset\\ d(C)\geq L/4\wedge|\mathcal{L}^{-1}C|\geq 2}} \tilde{\psi}_{\Gamma^l,K}(C)\\ &\equiv \delta\psi_{loc}(\Gamma^l,K)+\delta\psi_{nl}(\Gamma^l,K) \end{aligned} \tag{4.60}$$

In all of the non-local terms only sets C give a contribution for which $C\cap\mathcal{L}(\mathcal{L}^{-1}K)\neq\emptyset$, $d(C)\geq L/4$ and $|\mathcal{L}^{-1}C|\geq 2$. Moreover, for connected C with $d(C)>L/4$ we have that $|C|\leq const|C\backslash\overline{D(h)}|$ and hence (4.27) implies $\left|\tilde{\psi}_{\Gamma^l,K}(C)\right| \leq e^{-const\,\tilde{\beta}|C|}$. The inductive hypothesis yields a similar estimate for $\tilde{S}$ so that in fact $\left|\tilde{s}_{\Gamma^l,\Gamma'}(C)+\tilde{\psi}_{\Gamma^l,K}(C)\right| \leq e^{-const'\,\tilde{\beta}|C|}\equiv\tilde{f}(C)$.

In analogy to Lemma 4.2 we can therefore expand these contributions to get

$$\begin{aligned} e^{-\beta(\delta\tilde{S}_{nl}(\Gamma^l,\Gamma')+\delta\psi_{nl}(\Gamma^l,K))} &= R(K)\sum_{l=0}^{\infty}\frac{1}{l!} \sum_{\substack{C_1,\dots,C_l:\,C_i\neq C_j\\ C_i\cap\mathcal{L}(\mathcal{L}^{-1}K)\neq\emptyset\\ d(C)\geq L/4\wedge|\mathcal{L}^{-1}C|\geq 2}} \prod \Phi_{\Gamma',\Gamma^l,K}(C_i)\\ &\equiv \sum_{\substack{\mathcal{C}:\mathcal{C}\cap\mathcal{L}(\mathcal{L}^{-1}K)\neq\emptyset\\ d(\mathcal{C})\geq L/4\wedge|\mathcal{L}^{-1}\mathcal{C}|\geq 2}} \Phi_{\Gamma',\Gamma^l,K}(\mathcal{C}) \end{aligned} \tag{4.61}$$

where the activities Φ satisfy

$$0\leq \Phi_{\Gamma',\Gamma^l,K}(\mathcal{C})\leq e^{-g''\tilde{\beta}|\mathcal{C}|} \tag{4.62}$$

And $R(K)$ are non-random activities factoring over connected components of $\mathcal{L}(\mathcal{L}^{-1}K)$, satisfying, for a connected component,

$$1\geq R(K)\equiv\exp\left(-\sum_{\substack{C\cap\mathcal{L}(\mathcal{L}^{-1}K)\neq\emptyset\\ d(C)\geq L/4\wedge|\mathcal{L}^{-1}C|\geq 2}} \tilde{f}(C)\right)\geq e^{-|\mathcal{L}(\mathcal{L}^{-1}K)|e^{-\frac{L}{4}\beta''}} \tag{4.63}$$

Note that in these bounds the terms $\overline{D(h)}$ do no longer appear.

With these preparations we can now write down the blocked contour measures in the form

$$(\mathcal{R}T_1\hat{\mu})(\Gamma') = e^{-\beta L^{d-1-\alpha}(\tilde{S}',V(\Gamma')}\sum_{G'\supset\underline{\Gamma'}}\rho'(\Gamma',G') \tag{4.64}$$

where

$$
\begin{aligned}
&\rho'(\Gamma', G') \\
&\equiv \sum_{G' \supset K' \supset \underline{\Gamma'}} \sum_{C': C' \cup K' = G'} \sum_{\Gamma^l :\, T_2(\Gamma^l) = \Gamma'} \sum_{\substack{K \supset \underline{\Gamma^l} \\ \mathcal{L}^{-1} K = K'}} \sum_{\substack{\mathcal{C} : \mathcal{L}^{-1} \mathcal{C} = C' \\ d(\mathcal{C}) \geq L/4 \wedge |\mathcal{L}^{-1} \mathcal{C}| \geq 2}} \left(\prod_{\substack{\mathcal{D}_i(h) \subset V_h(\Gamma^l) \cap \overline{K}^c \\ \overline{\mathcal{D}_i(h)} \not\subset \overline{K}^c}} \frac{\tilde{\rho}'(h, \overline{\mathcal{D}_i(h)} \cap \overline{K}^c)}{\tilde{\rho}'(h, \overline{\mathcal{D}_i(h)})} \right) \\
&\times e^{-\beta\left(\delta \tilde{S}_{loc}(\Gamma', \Gamma^l) + \delta \psi_{loc}(\Gamma^l, K)\right)} R(K) \hat{\rho}'(\Gamma^l, K) \Phi_{\Gamma', \Gamma^l, K}(\mathcal{C})
\end{aligned}
\tag{4.65}
$$

Notice that by construction the C occurring in the local fields $\delta\tilde{S}$ and $\delta\psi$ cannot connect disconnected components of G', and therefore $\rho'(\Gamma', G')$ factorizes over connected components of G'. The main task that is left is to prove that ρ' yields a N' bounded contour measure for a suitably defined N'. As in Section III, we define the preliminary new control field by

$$
\tilde{N}'_y(h) \equiv L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}y \setminus \mathcal{D}(h)} \sup_{h' \in \mathbb{Z}} \left(N_x(h + h') - \frac{1}{8L} |h'| \right) \tag{4.66}
$$

where $\mathcal{N}$ was defined already in (4.1). We will now proof the following

LEMMA 4.6: *Let $\tilde{N}'$ be defined in (4.66) and set $\hat{D}' \equiv D(\tilde{N}')$. Then the activities ρ' defined in (4.65) factor over connected components of G' and for any connected G'*

$$
0 \leq \rho'(\Gamma,' G') \leq e^{-c_1 L^{d-1} \beta E_s(\Gamma') - c_2 L \tilde{\beta} |G' \setminus \overline{\hat{D}'(\Gamma')}| + L^{d-1-\alpha} \beta B(\tilde{N}', V(\Gamma') \cap G') + C_3 L^d |G'|} \tag{4.67}
$$

for some positive constants c_1, c_2, C_3. For $\Gamma' = (C, h_y \equiv h)$, with $C \subset \hat{D}'$ connected,

$$
\rho'(\Gamma', \underline{\Gamma'}) \geq e^{-L^{d-1-\alpha} \beta B(\tilde{N}', V(\Gamma')) - e^{-cont \tilde{\beta}} |C|} \tag{4.68}
$$

Proof: Let us first proof the upper bound. Notice first that of course

$$
\frac{\tilde{\rho}'(h, \overline{\mathcal{D}_i(h)} \cap \overline{K}^c)}{\tilde{\rho}'(h, \overline{\mathcal{D}_i(h)})} \leq 1 \tag{4.69}
$$

The sum over the C can first be estimated just as in Lemma 4.4, but taking into account the restrictions on the minimal size of the sets C; that is, we get that

$$
\begin{aligned}
\sum_{C' : C' \cup K' = G'} \sum_{\substack{\mathcal{C} : \mathcal{L}^{-1} \mathcal{C} = C' \\ d(\mathcal{C}) \geq L/4 \wedge |\mathcal{L}^{-1} \mathcal{C}| \geq 2}} \Phi_{\Gamma', \Gamma^l, K}(\mathcal{C}) &\leq e^{|\mathcal{L}(\mathcal{L}^{-1} K| e^{-g \tilde{\beta}(L/4)}} e^{-g \tilde{\beta} \min\left(|V| : \mathcal{L}^{-1} V = G' \setminus K' \wedge d(V) \geq L/4\right)} \\
&\leq e^{|K'| L^d e^{-L/4 \tilde{\beta}}} e^{-|G' \setminus K'| g \tilde{\beta} L/4}
\end{aligned}
\tag{4.70}
$$

The two local field terms, $\delta\tilde{S}_{loc}$ and $\delta\psi_{loc}$ will be dealt with differently: $\delta\tilde{S}_{loc}$ is only present for locally non-flat Γ^l and will thus be estimated against some fraction of the surface energy (just like in Section III), while for the $\delta\psi_{loc}$ we get

$$\delta\psi_{loc}(\Gamma^l, K) = \sum_{h\in\mathbb{Z}}\sum_{y\in K'}\sum_{\substack{C\subset V_h(\Gamma^l):\, C\cap\mathcal{L}y\neq\emptyset\\ C\cap K\neq\emptyset\\ d(C)<L/4\vee C\subset\mathcal{L}y}} \frac{\psi_C(h)}{|\mathcal{L}^{-1}C|} \leq |K'|L^d e^{-g\tilde{\beta}} \tag{4.71}$$

(in both formulas g stands for some geometric constant (less than one)). Using this, we get that

$$\begin{aligned}\rho'(\Gamma', G') \leq & \sum_{G'\supset K'\supset\underline{\Gamma'}}\ \sum_{\Gamma^l:\,\mathcal{R}(\Gamma^l)=\Gamma'}\ \sum_{\substack{K\supset\underline{\Gamma^l}\\ \mathcal{L}^{-1}K=K'}} e^{-\beta\left(\delta\tilde{S}_{loc}(\Gamma',\Gamma^l)\right)}\\ &\times e^{-\beta E_s(\Gamma^l)-\frac{g\tilde{\beta}}{2}|K\backslash\overline{D(\Gamma^l)}|+\beta B(N,V(\Gamma^l)\cap\underline{\Gamma^l})}25^{L^d|K'|}e^{-g\tilde{\beta}L|G'\backslash K'|+2L^d|K'|e^{-g\tilde{\beta}}}\end{aligned} \tag{4.72}$$

Now we may use the estimates of Section III.3 (in particular (3.57)) to estimate

$$\begin{aligned}&\beta\delta\tilde{S}_{loc}(\Gamma',\Gamma^l) + \beta E_s(\Gamma^l) + \tilde{\beta}'|K\backslash\overline{D(\Gamma^l)}| - \beta(N, V(\Gamma^l)\cap\underline{\Gamma^l})\\ &\geq \beta\frac{L^{d-1}}{16(d+1)}E_s(\Gamma') + \tilde{\beta}Lc_6|K'\backslash\overline{\hat{D}'(\Gamma')}| - \beta L^{d-1-\alpha}(\tilde{N}', V(\Gamma')\cap\underline{\Gamma'})\\ &+\frac{\beta}{2}E_s(\Gamma^l)\end{aligned} \tag{4.73}$$

The idea here is that the first three terms in the lower bound provide essentially the bound for the new activities, while remaining surface energy term suffices to control the convergence of the sums over K', K and Γ^l. Indeed inserting (4.73) into (4.72) we arrive at

$$\begin{aligned}\rho'(\Gamma', G') \leq\, & e^{-\beta\frac{L^{d-1}}{16(d+1)}E_s(\Gamma')-const\ \tilde{\beta}L|G'\backslash\overline{\hat{D}'(\Gamma')}|+\beta L^{d-1-\alpha}B(\tilde{N}',V(\Gamma')\cap\underline{\Gamma'})+C_{10}L^d|G'|}\\ &\times\sum_{G'\supset K'\supset\underline{\Gamma'}}\ \sum_{\Gamma^l:\,\mathcal{R}(\Gamma^l)=\Gamma'}\ \sum_{\substack{K\supset\underline{\Gamma^l}\\ \mathcal{L}^{-1}K=K'}} e^{-\frac{\beta}{2}E_s(\Gamma^l)}\end{aligned} \tag{4.74}$$

The sums in the last line are estimates as in the proof of Lemma 4.3 (see in particular (4.42) and yield

$$\begin{aligned}&\sum_{G'\supset K'\supset\underline{\Gamma'}}\ \sum_{\Gamma^l:\,\mathcal{R}(\Gamma^l)=\Gamma'}\ \sum_{\substack{K\supset\underline{\Gamma^l}\\ \mathcal{L}^{-1}K=K'}} e^{-\frac{\beta}{2}E_s(\Gamma^l)}\\ &\leq \sum_{\substack{K,\Gamma^l:\\ \mathcal{L}G'\supset K,K\supset\underline{\Gamma^l}}} e^{-\frac{\beta}{2L}\sum_{x\in\underline{\Gamma^l}}|h_x(\Gamma^l)-h_{\mathcal{L}^{-1}x}(\Gamma')|} \leq 3^{L^d|G'|}e^{e^{-\frac{\beta}{3L}}L^d|G'|}\end{aligned} \tag{4.75}$$

which inserted into (4.74) yields directly the upper bound in Lemma 4.6.

The proof of the lower bound (4.75) is again very easy. For $\Gamma' = (C, h_y \equiv h)$ we consider the contribution to $\rho'(\Gamma', \underline{\Gamma'})$ that arises from $\Gamma^l = (\mathcal{L}C\cap(D(h)\backslash\mathcal{D}(h)), h_x \equiv h)$, $K = \underline{\Gamma^l}$, $\mathcal{C} = \emptyset$. In this case the product of the ratios of the $\tilde{\rho}'$ is just equal to one, since $\underline{\Gamma}^l$ and $\mathcal{D}(h)$ are sufficiently far separated. The local term $\delta\tilde{S}_{loc}$ is zero and so

$$\rho'(\Gamma', \underline{\Gamma}') \geq e^{-\beta\delta\psi_{loc}(\Gamma^l, \underline{\Gamma}^l)} R(\underline{\Gamma}^l)\hat{\rho}'(\Gamma^l, \underline{\Gamma}^l) \tag{4.76}$$

With our bound (4.71) on ψ_{loc} and the bound (4.61) on R we get

$$\begin{aligned} \rho'(\Gamma', \underline{\Gamma}') &\geq e^{-|C|L^d e^{-const\,\beta}} e^{-|C|L^d e^{-const\,L\tilde{\beta}}} e^{-\beta B(N, V(\Gamma^l)\cap\underline{\Gamma}^l)} \\ &\geq e^{-L^{d-1-\alpha}\beta B(\tilde{N}', V(\Gamma')\cap C) - e^{-const'\,\tilde{\beta}} L^d|C|} \end{aligned} \tag{4.77}$$

which concludes the proof of Lemma 4.6. $\diamondsuit$

IV.4 Final shape up

Just as in section III we must make some final changes in the definition of the small and control fields and in the definition of the contours to recover the exact form of N'-bounded contour models. We also take care of the entropy terms that have been created in the estimates in Lemma 4.6.

The definition of the local small fields (3.59) and the control fields (3.60,61) remain unchanged. The nonlocal small fields will be left unaltered, i.e. we simply set $S'_{C'}(h) \equiv \tilde{S}'_{C'}(h)$. [7]

The centering has of course no effect on the contour measures, as the effect cancels with the partition functions (which are *not* invariant under this last part of the RG map), except for some boundary effects which can be easily dealt with as Section III. The final result is then the following

PROPOSITION 4.1: *Let $\mathcal{R}^{(N)} \equiv T_3T_2T_1 : \Omega_n(D(N)) \to \Omega_{n-1}(D(N'))$ with T_1, T_2 and T_3 defined above; let N' and S' and ρ' be defined as above and let μ be a N-bounded contour measure at temperatures β and $\tilde{\beta}$ of level k. Then $\mu' \equiv \mathcal{R}\mu$ is a N'-bounded contour measure with temperatures $\beta' = L^{d-1-\alpha}\beta$ and $\tilde{\beta}' = L^{1-\alpha}\tilde{\beta}$ of level $k+1$, for suitably chosen $\alpha > 0$.*

Proof: For any contour $\Gamma'' \in \Omega_{\Lambda'}(D')$ and $G'' \supset \underline{\Gamma}''$ we set

$$\begin{aligned} \rho''(\Gamma'', G'') = &\sum_{\substack{\Gamma', G' :\, \underline{\Gamma}' \subset G' \\ (D'(\Gamma'')\cap G'')\cup\underline{\Gamma}' = \underline{\Gamma}'' \\ (D'(\Gamma'')\cap G'')\cup G' = G''}} \rho'(\Gamma', G') e^{-\beta' \sum_{y\in\underline{\Gamma}''} \tilde{S}'_y(h_y(\Gamma''))\mathbb{1}_{|\tilde{S}'_y(h_y(\Gamma''))| > \frac{1}{16L}}} \\ &\times \exp\left(-\beta' \sum_{\substack{y\in\underline{\Gamma}'' \\ d(y,(\Lambda')^c)=1}} \left(I\!E\left[\tilde{S}'_y(h_y(\Gamma''))\mathbb{1}_{|\tilde{S}'_y(h_y(\Gamma''))| \leq \frac{1}{16L}}\right] - I\!E\left[\tilde{S}'_y(0)\mathbb{1}_{|\tilde{S}'_y(h_y(0)| \leq \frac{1}{16L}}\right]\right)\right) \end{aligned} \tag{4.78}$$

[7] One might have expected that the nonlocal small fields should also be centered. However, due to their extreme smallness, this is unnecessary. This does not lead to an accumulation to potentially dangerous 'drift' since after a certain number of iterations, any nonlocal contribution is eventually absorbed into a local one, and at this point the accumulated mean is subtracted.

Note that the term in the second line are boundary terms arising from the breaking of stationarity by the boundary conditions. This is completely analogous to the ground state situation, see (3.63). With this definition we have clearly that

$$\mu'(\Gamma'') = \frac{1}{Z'} e^{-\beta'(S', V(\Gamma''))} \sum_{G'' \supset \underline{\Gamma''}} \rho''(\Gamma'', G'') \tag{4.79}$$

which is the desired form of the renormalized contour measure. From Lemma 4.6 and estimating the boundary contribution against a small fraction of the surface energy, as explained in (3.67), we get for each term in the sum in (4.78) the bound

$$e^{-\bar{c}_1 L^{d-1} \beta E_s(\Gamma') - c_2 L \tilde{\beta} |G' \backslash \overline{\hat{D}'(\Gamma')}|} e^{\beta L^{-(d-1-\alpha)} B \sum_{y \in \underline{\Gamma''}} \left(\tilde{N}'_y(h_y(\Gamma'')) + |\tilde{S}'_y(h_y(\Gamma''))| 1_{|\tilde{S}'_y(h_y(\Gamma''))| > \frac{1}{16L}} \right) + C_3 L^d |G'|} \tag{4.80}$$

where the constant $\bar{c}_1$ is slightly smaller than c_1. Here only the volume term $C_3 L^d |G'|$ is still disturbing. To get rid of it, observe that (as already used in the proof of Lemma 4.2),

$$|G'| \leq |G' \backslash \overline{\hat{D}'(\Gamma')}| + C_{13} |\hat{D}'(\Gamma') \cap \underline{\Gamma'}| \tag{4.81}$$

Now $C_3 L^d |G'| \leq |G' \backslash \overline{\hat{D}'(\Gamma')}|$ is just a tiny fraction of $c_2 \tilde{\beta} L |G'| \leq |G' \backslash \overline{\hat{D}'(\Gamma')}|$ and can thus be absorbed in this term by slightly decreasing the constant c_2. The remaining term $C_3 L^d C_{13} |\hat{D}'(\Gamma') \cap \underline{\Gamma'}|$, on the other hand can be shown to be bounded by a tiny fraction of the control field N'. Here we will make use of the fact that a non-vanishing N' has a minimal size (see (3.90) in Proposition 3.3) and this is the real reason why N' is defined the way we do. The precise statement is given in the following

LEMMA 4.7: *Let N be a control field on the k-th level. and let $D = D(N)$ be the corresponding bad region. Then for every contour $\Gamma \in \Omega_\Lambda(D)$*

$$|\{x \in \underline{\Gamma}, N_x(h_x(\Gamma)) \neq 0\}| \leq \frac{\left(C_1 L^{d-3/2}\right)^k}{\sigma^2} (N, V(\Gamma) \cap \underline{\Gamma}) + C_2 E_s(\Gamma) \tag{4.82}$$

with dimension dependent constants C_1 and C_2.

Proof: Define for any Γ the set

$$F(\Gamma) \equiv \left\{ x \in \underline{\Gamma} \,\middle|\, h_y(\Gamma) = h_x(\Gamma) \;\; \forall y \in \overline{\underline{\Gamma}} : d(x,y) \leq 2 \right\} \tag{4.83}$$

i.e. $F(\Gamma)$ is the part of the support of Γ where Γ is 'flat'. Then

$$|\{x \in \underline{\Gamma}, N_x(h_x(\Gamma)) \neq 0\}| \leq |\{x \in F(\Gamma), N_x(h_x(\Gamma)) \neq 0\}| + |\underline{\Gamma} \backslash F(\Gamma)| \tag{4.84}$$

Obviously in the region $\underline{\Gamma} \backslash F(\Gamma)$ there must be a density of height differences and thus the second term in (4.84) can be bounded by a fraction of the surface energy, viz.

$$|\underline{\Gamma} \backslash F(\Gamma)| \leq C_2 E_s(\Gamma) \tag{4.85}$$

where $C_2 = |\bar{\bar{0}}|^2 = 5^{2d}$ is certainly a generous upper bound.

Now from (3.86) in Lemma 3.12 one obtains easily that

$$\sum_{x\in F(\Gamma)} \bar{N}_x(h_x(\Gamma)) \leq C_1^k \sum_{x\in\underline{\Gamma}} N_x(h_x(\Gamma)) \tag{4.86}$$

while (3.90) of Proposition 3.3 implies that

$$\begin{aligned}\sum_{x\in F(\Gamma)} \bar{N}_x(h_x(\Gamma)) &\geq L^{-(d-3/2)k}\sigma^2|\{x\in F(\Gamma), \bar{N}_x(h_x(\Gamma))\neq 0\}| \\ &\geq L^{-(d-3/2)k}\sigma^2|\{x\in F(\Gamma), N_x(h_x(\Gamma))\neq 0\}|\end{aligned} \tag{4.87}$$

and combining these relations we get

$$|\{x\in F(\Gamma), N_x(h_x(\Gamma))\neq 0\}| \leq\leq \frac{\left(C_1 L^{d-3/2}\right)^k}{\sigma^2}\sum_{x\in\underline{\Gamma}} N_x(h_x(\Gamma)) \tag{4.88}$$

This completes the proof of the lemma. ◊

To use Lemma 4.7 it remains to observe that $x\in\hat{D}'(h)$ indeed implies that $N'_x(h)\neq 0$ (recall that $\hat{D}'\equiv\mathcal{L}^{-1}(D\backslash\mathcal{D})$). Thus

$$\begin{aligned}|\hat{D}'(\Gamma')\cap\underline{\Gamma'}| = |\hat{D}'(\Gamma'')\cap\underline{\Gamma''}| &\leq |\{x\in\underline{\Gamma''}, N'_x(h_x(\Gamma''))\neq 0\}| \\ &\leq \frac{\left(C_1 L^{d-3/2}\right)^{k+1}}{\sigma}(N', V(\Gamma'')\cap\underline{\Gamma''}) + C_2 E_s(\Gamma'')\end{aligned} \tag{4.89}$$

This allows us to replace the bound (4.80) by

$$e^{-c_4 L^{d-1}\beta E_s(\Gamma') - c_5 L\tilde{\beta}|G'\backslash\overline{\hat{D}'(\Gamma')}| + L^{d-1-\alpha}\beta B'\sum_{y\in\underline{\Gamma''}} N'_y(h_y(\Gamma''))} \tag{4.90}$$

with positive constants c_4, c_5 and with $B' = C_{k+1}B$ where $C_{k+1} = 1 + C_3 C_{13}\frac{L^d}{\sigma\beta_0}\left(C_1 L^{-(\frac{1}{2}-\alpha)}\right)^{k+1}$ is such that $\prod_{k=1}^\infty \leq 2$, that is, in all levels k the constants B will remain bounded between one and two and are thus completely irrelevant. If now L is large enough, even for small $\alpha>0$ we can assure that the constants c_4, c_5 are majorized by L^α, and this gives, with $\beta' = L^{d-1-\alpha}\beta$ and $\tilde{\beta}' = L^{1-\alpha}\tilde{\beta}$ the final bound

$$e^{-\beta' E_s(\Gamma') - \tilde{\beta}'|G'\backslash\overline{\hat{D}'(\Gamma')}| + \beta' B'(N', V(\Gamma')\cap G')} \tag{4.91}$$

on the summands in (4.78). Therefore

$$\begin{aligned}\rho''(\Gamma'', G'') &\leq e^{-\beta' E_s(\Gamma'') - \tilde{\beta}'|G''\backslash\overline{D'(\Gamma'')}| + \beta' B'(N', V(\Gamma'')\cap\underline{\Gamma''})} \\ &\quad \sum_{\substack{\Gamma', G':\ \underline{\Gamma'}\subset G' \\ (D'(\Gamma'')\cap G'')\cup\underline{\Gamma'} = \underline{\Gamma''} \\ (D'(\Gamma'')\cap G'')\cup G' = G''}} e^{-\tilde{\beta}'\left(|G'\backslash\overline{\hat{D}'(\Gamma')}| - |G''\backslash\overline{D'(\Gamma'')}|\right)} \\ &\leq e^{-\beta' E_s(\Gamma'') - \tilde{\beta}'|G''\backslash\overline{D'(\Gamma'')}| + \beta' B'(N', V(\Gamma'')\cap\underline{\Gamma''}) + 2e^{-\tilde{\beta}'}|D'(\Gamma)\cup\underline{\Gamma}''|}\end{aligned} \tag{4.92}$$

Here we have estimated the sum over Γ', G' using that $\hat{D}'(\Gamma') \subset D'(\Gamma'')$. This is the desired form of the upper bound.

The final form of the lower bound is is obtained even more easily. Here we just choose, for a contour $\Gamma'' = (C, h_y \equiv h)$ the contribution to $\rho''(\Gamma'', G'')$ coming from $\Gamma' = \left(C \cap \hat{D}'(h), h_x \equiv h\right)$, $G' = \underline{\Gamma'}$. For this Lemma 4.6 gives

$$
\begin{aligned}
&\rho''(\Gamma'', \underline{\Gamma''}) \\
&\geq e^{-L^{d-1-\alpha}\beta(\tilde{N}', V(\Gamma') \cap C) - e^{-const\tilde{\beta}} |\underline{\Gamma''} \cap \hat{D}'(h)|} e^{-L^{d-1-\alpha}\beta \sum_{y \in \underline{\Gamma''}} |\tilde{S}'_y(h_y(\Gamma''))| 1_{|\tilde{S}'_y(h_y(\Gamma''))| > \frac{1}{16L}}} \\
&\geq e^{-\beta'(N', V(\Gamma'') \cap \underline{\Gamma''}) - e^{-const\tilde{\beta}} |\underline{\Gamma''} \cap \hat{D}'(h)|}
\end{aligned}
\tag{4.93}
$$

and a final application of Lemma 4.7 leads to the desired

$$
\rho''(\Gamma'', \underline{\Gamma''}) \geq e^{-\beta' B'(N', V(\Gamma'') \cap \underline{\Gamma''})} \tag{4.94}
$$

We have not made specific mention of all the locality and stationarity properties of the renormalized quantities. However, from our explicit formulas these are quite evident. This then proves the Proposition 4.1 and this concludes the control of the renormalization procedure, that is to say the construction of the renormalized contour measures. $\diamondsuit$

Remark: The reader that has carried with us to this point may justly feel fairly exhausted at this stage and may consider our construction extremely cumbersome and complicated. We would like him, however, to recover the main points of what we have done. We needed a form of a contour measure whose form remains invariant under renormalization and that otherwise should be as simple as possible. This is achieved with the form (4.1). The specific form of the bounds on the activities is not so important, but it must necessarily have three main elements: There must be a term ($E_s(\Gamma)$ in our case) that weights the renormalized configurations; a second term (the term $|G \backslash \overline{D(\Gamma)}|$ suppresses 'bad histories', that is contours that are images of 'unlikely' original configurations must retain a trace of this history (that is their (large) support) and these must be suppressed; finally, the the control field term must allow deviations from ground states in exceptional regions; the probabilistic estimates must then assure that such regions are becoming less and less prominent. Given such a proper form the estimate of the RG flow are somewhat cumbersome, but not really very delicate. In fact we have gotten away with rather crude estimates in many places and maybe an even cruder procedure would suffice. In particular, this procedure will work as a method in many related situations.

In the next section we will harvest the fruit of our labours and demonstrate how the control over the renormalized contour measures gives us control on the infinite volume Gibbs state.

IV.6 Proof of the main Theorem

From the definition of the renormalized small fields and control fields it is clear that the probabilistic estimates carried out in Section 3 apply unaltered at small temperatures provided the hypothesis of Proposition 4.1 hold, i.e. if the RG program can be carried through. We will now show how these estimates can be used to proof Theorem 1. The main idea here is of course that contours are suppressed outside the union of all the bad regions in all hierarchies and that this latter set is, by the estimates on the control fields, very sparse. Moreover, the randomness essentially only produces local deformations which are very weakly correlated over larger distances, and thus finite volume measures with constant boundary conditions should converge. While all this is intuitively clear, it remains a somewhat delicate, although not difficult, affair to turn this intuition into a rigorous proof, as we will see. In particular, we have to deal with the fact in our renormalized measures connected components of contours are not independent but interact non-locally. This will require us to derive a Peierls-type argument, i.e. an estimate on certain cylinder events.

Let us now assume that β is large enough, σ small enough and the parameters L, α and η chosen such that the preceeding results are all valid. Let $H = 0$; we denote by $\mu_\Lambda \equiv \mu_{\Lambda,\beta}$ the finite volume measure in Λ with zero boundary conditions.

A key point needed to proof Theorem 1 is that

LEMMA 4.8: *For all $y \in \mathbb{Z}^d$,*

$$\mathbb{P}\left[\limsup_{M\uparrow\infty} \mu_{\mathcal{L}^M 0,\beta}\left(|h_y|\right) < \infty\right] = 1 \tag{4.95}$$

Given Lemma 4.8, Theorem 1 follows by a standard compactness argument (see e.g. Georgii's text [Ge]). Namely, if the event considered in (4.95) holds, than there exist subsequences M_k tending to infinity such that $\lim_{k\uparrow\infty} \mu_{M_k,\beta}$ converges to a measure and satisfies the DLR equations. Since the lemma affirms that this condition is fulfilled with probability one, Theorem 1 will be proven if Lemma 4.8 is proven. $\diamondsuit\diamondsuit$

Thus all we have left to do now is to prove this lemma.

Proof: (of Lemma 4.8) We may set, without loss of generality, $y = 0$. To estimate the expectation of $|h_0|$, let us introduce, for any contour $\Gamma \subset \Omega_M(D)$ the notation γ_0 for the unique weakly connected component of Γ those interior contains the origin. If no such component exists, γ_0 is understood to be the empty set. Then

$$
\begin{aligned}
\mu_{\mathcal{L}^M 0}(|h_0|) = & \sum_{G\subset \mathcal{L}^M 0, G\ni 0, \vee G=\emptyset} \mu_{\mathcal{L}^M 0}\left(\ \mathrm{int}\, \underline{\gamma_0} = G\right) \times \mu_{\mathcal{L}^M 0}\left(|h_0| \Big|\ \mathrm{int}\, \underline{\gamma_0} = G\right) \\
& \leq \sum_{k=1}^{M} \sum_{\substack{G\subset \mathcal{L}^M 0, G\ni 0 \\ G\subset \mathcal{L}^k 0,\, G\not\subset \mathcal{L}^{k-1} 0}} \mu_{\mathcal{L}^M 0}\left(\ \mathrm{int}\, \underline{\gamma_0} = G\right) \\
& \quad \times \sup_{\substack{G\subset \mathcal{L}^k 0 \\ G\ni 0}} \mu_{\mathcal{L}^M 0}\left(|h_0| \Big|\ \mathrm{int}\, \underline{\gamma_0} = G\right)
\end{aligned}
\tag{4.96}
$$

The final estimate in (4.96) can be rewritten in the form

$$
\mu_{\mathcal{L}^M 0}(|h_0|) \leq \sum_{k=1}^{M} \alpha_M^{(k-1)} s_M^{(k)}
\tag{4.97}
$$

where

$$
\begin{aligned}
\alpha_M^{(k)} &\equiv \mu_{\mathcal{L}^M 0}\left(\ \mathrm{int}\, \underline{\gamma_0} \not\subset \mathcal{L}^k 0\right) \quad \text{and} \\
s_M^{(k)} &\equiv \sup_{G\subset \mathcal{L}^k 0, G\ni 0} \mu_{\mathcal{L}^M 0}\left(|h_0| \Big|\ \mathrm{int}\, \underline{\gamma_0} = G\right)
\end{aligned}
\tag{4.98}
$$

We must proof that $\alpha_M^{(k)}$ decays rapidly with k; a rather crude estimate on $s_M^{(k)}$ will then suffice. The estimate on $\alpha_M^{(k)}$ is, of course the Peierls type estimate we alluded to before. It will tell us that it is indeed unlikely that a connected component with large support encircles the origin. Of course such an estimate has to be conditioned on the environments. The precise form is the

LEMMA 4.9:

Let $0 \leq k \leq M-1$ and let denote $F_{l,M} \subset \mathring{A}$ the event

$$
F_{l,M} \equiv \left\{ d\left(D^{(l)}(0), 0\right) \leq \frac{L}{2} \right\}
\tag{4.99}
$$

Then there exists a constant $b > 0$ s.t.

$$
\left\{ \alpha_M^{(k)} \geq e^{-b\tilde{\beta}^{(k)}} \right\} \subset \bigcup_{l=k}^{M} F_{l,M}
\tag{4.100}
$$

The proof of this lemma will be given later. For the $s_M^{(k)}$ we have

LEMMA 4.10: *For $k \geq 1$ the upper bound*

$$
I\!P\left[\sup_{M\geq k} s_M^{(k)} \geq \delta \right] \leq e^{-\frac{\delta^2}{Const\sigma^2 L^{dk}}} + e^{-\frac{\delta}{Const\sigma^2 L^k}}
\tag{4.101}
$$

holds, if $\delta \geq \delta_k \equiv Const \max\left\{ L^{dk}, \sigma\sqrt{\ln(1+\sigma L^k)} L^{dk}, \frac{1}{\beta} L^{dk} \ln L^k \right\}$. Here $Const > 0$ is a d-dependent constant.

The proof of this lemma will also be postponed. Assuming them for the moment, it is now easy to prove Lemma 4.8.

Note first that the events $F_{l,M}$ are independent of M (recall that $D^{(k)}$ depends on the finite volume only near the boundary). Therefore,

$$\sum_{k=0}^{\infty} I\!P\left[\sup_{M\geq k} \alpha_M^{(k)} \geq e^{-b\tilde{\beta}^{(k)}}\right] \leq \sum_{k=0}^{\infty} I\!P\left[F_{k,\infty}\right] + \sum_{k=0}^{\infty} I\!P\left[F_{k,k}\right] \tag{4.102}$$

Moreover, the probabilities of the events $F_{l,M}$ satisfy

$$I\!P[F_{k,M}] \leq L^d \exp\left(-L^{\left(\frac{d-2}{2(d-1)}-\eta\right)k} \frac{\delta^2}{a\sigma^{2-\frac{d-2}{d-1}}}\right) \tag{4.103}$$

as follows from the proof of Corollary 3.1. Thus the right-hand side of (4.102) is finite and therefore, by the Borel-Cantelli Lemma the event $\sup_{M\geq k} \alpha_M^{(k)} \geq e^{-b\tilde{\beta}^{(k)}}$ occurs only for a finite number of indices k, $I\!P$- almost surely. By the estimates of Lemma 4.10 and the same kind of reasoning, we also find that the event $s^{(k)} \geq \delta_k$ occurs only finitely often, almost surely, and hence, since $\sum_{k=1}^{\infty} e^{-b\tilde{\beta}^{(k-1)}}\delta_k < \infty$,

$$\sum_{k=1}^{\infty} \sup_{M\geq k} \alpha_M^{(k-1)} \times \sup_{M\geq k} s_M^{(k)} < \infty \quad I\!P\text{-a.s.} \tag{4.104}$$

By (4.97), this proves Lemma 4.8.$\diamondsuit$

Remark: It is of course an easy matter now to get more quantitative results. One may for instance use the Schwartz inequality to show that

$$I\!E\left[\limsup_{M\uparrow\infty} \mu_{\mathcal{L}^M 0,\beta}\left(|h_y|\right)\right] \leq e^{-\frac{1}{\sigma^{\kappa_d}}} + e^{-C_d\beta} \tag{4.105}$$

for some dimension dependent constants C_d and κ_d.

We now turn to the proof of Lemma 4.9. This is the more intricate, but also the more interesting proof of this subsection.

Proof: (of Lemma 4.9) Let us fix for simplicity $\Lambda \equiv \mathcal{L}^M 0$ and let us write $\mu^{(k)} \equiv \mathcal{R}^k \mu_{\Lambda,\beta}$ for the renormalized measures. The key observation allowing the use of the RG in this estimate is that if Γ is such that $\gamma_0(\Gamma) \not\subset \mathcal{L}^k 0$, then $\operatorname{int} \underline{\mathcal{R}^k(\Gamma)} \ni 0$ (simply because a connected component of such a size cannot have become "small" in only $k-1$ RG steps). But this implies that

$$\mu\left(\gamma_0 \not\subset \mathcal{L}^k 0\right) \leq \mu^{(k)}\left(\operatorname{int} \underline{\Gamma} \ni 0\right) \tag{4.106}$$

To analyse the right hand side of this bound, we decompose the event $\operatorname{int} \underline{\Gamma} \ni 0$ according to decomposition of contours in small and large parts: either 0 is contained in the interior of the support of Γ^l, or else it is in the interior of the support of Γ^s and *not* in that of Γ^l. That is

$$\leq \mu^{(k)}\left(\operatorname{int} \underline{\Gamma} \ni 0\right) \leq \mu^{(k)}\left(\operatorname{int} \underline{\Gamma^l} \ni 0\right) + \mu^{(k)}\left(\operatorname{int} \underline{\Gamma^s} \ni 0 \;\;,\;\; \operatorname{int} \underline{\Gamma^l} \not\ni 0\right) \tag{4.107}$$

If $\mathrm{int}\,\underline{\Gamma^l} \ni 0$, then obviously $\mathrm{int}\,\underline{\mathcal{R}\Gamma} \ni 0$, which allows us to push the estimation of the first term in (4.107) into the next hierarchy; the second term concerns, on the other hand an event that is sufficiently 'local' to be estimated, s we will see. Iterating this procedure, we arrive at the bound

$$\mu\left(\gamma_0 \not\subset \mathcal{L}^k 0\right) \leq \sum_{l=k}^{M-1} \mu^{(l)}\left(\mathrm{int}\,\underline{\Gamma^s} \ni 0 \;,\quad \mathrm{int}\,\underline{\Gamma^l} \not\ni 0\right) + \mu^{(M)}\left(\underline{\Gamma} \ni 0\right) \tag{4.108}$$

The last term in (4.108) concerns a single-site measure and will be very easy to estimate. To bound the other terms, we have to deal with the nonlocality of the contour measures. To do so, we introduce the (non-normalized measure)

$$\nu(\Gamma) \equiv \frac{1}{Z} e^{-\beta(S,V(\Gamma))} \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) \mathbb{I}_{G \ni 0} \tag{4.109}$$

where Z is the usual partition function (i.e. the normalization factor for the measure μ). For all G contributing to ν (i.e. containing the origin) we write $G_0 \equiv G_0(G)$ for the connected component of G that contains the origin. We then define further

$$\nu_s(\Gamma) \equiv \frac{1}{Z} e^{-\beta(S,V(\Gamma))} \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) \mathbb{I}_{G \ni 0} \mathbb{I}_{G_0 \cap \underline{\Gamma^l} = \emptyset} \tag{4.110}$$

and

$$\nu_l(\Gamma) \equiv \frac{1}{Z} e^{-\beta(S,V(\Gamma))} \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) \mathbb{I}_{G \ni 0} \mathbb{I}_{G_0 \cap \underline{\Gamma^l} \neq \emptyset} \tag{4.111}$$

Of course, $\nu = \nu_s + \nu_l$. Let us further set

$$m_s \equiv \frac{1}{Z} \sum_{\Gamma} e^{-\beta(S,V(\Gamma))} \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) \mathbb{I}_{\mathrm{int}\,\underline{\Gamma^s} \ni 0} \mathbb{I}_{\mathrm{int}\,\underline{\Gamma^l} \not\ni 0} \mathbb{I}_{g_0 \cap \underline{\Gamma^l} = \emptyset} \tag{4.112}$$

and

$$m_l \equiv \frac{1}{Z} \sum_{\Gamma} e^{-\beta(S,V(\Gamma))} \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) \mathbb{I}_{\mathrm{int}\,\underline{\Gamma^s} \ni 0} \mathbb{I}_{\mathrm{int}\,\underline{\Gamma^l} \not\ni 0} \mathbb{I}_{g_0 \cap \underline{\Gamma^l} \neq \emptyset} \tag{4.113}$$

where $g_0 \equiv g_0(G, \Gamma)$ denotes the connected component of G that contains the maximal connected component of Γ_s whose interior contains the origin. (Note that in general $g_0 \neq G_0$). The point here is that

$$\mu\left(\mathrm{int}\,\underline{\Gamma^s} \ni 0,\; \mathrm{int}\,\underline{\Gamma^l} \not\ni 0\right) = m_s + m_l \tag{4.114}$$

We will shortly see that we can easily estimate m_s. On the other hand, the estimation of m_l can be pushed to the next RG level. Namely,

$$\sum_{\Gamma\,:T(\Gamma)=\Gamma'} \nu_l(\Gamma) \leq \nu'(\Gamma') \tag{4.115}$$

and

$$m_l \leq \nu'(\Omega) \tag{4.116}$$

To see why (4.115) holds, just consider the first two steps of the RG procedure. The point is that the G_0 contributing to ν_l, as they contain the support of a large component of Γ are never summed over in the first RG step. In the second step (the blocking) they contribute to terms in which G' is such that $\mathcal{L}G' \supset G \ni 0$, and in particular $G' \ni 0$. Therefore

$$Z \times \sum_{\Gamma : T_2 T_1(\Gamma) = \tilde{\Gamma}'} \nu_l(\Gamma) \leq e^{-\beta'(\tilde{S}', V(\Gamma'))} \sum_{G' \supset \underline{\tilde{\Gamma}'}} \rho'(\tilde{\Gamma}', G') \mathbb{1}_{G' \ni 0} \tag{4.117}$$

In the third step, finally, the number of terms on the right can only be increased, while the constant produced by centering the small fields cancels against corresponding change of the partition function. This then yields (4.115).

(4.116) is understood in much the same way. The set γ_0 is not summed away in the first step. On the other hand, g_0 contains a small connected component γ_0 whose interior contains the origin. By the geometric smallness of such a component, $\ll^{-1} \underline{\Gamma_0} = \{0\}$ and so $\ll^{-1} G_0 \ni 0$, which implies (4.116).

Iterating these two relations, we get, in analogy to (4.108)

$$\nu^{(l+1)}(\mathbb{1}) \leq \sum_{j=l+1}^{M-1} \nu_s^{(j)}(\mathbb{1}) + \nu^{(M)}(\mathbb{1}) \tag{4.118}$$

where the superscripts of course refer to the RG level. Combining all this, we get

$$\begin{aligned} &\mu_{\mathcal{L}^M 0}\left(\gamma_0 \not\subset \mathcal{L}^k 0\right) \\ &\leq \sum_{l=k}^{M-1} \left[m^{(l)}_{s,\mathcal{L}^M 0} + \sum_{j=l+1}^{M-1} \nu^{(j)}_{s,\mathcal{L}^M 0}(\mathbb{1}) + \nu^{(M)}_{\mathcal{L}^M 0}(1) \right] + \mu^{(M)}_{\mathcal{L}^M 0}\left(\underline{\Gamma} \ni 0\right) \\ &= \sum_{l=k}^{M-1} m^{(l)}_{s,\mathcal{L}^M 0} + \sum_{j=k+1}^{M-1} (j-k) \nu^{(j)}_{s,\mathcal{L}^M 0}(\mathbb{1}) + (M-k) \nu^{(M)}_{\mathcal{L}^M 0}(1) + \mu^{(M)}_{\mathcal{L}^M 0}\left(\underline{\Gamma} \ni 0\right) \end{aligned} \tag{4.119}$$

All the terms appearing in this final bound can be estimated directly, i.e. without recourse to further renormalization. Again, of course these bounds are probabilistic. WE formulate them as

Lemma 4.11: *Let $F_{l,M} \subset \mathring{A}$ be defined as in Lemma 4.9. Then there exist a positive constant $\bar{b} > 0$ such that*

$$\begin{aligned} &\left\{ m^{(l)}_{s,\mathcal{L}^M 0} \geq e^{-\bar{b}\tilde{\beta}^{(l)}} \right\} \subset F_{l,M} \\ &\left\{ \nu^{(l)}_{s,\mathcal{L}^M 0}(1) \geq e^{-\bar{b}\tilde{\beta}^{(l)}} \right\} \subset F_{l,M} \end{aligned} \tag{4.120}$$

and

$$
\begin{aligned}
&\left\{\nu^{(M)}_{\mathcal{L}^M 0}(1) \geq e^{-\bar{b}\tilde{\beta}^{(M)}}\right\} \subset F_{M,M} \\
&\left\{\mu^{(M)}_{\mathcal{L}^M 0}(\underline{\Gamma} \ni 0) \geq e^{-\bar{b}\tilde{\beta}^{(M)}}\right\} \subset F_{M,M}
\end{aligned}
\tag{4.121}
$$

Proof: Relations (4.121) are trivial to verify as they refer to systems with a single lattice site. The proof of the two relations (4.120) are very similar and we will present the details only for the first relation. It is very much in the spirit of a Peierls argument or Ruelle's superstability estimates. We suppress again the level index l in our natation. Clearly

$$
m_s = \frac{1}{Z} \sum_{\substack{\gamma_0 \text{ small} \\ \text{int } \underline{\gamma_0} \ni 0}} \sum_{\substack{G_0 \supset \underline{\gamma_0} \\ G_0 \text{ conn.}}} \sum_{\substack{\Gamma_0^s : \underline{\Gamma_0^s} \subset G_0 \\ \gamma_0 \subset \Gamma_0^s}} \rho(\Gamma_0^s, G_0) \times \sum_{G \cap \overline{G_0} = \emptyset} \sum_{\substack{\Gamma : \underline{\Gamma} \subset G \\ \text{int } \underline{\Gamma} \not\supset G_0}} \rho(\Gamma, G) e^{-\beta(S, V(\Gamma \cup \Gamma_0^s))}
\tag{4.122}
$$

Note that the second line almost reconstitutes a partition function outside the region G_0, except for the (topological) constraint on the support of Γ and the fact that the field term is not the correct one. This latter problem can be repaired by noting that

$$
(S, V(\Gamma \cup \Gamma_0^s)) = (S, V(\Gamma) \backslash G_0) + \sum_{h \in \mathbb{Z}} \sum_{\substack{C \subset V_h(\Gamma \cup \Gamma_0^s) \\ C \cap G_0 \neq \emptyset}} S_C(h)
\tag{4.123}
$$

The second term on the right consists of a local term (i.e. involving only C consisting of a single site x) which depends only on Γ_0^s, and the non-local one, which as in the previous instances is very small, namely

$$
\Big| \sum_{h \in \mathbb{Z}} \sum_{\substack{C \subset V_h(\Gamma \cup \Gamma_0^s),\ |C| \geq 2 \\ C \cap G_0 \neq \emptyset}} S_C(h) \Big| \leq Const\ |G_0| e^{-\tilde{\beta}}
\tag{4.124}
$$

Thus we get the upper bound

$$
\begin{aligned}
m_s \leq & \sum_{\substack{\gamma_0 \text{ small} \\ \text{int } \underline{\gamma_0} \ni 0}} \sum_{\substack{G_0 \supset \underline{\gamma_0} \\ G_0 \text{ conn.}}} \sum_{\substack{\Gamma_0^s : \underline{\Gamma_0^s} \subset G_0 \\ \gamma_0 \subset \Gamma_0^s}} \rho(\Gamma_0^s, G_0) e^{-\beta(S_{loc}, V(\Gamma_0^s) \cap G_0)} e^{Const\ |G_0| e^{-\tilde{\beta}}} \\
& \times \frac{1}{Z} \sum_{G \cap \overline{G_0} = \emptyset} \sum_{\substack{\Gamma : \underline{\Gamma} \subset G \\ \text{int } \underline{\Gamma} \not\supset G_0}} \rho(\Gamma, G) e^{-\beta(S, V(\Gamma) \backslash G_0)}
\end{aligned}
\tag{4.125}
$$

Now the last line has the desired form. A slight problem here is that the contours contributing to the denominator are not (in general) allowed to have empty support in G_0, as the support of any Γ must contain $D(\Gamma)$. Note however that G_0 is necessarily such that $D(0) \cap G_0 \subset \mathcal{D}(0)$, as otherwise G_0 would have to contain support from large contours. Thus for given G_0, we may bound the partition function from below by summing only over such contours that within G_0 have $h_x(\Gamma) \equiv 0$

and those have support in G_0 is exactly given by $\mathcal{D}(0) \cap G_0$. Treating the small-field term as above this gives the lower bound on the partition function

$$
\begin{aligned}
Z \geq & \prod_{i: \mathcal{D}_i(0) \subset G_0} \rho(\mathcal{D}_i(0), \mathcal{D}_i(0)) e^{-\beta \sum_{x \in G_0} S_x(0)} e^{-Const\ |G_0| e^{-\tilde{\beta}}} \\
& \times \sum_{G \cap \overline{G_0} = \emptyset} \sum_{\substack{\Gamma : \underline{\Gamma} \subset G \\ \text{int}\, \underline{\Gamma} \not\supset G_0}} \rho(\Gamma, G) e^{-\beta(S, V(\Gamma) \setminus G_0)}
\end{aligned}
\tag{4.126}
$$

and so

$$
\begin{aligned}
m_s \leq & \frac{1}{Z} \sum_{\substack{\gamma_0 \text{ small} \\ \text{int}\, \underline{\gamma_0} \ni 0}} \sum_{\substack{G_0 \supset \underline{\gamma_0} \\ G_0 \text{ conn.}}} \sum_{\substack{\Gamma_0^s : \underline{\Gamma_0^s} \subset G_0 \\ \gamma_0 \subset \Gamma_0^s}} e^{2\ Const\ |G_0| e^{-\tilde{\beta}}} e^{-\beta(S_{loc}, V(\Gamma_0^s) \cap G_0) + \beta \sum_{x \in G_0} S_x(0)} \\
& \times \frac{\rho(\Gamma_0^s, G_0)}{\prod_{i: \mathcal{D}_i(0) \subset G_0} \rho(\mathcal{D}_i(0), \mathcal{D}_i(0))}
\end{aligned}
\tag{4.127}
$$

Here the ρ's appearing in the denominator are exactly those for which we have lower bounds. Note that for this reason we could not deal directly we expressions in which G_0 is allowed to contain also large components of Γ. The estimation of the sums in (4.128) is now performed just like in the absorbtion of small contours RG step. Γ_0^s with non-constant heights give essentially no contribution, and due to the separatedness of the components $\mathcal{D}_i(0)$, and the smallness of the total control field on one such component, the main contribution comes from the term where Γ_0^s has support in only one component $\mathcal{D}_i(0)$. If there is such a component which surrounds 0, this could of course give a contribution of order one. But on $F_{l,M}$ this precisely is excluded, so that G_0 cannot be contained in $\mathcal{D}(0)$ and therefore

$$
m_{s,\mathcal{L}^M 0}^{(l)} \leq Const\ e^{-\tilde{\beta}^{(l)}}
\tag{4.128}
$$

as claimed. $\diamondsuit$

From Lemma 4.11 and the bound (4.18) Lemma 4.9 follows immediately. $\diamondsuit$

To conclude, we give the proof of Lemma 4.10.

Proof: (of Lemma 4.10) Let $G \subset \mathcal{L}^k 0$ be a subset containing the origin. Denote by $\mathcal{H}_{G,M} \subset \mathbb{Z}^G$ the set of height configurations $h_G \equiv \{h_x\}_{x \in G}$ on G, s.t. the restriction of the associated [8]contour on $\mathcal{L}^M 0$ has a connected component whose interior is G. We need only consider the case of $\mathcal{H}_{G,M} \neq \emptyset$. Then

$$
\begin{aligned}
& \mu_{\mathcal{L}^M 0} \left(|h_0| \middle| \exists \gamma \subset \Gamma,\ \text{int}\, \underline{\gamma} = G, h_\gamma = 0 \right) \\
& = \frac{\sum_{\{h_x\}_{x \in G} \in \mathcal{H}_{G,M}} |h_0| e^{-\beta \left(E_s(\{h_x\}_{x \in G}) + \sum_{x \in G} J_x(h_x) \right)}}{\sum_{\{h_x\}_{x \in G} \in \mathcal{H}_{G,M}} e^{-\beta \left(E_s(\{h_x\}_{x \in G}) + \sum_{x \in G} J_x(h_x) \right)}}
\end{aligned}
\tag{4.129}
$$

[8] Here we mean of course the initial mapping of the SOS-model to the contour model as described in Section II.3.

Hence we can write for the associated events in the probability space of the disorder

$$\begin{aligned}
&\left\{\mu_{\mathcal{L}^M 0}\left(|h_0|\Big|\exists\gamma\subset\Gamma,\ \mathrm{int}\,\underline{\gamma}=G, h_\gamma=0\right)\geq\delta\right\}\\
&\subset\left\{\sum_{\{h_x\}_{x\in G}\in\mathcal{H}_{G,M},|h_0|>\delta}\left(|h_0|-\delta\right)e^{-\beta\left(E_s(\{h_x\}_{x\in G})+\sum_{x\in G}J_x(h_x)\right)}\right.\\
&\qquad\left.\geq\delta\sum_{\{h_x\}_{x\in G}\in\mathcal{H}_{G,M},|h_0|<\delta}e^{-\beta\left(E_s(\{h_x\}_{x\in G})+\sum_{x\in G}J_x(h_x)\right)}\right\}
\end{aligned}\tag{4.130}$$

We estimate the last sum by the term for the 'checkerboard-like' height configuration $\{h_x(G)\}_{x\in G}\in\mathcal{H}_{G,M}$ which is given by $h_x(G)=(1+d(x,G^c))\bmod 2$. To estimate this term, note further that

$$E_s\left(\{h_x(G)\}_{x\in G}\right)+\sum_{x\in G}J_x(h_x(G))\leq 2dL^{dk}+\sum_{x\in G}\sup\{J_x(0),J_x(1)\}\tag{4.131}$$

Using Lemma 3.3 to estimate the surface energy term as in (3.116) one obtains from (4.128) that

$$\left\{\mu_{\mathcal{L}^M 0}\left(|h_0|\Big|\exists\gamma\subset\Gamma,\ \mathrm{int}\,\underline{\gamma}=G, h_\gamma=0\right)>\delta\right\}\subset\left\{\sum_{x\in G}I_x^k\leq-\left(\frac{\delta}{2}-2dL^{dk}\right)\right\}\tag{4.132}$$

where

$$I_x^k\equiv-\frac{1}{\beta}\ln\left(\sum_{h\in\mathbb{Z}}e^{-\beta\left(\frac{d|h|}{L^k}+J_x(h)-\sup\{J_x(0),J_x(1)\}\right)}\right)\leq 0\tag{4.133}$$

for $x\neq 0$ and

$$I_0^k\equiv I_0^k(\delta)\equiv-\frac{1}{\beta}\ln\left(\sum_{h\in\mathbb{Z},\ |h|>\delta}\left(\frac{|h|}{\delta}-1\right)e^{-\beta(|h|/2+J_0(h)-\sup\{J_0(0),J_0(1)\})}\right)\tag{4.134}$$

We choose the constant in Lemma 4.10 such that $\frac{\delta}{2}-2dL^{dk}\geq\frac{\delta}{4}$. Since $I_x^k\leq 0$ from (4.132) follows then

$$\begin{aligned}
&\mathbb{P}\left[\sup_{M\geq k}\sup_{\substack{G\subset\mathcal{L}^k 0, G\ni 0\\ \exists\gamma\subset\Gamma,\mathrm{int}\,\underline{\gamma}=G,h_\gamma=0}}\mu_{\mathcal{L}^M 0}\left(|h_0|\Big|\exists\gamma\subset\Gamma,\ \mathrm{int}\,\underline{\gamma}=G, h_\gamma=0\right)\geq\delta\right]\\
&\leq\mathbb{P}\left[\bigcup_{\substack{G\subset\mathcal{L}^k 0\\ G\ni 0}}\left\{\sum_{x\in G}I_x^k\leq-\frac{\delta}{4}\right\}\right]\leq\mathbb{P}\left[\sum_{x\in\mathcal{L}^k 0\setminus\{0\}}I_x^k\leq-\frac{\delta}{8}\right]+\mathbb{P}\left[I_0^k\leq-\frac{\delta}{8}\right]
\end{aligned}\tag{4.135}$$

The random variables I_x^k can be controlled in a standard way. As an example we treat the first term. For positive α we introduce the constants $\alpha_h=\tanh\frac{\alpha}{2}\cdot e^{-\alpha|h|}$ where the tanh is just the proper normalization to have $\sum_{h\in\mathbb{Z}}\alpha_h=1$. Then, for $x\neq 0$,

$$\begin{aligned}
&\mathbb{P}\left[I_x^k\leq-\epsilon\right]\leq\sum_{h\in\mathbb{Z}}\mathbb{P}\left[e^{-\beta\left(\frac{d|h|}{L^k}+J_x(h)-\sup\{J_x(0),J_x(1)\}\right)}\geq\alpha_h e^{\beta\epsilon}\right]\\
&=\sum_{h\in\mathbb{Z}}\mathbb{P}\left[J_x(h)-\sup\{J_x(0),J_x(1)\}\leq-\epsilon-\left(\frac{d}{2L^k}-\frac{\alpha}{\beta}\right)|h|-\frac{1}{\beta}\ln\left(\tanh\frac{\alpha}{2}\right)\right]
\end{aligned}\tag{4.136}$$

Setting $\alpha = \frac{d\beta}{4L^k}$ this gives, for $\epsilon \geq -\frac{2}{\beta}\ln\left(\tanh\frac{d\beta}{8L^k}\right) \equiv \epsilon_{min}$,

$$I\!\!P\left[I_x^k \leq -\epsilon\right] \leq \sum_{h\in\mathbb{Z}} I\!\!P\left[J_x(h) - \sup\{J_x(0), J_x(1)\} \leq -\frac{\epsilon}{2} - \frac{d|h|}{4L^k}\right] \tag{4.137}$$

We use our hypothesis on the distribution of J, and, to treat the x-sum, convert the resulting bounds on probabilities into bounds on the Laplace transform. Doing so we obtain

$$\begin{aligned} I\!\!E[e^{-I_x^k t}] &\leq e^{\epsilon_{min} t} + t\int_{\epsilon_{min}}^{\infty} e^{t\epsilon} I\!\!P\left[I_x^k < -\epsilon\right] d\epsilon \\ &\leq e^{\epsilon_{min} t} + Const\,\left(1+\sigma L^k\right)\sigma t e^{8\sigma^2 t^2} \end{aligned} \tag{4.138}$$

for $t \geq 0$. Setting $t_0 = \frac{z}{4\sigma^2 L^{dk}}$ the exponential Markov inequality gives for the x-sum of the variables

$$\begin{aligned} I\!\!P\left[\sum_{x\in\mathcal{L}^k 0\setminus\{0\}} I_x^k \leq -z\right] &\leq e^{-zt_0} \prod_{x\in\mathcal{L}^k 0\setminus\{0\}} I\!\!E[e^{-I_x^k t_0}] \\ \leq e^{-\frac{z^2}{8\sigma^2 L^{dk}}} &\left(e^{-\frac{1}{\beta}\ln\left(\tanh\frac{d\beta}{8L^k}\right)\frac{z}{2\sigma^2 L^{dk}}} + Const\,\left(1+\sigma L^k\right)\frac{z}{4\sigma L^{dk}} e^{\frac{z^2}{2\sigma^2 L^{2dk}}}\right)^{L^{dk}} \end{aligned} \tag{4.139}$$

for $z \geq 0$. For $z \geq Const\max\left\{\sigma L^{dk}, \sigma\sqrt{\ln\left(1+\sigma L^k\right)}L^{dk}\right\}$ the prefactor of the second exponential in the brackets can be absorbed in the exponential with a slightly enlarged σ^2. For $z \geq Const\frac{1}{\beta}L^{dk}\ln\left(L^k\right)$ the first exponential is dominated by the second one. Thus, for $\delta \geq Const\max\left\{\sigma L^{dk}, \sigma\sqrt{\ln\left(1+\sigma L^k\right)}L^{dk}, \frac{1}{\beta}L^{dk}\ln L^k\right\}$, the desired bound is obtained:

$$I\!\!P\left[\sum_{x\in\mathcal{L}^k 0\setminus\{0\}} I_x^k \leq -\frac{\delta}{8}\right] \leq e^{-\frac{\delta^2}{Const\sigma^2 L^{dk}}} \tag{4.140}$$

Thus the proof is nearly finished; using the same arguments the reader will in fact easily check that for δ as in the statement

$$I\!\!P\left[I_0^k \leq -\frac{\delta}{8}\right] \leq e^{-\frac{\delta^2}{Const\sigma^2}} \tag{4.141}$$

Thus the two bounds (4.140) and (4.141) can be summed to give (4.101).$\diamondsuit$

V. Concluding remarks

We have presented a renormalization group method suitable to prove the existence of low-temperature Gibbs states describing 'flat' interfaces in a SOS-model for surfaces in weakly random media in dimension $D \geq 4$. This consisted on realizing the SOS-model as a certain example of a class of contour models in dimension $d = D-1$ with non-compact state spaces. We have shown that we can construct, in an algorithmic way, RG maps that leave this class of models invariant. We have controlled the flow of the iterative application of these maps and have shown that under certain conditions on the initial model, this flow tends to a 'trivial' limit. This situation corresponds, in the language of classical probability theory, to a 'strong law of large numbers' type result. We have then shown how the control on this sequence of image systems can be used to obtain relevant information on the initial system.

It should be stressed that in a certain sense the situation we were dealing with is 'trivial' – in spite of the rather heavy machinery we needed to employ. What we mean by trivial here is that a single iteration of the RG brings the system much closer to the trivial one: temperatures are being reduced, variances of random fields are reduced, etc., so that, provided we can carry out *one* RG transformation, the subsequent iterations become more and more easy. For this reason, we could be rather generous in many of our estimates and even in the way we defined the RG maps themselves. It should be clear that there is a lot of room to improve things, if this is necessary for other applications or different models. One point, for instance, that may have annoyed some readers, is the appearance of the two inverse 'temperatures' β and $\tilde{\beta}$ that scale with different speeds to infinity. This is essentially due the way the 'coarse graining' step, or the absorption of small contours is performed, which still leaves untouched 'long and thin' contours as well as thin spikes emerging from fat contours, although such configurations cannot be provoked by 'small' random fields. A more extensive coarse graining could thus remove this artefact, if necessary or desired.

There are a number of direction to further generalize and develop this method. For one thing, one would like to prove the existence of Dobrushin states [Do] in the full-fledged disordered Ising model. We believe that such a proof is now actually within range. Another type of questions concerns systems with less 'symmetry', a simple example being already this SOS-model restricted to a half space, or other 'wetting-type' problems in disordered media. As mentioned in the introduction, this may be possible by merging in ideas from Pirogov-Sinai theory [Za1,Za2].

A particularly challenging problem is of course the analysis of the situation in lower dimension. Here one would no longer expect to have an infinite volume Gibbs or even ground state for the interface, but there should be some kind of scaling law for the interface fluctuation in finite volumes (a celebrated result of this type is the supposedly exact $L^{2/3}$ law in dimension $D = 2$ [FHH,KN]). One should say that the non-existence of a Gibbs state is likely to follow from arguments of Aizenman and Wehr [AW], but no formal proof has been given. An analysis of this regime through the RG method appears technically very hard, in that much sharper estimates would be required, but not entirely hopeless.

In the same spirit, the analysis of systems with genuinely 'strong' disorder remains a desideratum; here we have in mind in particular spin glass models. Although we are very far yet from treating such cases, the RG approach may prove a useful tool also there.

In conclusion, we hope that the present exposition of the RG method for disordered systems is convincing evidence for the power and flexibility of this technique and will help to make it a useful tool with numerous applications in this domain.

Appendix

This appendix contains the proofs of the four main geometrical lemmas used in Sections III and IV.

LEMMA 3.3: *Let γ be a weakly connected contour s.t. $d(\ \mathrm{int}\underline{\gamma}) \leq L$. Let h_γ denote the height of γ on $\partial\,\mathrm{int}\,\underline{\gamma}$. Then*

$$E_s(\gamma) \geq \frac{2d}{L} \sum_{x\in\,\mathrm{int}\,(\underline{\gamma})} |h_x(\gamma) - h_\gamma| \tag{A.1}$$

Proof: Without loss of generality we may assume that $h_\gamma = 0$ and that $\mathrm{int}\,\underline{\gamma}$ is contained in the cube $C_L \equiv [1, L]^d$. To prove the lemma, we then have to prove a lower bound on $\sum_{x\in C_L} |h_x|$ for any function h that vanishes outside this cube in terms of the surface energy. Let us write $x = (x_1, \ldots, x_d)$, let e_i denote the positive unit vectors in $I\!R^d$ and let $\hat{x}_i \equiv x - x_i e_i$ (i.e. the vector x with the i-th component set to zero. With this notation we have

$$\begin{aligned}
\sum_{x\in C_L} |h_x| &= \sum_{x\in C_L} \frac{1}{d}\sum_{i=1}^{d}\frac{1}{2}\left|\sum_{z_i=1}^{x_i}\left(h_{\hat{x}_i+z_ie_i} - h_{\hat{x}_i+(z_i-1)e_i}\right) + \sum_{z_i=x_i+1}^{L+1}\left(-h_{\hat{x}_i+z_ie_i} + h_{\hat{x}_i+(z_i-1)e_i}\right)\right| \\
&\leq \sum_{x\in C_L} \frac{1}{d}\sum_{i=1}^{d}\frac{1}{2}\left[\sum_{z_i=1}^{x_i}\left|h_{\hat{x}_i+z_ie_i} - h_{\hat{x}_i+(z_i-1)e_i}\right| + \sum_{z_i=x_i+1}^{L+1}\left|-h_{\hat{x}_i+z_ie_i} + h_{\hat{x}_i+(z_i-1)e_i}\right|\right] \\
&= \sum_{x\in C_L} \frac{1}{2d}\sum_{i=1}^{d}\sum_{z_i=1}^{L+1}\left|h_{\hat{x}_i+z_ie_i} - h_{\hat{x}_i+(z_i-1)e_i}\right| \\
&= \frac{L}{2d}\sum_{<x,y>\in\overline{C_L}} |h_x - h_y|
\end{aligned} \tag{A.2}$$

where we have used the fact that the i-th term in the one-but-last line is independent of x_i; thus the summation over x_i gives a factor L, while the remaining sums together with the sum over z_i gives the part of the surface energy coming from the steps in the i-th direction. All terms together then yield the entire surface energy. This obviously proves the lemma.$\diamondsuit$

LEMMA 3.6: *Let h be any integer-valued height-function, and set $h' \equiv \mathrm{Rnd}\left(\overline{h}\right)$ where $\overline{h} \equiv L^{-d}\sum_{x\in\mathcal{R}0} h_x$. Then*

$$\sum_{<x,y>:x,y\in\mathcal{R}0} |h_x - h_y| \geq \frac{1}{L}\sum_{x\in\mathcal{L}0} |h_x - h'| \tag{A.3}$$

Proof: To prove this lemma, we will first proof that

$$\sum_{<x,y>:x,y\in\mathcal{R}0} |h_x - h_y| \geq \frac{2}{L}\sum_{x\in\mathcal{L}0} |h_x - \overline{h}| \tag{A.4}$$

for any function h (not necessarily integer-valued. Note that (A.3) follows immediately from (A.4) for integer valued h: By definition, h' is the integer closest to $\overline{h}$, so in particular for any integer h_x, $|h_x - h'| \leq |\overline{h} - h'|$. Thus $|h_x - h'| \leq |h_x - \overline{h}| + |\overline{h} - h'| \leq 2||\overline{h} - h'|$, which inserted into (A.4) gives (A.3).

We are thus left with proving (A.4). This will be done by induction over the dimension. Let first $d = 1$. Without loss of generality, we may assume $\overline{h} = 0$. Let $n_\pm$ denote the number of sites where h_x is positive or negative, respectively; set $\overline{h}_+ \equiv \frac{1}{n_+}\sum_{x=1}^L h_x \mathbb{I}_{h_x>0}$ and define $\overline{h}_-$ analogously. Then

$$\begin{aligned}\sum_{x=1}^L |h_x| &= n_+\overline{h}_+ - n_-\overline{h}_- = n_+\left(\overline{h}_+ - \frac{n_+\overline{h}_+}{L} - \frac{n_-\overline{h}_-}{L}\right) - n_-\left(\overline{h}_- - \frac{n_+\overline{h}_+}{L} - \frac{n_-\overline{h}_-}{L}\right) \\ &= \frac{2}{L}n_+n_-(\overline{h}_+ - \overline{h}_-) \leq \frac{L}{2}(h_{max} - h_{min})\end{aligned} \tag{A.5}$$

where we have used that $n_+\overline{h}_+ + n_-\overline{h}_- = L\overline{h} = 0$, that $n_+n_- = n_+(L - n_-) \leq \frac{L^4}{4}$ and that $\overline{h}_+ - \overline{h}_- \leq h_{max} - h_{min}$. Now, obviously,

$$h_{max} - h_{min} \leq \sum_{x=2}^L |h_x - h_{x-1}| \tag{A.6}$$

which gives (A.4) for $d = 1$.

Assume now that (A.4) holds for $d - 1$. We will show that it holds for d. Let us write for $x \in \mathbb{Z}^d$, $x = (\hat{x}, t)$ with $\hat{x} \in \mathbb{Z}^{d-1}$, $t \in \mathbb{Z}$. Define $\overline{h}_t \equiv L^{1-d}\sum_{\hat{x}\in\{1,\dots,L\}^{d-1}} h_{\hat{x},t}$. Then clearly

$$\begin{aligned}\sum_{x\in C_L} |h_x - \overline{h}| &= \sum_{t=1}^L \sum_{\hat{x}\in\{1,\dots,L\}^{d-1}} \left|h_{\hat{x},t} - \overline{h}_t + \overline{h}_t - \overline{h}\right| \\ &\leq \sum_{t=1}^L \left(\sum_{\hat{x}\in\{1,\dots,L\}^{d-1}} \left|h_{\hat{x},t} - \overline{h}_t\right|\right) + L^{d-1}\sum_{t=1}^L \left|\overline{h}_t - \overline{h}\right|\end{aligned} \tag{A.7}$$

Now in the first term, for each fixed t we may apply (A.4) for $d - 1$, so that

$$\sum_{t=1}^L \left(\sum_{\hat{x}\in\{1,\dots,L\}^{d-1}} \left|h_{\hat{x},t} - \overline{h}_t\right|\right) \leq \frac{L}{2}\sum_{t=1}^L \sum_{<\hat{x},\hat{y}>} |h_{\hat{x},t} - h_{\hat{y},t}| \tag{A.8}$$

while for the second term the one-dimensional version of (A.4) can be applied, giving

$$\begin{aligned}L^{d-1}\sum_{t=1}^L \left|\overline{h}_t - \overline{h}\right| &\leq L^{d-1}\frac{L}{2}\sum_{t=2}^L \left|\overline{h}_t - \overline{h}_{t-1}\right| \\ &\leq \frac{L}{2}\sum_{\hat{x}\in\{1,\dots,L\}^{d-1}} \sum_{t=2}^L |h_{\hat{x},t} - h_{\hat{x},t-1}|\end{aligned} \tag{A.9}$$

Obviously, the sum of the terms in (A.8) and (A.9) is bounded by $\frac{L}{2}\sum_{<x,y>}|h_x - h_y|$, which gives (A.4) for d and concludes the proof of the lemma. $\diamondsuit$

Note that the bounds given by the previous two lemmas are optimal since it is not difficult to construct configurations for which equality holds.

LEMMA 3.7: *Let $\Gamma \in \mathcal{R}^{-1}\gamma'$. Then*

$$E_s(\Gamma) \geq \frac{L^{d-1}}{d+1} E_s(\gamma') \tag{A.10}$$

Proof: Set $\overline{h}'_y = L^{-d}\sum_{x\in\mathcal{L}y} h_x$ and $E_s(\overline{\gamma}') = \sum_{<z,w>:z,w\in\mathcal{L}\underline{\overline{\gamma'}}} |\overline{h}'_z - \overline{h}'_w|$. We will first show that

$$E_s(\Gamma) \geq L^{d-1} E_s(\overline{\gamma}') \tag{A.11}$$

In fact, this is quite easy. Just write

$$E_s(\overline{\gamma}') = \sum_{i=1}^{d}\sum_{\hat{y}_i}\sum_{y_i}\left|\overline{h}'_{\hat{y}_i,y_i} - \overline{h}'_{\hat{y}_i,y_i-1}\right| \tag{A.12}$$

By an argument quite similar to the one used in the previous proof, we have that

$$\sum_{y_i}\left|\overline{h}'_{\hat{y}_i,y_i} - \overline{h}'_{\hat{y}_i,y_i-1}\right| \leq \sum_{y_i}\sum_{x_i=L(y_i)-1}^{Ly_i}\left|\tilde{h}_{\hat{y}_i,x_i} - \tilde{h}_{\hat{y}_i,x_i-1}\right| \tag{A.13}$$

where $\tilde{h}_{\hat{y}_i,x_i} \equiv L^{-d+1}\sum_{\hat{x}_i\in\mathcal{L}\hat{y}_i} h_{\hat{x}_i,x_i}$ with the obvious meaning of the notation for the summation range. From (A.13) (A.11) follows now simply by inserting this definition and using the triangle inequality.

We now have to cope with the fact that in $E_s(\gamma)$ enter the rounded means of the heights rather than the block means themselves. The basic idea here is that this may cause a problem only if these means are far from integers, in which case the height within such a block has been very non-constant. Indeed, using lemma 3.6, we may get another lower bound on $E_s(\Gamma)$, namely

$$\begin{aligned} E_s(\Gamma) &\geq \frac{2}{L}\sum_{y\in\underline{\overline{\gamma'}}}\sum_{x\in\mathcal{L}y}|h_x - \overline{h}'_y| \geq \frac{2}{L}\sum_{y\in\underline{\overline{\gamma'}}}\sum_{x\in\mathcal{L}y}|h'_y - \overline{h}'_y| \\ &= 2L^{d-1}\sum_{y\in\underline{\overline{\gamma'}}}|h'_y - \overline{h}'_y| = 2L^{d-1}\frac{1}{2d}\sum_{<y,z>\in\underline{\overline{\gamma'}}}|h'_y - \overline{h}'_y| + |h'_z - \overline{h}'_z| \end{aligned} \tag{A.14}$$

where we have again used that h'_y is the closest integer to $\overline{h}'_y$. Now

$$\begin{aligned}
E_s(\Gamma) &= \frac{1}{d+1}\frac{d}{d+1}E_s(\Gamma) + \frac{d}{d+1}E_s(\Gamma) \\
&\geq \frac{L^{d-1}}{d+1} \sum_{<y,z>:y,z\in\mathcal{L}\underline{\overline{\gamma'}}} \left| h'_y - h'_z - h'_y + \overline{h}'_y + h'_z - \overline{h}'_z \right| \\
&+ L^{d-1}\frac{1}{d+1} \sum_{<y,z>\in\underline{\overline{\gamma'}}} |h'_y - \overline{h}'_y| + |h'_z - \overline{h}'_z| \\
&\geq \frac{L^{d-1}}{d+1} \sum_{<y,z>:y,z\in\mathcal{L}\underline{\overline{\gamma'}}} \left| h'_y - h'_z \right|
\end{aligned} \tag{A.15}$$

using again the triangle inequality for the last inequality. This proves the lemma. $\diamondsuit$

LEMMA 3.8: *Let γ be connected and large. Then*

$$L^{(1-\alpha)/2}E_s(\gamma) + |\underline{\gamma}\backslash\overline{D}(\gamma)| \geq \frac{1}{2}|\underline{\gamma}\cap\overline{\mathcal{D}}(\gamma)| \tag{A.16}$$

Proof: The following two properties of the sets $\mathcal{D}(h)$ are the essential ingredients in the proof of this lemma:

(i) (sparsity of $\mathcal{D}(h)$): If $C \subset \Lambda$ is connected and $d(C) \geq L^{\frac{1-\alpha/2}{2}}$ then $|C\backslash\overline{\mathcal{D}}(h)| \geq |C|/2$ for L large enough.

This follows from the definition of $\mathcal{D}(h)$ as a union of $L^{\frac{1}{2}}$-components each of which has a maximum volume of $L^{\frac{1-\alpha}{2}}$.

(ii) (separation of $\overline{\mathcal{D}}(h)$ and $\overline{D}(h)\backslash\overline{\mathcal{D}}(h)$): If $C \subset \Lambda$ is connected s.t. $C \cap \overline{\mathcal{D}}(h) \neq \emptyset$ and $|C \cap (\overline{\overline{D}}(h)\backslash\overline{\mathcal{D}}(h))| \neq \emptyset$ then $d(C) \geq L^{\frac{1-\alpha/2}{2}}$ and hence the conclusion of (i) holds for C.

This follows from the definition of $\mathcal{D}(h)$ as a union of $L^{\frac{1}{2}}$-components which implies that

$$d(D(h)\backslash\mathcal{D}(h), \mathcal{D}(h)) \geq L^{\frac{1}{2}} \tag{A.17}$$

We must now distinguish the cases where γ is flat or not. In the first case, the proof is in fact identical to the one given in [BK] and we repeat it here only for the convenience of the reader.

In this case we write $\gamma = (\underline{\gamma}, h_x \equiv h)$. Assume first that the set $\underline{\gamma}\backslash(\overline{D}(h)\backslash\overline{\mathcal{D}}(h))$ is nonempty and denote its connected components by C_i. Then

$$\begin{aligned}
|\underline{\gamma}\backslash\overline{D}(h)| &= \sum_i |C_i\backslash\overline{D}(h)| \geq \sum_{i:C_i\cap\overline{\mathcal{D}}(h)\neq\emptyset} |C_i\backslash\overline{D}(h)| \\
&\geq \sum_{i:C_i\cap\overline{\mathcal{D}}(h)\neq\emptyset} \frac{1}{2}|C_i\cap\overline{\mathcal{D}}(h)| = \frac{1}{2}|\underline{\gamma}\cap\overline{\mathcal{D}}(h)|
\end{aligned} \tag{A.18}$$

where property (ii) was used.

Assume next that $|\underline{\gamma}\backslash(\overline{D}(h)\backslash\overline{\mathcal{D}}(h))| = \emptyset$. Now if $\underline{\gamma}\cap\overline{\mathcal{D}}(h) = \emptyset$, then there (A.16) is trivial. Thus we may assume the contrary. Now since γ is large, either $d(\underline{\gamma}) \geq L-2$ or $(D(\gamma)\backslash\mathcal{D}(\gamma))\cap V_i(\gamma) \neq \emptyset$. In the first case, (A.16) follows by property (i) while in the second it follows from property (ii). Thus, (A.16) is proven for flat contours.

Now consider the case that γ is not flat. We cannot say anything a priori about the sets $D(\gamma)$ and $\mathcal{D}(\gamma)$ for general γ since the defining geometrical properties of $\mathcal{D}$ only refer to the slices $D(h)$ at fixed height. However, all fluctuations in the heights introduce surface energy terms which ensure the validity of (A.16). Define $V_{h,i}(\gamma)$ to be the connected components of $V_h(\gamma)\cap\underline{\gamma}$. Notice that $E_s(\gamma)$ is clearly bounded from below by one-half times the number of such connected components Then,

$$
\begin{aligned}
&L^{(1-\alpha)/2}E_s(\gamma) + |\underline{\gamma}\backslash\overline{D}(\gamma)| \\
&\geq \sum_{h,i:d(V_{h,i}(\gamma))\leq L^{1/2}} \frac{1}{2}L^{(1-\alpha)/2} + \sum_{h,i:d(V_{h,i}(\gamma))>L^{1/2}} |V_{h,i}(\gamma)\backslash\overline{D}(h)| \\
&\geq \frac{1}{2}\sum_{h,i:d(V_{h,i}(\gamma))\leq L^{1/2}} |V_{h,i}(\gamma)\cap\overline{\mathcal{D}}(h)| + \frac{1}{2}\sum_{h,i:d(V_{h,i}(\gamma))>L^{1/2}} |V_{h,i}(\gamma)\cap\overline{\mathcal{D}}(h)| \\
&= \frac{1}{2}|\underline{\gamma}\cap\overline{\mathcal{D}}(\gamma)|
\end{aligned}
\tag{A.19}
$$

Here, the estimation in the second last line follows since, for $d(V_{h,i}(\gamma)) \leq L^{1/2}$, $|V_{h,i}(\gamma)\cap\overline{\mathcal{D}}(h)|$ contains at most $L^{(1-\alpha)/2}$ sites, due to the definition of $\mathcal{D}$ (definition 3.4), while for $d(V_{h,i}(\gamma)) > L^{1/2}$ we can apply the arguments of the flat case to obtain $|V_{h,i}(\gamma)\backslash\overline{D}(h)| \geq \frac{1}{2}|V_{h,i}(\gamma)\cap\overline{\mathcal{D}}(h)|$. This in fact concludes the proof of the lemma. $\diamondsuit$

LEMMA 3.9: *Let $\Gamma \in {T_2}^{-1}\gamma'$. Then there exists a constant $c_6 > 0$ s.t.*

$$
LE_s(\Gamma) + |\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma))| \geq c_6 L|\underline{\gamma'}\backslash\overline{\hat{D}'}(\gamma')|
\tag{A.20}
$$

where $\hat{D}' \equiv \mathcal{L}{-}1(D\backslash\mathcal{D})$.

Proof: Let us define

$$
\begin{aligned}
\hat{\mathcal{D}}'(h) &\equiv \mathcal{L}^{-1}\left(D(h)\backslash\mathcal{D}(h)\right) \\
\hat{\mathcal{D}}' &\equiv \bigcup_{h\in\mathbb{Z}}\left(\hat{\mathcal{D}}'(h)\times\{h\}\right)
\end{aligned}
\tag{A.21}
$$

We consider the partition into connected components

$$
\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma)) = \bigcup_j X_j
\tag{A.22}
$$

The idea of the proof is to write the l.h.s. of (A.20) as a sum over the X_j's and use for those parts of them which are 'well inside' the flat regions of Γ the same arguments as [BK], while for those parts which are 'near' to a region of the contour which is not entirely flat the surface energy will provide a sufficiently large contribution. Denote the set of blocks in which Γ has no constant height by $T(\Gamma)$, i.e.

$$T(\Gamma) \equiv \{y \in \underline{\gamma}', \nexists h \in \mathbb{Z} : \mathcal{L}y \subset V_h(\Gamma)\} \tag{A.23}$$

Now, for X_j such that $X_j \cap \mathcal{L}\overline{T(\Gamma)} = \emptyset$ with $d(X_j) \leq \frac{L}{4}$, we can apply the same arguments as [BK] to show that they are contained in $\mathcal{L}\hat{\overline{D}}'(\gamma')$ and thus do not contribute to the right-hand side of (A.20). Namely, let γ be a large connected component of Γ s.t. $X_j \subset \underline{\gamma}$. Then if $X_j \cap \mathcal{L}\overline{T(\Gamma)} = \emptyset$, then there is some $h \in \mathbb{Z}$ such that $h_x(\gamma') = h$ for $x \in \mathcal{L}(\mathcal{L}^{-1}X_j)$. We claim that this implies $\overline{X_j} \cap \left(\overline{D}(h)\backslash\overline{\mathcal{D}}(h)\right) \neq \emptyset$. For, if this was not true, since X_j is a component of $\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma))$, we would have $\gamma = (\underline{\gamma}, h_x \equiv h)$ s.t. $\underline{\gamma} \subset \overline{X_j}$. But since $d(X_j) \leq \frac{L}{4}$ this would be in contradiction to the assumed largeness of γ. Now from the fact that $\overline{X_j} \cap \left(\overline{D}(h)\backslash\overline{\mathcal{D}}(h)\right) \neq \emptyset$ it follows obviously that $X_j \subset \mathcal{L}\hat{\overline{D}}'(\gamma')$. We remark here that we have written $\overline{T(\Gamma)}$ to ensure that the parts of $\hat{\overline{D}}'(\gamma')$ which absorb such X_j are in fact at the same uniform height as Γ is on X_j.

Next, consider the components X_j s.t. $X_j \cap \mathcal{L}\overline{T(\Gamma)} \neq \emptyset$. If $X_j \subset \mathcal{L}\overline{\overline{\overline{T(\Gamma)}}}$, [9]we will forget its contribution to the second term on the l.h.s. of (A.20), but only use the surface energy term to estimate is contribution to the r.h.s. of (A.20) from above. If $X_j \not\subset \mathcal{L}\overline{\overline{\overline{T(\Gamma)}}}$, we decompose

$$X_j \backslash \mathcal{L}\overline{T(\Gamma)} = \bigcup_k Z_{j,k} \tag{A.24}$$

Note that for those $Z_{j,k}$ with $Z_{j,k} \backslash \mathcal{L}\overline{\overline{\overline{T(\Gamma)}}} \neq \emptyset$, we have $d(Z_{j,k}) \geq L$. Thus we obtain

$$\begin{aligned} |\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma))| &\geq \sum_{X_j : X_j \cap \mathcal{L}\overline{T(\Gamma)} = \emptyset, d(X_j) \geq L/4} |X_j| + \sum_{Z_{j,k} : Z_{j,k} \backslash \mathcal{L}\overline{\overline{\overline{T(\Gamma)}}} \neq \emptyset} |Z_{j,k}| \\ &\geq \bar{c}L \left(\sum_{X_j : X_j \cap \mathcal{L}\overline{T(\Gamma)} = \emptyset, d(X_j) \geq L/4} |\mathcal{L}^{-1}X_j| + \sum_{Z_{j,k} : Z_{j,k} \backslash \mathcal{L}\overline{\overline{\overline{T(\Gamma)}}} \neq \emptyset} |\mathcal{L}^{-1}Z_{j,k}| \right) \\ &\geq \bar{c}|(\underline{\gamma}' \backslash \hat{\overline{D}}'(\gamma')) \backslash \overline{\overline{\overline{T(\Gamma)}}}| \end{aligned} \tag{A.25}$$

with some constant $\bar{c}$, where the second inequality follows since the diameter of all sets involved is or the order L and the third inequality follows from the previous arguments. To get rid of $\overline{\overline{\overline{T(\Gamma)}}}$ we now use the surface energy. Obviously

$$|\overline{\overline{\overline{T(\Gamma)}}}| \leq c_d E_s(\Gamma) \tag{A.26}$$

[9] The double bar on a set really means the set of all points those distance to the set is less than or equal to two.

with $c_d = \frac{\#\{y\in\mathbb{Z}^d, d(y,0)\leq 2\}}{2d} = \frac{5^d}{2d}$. Hence we can finish the proof by

$$|\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma))| + LE_s(\Gamma) \geq c_6 L\left(|(\underline{\gamma'}\backslash\overline{\hat{D'}}(\gamma'))\backslash\overline{\overline{T(\Gamma)}}| + |\overline{\overline{T(\Gamma)}}|\right) \geq c_6 L|\underline{\gamma'}\backslash\overline{\hat{D'}}(\gamma')| \tag{A.27}$$

with $c_6 = \min\{\bar{c}, c_d^{-1}\}$.$\diamondsuit$

Remark: We would like to give an example which shows that the L in front of $E_s(\Gamma)$ is really necessary. Let Γ be defined by $h_x(\Gamma) = 1_{x=x_0}$ and $\underline{\Gamma} = \{x_0\}$ and assume that $\{x_0\} = D(\Gamma)$ and $x_0 \notin \overline{\mathcal{D}(\Gamma)}$. Then $|\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma))| = 0$ but $|\underline{\gamma'}\backslash\overline{\hat{D'}}(\gamma')| = 1$. Hence we really need a factor of the order L to ensure the validity of (A.20).

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