THE PINWHEEL TILINGS OF THE PLANE*

by
Charles Radin

Mathematics Department
University of Texas
Austin, TX 78712

radin@math.utexas.edu

Abstract

We give the first example of a finite set of topological disks (in fact polygons) in the Euclidean plane, isometric copies of which can tile the plane but only using infinitely many orientations of the disks.

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0. Introduction

The subject of forced tilings (also called aperiodic tilings) was created by the philosopher Hao Wang in 1961 [8-11], as a tool in the study of certain decidability problems in propositional calculus. It is concerned with the patterns generated by "tiles" as they are used to tile space; the formalism is the following. One defines a fixed, finite number of shapes (henceforth called "prototiles") in Euclidean n-space, E^n, for $n \geq 2$. The prototiles are usually required to be rather nice topologically, at least homeomorphs of the closed unit ball. One then makes arbitrarily many congruent copies, called "tiles", of these prototiles, and considers all ways (called "tilings") that such tiles may provide a simultaneous covering and packing of E^n; a tiling is thus an unordered collection of tiles for which the union is all of E^n, but such that the interiors of each pair of tiles do not intersect.

Wang's original problem was to determine if it was possible to have a finite set of prototiles, with associated tilings of space, for which **all** the tilings were **nonperiodic**. (A tiling is "nonperiodic" if it is not invariant under any single simultaneous translation of its tiles.) This was settled in the affirmative in the thesis of his student Robert Berger [1].

Slowly over the years, due to the efforts of Raphael Robinson [7], Roger Penrose [2], Shahar Mozes [4] and many others (see [3,6,12] for other references), Wang's problem has been generalized to ask whether it was possible for a fixed, finite set of prototiles to generate tilings of space **but only very complicated tilings**, where "very complicated" is interpreted as appropriate, but always implying nonperiodicity. In some interpretations, particularly the first ones associated with physical models of quasicrystals, "very complicated" means "without crystallographic symmetry". In later work, "very complicated" often means "disordered" in the probabilistic sense used to study patterns in nonlinear dynamics. Within the field of logic, there was a brief line of development in which "very complicated" means "nonrecursive".

All published examples, of finite sets of prototiles which can only tile space nonperiodically, have the feature that in every tiling each prototile only appears in finitely many orientations. Therefore, for these examples one could add the requirement that copies of a prototile be not just congruent to the prototile, but congruent by a translation; to recover the tilings of these previous examples one might then need to increase the sets of prototiles to larger but still finite sets, effectively declaring certain rotated or reflected prototiles as new distinct prototiles. In other words, the use of congruence in the production of copies of the prototiles was effectively replaceable by translation in all these examples, at the expense of increasing the number of prototiles to a larger finite number. This is not true of the example of this paper. In this example there is a finite set of prototiles, and in **every** associated tiling of the plane tiles appear in **infinitely** many orientations; all the connected part of the Euclidean group is needed to analyze the tilings by this set of prototiles, not just its translation subgroup. This constitutes a major advance in the field, which began with models only requiring discrete translations (using "Wang dominoes"), and expanded to continuous translations with models such as those of Penrose; this introduction of rotations adds a distinctly new element to the subject, of particular interest to certain applications such as the physics of quasicrystals. There are other features of this example which are novel, but we postpone discussion of them to the end of the paper.

I. The substitution tessellation of the plane

I a). The tessellation and its Levels

We begin with a certain hieracrchical structure, a tessellation of the plane which motivated our tiling example.

We define the "unmarked prototile" as the right triangle with the following vertices in the Cartesian plane: (0,0), (2,0), (2,1). (To be precise, it is the closed convex hull of these points.) The interior angles at the vertices of this triangle (and any similar triangle) are of three sizes, small, medium and large, so the vertices are denoted S, M and L. Similarly the edges of this triangle (and any similar triangle) are of three sizes and are denoted S, M and L.

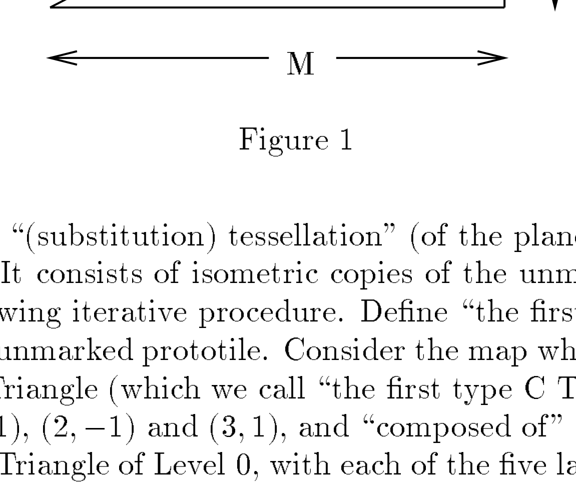


Figure 1

We now define the "(substitution) tessellation" (of the plane) based on this unmarked prototile. It consists of isometric copies of the unmarked prototile, obtained by the following iterative procedure. Define "the first type C Triangle of Level 0" to be the unmarked prototile. Consider the map which takes this Triangle into a similar Triangle (which we call "the first type C Triangle of Level 1"), with vertices $(-2,1)$, $(2,-1)$ and $(3,1)$, and "composed of" five isometric copies of the first type C Triangle of Level 0, with each of the five labeled with a "type" A, B, C, D, or E, as follows: a Triangle of type A with vertices $(-2,1)$, $(0,1)$ and $(0,0)$; a Triangle of type B with vertices $(0,1)$, $(0,0)$ and $(2,1)$; a Triangle of type C with vertices $(0,0)$, $(2,0)$ and $(2,1)$ (coinciding with the first type C

Triangle of Level 0); a Triangle of type D with vertices $(0,0)$, $(2,0)$ and $(2,1)$; and a Triangle of type E with vertices $(2,1)$, $(3,1)$ and $(2,-1)$. In summary, the first type C Triangle of Level 1 is a certain collection of five "component" Triangles.

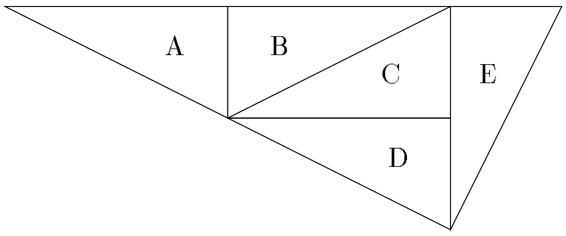


Figure 2

Next we repeat this process, mapping the first type C Triangle of Level 1 into the first type C Triangle of Level 2, with vertices $(-5,5), (1,-3)$ and $(5,0)$; it is similar to the former and composed of five Triangles (each composed of five subtriangles) isometric to the former, which are in the same geometric relation to the former as are the five created at the first step, with the first type C Triangle of Level 1 as the C in this Level 2 Triangle. See Figure 3.

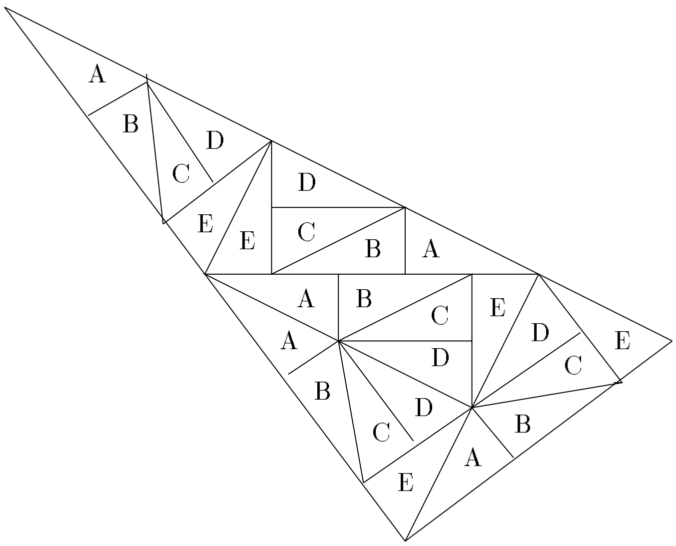


Figure 3

Continuing this process leads to the desired tessellation of the plane. See Figure 4.

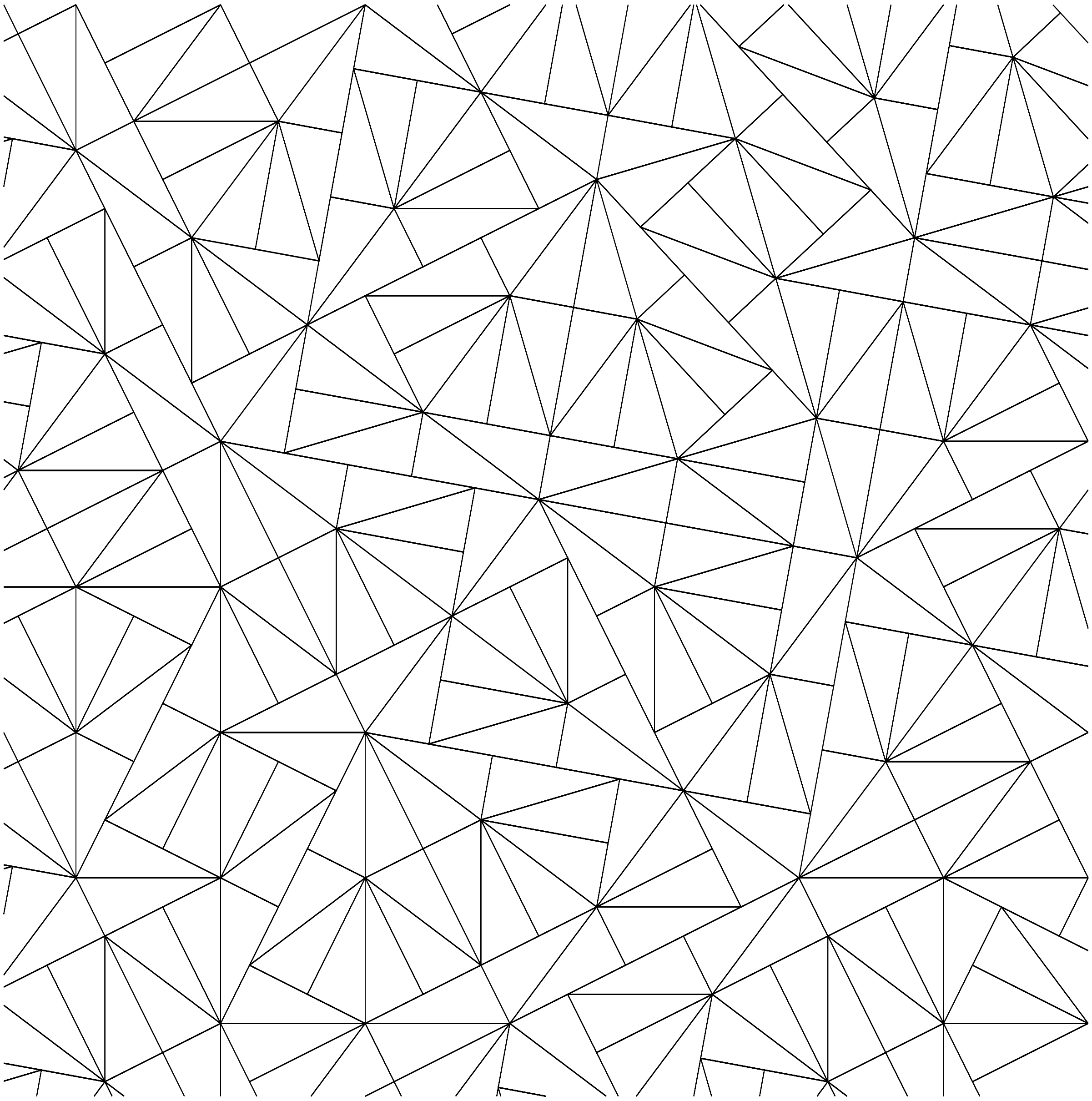


Figure 4

We define Triangles of Level n as those Triangles, created above, which are isometric to the first type C Triangle of Level n; each has a "type" (A $\cdots$ E) defined by its relative position in the unique Level $n+1$ Triangle containing it. We also define a "class" (A1, B1,$\cdots$, E1, A2, B2,$\cdots$, E2) for each Triangle; the first component in the symbol represents the type, and the second component, either 1 or 2, distinguishes between Triangles which are reflections of one another.

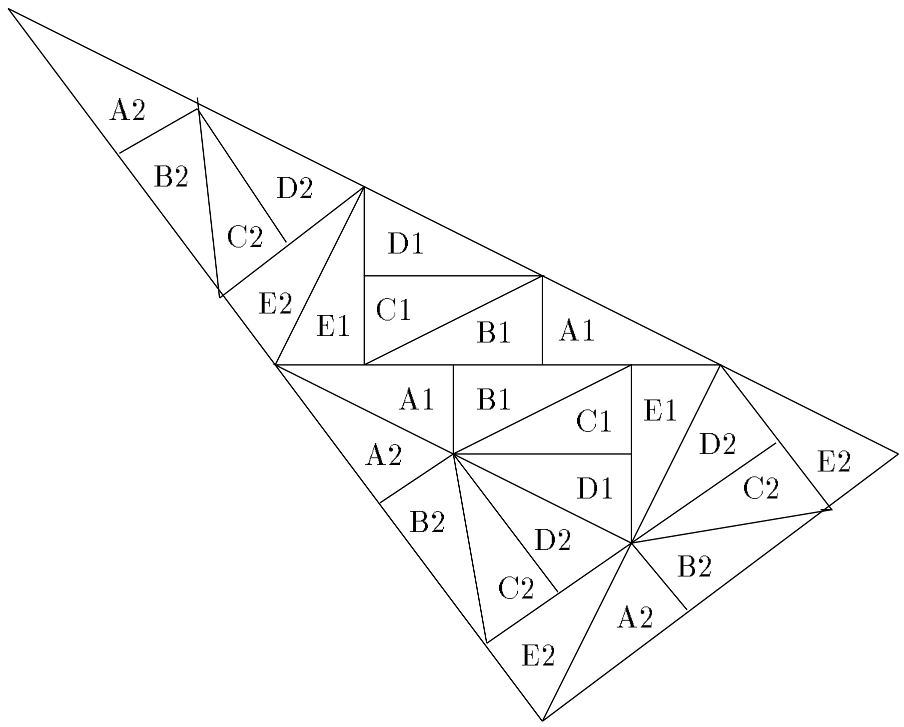


Figure 5

(This pattern of Triangles of all Levels is the key feature of the tessellation on which we will focus our attention. Also, we emphasize that by definition the above Triangles are each fixed sets in the plane, and not movable in any sense.) It is of prime importance to this paper that the Triangles of each Level appear at infinitely many orientations in this tessellation, as follows from the following simple lemma applied to the small angle $S = \tan^{-1}(1/2)$.

Lemma. (Theorem 6.15 in [5]) For any rational number r other than 0 and ± 1, $\tan^{-1}(r)$ is irrational with respect to π.

I b). Marks in the tessellation

Our main objective is to define a finite set of prototiles which can tile the plane but only with tilings that have some of the geometric relationships of this tessellation; in particular, the hierarchical relationships between consecutive

Levels of Triangles. This will be done in section II. As preparation, we begin by adding "marks" to the (smallest) Triangles in this tessellation. Each Triangle of Level 0 in the tessellation will be called a Tile, and will have a mark associated with (but not necessarily located at) each of its vertices, $\mathcal{V}_S, \mathcal{V}_M, \mathcal{V}_L$, where the subscript refers to the angle of the vertex. (At this point it is difficult to discuss the reason for each kind of mark we introduce; this will be clarified in our proofs below, and the Remarks afterward. We can say that the basic idea will be to record enough information in the marks of the Tiles so that similarly defined but movable "tiles" will only have tilings with an analogous hierarchical structure of Triangles.) Each of the three vertex marks of a Tile will contain a variety of information about the edges of those Triangles in the tessellation which are related in some sense to the vertex. In particular, with each vertex V of each Tile T there will be recorded information concerning each edge of each Tile which either has a coinciding vertex, or which has an edge with V at its center. Since (by definition) abutting Tiles have edges which coincide in an interval, the lines representing Tile edges will be separated into double lines, as follows. At vertex V of Tile T, the information at V is given in linear order. There are two edges of T meeting at V. Measuring angles as positive when referring to clockwise rotation, we call that edge of T at V the "first" one which is a positive angle from the other, the other being called the "last" edge of T at V. We will record the angle between edges at V. The first angle will be that of Tile T at V, and we use the abbreviations S, M and L noted above. There must be some Tile T' abutting T at the first edge of T at V. Since this edge of T' at V and the first edge of T at V coincide in an interval, the angle between them is zero, and does not need to be recorded. Therefore, we include in the information at V: the angle of T (called A_1), then some information about the first edge of T (called $\mathcal{E}_1^1$), then information about the abutting edge of T' (called $\mathcal{E}_1^2$), then the angle of T' (called A_2), information about the other edge of T' (called $\mathcal{E}_2^1$), ..., and finally, information about the last edge of T (called $\mathcal{E}_e^2$; e is used to denote the last two edges). So the information at V is a sequence of triples, each beginning with an angle and then information about two edges which coincide in an interval; the k^{th} such triple is referred to as A_k, $\mathcal{E}_k^1$ and $\mathcal{E}_k^2$. Edge-markers $\mathcal{E}_k^j$ will be said to "refer to" or "lie on" the appropriate edges of a Tile (and are denoted by thick intervals in some figures). Now we need to specify what the information is in each $\mathcal{E}_k^j$, where $j = 1, 2$ and $1 \leq k \leq e$.

We are concerned with the edges not just of Tiles, but of Triangles of all Levels. We call a "complete" edge only the following: the small edges of Triangles of type A, B, C, and D; the medium edges of Triangles of type C, D and E; and the large edges of Triangles of type B and C. Note that an edge of a Triangle is complete if and only if it is not part or all of an edge of a higher Level Triangle. For each of the three edges of each Tile, we will record information about the complete edge of highest level of which it is a part; we record the "size" of that edge (S, M or L), what type Triangle ($A \cdots E$) it is an edge of, and with what type Triangle ($A \cdots E$) this last Triangle is "associated"; within each $\mathcal{E}_k^j$ we use variable names $\mathcal{S}$, $\mathcal{N}$ and $\mathcal{P}$ for the three pieces of information just described – $\mathcal{S} \in \{S, M, L\}$ for the size of the complete edge; $\mathcal{N} \in \{\text{A}, \ldots, \text{E}\}$ for the type of Triangle of which the complete edge is a part; and $\mathcal{P} \in \{\text{A}, \ldots, \text{E}\}$ for the type of Triangle of which that of $\mathcal{N}$ is a part.

As an example, the vertex $\mathcal{V}_L$ of the Tile of type D in Figure 6 has the following structure: $\mathcal{V}_L = \{A_1, \mathcal{E}_1^1, \mathcal{E}_1^2, A_2, \mathcal{E}_2^1, \mathcal{E}_2^2, A_e, \mathcal{E}_e^1, \mathcal{E}_e^2\}$, where $\mathcal{E}_k^j = (\mathcal{S}, \mathcal{N}, \mathcal{P}, \cdots)$ and so:

$$\mathcal{V}_L = \{\pi/2, (M, D, C, \cdots), (M, C, C, \cdots), \pi/2, (S, C, C, \cdots), (M, E, C, \cdots), \\ \pi, (M, E, C, \cdots), (S, D, C, \cdots)\}$$

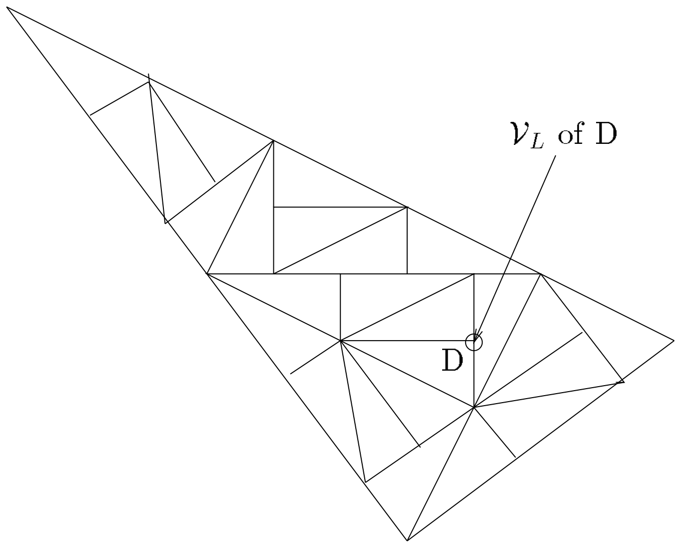


Figure 6

There are three other pieces of information recorded in each $\mathcal{E}_k^j$, described by variables $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. Using the notation $\mathcal{E}_k^j[\alpha]$ for the edge-mark $\mathcal{E}_m^j$ at angle α beyond that of $\mathcal{E}_k^j$, we assign $\mathcal{F} \in \{S, M, L, Z, R\}$ the value Z in $\mathcal{E}_k^1$ (resp. $\mathcal{E}_k^2$) if there is an edge $\mathcal{E}_k^2[+\pi]$ (resp. $\mathcal{E}_k^1[+\pi]$) at V (referring of course to a different Tile than does $\mathcal{E}_k^1$ (resp. $\mathcal{E}_k^2$)), which is part of the same complete edge as that of $\mathcal{E}_k^1$ (resp. $\mathcal{E}_k^2$). $\mathcal{F} = S$ (resp. M, resp. L) if the complete edge of $\mathcal{E}_k^j$ ends at V at an angle of S (resp. M, resp. L) with the rest of its Triangle. $\mathcal{F} = R$ in $\mathcal{E}_k^j$ if a different value has not been specified above. (As one sees in the above example, it sometimes happens that in a vertex of a Tile T information must be given about an abutting Tile T' which does not share this vertex, as is the case where the type E triangle abuts the type D; in such a case we use the value R for $\mathcal{F}$.) To expand on the above example then, we have: $\mathcal{E}_k^j = (\mathcal{S}, \mathcal{N}, \mathcal{P}, \mathcal{F}, \cdots)$ and

$$\mathcal{V}_L = \{\pi/2, (M,D,C,L,\cdots), (M,C,C,L,\cdots), \pi/2, (S,C,C,L,\cdots),$$
$$(M,E,C,R,\cdots), \pi, (M,E,C,R,\cdots), (S,D,C,L,\cdots)\}$$

In order to discuss variables $\mathcal{G}$ and $\mathcal{H}$ we need the following convention. For each Triangle T of Level $n \geq 1$ we define three Tiles in the Triangle, T_S, T_M and T_L by: T_S shares its small vertex $V_{SS} \equiv \mathcal{V}_S$ with that of the Triangle, another vertex V_{SM} of T_S lies on the large edge of the Triangle, and the third vertex V_{SM} of T_S lies on the medium edge of the Triangle; T_M shares its medium vertex $V_{MM} \equiv \mathcal{V}_M$ with that of the Triangle, another vertex V_{MS} of T_M lies on the large edge of the Triangle, and the third vertex V_{ML} of T_S lies on the small edge of the Triangle; T_L shares its small vertex $V_{LL} \equiv \mathcal{V}_S$ with the large vertex of the Triangle, and another vertex V_{LS} of T_L lies on the small edge of the Triangle. For Level 0 Triangles, which are Tiles, we define T_S, T_M and T_L as the Tile itself, with V_{SS} as $\mathcal{V}_S$, V_{MM} as $\mathcal{V}_M$, and V_{LL} as $\mathcal{V}_L$. Note that this generalizes the concept of $\mathcal{V}_S$, $\mathcal{V}_M$ and $\mathcal{V}_L$ of Tiles to V_{SS}, V_{MM} and V_{LL} of Triangles.

The variable $\mathcal{G}$ has the following structure:
$\mathcal{G} = (G_A^{\alpha 1M}, G_E^{\alpha 1S}, G_B^{\alpha 1S}, G_D^{\alpha 1S}, G_D^{\alpha 1M}, G_A^{\alpha 2M}, G_E^{\alpha 2S}, G_B^{\alpha 2S}, G_D^{\alpha 2S}, G_D^{\alpha 2M}, G_A^{\epsilon 1M}, G_E^{\epsilon 1S}, G_B^{\epsilon 1S}, G_D^{\epsilon 1S}, G_D^{\epsilon 1M}, G_A^{\epsilon 2M}, G_E^{\epsilon 2S}, G_B^{\epsilon 2S}, G_D^{\epsilon 2S}, G_D^{\epsilon 2M})$, where each G_m^{jkn} takes values in $\{+1,-1\}$. We will now define these variables on each complete edge, assigning a value at the edge-marks at each end of the edge and prescribing an algorithm for assigning values at the other edge-marks on the edge.

M1) $[G_X^{jkM}]$. Let $X = A$ or D. Consider any Tile which is part of a Triangle T, with vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ of the Tile lying on an edge of T, and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ and $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ referring to the edge of T. If $\mathcal{F} = Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then G_X^{jkM} has the same value in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ as it has in $\mathcal{E}_1^2[-\pi]$ (resp. $\mathcal{E}_e^1[+\pi]$) of $\mathcal{V}_2$, unless $\mathcal{V}_2$ has an edge-mark $\mathcal{E}_1^2[-\pi/2]$ (resp. $\mathcal{E}_e^1[+\pi/2]$) which contains the values $(\mathcal{S},\mathcal{N},\mathcal{F},\mathcal{P}) = (S,A,L,A)$ or (S,A,L,D), in which case the two $\mathcal{G}$ values have opposite sign. (In Figure 7, numbers 1 and 2 each label a pair of edge-marks; for each pair there is the $\mathcal{E}_e^2[-\pi]$ of $\mathcal{V}_L$ and the $\mathcal{E}_e^2$ of $\mathcal{V}_S$ of a Tile of type B. For pair 1, G_D^{jkn} has different values in the two edge-marks, and for pair 2, G_D^{jkn} has the same values in the two edge-marks.) If $\mathcal{F} \neq Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, and we do not have: $k = 2$ (resp. $k = 1$) and $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,X,L)$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then $G_X^{\epsilon kM} = +1$ in that edge-mark; $k = 1$ (resp. $k = 2$) and $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,A,L)$ or (S,D,L) in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then $G_X^{\alpha kM} = +1$ in that edge-mark. (In Figure 7, label 3 indicates an edge-mark $\mathcal{E}_1^1$ of vertex $\mathcal{V}_M$ of a Tile of type D, in which $G_X^{\alpha kM} = G_X^{\epsilon kM} = +1$ from above.)

M2) $[G_X^{\epsilon kM}$ in $X]$. Let T be a Triangle of class $X1 = A1$ or $D1$ (resp. $X2 = A2$ or $D2$). In edge-mark $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of vertex V_{LL}, $G_X^{\epsilon 2M}$ (resp. $G_X^{\epsilon 1M}$) has the same value as that of $G_X^{\alpha 2M}$ (resp. $G_X^{\alpha 1M}$) in the edge-mark $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of vertex V_{SS}. See Figure 8a.

M3) $[G_X^{\alpha kM}$ in $Y]$. Let T be a Triangle of class $Y1 = A1$ or $D1$ (resp. $Y2 = A2$ or $D2$), and let $X = A$ or D. In edge-mark $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of vertex V_{LL}, $G_X^{\alpha 1M}$ (resp. $G_X^{\alpha 2M}$) has the negative of the value of $G_X^{\epsilon 1M}$ (resp. $G_X^{\epsilon 2M}$) in the edge-mark $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of vertex V_{SS}. See Figure 8a.

M4) $[G_X^{jkS}]$. Let $X = D$ or E. Consider any Tile which is part of a Triangle T, with vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ of the Tile lying on an edge of T, and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ and $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ referring to the edge of T. If $\mathcal{F} = Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then G_X^{jkS} has the same value in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ as it has in $\mathcal{E}_1^2[-\pi]$ (resp. $\mathcal{E}_e^1[+\pi]$) of $\mathcal{V}_2$, unless: $\mathcal{V}_2$ has an edge-mark $\mathcal{E}_1^2[-\pi/2]$ (resp. $\mathcal{E}_e^1[+\pi/2]$) which contains the values $(\mathcal{S}, \mathcal{N}, \mathcal{F}, \mathcal{P}) = (S, A, L, E)$ or (S, A, L, D); or $\mathcal{V}_2$ has an edge-mark $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) which contains the values $(\mathcal{S}, \mathcal{N}, \mathcal{F}, \mathcal{P}) = (S, A, M, B)$, in which case the two $\mathcal{G}$ values have opposite sign. If $\mathcal{F} \neq Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, and we do not have: $k = 2$ (resp. $k = 1$) and $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (M, X, L)$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then $G_X^{\epsilon kS} = +1$ in that edge-mark; $k = 1$ (resp. $k = 2$) and $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (M, E, L)$ or (M, D, L) in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then $G_X^{\alpha kS} = +1$ in that edge-mark. If $\mathcal{F} = Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ and we do not have $k = 1$ (resp. $k = 2$) and $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (L, B, S)$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$, then $G_X^{\alpha kS} = +1$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$.

M5) $[G_X^{\epsilon kS}$ in $X]$. Let T be a Triangle of class $X1 = E1$ or $D1$ (resp. $X2 = E2$ or $D2$). In edge-mark $\mathcal{E}_e^1[+\pi/2]$ (resp. $\mathcal{E}_1^2[-\pi/2]$) of vertex V_{LL}, $G_X^{\epsilon 2S}$ (resp. $G_X^{\epsilon 1S}$) has the same value as that of $G_X^{\alpha 2S}$ (resp. $G_X^{\alpha 1S}$) in the edge-mark $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of vertex V_{SM}. See Figure 8a.

M6) $[G_X^{\alpha kS}$ in $Y]$. Let T be a Triangle of class $Y1 = E1$ or $D1$ (resp. $Y2 = E2$ or $D2$), and let $X = E$ or D. In edge-mark $\mathcal{E}_e^1[+\pi/2]$ (resp. $\mathcal{E}_1^2[-\pi/2]$) of vertex V_{LL}, $G_X^{\alpha 1S}$ (resp. $G_X^{\alpha 2S}$) has the negative of the value of $G_X^{\epsilon 1S}$ (resp. $G_X^{\epsilon 2S}$) in the edge-mark $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of vertex V_{SM}.

M7) $[G_X^{\alpha kS}$ in $B]$. Let T be a Triangle of class $B1$ (resp. $B2$), and let $X = E$ or D. In edge-mark $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of vertex V_{SM}, $G_X^{\alpha 1S}$ (resp. $G_X^{\alpha 2S}$) has the negative of the value of $G_X^{\epsilon 1S}$ (resp. $G_X^{\epsilon 2S}$) in the edge-mark $\mathcal{E}_1^2[-\pi/2]$ (resp. $\mathcal{E}_e^1[+\pi/2]$) of vertex V_{LL}. See Figure 8b.

M8) $[G_B^{jkS}]$. Consider any Tile which is part of a Triangle T, with vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ of the Tile lying on an edge of T, and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ and $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ referring to the edge of T. If $\mathcal{F} = Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then G_B^{jkS} has the same value in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ as it has in $\mathcal{E}_1^2[-\pi]$ (resp. $\mathcal{E}_e^1[+\pi]$) of $\mathcal{V}_2$, unless: $\mathcal{V}_2$ has an edge-mark $\mathcal{E}_1^2[+\pi/2]$ (resp. $\mathcal{E}_e^1[-\pi/2]$) which contains the values $(\mathcal{S}, \mathcal{N}, \mathcal{F}, \mathcal{P}) = (S, A, L, E)$ or (S, A, L, D); or $\mathcal{V}_2$ has an edge-mark $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) which contains the values $(\mathcal{S}, \mathcal{N}, \mathcal{F}, \mathcal{P}) = (S, A, M, B)$, in which case the two $\mathcal{G}$ values have opposite sign. If $k = 2$ (resp. $k = 1$) and we do not have $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (M, E, L)$ or (M, D, L) in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then $G_X^{\alpha kS} = +1$ in that edge-mark. If $k = 1$ (resp. $k = 2$) and we do not have $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (L, B, S)$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$, then $G_X^{\alpha kS} = G_X^{\epsilon kS} = +1$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$.

M9) $[G_B^{\epsilon kS}$ in $Y]$. Let T be a Triangle of class $Y1 = E1$ or $D1$ (resp. $Y2 = E2$ or $D2$). In edge-mark $\mathcal{E}_e^1[+\pi/2]$ (resp. $\mathcal{E}_1^2[-\pi/2]$) of vertex V_{LL}, $G_B^{\epsilon 2S}$ (resp. $G_B^{\epsilon 1S}$) has the negative of the value of $G_B^{\alpha 2S}$ (resp. $G_B^{\alpha 1S}$) in the edge-mark $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of vertex V_{SM}. See Figure 8b.

M10) $[G_B^{\epsilon kS}$ in $B]$. Let T be a Triangle of class $B1$ (resp. $B2$). In edge-mark $\mathcal{E}_1^1$

(resp. $\mathcal{E}_e^2$) of vertex V_{SM}, $G_B^{\epsilon 1S}$ (resp. $G_B^{\epsilon 2S}$) has the negative of the value of $G_B^{\alpha 1S}$ (resp. $G_B^{\alpha 2S}$) in the edge-mark $\mathcal{E}_1^2[-\pi/2]$ (resp. $\mathcal{E}_e^1[+\pi/2]$) of vertex V_{LL}. See Figure 8c.

M11) $[G_B^{\alpha kS}$ in $B]$. Let T be a Triangle of class $B1$ (resp. $B2$). In edge-mark $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of vertex V_{SM}, $G_B^{\alpha 1S}$ (resp. $G_B^{\alpha 2S}$) has the same value as that of $G_B^{\epsilon 1S}$ (resp. $G_B^{\epsilon 2S}$) in the edge-mark $\mathcal{E}_1^2[-\pi/2]$ (resp. $\mathcal{E}_e^1[+\pi/2]$) of vertex V_{LL}. See Figure 8c.

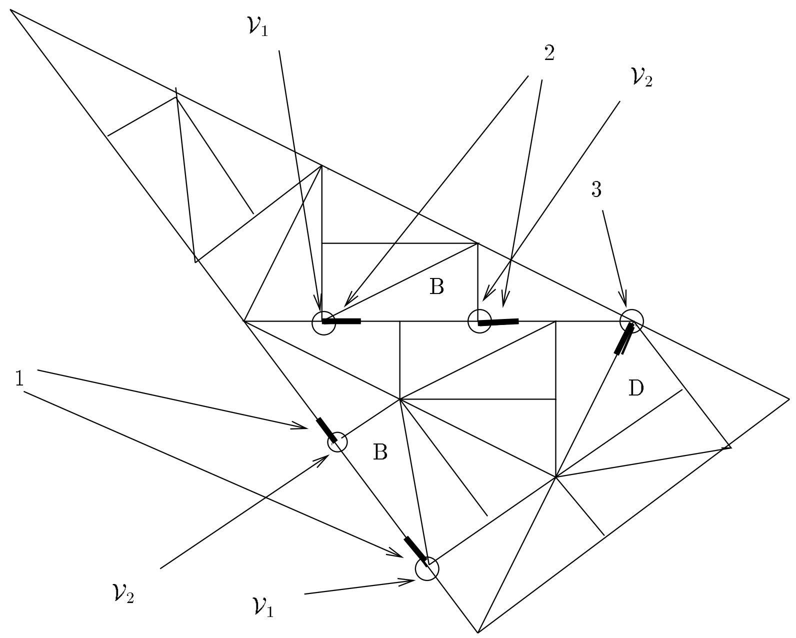


Figure 7

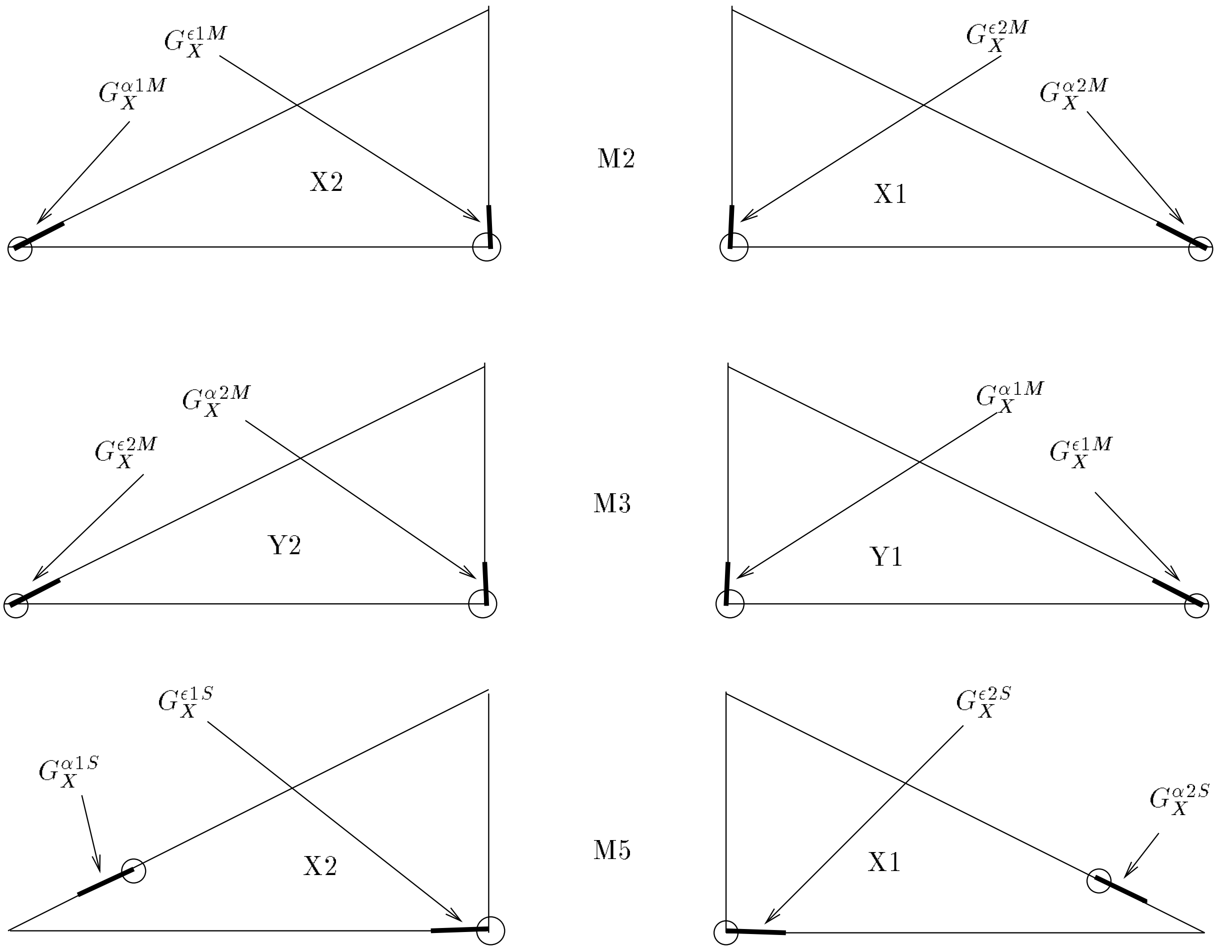

$G_X^{\epsilon 1M}$
$G_X^{\alpha 1M}$
X2
M2
$G_X^{\epsilon 2M}$
$G_X^{\alpha 2M}$
X1
$G_X^{\alpha 2M}$
$G_X^{\epsilon 2M}$
Y2
M3
$G_X^{\alpha 1M}$
$G_X^{\epsilon 1M}$
Y1
$G_X^{\epsilon 1S}$
$G_X^{\alpha 1S}$
X2
M5
$G_X^{\epsilon 2S}$
$G_X^{\alpha 2S}$
X1

Figure 8a

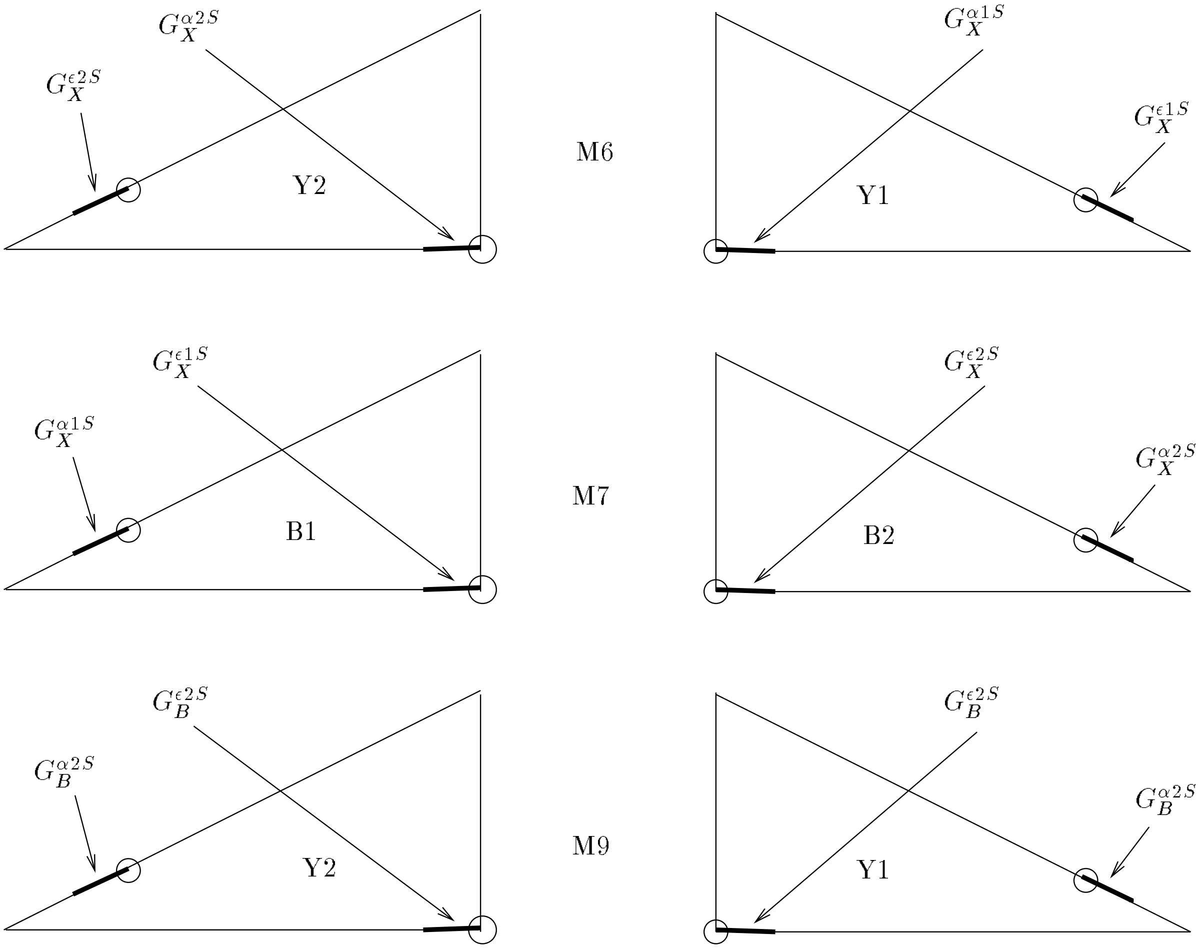

$G_X^{\alpha 2S}$
$G_X^{\epsilon 2S}$
Y2
M6
$G_X^{\alpha 1S}$
$G_X^{\epsilon 1S}$
Y1
$G_X^{\epsilon 1S}$
$G_X^{\alpha 1S}$
B1
M7
$G_X^{\epsilon 2S}$
$G_X^{\alpha 2S}$
B2
$G_B^{\epsilon 2S}$
$G_B^{\alpha 2S}$
Y2
M9
$G_B^{\epsilon 2S}$
$G_B^{\alpha 2S}$
Y1

Figure 8b

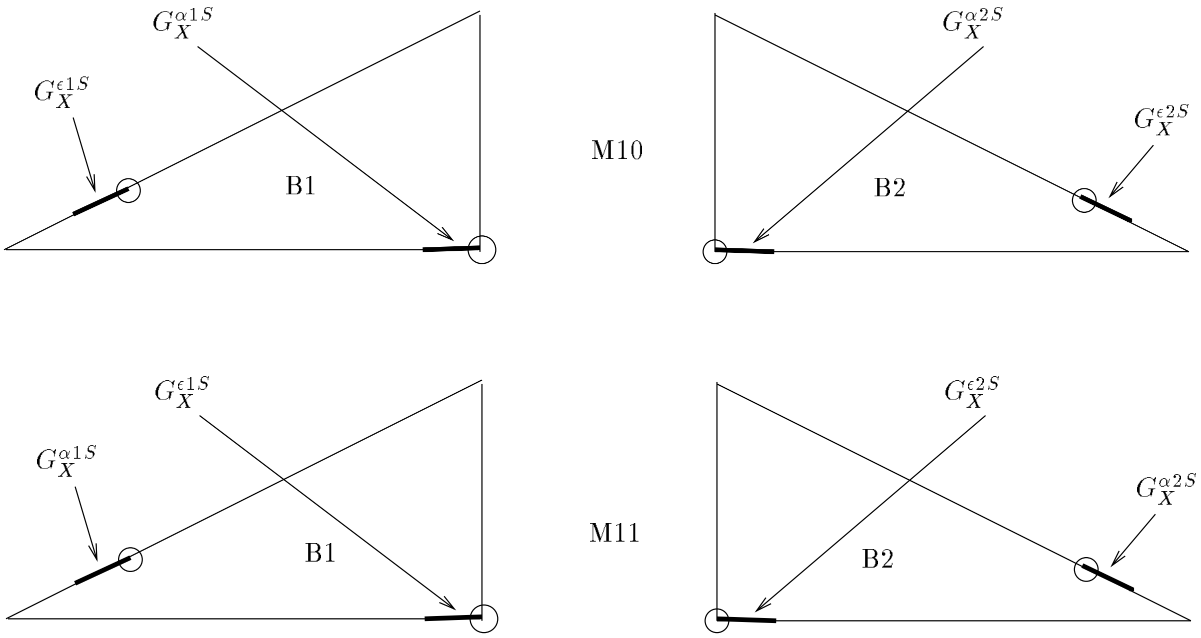


Figure 8c

(Note: the above values of $\mathcal{G}$ are complicated but well defined. To see that they are well defined just note even though one value is sometimes defined in terms of another, this process always ends since after at most two such indirect definitions one is referred to a variable on a larger edge; in the tessellation every Triangle is contained in infinitely many Triangles of type C, on the edges of which all $\mathcal{G}$ variables are explicitly defined, and the process cannot take us outside any Triangle of type C.)

$\mathcal{H} \in \{+1, -1\}$ basically maintains a "parity" between two specific points in each Triangle of Level $n \geq 1$, marked V1 and V5 in Figure 12a, and V6 and V10 in Figure 12b. The usefulness of this parity must remain obscure until the proof of Theorem 2.

The variable $\mathcal{H}$ has the following structure:
$\mathcal{H} = (H_A^{1M}, H_E^{1S}, H_B^{1S}, H_D^{1S}, H_D^{1M}, H_A^{2M}, H_E^{2S}, H_B^{2S}, H_D^{2S}, H_D^{2M})$, where each H_m^{kn} takes values in $\{+1, -1\}$, as follows.

M12) $H_m^{kn} = G_m^{\epsilon kn}$ in an edge-mark if in that edge-mark we have $(\mathcal{S}, \mathcal{N}) \neq (S, A)$, (S, D) or (M, D), and the Triangle of type A or D is of class Ak or Dk.

M13) Consider any vertex V of a prototile with edge-mark $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) in which $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,D,M)$; in particular, the edge-mark is lying in the small edge of a Triangle of class D1 (resp. D2). Let $\mathcal{I}$ be the value of $G_m^{\epsilon 1n}$ (resp. $G_m^{\epsilon 2n}$) in $\mathcal{E}_2^1$ (resp. $\mathcal{E}_{e-1}^2$) of V. See Figure 9. Then the value of H_m^{1M} (resp. H_m^{2M}) in any edge-mark lying in the same small edge of the Triangle of type D is the product of $G_m^{\epsilon 1M}$ (resp. $G_m^{\epsilon 2M}$) in that edge-mark with $\mathcal{I}$ if $\mathcal{P}=m$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of V. Let $\mathcal{I}'$ be the value of $G_m^{\epsilon 1S}$ (resp. $G_m^{\epsilon 2S}$) in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of V. Then the value of H_m^{1S} (resp. H_m^{2S}) in any edge-mark lying in the same small edge of the Triangle of type D is the product of $G_m^{\epsilon 1S}$ (resp. $G_m^{\epsilon 2S}$) in that edge-mark with $\mathcal{I}'$ if $\mathcal{P}=m$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$). (And so, for all edge-marks lying in that same small edge of the Triangle of type D, the quotient of H_m^{1n} (resp. H_m^{2n}) with $G_m^{\epsilon 1n}$ (resp. $G_m^{\epsilon 2n}$) has the same value.)

M14) Consider any vertex of a prototile with edge-mark $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) in which $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,D,L)$; in particular, the edge-mark is lying in the medium edge of a Triangle of type D. See Figure 9. The value of H_m^{kn} in that edge-mark is equal to its value in the edge-mark $\mathcal{E}_1^2[-\pi/2]$ (resp. $\mathcal{E}_e^1[+\pi/2]$) of the same vertex. Then for all edge-marks lying in the same medium edge of that Triangle of type D, the quotient of H_m^{1n} (resp. H_m^{2n}) with $G_m^{\epsilon 1n}$ (resp. $G_m^{\epsilon 2n}$) has the same value.

M15) Consider any vertex of a prototile with edge-mark $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) in which $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,A,M)$; in particular, the edge-mark is lying in the small edge of a Triangle of type A. See Figure 9. The value of H_m^{kn} in that edge-mark is equal to its value in the edge-mark $\mathcal{E}_1^2[+\pi/2]$ (resp. $\mathcal{E}_e^1[-\pi/2]$) of the same vertex. Then for all edge-marks lying in the same medium edge of that Triangle of type A, the quotient of H_m^{1n} (resp. H_m^{2n}) with G_m^{j1n} (resp. G_m^{j2n}) has the same value.

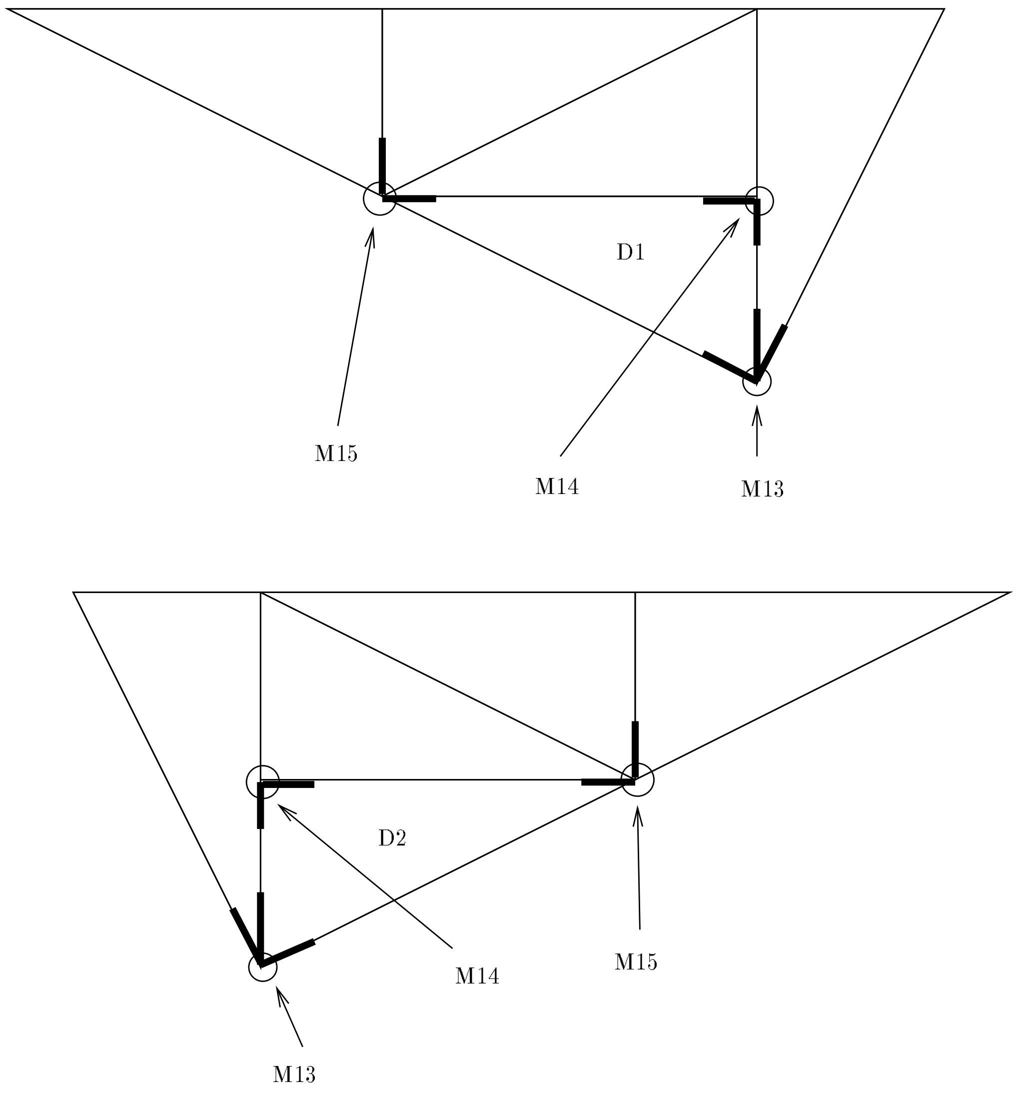


Figure 9

To summarize: the values of most of the vertex variables are easy to read off from the tessellation. The exceptions are some of the values of $\mathcal{G}$ and $\mathcal{H}$, which are designed to analyze certain key vertices in the Triangles of the tessellation, as will be seen in the proof of Theorem 2.

II. Tiling the plane

II a). Prototiles

We now consider the problem of trying to reproduce the structure of the substitution tessellation by the tilings of a finite set of prototiles. Before we begin, we need to introduce a simple technique used almost universally in forced tiling.

As stated above, tiling consists of simultaneous covering and packing; the packing condition intuitively requires (given the covering condition) that the tiles must fit together like a jig-saw puzzle, the boundary of one tile nestled into the boundaries of its neighbors. Consider for example the system in Figure 10a consisting of two prototiles in the plane which are both roughly square, with edges alligned. (They can only tile the plane in checkerboard fashion.)

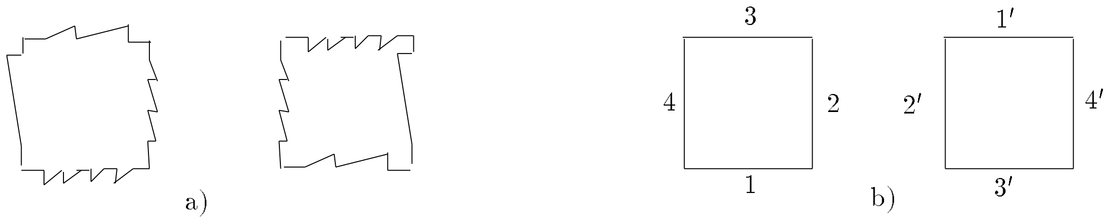


Figure 10

Now consider in Figure 10b the set of two perfect squares, with edges alligned, which have "colors" $(1, 2, 3, 4, 1', 2', 3', 4')$ assigned to their edges; the colors come in "complementary" pairs: j and j' are complements. If we consider this a pair of prototiles, and allow translated copies (called tiles) to be made, preserving the assignment of colors to edges, and add the "matching rule" that in a tiling tiles must abut full edge to full edge and with complementary colors meeting, then it is clear that we will reproduce, in any reasonable sense, the previous example. In other words, one can often simplify an example of prototiles by using simpler shapes, but with added "colors" and "matching rules" to make up for the simplified boundary. Reversing this process, if one is given a set of prototiles with colors and matching rules, it is straightforward to replace it by a set of prototiles with more complicated boundaries and no colors or matching rules. This is what we will do. We will have in fact only one basic shape (and its reflection), the unmarked prototile of the tessellation, but we will define a set of colors, which we will call "markings", with a notion of complementary pairs, and a set of "matching rules", and we will analyze those tilings of the plane by tiles which satisfy the matching rules, the object being to show that these tilings all exhibit the basic structure of Triangles of all Levels which we see in the tessellation. We begin by introducing the markings.

The markings were already introduced for the Tiles of the tessellation. Intuitively, we would like to define our prototile set as all the distinct versions of the unmarked prototile and its reflection (reflecting about its small edge say) obtained by adding to them those markings which appear on the Tiles which are obtained from these two unmarked prototiles by orientation preserving isometries. This could be done, but it is somewhat unsatisfying in that it would then be difficult to determine precisely which of the conceivable marking combinations are used. So we instead define our prototiles as those obtained by using all possible markings, but satisfying an explicit list of "restrictions".

A "(marked) prototile" consists of the unmarked prototile, or its reflection about its small edge (which we will now call another unmarked prototile), together with three (vertex) marks $\mathcal{V}_S, \mathcal{V}_M, \mathcal{V}_L$, each of the three of the form $(\mathcal{A}_1, \mathcal{E}_1^1, \mathcal{E}_1^2, \mathcal{A}_2, \mathcal{E}_2^1, \mathcal{E}_2^2, \ldots, \mathcal{A}_e, \mathcal{E}_e^1, \mathcal{E}_e^2)$, where e varies between 3 and 8 (e stands for "end") and where:

a) $\mathcal{A}_j \in \{S, M, L, \pi\}$
b) each $\mathcal{E}_j^k$ is of the form $(\mathcal{S}, \mathcal{N}, \mathcal{P}, \mathcal{F}, \mathcal{G}, \mathcal{H})$, where:
c) $\mathcal{S} \in \{S, M, L\}$
d) $\mathcal{N} \in \{\text{A, B, C, D, E}\}$
e) $\mathcal{P} \in \{\text{A, B, C, D, E}\}$
f) $\mathcal{F} \in \{S, M, L, R, Z\}$
g) $\mathcal{G} = (G_A^{\alpha 1M}, G_E^{\alpha 1S}, G_B^{\alpha 1S}, G_D^{\alpha 1S}, G_D^{\alpha 1M}, G_A^{\alpha 2M}, G_E^{\alpha 2S}, G_B^{\alpha 2S}, G_D^{\alpha 2S}, G_D^{\alpha 2M}, G_A^{\epsilon 1M}, G_E^{\epsilon 1S}, G_B^{\epsilon 1S}, G_D^{\epsilon 1S}, G_D^{\epsilon 1M}, G_A^{\epsilon 2M}, G_E^{\epsilon 2S}, G_B^{\epsilon 2S}, G_D^{\epsilon 2S}, G_D^{\epsilon 2M})$, where each G_m^{jkn} takes values in $\{+1, -1\}$.
h) $\mathcal{H} = (H_A^{1M}, H_E^{1S}, H_B^{1S}, H_D^{1S}, H_D^{1M}, H_A^{2M}, H_E^{2S}, H_B^{2S}, H_D^{2S}, H_D^{2M})$, where each H_m^{kn} takes values in $\{+1, -1\}$.

(We note that although these are called "vertex marks", they could easily be implemented by encoding their information in bumps and dents in the edges of the tiles, away from the vertices.) As mentioned above, we do not allow our prototiles to have all possible values of the above marks; we allow precisely those combinations satisfying the following set of restrictions.

II b). Restrictions.

Every prototile must contain marks of one of the following (mutually exclusive) ten "classes", A1 $\cdots$ E2, (and thus five "types", $A \cdots E$, where the type of class Xj is denoted X), the "original " ones, with vertices at (0,0), (2,0), and (2,1), satisfying one of A2, B1, C1, D2, E2, and the "reflected" ones, with vertices at (2,0), (4,0) and (2,1), satisfying one of A1, B2, C2, D1, E1, as shown in Figures 11a,b.

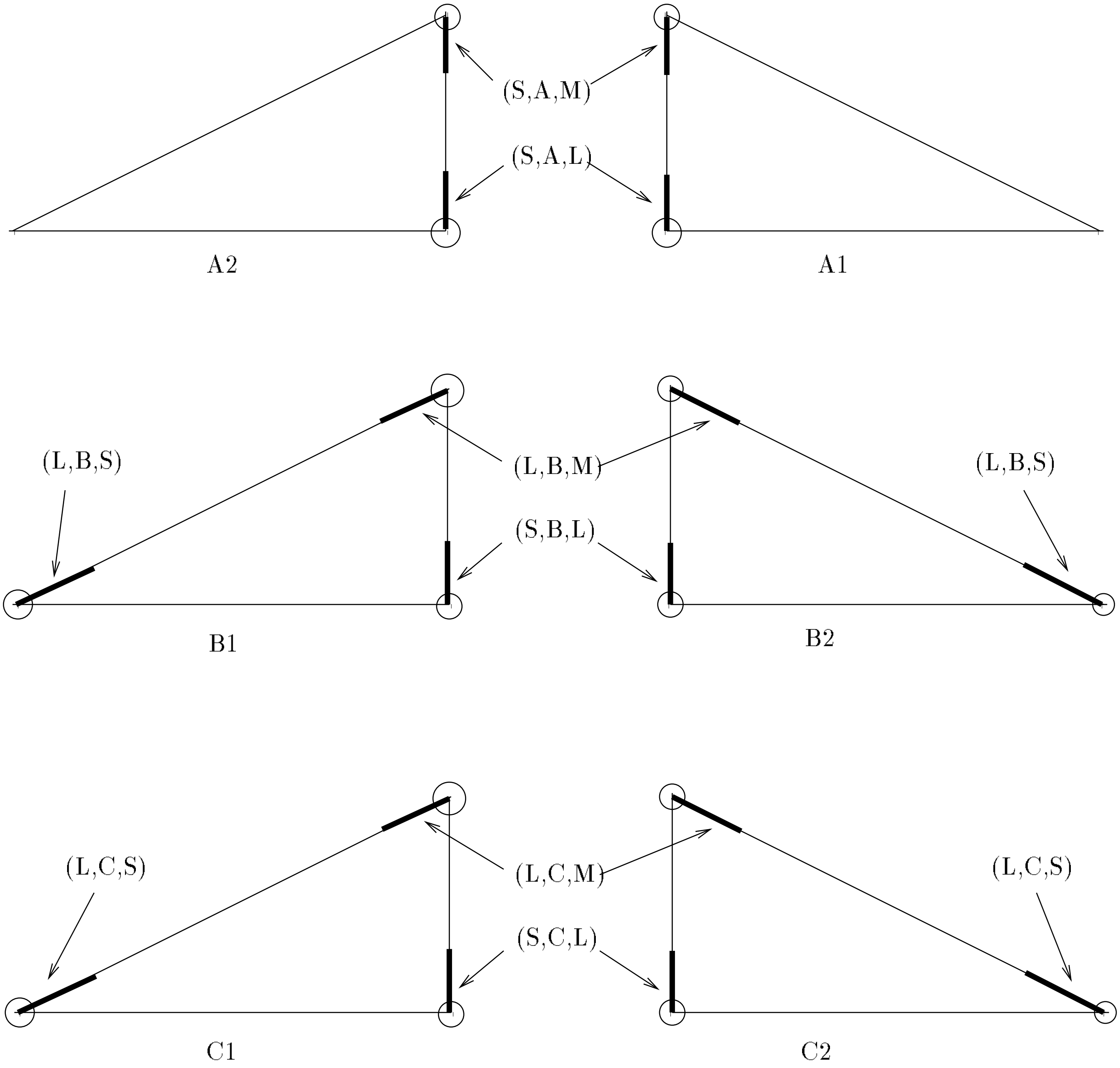


Figure 11a

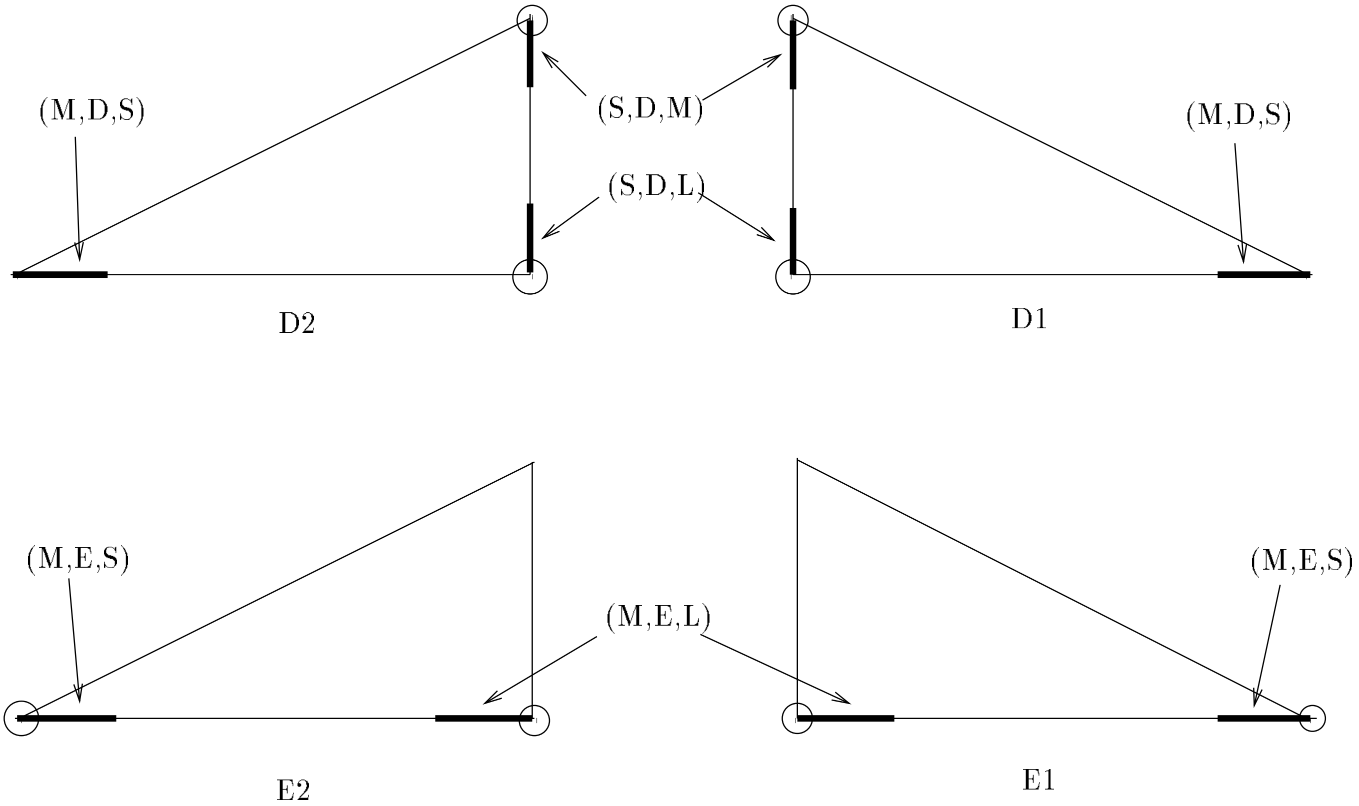


Figure 11b

A1) In $\mathcal{V}_L$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,A,L)$, and in $\mathcal{V}_M$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,A,M)$.

B1) In $\mathcal{V}_L$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,B,L)$; in $\mathcal{V}_M$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (L,B,M)$; and in $\mathcal{V}_S$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (L,B,S)$.

C1) In $\mathcal{V}_S$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,C,S)$; in $\mathcal{V}_M$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (L,C,M)$; and in $\mathcal{V}_L$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,C,L)$.

D1) In $\mathcal{V}_S$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,D,S)$; in $\mathcal{V}_L$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,D,L)$; and in $\mathcal{V}_M$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,D,M)$.

E1) In $\mathcal{V}_S$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,E,S)$, and in $\mathcal{V}_L$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,E,L)$.

A2) In $\mathcal{V}_L$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,A,L)$, and in $\mathcal{V}_M$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,A,M)$.

B2) In $\mathcal{V}_L$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,B,L)$; in $\mathcal{V}_M$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,B,M)$; and in $\mathcal{V}_S$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (L,B,S)$.

C2) In $\mathcal{V}_S$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (L,C,S)$; in $\mathcal{V}_M$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,C,M)$; and in $\mathcal{V}_L$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,C,L)$.

D2) In $\mathcal{V}_S$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,D,S)$; in $\mathcal{V}_L$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,D,L)$; and in $\mathcal{V}_M$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,D,M)$.

E2) In $\mathcal{V}_L$, $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,L)$, and in $\mathcal{V}_S$, $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,S)$. The following restrictions can be understood from Figures 12a-12f; we record in the restrictions V1 $\cdots$ V10 almost everything that can be easily deduced as occuring in the tessellation at such vertices. The references to variable $\mathcal{H}$ in V1 and V6 will need special analysis.

V1) Consider a prototile T, with vertices $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$, such that $\mathcal{E}_1^1$ of $\mathcal{V}_2$ and $\mathcal{E}_e^2$ of $\mathcal{V}_1$ refer to the same edge of T. If in $\mathcal{V}_1$ either of the following conditions are satisfied: $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,A,L)$, or $\mathcal{E}_e^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,B,L)$, then so is the other, and also: $\mathcal{F}=Z$ in $\mathcal{E}_e^1[+\pi/2]$ and $\mathcal{E}_e^2[-\pi/2]$; $\mathcal{P}$ has the same value in $\mathcal{E}_e^1$ and $\mathcal{E}_e^2$; the quotient of H_m^{1M} by $G_m^{\epsilon 1M}$ in $\mathcal{E}_1^1$ of $\mathcal{V}_2$ equals $G_m^{\alpha 1M}$ in $\mathcal{E}_e^2[-\pi/2]$ of $\mathcal{V}_1$ if $\mathcal{P}=m$ in $\mathcal{E}_e^2$ of $\mathcal{V}_1$; and the quotient of H_m^{1S} by $G_m^{\epsilon 1S}$ in $\mathcal{E}_1^1$ of $\mathcal{V}_2$ equals $G_m^{\alpha 1S}$ in $\mathcal{E}_e^1[+\pi/2]$ of $\mathcal{V}_1$ if $\mathcal{P}=m$ in $\mathcal{E}_e^2$. And if in $\mathcal{V}_1$ we have, in $\mathcal{E}_e^2$, that $\mathcal{P}$ has value: B, then $(\mathcal{S},\mathcal{N})=(L,B)$ in $\mathcal{E}_e^1[+\pi/2]$, and $(\mathcal{S},\mathcal{N})=(L,B)$ in $\mathcal{E}_e^2[-\pi/2]$; C, then $(\mathcal{S},\mathcal{N})=(L,C)$ in $\mathcal{E}_e^1[+\pi/2]$ and in $\mathcal{E}_e^2[-\pi/2]$.

V2) If a vertex of a prototile satisfies any of the following conditions: $\mathcal{E}_k^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,A,M)$; $\mathcal{E}_k^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,B,M)$; $\mathcal{E}_{k+1}^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(L,B,M)$; $\mathcal{E}_{k+1}^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(L,C,S)$; $\mathcal{E}_{k+2}^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,C,S)$; $\mathcal{E}_{k+2}^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,D,S)$, then it satisfies the others, and also: $\mathcal{F}=Z$ in $\mathcal{E}_{k-1}^2$ and in $\mathcal{E}_{k+3}^1$; $\mathcal{P}$ has the same value in $\mathcal{E}_k^1$, $\mathcal{E}_k^2$, $\mathcal{E}_{k+1}^1$, $\mathcal{E}_{k+1}^2$, $\mathcal{E}_{k+2}^1$ and $\mathcal{E}_{k+2}^1$.

V3) If a vertex of a prototile satisfies any of the following conditions: $\mathcal{E}_k^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(L,B,S)$; $\mathcal{E}_k^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(L,C,M)$; $\mathcal{E}_{k-1}^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,C,M)$; $\mathcal{E}_{k-1}^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,L)$, then it satisfies the others, and also: $\mathcal{F}=Z$ in $\mathcal{E}_{k-1}^2[-\pi/2]$ and in $\mathcal{E}_{k+1}^1$; $\mathcal{P}$ has the same value in $\mathcal{E}_k^2$, $\mathcal{E}_k^1$, $\mathcal{E}_{k-1}^2$ and $\mathcal{E}_{k-1}^1$ in $\mathcal{V}_S$.

V4) If a vertex of a prototile satisfies any of the following conditions: $\mathcal{E}_k^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,C,L)$; $\mathcal{E}_k^2[-\pi/2]$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,C,L)$; $\mathcal{E}_k^1[-\pi/2]$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,D,L)$; $\mathcal{E}_k^2[-\pi]$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,D,L)$, then it satisfies the others, and also: $\mathcal{E}_k^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,R)$; $\mathcal{E}_k^1[-\pi]$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,R)$; $\mathcal{P}$ has the same value in $\mathcal{E}_k^2$, $\mathcal{E}_k^1$, $\mathcal{E}_k^2[-\pi/2]$, $\mathcal{E}_k^1[-\pi/2]$, $\mathcal{E}_k^2[-\pi]$ and $\mathcal{E}_k^1[-\pi]$.

V5) If a vertex of a prototile satisfies either of the following conditions: $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,D,M)$, or $\mathcal{E}_1^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,S)$, then it satisfies the other and: $\mathcal{P}$ has the same value in $\mathcal{E}_1^1$ and $\mathcal{E}_1^2$; the quotient of H_m^{1M} by $G_m^{\epsilon 1M}$ in $\mathcal{E}_1^1$ equals $G_m^{\epsilon 1M}$ in $\mathcal{E}_2^1$ if $\mathcal{P}=m$ in $\mathcal{E}_1^1$; the quotient of H_m^{1S} by $G_m^{\epsilon 1S}$ in $\mathcal{E}_1^1$ equals $G_m^{\epsilon 1M}$ in $\mathcal{E}_e^2$ if $\mathcal{P}=m$ in $\mathcal{E}_1^1$. Also, if, in $\mathcal{E}_1^1$, $\mathcal{P}$ has value: B, then $\mathcal{F}=Z$ in $\mathcal{E}_e^2$, and $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,B,L)$ in $\mathcal{E}_2^1$; C, then $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,C,L)$ in $\mathcal{E}_e^2$ and $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,C,L)$ in $\mathcal{E}_2^1$; A, then $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,A,L)$ in $\mathcal{E}_2^1$, and $\mathcal{F}=Z$ in $\mathcal{E}_e^2$; D, then $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,D,L)$ in $\mathcal{E}_e^2$, and $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,D,L)$ in $\mathcal{E}_2^1$; E, then $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,L)$ in $\mathcal{E}_e^2$, and $\mathcal{F}=Z$ in $\mathcal{E}_2^1$ and in $\mathcal{E}_e^2[+\pi]$.

V6) Consider a prototile T, with vertices $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$, such that $\mathcal{E}_1^1$ of $\mathcal{V}_1$

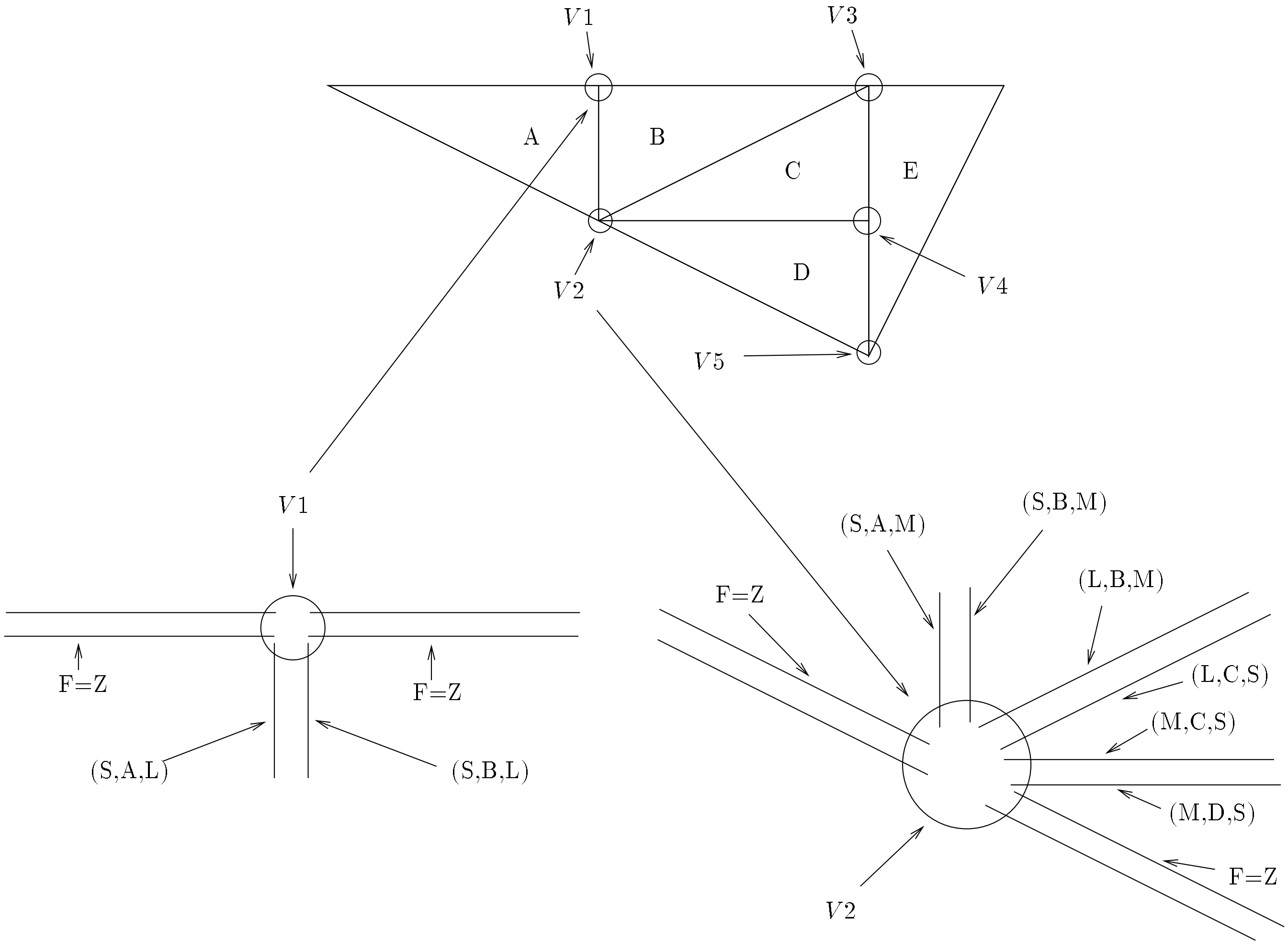
and $\mathcal{E}_e^2$ of $\mathcal{V}_2$ refer to the same edge of T. If in $\mathcal{V}_1$ either of the following conditions are satisfied: $\mathcal{E}_1^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,A,L)$, or $\mathcal{E}_1^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,B,L)$, then so is the other, and also: $\mathcal{F}=Z$ in $\mathcal{E}_1^1[+\pi/2]$ and $\mathcal{E}_1^2[-\pi/2]$; $\mathcal{P}$ has the same value in $\mathcal{E}_1^1$ and $\mathcal{E}_1^2$; the quotient of $H_m^{\alpha 2M}$ by $G_m^{\alpha 2M}$ in $\mathcal{E}_e^2$ of $\mathcal{V}_2$ equals $G_m^{\alpha 2M}$ in $\mathcal{E}_1^1[+\pi/2]$ of $\mathcal{V}_1$ if $\mathcal{P}=m$ in $\mathcal{E}_1^1$ of $\mathcal{V}_1$; and the quotient of $H_m^{\alpha 2S}$ by $G_m^{\alpha 2S}$ in $\mathcal{E}_e^2$ of $\mathcal{V}_2$ equals $G_m^{\alpha 2S}$ in $\mathcal{E}_1^2[-\pi/2]$ of $\mathcal{V}_1$ if $\mathcal{P}=m$ in $\mathcal{E}_1^1$. And if in $\mathcal{V}_1$ we have, in $\mathcal{E}_1^1$, that $\mathcal{P}$ has value: B, then $(\mathcal{S},\mathcal{N})=(L,B)$ in $\mathcal{E}_1^2[-\pi/2]$ and $\mathcal{E}_1^1[+\pi/2]$; C, then $(\mathcal{S},\mathcal{N})=(L,C)$ in $\mathcal{E}_1^2[-\pi/2]$ and $\mathcal{E}_1^1[+\pi/2]$.

V7) If a vertex of a prototile satisfies any of the following conditions: $\mathcal{E}_k^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,A,M)$; $\mathcal{E}_k^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,B,M)$; $\mathcal{E}_{k-1}^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(L,B,M)$; $\mathcal{E}_{k-1}^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(L,C,S)$; $\mathcal{E}_{k-2}^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,C,S)$; $\mathcal{E}_{k-2}^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,D,S)$, then it satisfies the others, and also: $\mathcal{F}=Z$ in $\mathcal{E}_{k+1}^1$ and $\mathcal{E}_{k-3}^2$; $\mathcal{P}$ has the same value in $\mathcal{E}_k^1$, $\mathcal{E}_k^2$, $\mathcal{E}_{k+1}^1$, $\mathcal{E}_{k+1}^2$, $\mathcal{E}_{k+2}^1$ and $\mathcal{E}_{k+2}^1$.

V8) If a vertex of a prototile satisfies any of the following conditions: $\mathcal{E}_k^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(L,B,S)$; $\mathcal{E}_k^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(L,C,M)$; $\mathcal{E}_{k+1}^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,C,M)$; $\mathcal{E}_{k+1}^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,L)$, then it satisfies the others, and also: $\mathcal{F}=Z$ in $\mathcal{E}_{k+1}^1[+\pi/2]$ and $\mathcal{E}_{k-1}^2$; $\mathcal{P}$ has the same value in $\mathcal{E}_k^1$, $\mathcal{E}_k^2$, $\mathcal{E}_{k+1}^1$ and $\mathcal{E}_{k+1}^2$.

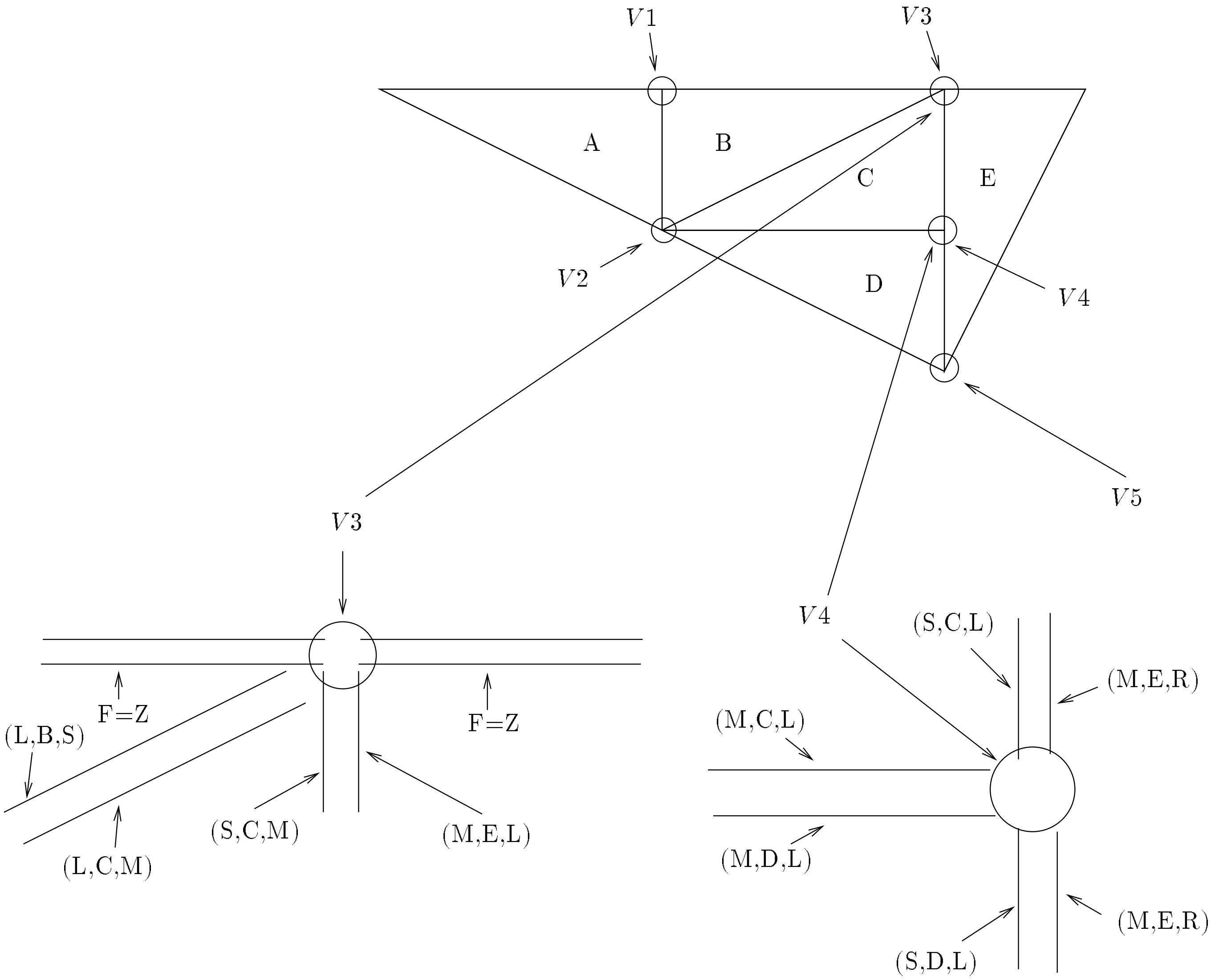
V9) If a vertex of a prototile satisfies any of the following conditions: $\mathcal{E}_k^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,C,L)$; $\mathcal{E}_k^1[+\pi/2]$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,C,L)$; $\mathcal{E}_k^2[+\pi/2]$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,D,L)$; $\mathcal{E}_k^1[+\pi]$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,D,L)$, then it satisfies the others, and also: $\mathcal{E}_k^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,R)$; $\mathcal{E}_k^2[+\pi]$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,R)$; $\mathcal{P}$ has the same value in $\mathcal{E}_k^2$, $\mathcal{E}_k^1$, $\mathcal{E}_k^2[+\pi/2]$, $\mathcal{E}_k^1[+\pi/2]$, $\mathcal{E}_k^2[+\pi]$ and $\mathcal{E}_k^1[+\pi]$.

V10) If a vertex of a prototile satisfies either of the following conditions: $\mathcal{E}_e^2$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,D,M)$, or $\mathcal{E}_e^1$ contains $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,S)$; then it satisfies the other, and: $\mathcal{P}$ has the same value in $\mathcal{E}_e^2$ and $\mathcal{E}_e^1$; the quotient of H_m^{2M} by $G_m^{\epsilon 2M}$ in $\mathcal{E}_e^2$ equals $G_m^{\epsilon 2M}$ in $\mathcal{E}_{e-1}^2$ if $\mathcal{P}=m$ in $\mathcal{E}_e^2$; and the quotient of H_m^{2S} by $G_m^{\epsilon 2S}$ in $\mathcal{E}_e^2$ equals $G_m^{\epsilon 2S}$ in $\mathcal{E}_1^1$ if $\mathcal{P}=m$ in $\mathcal{E}_e^2$. Also if, in $\mathcal{E}_e^2$, $\mathcal{P}$ has value: A, then $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,A,L)$ in $\mathcal{E}_{e-1}^2$, and $\mathcal{F}=Z$ in $\mathcal{E}_1^1$; D, then $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,D,L)$ in $\mathcal{E}_1^1$ and $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,D,L)$ in $\mathcal{E}_{e-1}^2$; B, then $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,B,L)$ in $\mathcal{E}_{e-1}^2$, and $\mathcal{F}=Z$ in $\mathcal{E}_1^1$; C, then $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,C,L)$ in $\mathcal{E}_1^1$ and $(\mathcal{S},\mathcal{N},\mathcal{F})=(S,C,L)$ in $\mathcal{E}_{e-1}^2$; E, then $(\mathcal{S},\mathcal{N},\mathcal{F})=(M,E,L)$ in $\mathcal{E}_1^1$, and $\mathcal{F}=Z$ in $\mathcal{E}_{e-1}^2$ and in $\mathcal{E}_{e-1}^1[+\pi]$.



$V1$
$V3$
A
B
C
E
$V2$
D
$V4$
$V5$
$V1$
F=Z
F=Z
(S,A,L)
(S,B,L)
(S,A,M)
(S,B,M)
F=Z
(L,B,M)
(L,C,S)
(M,C,S)
(M,D,S)
F=Z
$V2$

Figure 12a



V1
V3
A
B
C
E
V2
D
V4
V5
V3
F=Z
(L,B,S)
F=Z
(S,C,M)
(M,E,L)
(L,C,M)
V4
(S,C,L)
(M,E,R)
(M,C,L)
(M,D,L)
(M,E,R)
(S,D,L)

Figure 12b

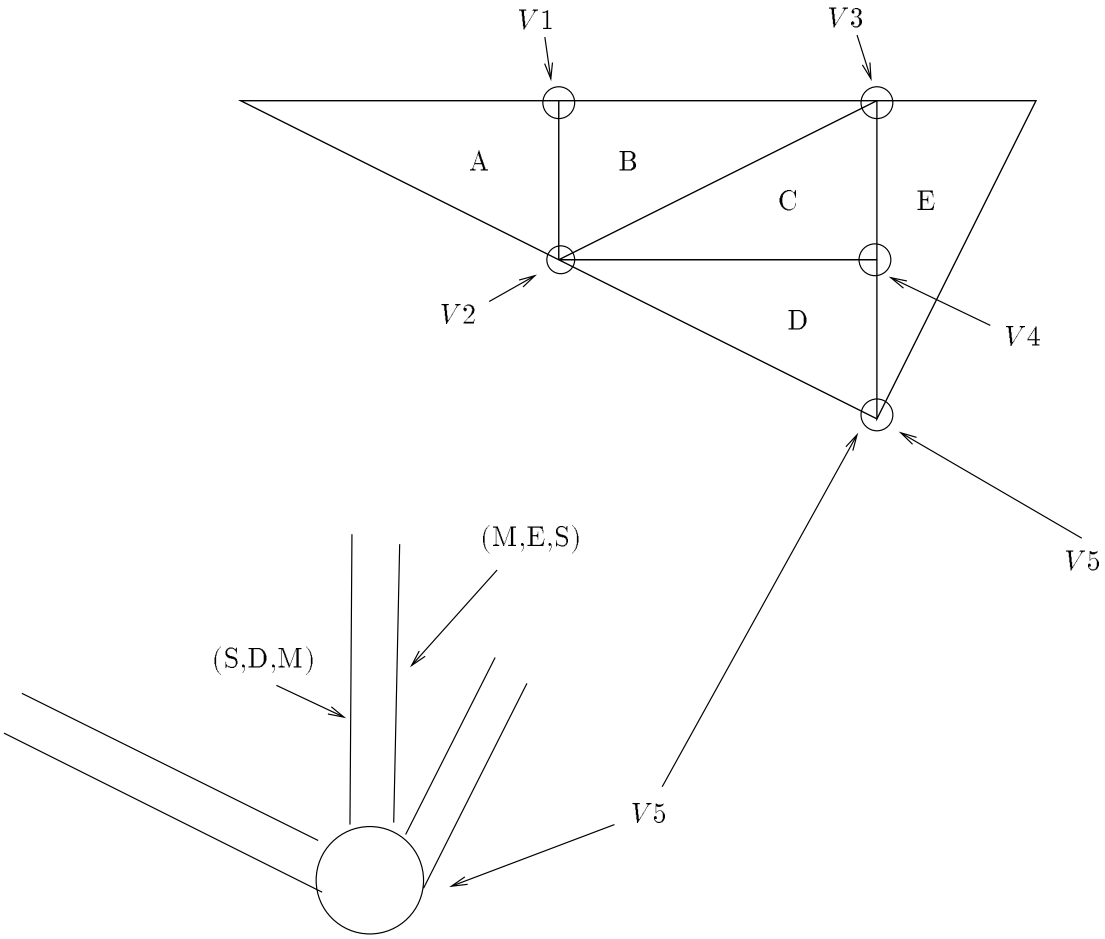
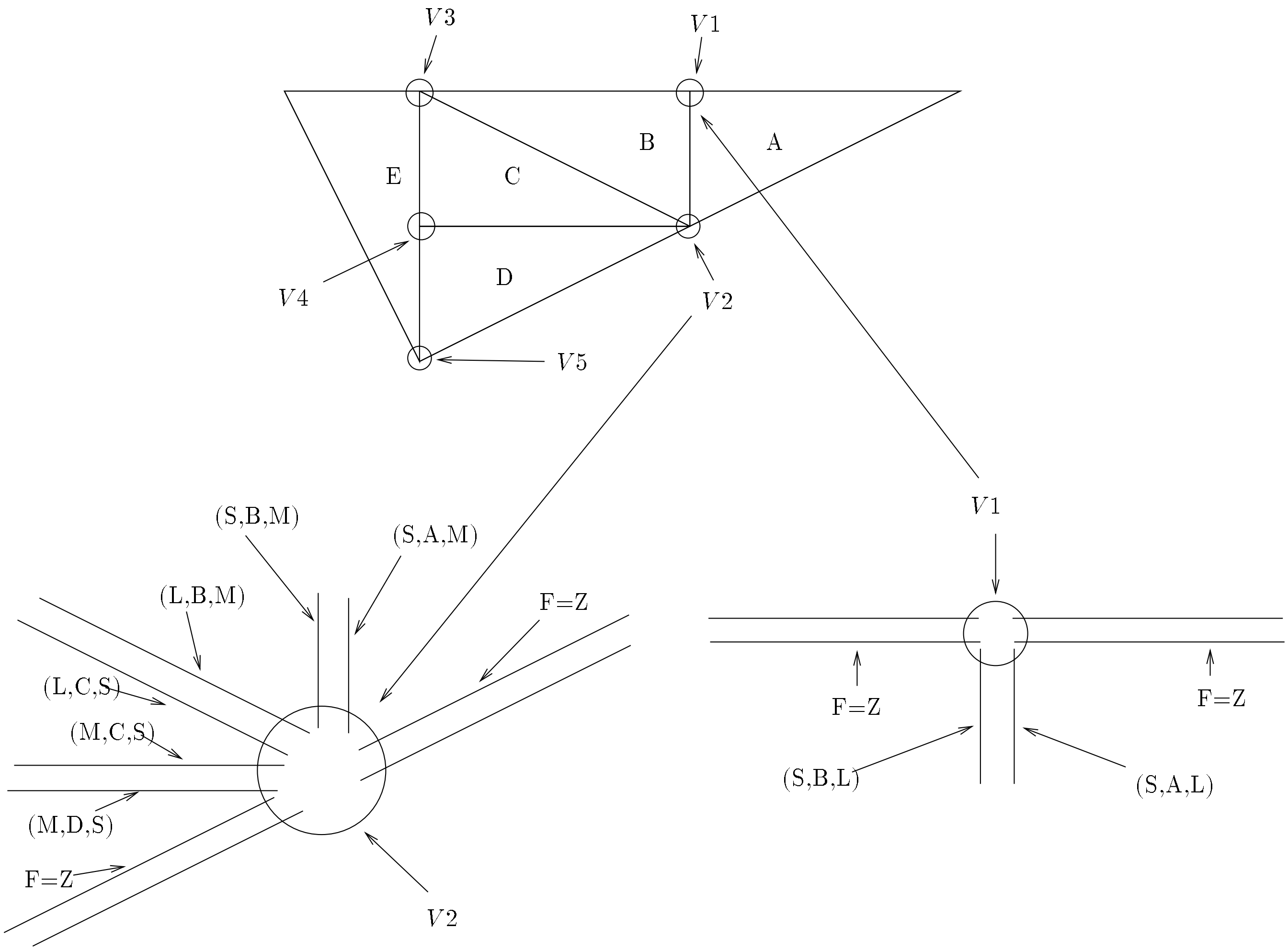


Figure 12c



$V3$
$V1$
B
A
E
C
D
$V4$
$V2$
$V5$
$V1$
(S,B,M)
(S,A,M)
(L,B,M)
F=Z
(L,C,S)
(M,C,S)
(M,D,S)
F=Z
$V2$
F=Z
F=Z
(S,B,L)
(S,A,L)

Figure 12d

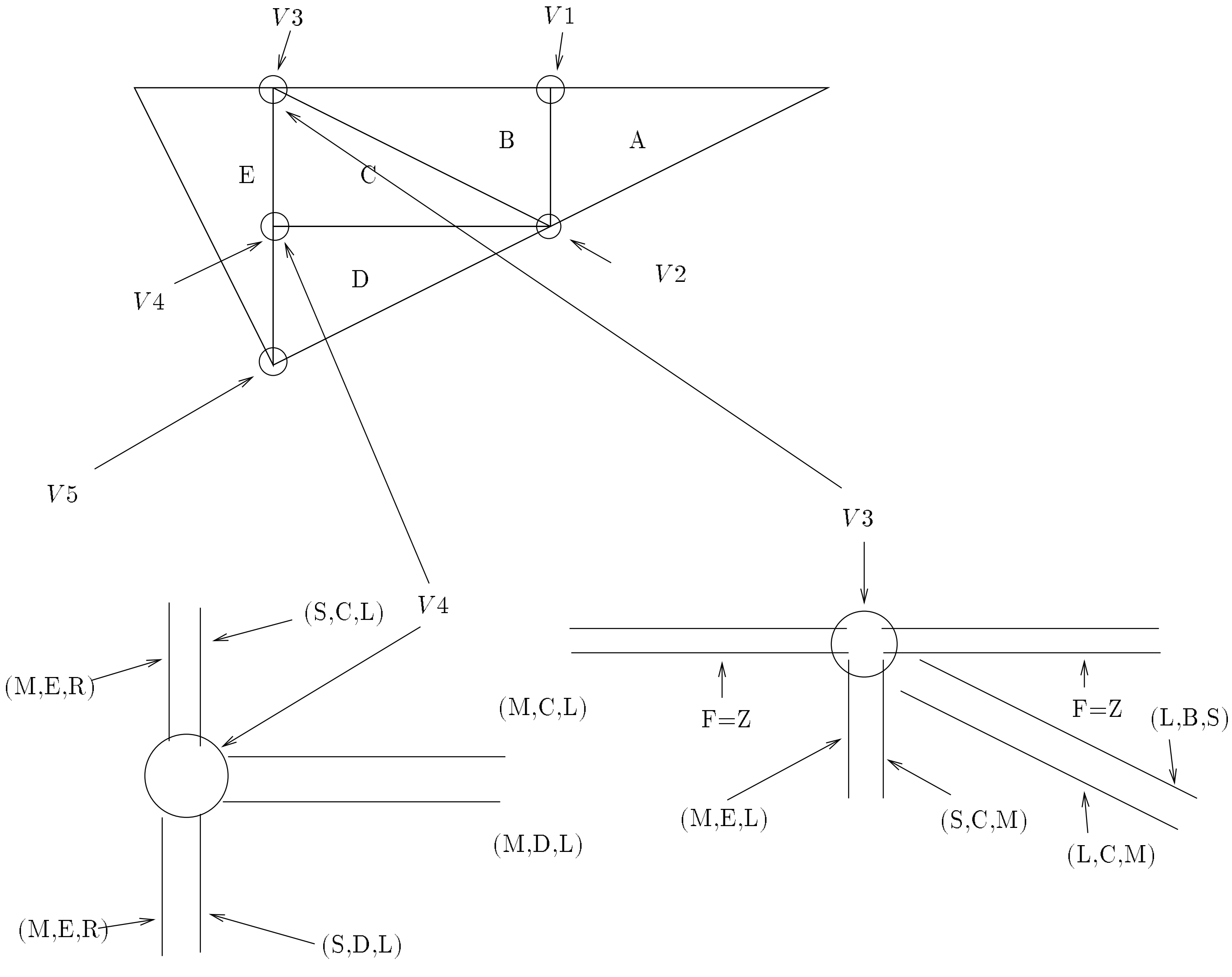

$V3$
$V1$
B
A
E
C
$V2$
$V4$
D
$V5$
$V3$
$V4$
(S,C,L)
(M,E,R)
(M,C,L)
F=Z
F=Z
(L,B,S)
(M,E,L)
(S,C,M)
(L,C,M)
(M,D,L)
(M,E,R)
(S,D,L)

Figure 12e

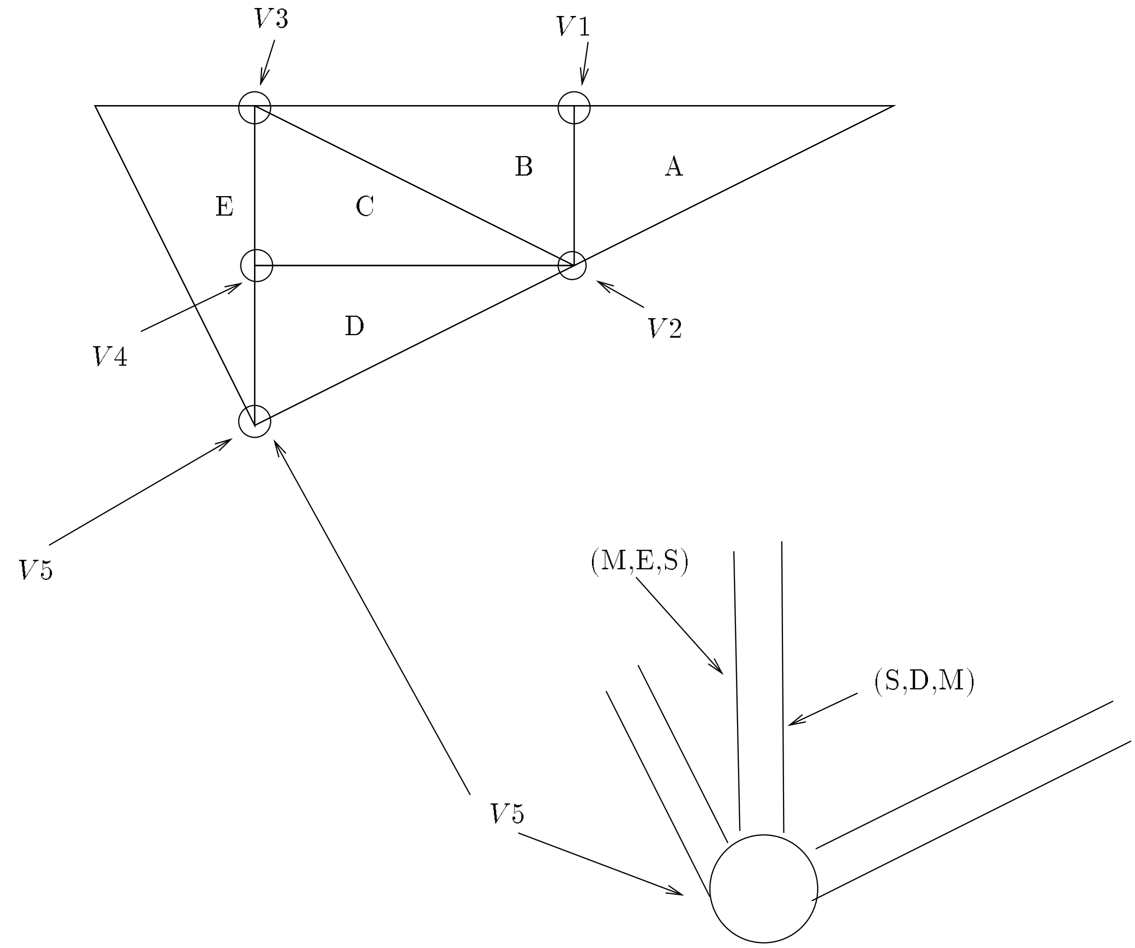


Figure 12f

Finally we make some restrictions which follow easily from examining Figures 3, 13, 14 and 15; R9 and R10, are less obvious, and we will analyze them carefully.

R1) In any $\mathcal{E}_k^j$, $(\mathcal{S}, \mathcal{N}) \neq (M, A), (L, A), (M, B), (L, D), (S, E), (L, E)$, and $\mathcal{F} \neq \mathcal{S}$.
R2) In any $\mathcal{V}_k$, $\sum_{j=1}^{e} Size(\mathcal{A}_j) = 2\pi$, where $Size(R) = \pi$.
R3) $\mathcal{A}_1$ must have the value of the vertex of the $\mathcal{V}$ component to which it belongs.
R4) If $\mathcal{E}_n^m$ and $\mathcal{E}_q^p$ belong to two different vertices and refer to the same edge of

a prototile, then $\mathcal{S}$ (resp. $\mathcal{N}$, resp. $\mathcal{P}$) must have the same value in $\mathcal{E}_n^m$ as in $\mathcal{E}_q^p$.

R5) In any vertex of a prototile, if one of a pair $\mathcal{E}_k^1, \mathcal{E}_k^2$ of edge-marks has $(\mathcal{S}, \mathcal{N})$ values: (S, A), then for the other $(\mathcal{S}, \mathcal{N}) = (S, B)$, and then one has $\mathcal{F} = L$ (resp. $\mathcal{F} = Z$, resp. $\mathcal{F} = M$) if and only if the other has $\mathcal{F} = L$ (resp. $\mathcal{F} = Z$, resp. $\mathcal{F} = M$); (S, B), then for the other $(\mathcal{S}, \mathcal{N}) = (S, A)$; (L, B), then for the other $(\mathcal{S}, \mathcal{N}) = (L, C)$, and then one has $\mathcal{F} = S$ (resp. $\mathcal{F} = Z$) if and only if the other has $\mathcal{F} = M$ (resp. $\mathcal{F} = Z$); (L, C), then for the other $(\mathcal{S}, \mathcal{N}) = (L, B)$; (M, C), then for the other $(\mathcal{S}, \mathcal{N}) = (M, D)$, and then one has $\mathcal{F} = L$ (resp. $\mathcal{F} = S$, resp. $\mathcal{F} = Z$) if and only if the other has $\mathcal{F} = L$ (resp. $\mathcal{F} = S$, resp. $\mathcal{F} = Z$); (M, D), then for the other $(\mathcal{S}, \mathcal{N}) = (M, C)$; (S, C), then for the other $(\mathcal{S}, \mathcal{N}) = (M, E)$; and then the former has $\mathcal{F} = M$ if and only if the other has $\mathcal{F} = L$, and if the former has $\mathcal{F} = L$ (resp. $\mathcal{F} = Z$) then the other has $\mathcal{F} = Z$ (resp. $\mathcal{F} = Z$); (S, D), then for the other $(\mathcal{S}, \mathcal{N}) = (M, E)$, and then the former has $\mathcal{F} = M$ if and only if the other has $\mathcal{F} = S$, and if the former has $\mathcal{F} = L$ (resp. $\mathcal{F} = Z$) then the other has $\mathcal{F} = Z$ (resp. $\mathcal{F} = Z$); (M, E), then for the other $(\mathcal{S}, \mathcal{N}) = (S, C)$ or (S, D), and if the former has $\mathcal{F} = Z$ then the other has $\mathcal{F} = Z$ or $\mathcal{F} = L$.

R6) If $\mathcal{F} = S$ (resp. M) in $\mathcal{E}_k^1$ then $A_k = S$ (resp. M). If $\mathcal{F} = S$ (resp. M) in $\mathcal{E}_k^2$ then $A_{k+1} = S$ (resp. M).

R7) Assume for an edge-mark $\mathcal{E}_k^1$ (resp. $\mathcal{E}_k^2$) of any $\mathcal{V}_m$ that $\mathcal{F} = R$. Then there is an edge-mark $\mathcal{E}_k^2[+\pi]$ (resp. $\mathcal{E}_k^1[+\pi]$) of $\mathcal{V}_m$, and $\mathcal{F} = R$ in that edge too. The value $\mathcal{F} = R$ may not appear in any edge-mark numbered $\mathcal{E}_1^1$ or $\mathcal{E}_e^2$. Also, the values of $\mathcal{S}$ are the same in $\mathcal{E}_k^1$ and $\mathcal{E}_k^2[+\pi]$ (resp. $\mathcal{E}_k^2$ and $\mathcal{E}_k^1[+\pi]$), the values of $\mathcal{N}$ are the same in $\mathcal{E}_k^1$ and $\mathcal{E}_k^2[+\pi]$ (resp. $\mathcal{E}_k^2$ and $\mathcal{E}_k^1[+\pi]$), and the values of $\mathcal{P}$ are the same in $\mathcal{E}_k^1$ and $\mathcal{E}_k^2[+\pi]$ (resp. $\mathcal{E}_k^2$ and $\mathcal{E}_k^1[+\pi]$). Finally, it cannot happen that $\mathcal{F} = R$ for another edge-mark $\mathcal{E}_q^p$ of $\mathcal{V}_m$.

R8) Assume for an edge-mark $\mathcal{E}_k^1$ (resp. $\mathcal{E}_k^2$) of any $\mathcal{V}_m$ that $\mathcal{F} = Z$. Then there is an edge-mark $\mathcal{E}_k^2[+\pi]$ (resp. $\mathcal{E}_k^1[+\pi]$) of $\mathcal{V}_m$, and $\mathcal{F} = Z$ in that edge also. Also, the values of $\mathcal{S}$ are the same in $\mathcal{E}_k^1$ and $\mathcal{E}_k^2[+\pi]$ (resp. $\mathcal{E}_k^2$ and $\mathcal{E}_k^1[+\pi]$), the values of $\mathcal{N}$ are the same in $\mathcal{E}_k^1$ and $\mathcal{E}_k^2[+\pi]$ (resp. $\mathcal{E}_k^2$ and $\mathcal{E}_k^1[+\pi]$), and the values of $\mathcal{P}$ are the same in $\mathcal{E}_k^1$ and $\mathcal{E}_k^2[+\pi]$ (resp. $\mathcal{E}_k^2$ and $\mathcal{E}_k^1[+\pi]$). Finally, it can only happen that $\mathcal{F} = Z$ for another edge-mark $\mathcal{E}_q^p$ of $\mathcal{V}_m$ if $\mathcal{E}_q^p$ has angle zero with one of the above pair.

R9) Let $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ be the three vertices of a prototile, with $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ both referring to the same physical edge, and $2 = S$. See Figure 13. If $G_m^{\epsilon 1 S}$ in $\mathcal{E}_e^2$ (resp. $G_m^{\alpha 2 S}$ in $\mathcal{E}_1^1$) of $\mathcal{V}_1$, equals $G_m^{\alpha 1 S}$ in $\mathcal{E}_1^1$ (resp. $G_m^{\epsilon 2 S}$ in $\mathcal{E}_e^2$) of $\mathcal{V}_3$, then in $\mathcal{V}_2$: if $m = D$, and $(\mathcal{S}, \mathcal{N}) = (M, D)$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$), then $\mathcal{F} = S$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$); if $m = E$, and $(\mathcal{S}, \mathcal{N}) = (M, E)$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$), then $\mathcal{F} = S$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$); if $m = B$, and $(\mathcal{S}, \mathcal{N}) = (L, B)$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$), then $\mathcal{F} = S$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$).

R10) Let $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ be the three vertices of a prototile, with $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ both referring to the same physical edge, and $2 = M$. See Figure 13. If $G_m^{\alpha 1 M}$ in $\mathcal{E}_e^2$ (resp. $G_m^{\alpha 2 M}$ in $\mathcal{E}_1^1$) of $\mathcal{V}_1$, equals $G_m^{\epsilon 1 M}$

in $\mathcal{E}_1^1$ (resp. $G_m^{\epsilon 2M}$ in $\mathcal{E}_e^2$) of $\mathcal{V}_3$, then in $\mathcal{V}_2$: if $m = D$, and $(\mathcal{S}, \mathcal{N}) = (S, D)$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_e^2$), then $\mathcal{F} = M$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_e^2$); if $m = A$, and $(\mathcal{S}, \mathcal{N}) = (S, A)$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_e^2$), then $\mathcal{F} = M$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_e^2$).

R11) Let $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ be the three vertices of a prototile, with $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ both referring to the same physical edge, and $2 = M$. See Figure 14. If $(\mathcal{S}, \mathcal{N}) = (L, B)$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$, and $(\mathcal{S}, \mathcal{N}) = (S, B)$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_3$, then in $\mathcal{V}_2$ we have $\mathcal{F} = M$ in both $\mathcal{E}_1^1$ and $\mathcal{E}_e^2$.

R12) Let $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ be the three vertices of a prototile, with $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ both referring to the same physical edge, and $2 = M$. See Figure 14. If $(\mathcal{S}, \mathcal{N}) = (L, C)$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$, and $(\mathcal{S}, \mathcal{N}) = (S, C)$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_3$, then in $\mathcal{V}_2$ we have $\mathcal{F} = M$ in both $\mathcal{E}_1^1$ and $\mathcal{E}_e^2$.

R13) Let $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ be the three vertices of a prototile, with $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ both referring to the same physical edge, and $2 = S$. See Figure 14. If $(\mathcal{S}, \mathcal{N}) = (L, C)$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$, and $(\mathcal{S}, \mathcal{N}) = (M, C)$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_3$, then in $\mathcal{V}_2$ we have $\mathcal{F} = S$ in both $\mathcal{E}_1^1$ and $\mathcal{E}_e^2$.

R14) If in two of the vertices of a prototile there are $\mathcal{F}$ values indicating that the edge between the vertices ends at the vertices, then the corresponding $\mathcal{N}$ value must agree with the class of prototile that it is.

R15) Let $X = A$ or D. Assume we have a prototile T, with vertices $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$, and that $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ and $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ refer to the same edge of T. Then if $\mathcal{F} = Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, G_X^{jkM} has the same value in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ as it has in $\mathcal{E}_1^2[-\pi]$ (resp. $\mathcal{E}_e^1[+\pi]$) of $\mathcal{V}_2$, unless $\mathcal{V}_2$ has an edge-mark $\mathcal{E}_1^2[-\pi/2]$ (resp. $\mathcal{E}_e^1[+\pi/2]$) which contains the values $(\mathcal{S}, \mathcal{N}, \mathcal{F}, \mathcal{P}) = (S, A, L, A)$ or (S, A, L, D), in which case the two $\mathcal{G}$ values have opposite sign. If $\mathcal{F} \neq Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, and we do not have: $k = 2$ (resp. $k = 1$) and $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (S, X, L)$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then $G_X^{\epsilon kM} = +1$ in that edge-mark; $k = 1$ (resp. $k = 2$) and $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (S, A, L)$ or (S, D, L) in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then $G_X^{\alpha kM} = +1$ in that edge-mark. See M1.

R16) Let $X = D$ or E. Assume we have a prototile T, with vertices $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$, and that $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ and $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ refer to the same edge of T. If $\mathcal{F} = Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then G_X^{jkS} has the same value in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ as it has in $\mathcal{E}_1^2[-\pi]$ (resp. $\mathcal{E}_e^1[+\pi]$) of $\mathcal{V}_2$, unless: $\mathcal{V}_2$ has an edge-mark $\mathcal{E}_1^2[+\pi/2]$ (resp. $\mathcal{E}_e^1[-\pi/2]$) which contains the values $(\mathcal{S}, \mathcal{N}, \mathcal{F}, \mathcal{P}) = (S, A, L, E)$ or (S, A, L, D); or $\mathcal{V}_2$ has an edge-mark $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) which contains the values $(\mathcal{S}, \mathcal{N}, \mathcal{F}, \mathcal{P}) = (S, A, M, B)$, in which case the two $\mathcal{G}$ values have opposite sign. If $\mathcal{F} \neq Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, and we do not have: $k = 2$ (resp. $k = 1$) and $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (M, X, L)$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then $G_X^{\epsilon kS} = +1$ in that edge-mark; $k = 1$ (resp. $k = 2$) and $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (M, E, L)$ or (M, D, L) in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then $G_X^{\alpha kS} = +1$ in that edge-mark. If $\mathcal{F} = Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ and we do not have $k = 1$ (resp. $k = 2$) and $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (L, B, S)$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$, then $G_X^{\alpha kS} = +1$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$. See M4.

R17) Assume we have a prototile T, with vertices $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$, and that $\mathcal{E}_1^1$ (resp.

$\mathcal{E}_e^2$) of $\mathcal{V}_2$ and $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ refer to the same edge of T. If $\mathcal{F} = Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then G_B^{jkS} has the same value in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ as it has in $\mathcal{E}_1^2[-\pi]$ (resp. $\mathcal{E}_e^1[+\pi]$) of $\mathcal{V}_2$, unless: $\mathcal{V}_2$ has an edge-mark $\mathcal{E}_1^2[+\pi/2]$ (resp. $\mathcal{E}_e^1[-\pi/2]$) which contains the values $(\mathcal{S}, \mathcal{N}, \mathcal{F}, \mathcal{P}) = (S, A, L, E)$ or (S, A, L, D); or $\mathcal{V}_2$ has an edge-mark $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) which contains the values $(\mathcal{S}, \mathcal{N}, \mathcal{F}, \mathcal{P}) = (S, A, M, B)$, in which case the two $\mathcal{G}$ values have opposite sign. If $k = 2$ (resp. $k = 1$) and we do not have $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (M, E, L)$ or (M, D, L) in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then $G_B^{\alpha kS} = +1$ in that edge-mark. If $k = 1$ (resp. $k = 2$) and we do not have $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (L, B, S)$ in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$, then $G_B^{\alpha kS} = G_B^{\epsilon kS} = +1$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$. See M8.

R18) $H_m^{jkn} = G_m^{jkn}$ in an edge-mark if in that edge-mark we have $(\mathcal{S}, \mathcal{N}) \neq (S, A), (S, D), (M, D)$, or if $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (S, A, L), (S, D, L)$ or (M, D, S).

R19) Assume we have a prototile T, with vertices $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$, and that $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ and $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ refer to the same edge of T. If $\mathcal{F} = Z$ in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$, then the ratio of H_m^{jkn} and G_m^{jkn} is the same in $\mathcal{E}_1^2[+\pi]$ (resp. $\mathcal{E}_e^1[+\pi]$) of $\mathcal{V}_2$ as it is $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$. See Figure 15.

R20) Consider any vertex of a prototile with edge-mark $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) in which $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (S, D, M)$. Then the value of H_m^{j1n} (resp. H_m^{j2n}) in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) is the product of G_m^{j1n} (resp. G_m^{j2n}) in that edge-mark with G_m^{j1n} (resp. G_m^{j2n}) in $\mathcal{E}_2^1$ (resp. $\mathcal{E}_{e-1}^2$). See M13.

R21) Consider any vertex of a prototile with edge-mark $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) in which $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (M, D, L)$. The value of $\mathcal{H}_m^{jkn}$ in that edge-mark is equal to its value in the edge-mark $\mathcal{E}_1^2[-\pi]$ (resp. $\mathcal{E}_e^1[+\pi]$). See M14.

R22) Consider any vertex of a prototile with edge-mark $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) in which $(\mathcal{S}, \mathcal{N}, \mathcal{F}) = (S, A, M)$. The value of $\mathcal{H}_m^{jkn}$ in that edge-mark is equal to its value in the edge-mark $\mathcal{E}_1^2[+\pi]$ (resp. $\mathcal{E}_e^1[-\pi]$). See M15.

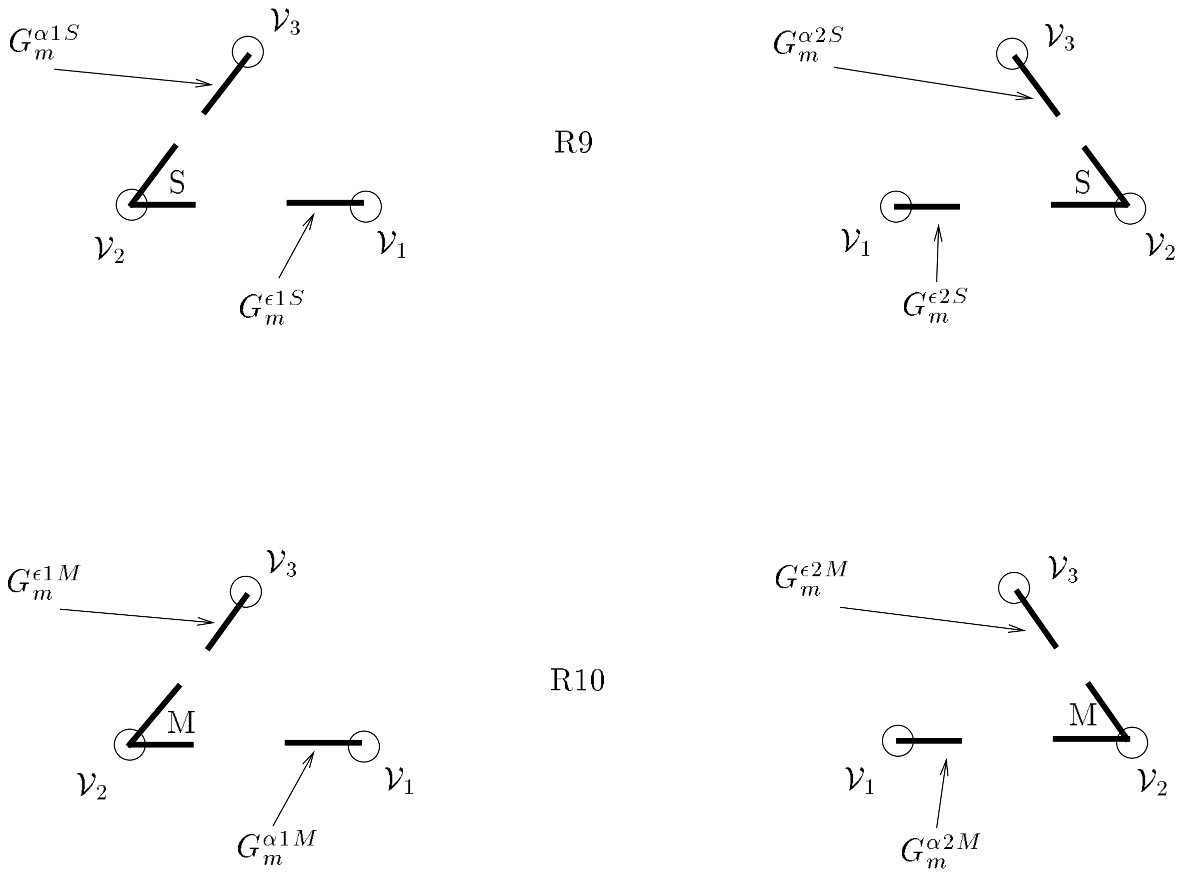

$G_m^{\alpha 1S}$
$\mathcal{V}_3$
$\mathcal{V}_2$
S
$\mathcal{V}_1$
$G_m^{\epsilon 1S}$
R9
$G_m^{\alpha 2S}$
$\mathcal{V}_3$
$\mathcal{V}_1$
S
$\mathcal{V}_2$
$G_m^{\epsilon 2S}$
$G_m^{\epsilon 1M}$
$\mathcal{V}_3$
$\mathcal{V}_2$
M
$\mathcal{V}_1$
$G_m^{\alpha 1M}$
R10
$G_m^{\epsilon 2M}$
$\mathcal{V}_3$
$\mathcal{V}_1$
M
$\mathcal{V}_2$
$G_m^{\alpha 2M}$

Figure 13

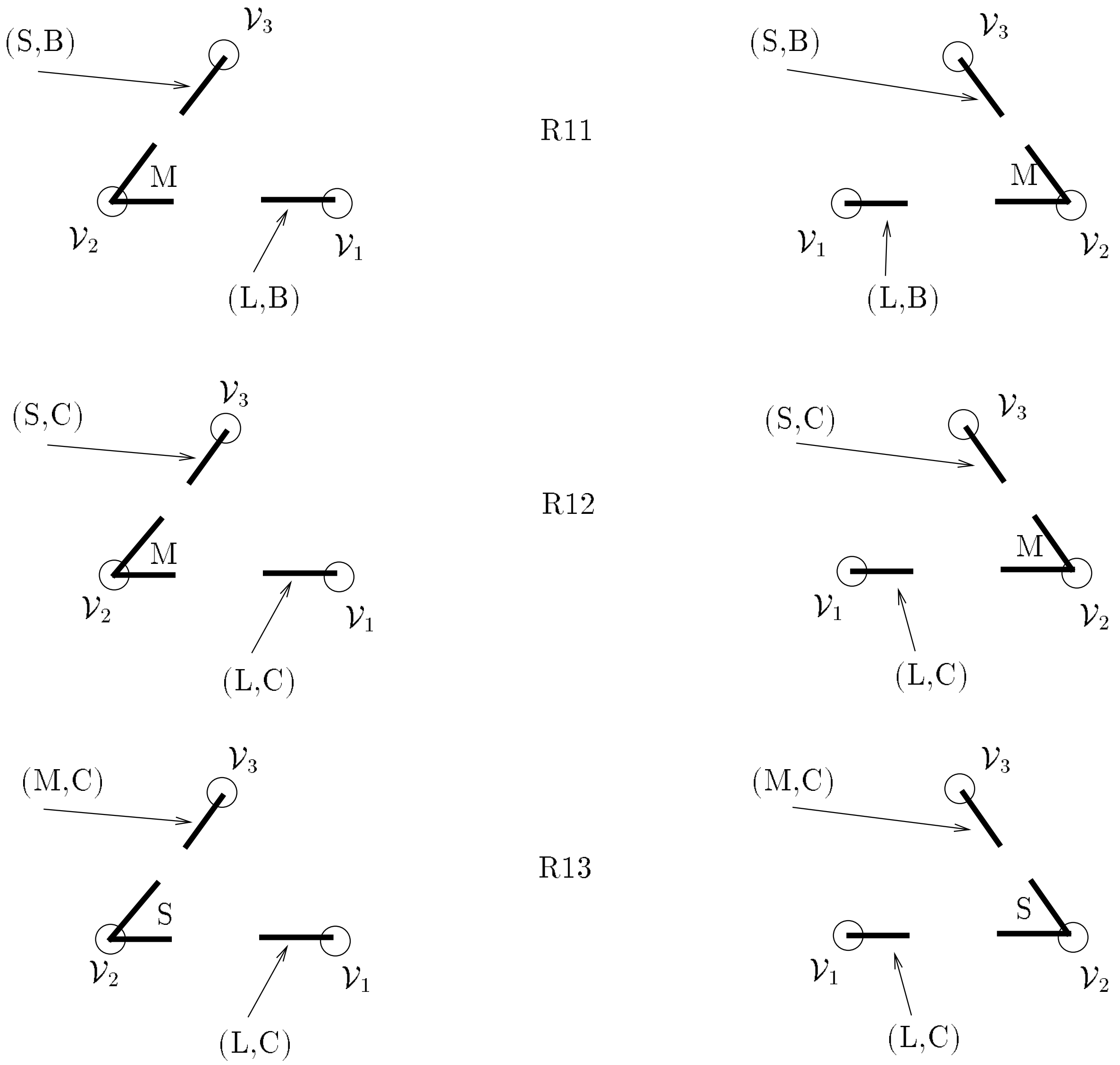

(S,B)
$\mathcal{V}_3$
M
$\mathcal{V}_2$
$\mathcal{V}_1$
(L,B)
R11
(S,B)
$\mathcal{V}_3$
M
$\mathcal{V}_1$
$\mathcal{V}_2$
(L,B)
(S,C)
$\mathcal{V}_3$
M
$\mathcal{V}_2$
$\mathcal{V}_1$
(L,C)
R12
(S,C)
$\mathcal{V}_3$
M
$\mathcal{V}_1$
$\mathcal{V}_2$
(L,C)
(M,C)
$\mathcal{V}_3$
S
$\mathcal{V}_2$
$\mathcal{V}_1$
(L,C)
R13
(M,C)
$\mathcal{V}_3$
S
$\mathcal{V}_1$
$\mathcal{V}_2$
(L,C)

Figure 14

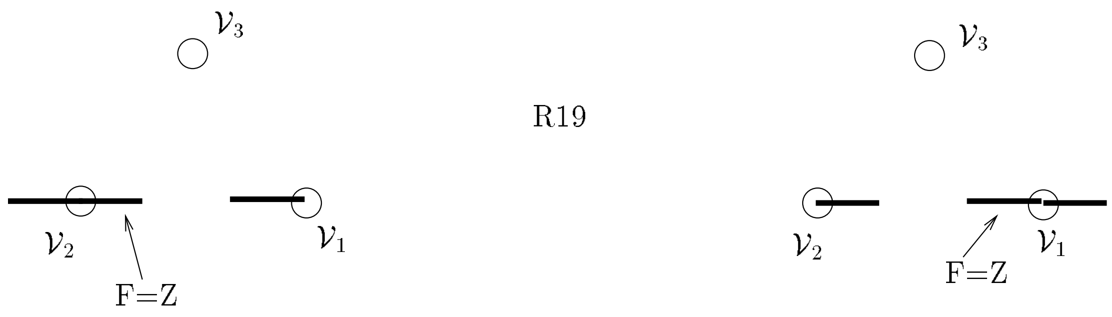


Figure 15

II c). Tiles and Matching rules

We define a "tile" as an (orientation preserving) isometric image of one of the two unmarked prototiles, together with its marks. (We could define the tiles of class X2 to be reflections – with appropriate changes of markings – of those of class X1 if it was desired to minimize the number of prototiles.) In order to define "matching rules" for these tiles, we first define what it means for two tiles to be "neighbors". Two tiles are neighbors if their intersection consists of an edge of each. Also, a tile of type E will be called the neighbor of a tile of type C if they intersect precisely in the small edge of the C and half the medium edge of the E, with the medium vertex of the C and the large vertex of the E coinciding. A tile of type E is a neighbor of a tile of type D if they intersect precisely in the small edge of the D and half the medium edge of the E, with the medium vertex of the D and the small vertex of the E coinciding. A tile of type E is a neighbor of a tile of type A if they intersect precisely in the small edge of the E and half the medium edge of the A, with the small vertex of the A and the medium vertex of the E coinciding. A tile of type A is a neighbor of a tile of type B if they intersect precisely in half the medium edges of each, with the small vertex of the B in the middle of the edge of the A, and the large vertex of the A in the middle of the edge of the B. And finally, a tile of type B is a neighbor of a tile of type B if they intersect precisely in half the medium edges of each, with the large vertex of each B in the middle of the edge of the other B. See the following Figure 16.

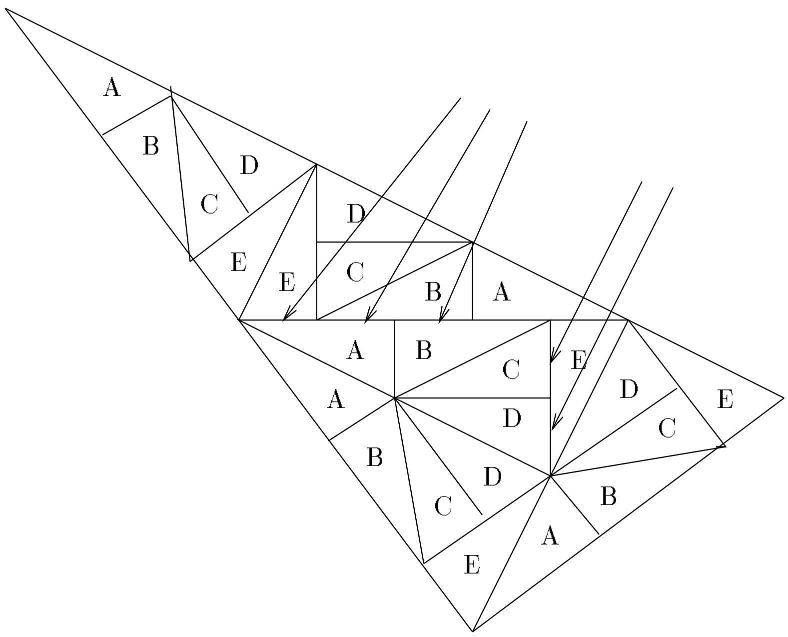


Figure 16

Using the above, the matching rules for tiles are: in a tiling two tiles T_m and T_n may abut if they are neighbors, and whenever a vertex $\mathcal{V}_m$ of T_m coincides with a vertex $\mathcal{V}_n$ of T_n, and an edge-mark $\mathcal{E}_1^1$ of $\mathcal{V}_m$ overlaps in an interval an edge-mark $\mathcal{E}_e^2$ of $\mathcal{V}_n$, then $\mathcal{E}_k^j$ of $\mathcal{V}_n$ equals $\mathcal{E}_{k+1}^j$ of $\mathcal{V}_m$, and A_k of $\mathcal{V}_n$ equals A_{k+1} of $\mathcal{V}_m$, where $e + 1 \equiv 1$.

III. Structure in tilings and results

III a). Structure in tilings

We define a triangle of level 0 (note the distinction from "Triangle" and "Level") as a tile; its class and type are those of the tile. We inductively define a triangle of level $n \geq 1$ as an "appropriate collection" (defined below) of five triangles of level $n-1$ together with certain conditions on values of $\mathcal{F}$, $\mathcal{N}$ and $\mathcal{S}$ as noted below.

An appropriate collection of five triangles of level n is the image, by a single isometry of the plane, of: a triangle of type A with vertices $5^{n/2}(-2,1)$, $5^{n/2}(0,1)$ and $(0,0)$; a triangle of type B with vertices $5^{n/2}(0,1)$, $(0,0)$ and $5^{n/2}(2,1)$; a triangle of type C with vertices $(0,0)$, $5^{n/2}(2,0)$ and $5^{n/2}(2,1)$; a triangle of type D with vertices $(0,0)$, $5^{n/2}(2,0)$ and $5^{n/2}(2,1)$; and a triangle of type E with vertices $5^{n/2}(2,1)$, $5^{n/2}(3,1)$ and $5^{n/2}(2,-1)$. For any such collection we carry over the definition we used for Triangles of tiles T_S, T_M and T_L, and vertices V_{SS}, V_{SM} etc.

To be a triangle of level $n \geq 1$ we require moreover, $\mathcal{F} = Z$ in all edge-marks referring to tile edges which coincide in an interval with one of the three edges of the collection, except possibly for edge-marks in those vertices coinciding with the three vertices of the collection, which however must satisfy the following. The large triangle must belong to one of these (mutually exclusive) "classes" (see Figure 17):

• A1 (resp. A2), in V_{MM}, $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,A,M)$, and $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) contains $\mathcal{F} = Z$, and in V_{SS}, $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,A,L)$, and $\mathcal{E}_2^1$ (resp. $\mathcal{E}_{e-1}^1$) contains $\mathcal{F} = Z$;
• B1 (resp. B2), in V_{MM}, $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,B,M)$, and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (L,B,M)$, and in V_{SS}, $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,B,L)$, and $\mathcal{E}_{e-1}^2$ (resp. $\mathcal{E}_2^1$) contains $\mathcal{F} = Z$, and in V_{SS}, $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (L,B,S)$, and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) contains $\mathcal{F} = Z$;
• C1 (resp. C2), in V_{MM}, $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,C,M)$, and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (L,C,M)$, and in V_{SS}, $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,C,L)$, and $\mathcal{E}_{e-1}^2$ (resp. $\mathcal{E}_2^1$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,C,L)$, and in V_{SS}, $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (L,C,S)$, and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,C,S)$;
• D1 (resp. D2), in V_{MM}, $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,D,M)$, and in V_{SS}, $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (S,D,L)$, and $\mathcal{E}_2^1$ (resp. $\mathcal{E}_{e-1}^2$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,D,L)$, and in V_{SS}, $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) contains $\mathcal{F} = Z$, and $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,D,S)$;
• E1 (resp. E2), in V_{SS}, $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) contains $\mathcal{F} = Z$, and $\mathcal{E}_2^1$ (resp. $\mathcal{E}_{e-1}^2$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,E,L)$, and in V_{SS}, $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) contains $(\mathcal{S},\mathcal{N},\mathcal{F}) = (M,E,S)$.

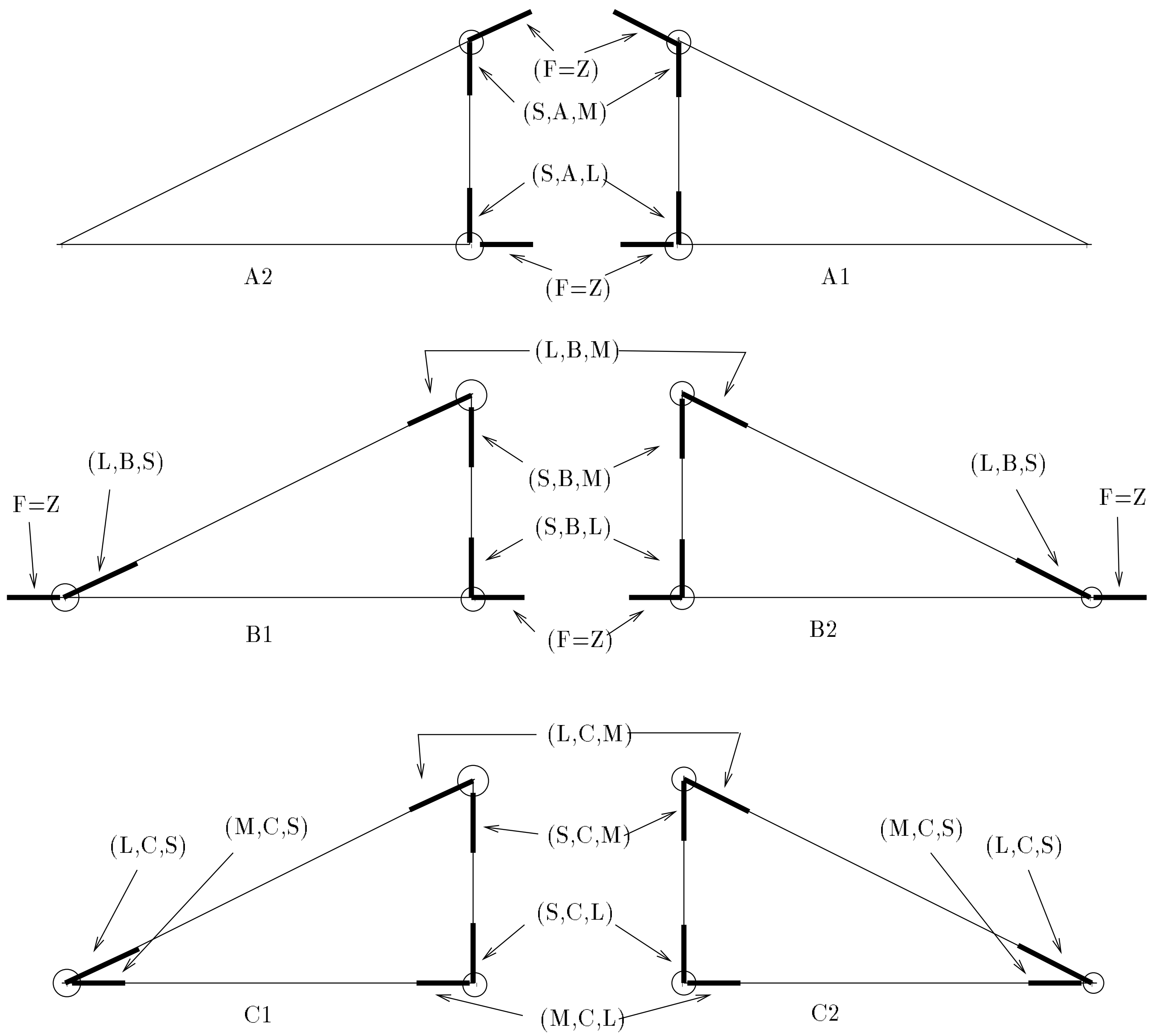

(F=Z)
(S,A,M)
(S,A,L)
A2
(F=Z)
A1
(L,B,M)
(L,B,S)
F=Z
(S,B,M)
(S,B,L)
(L,B,S)
F=Z
B1
(F=Z)
B2
(L,C,M)
(M,C,S)
(L,C,S)
(S,C,M)
(M,C,S)
(L,C,S)
(S,C,L)
C1
(M,C,L)
C2

Figure 17a

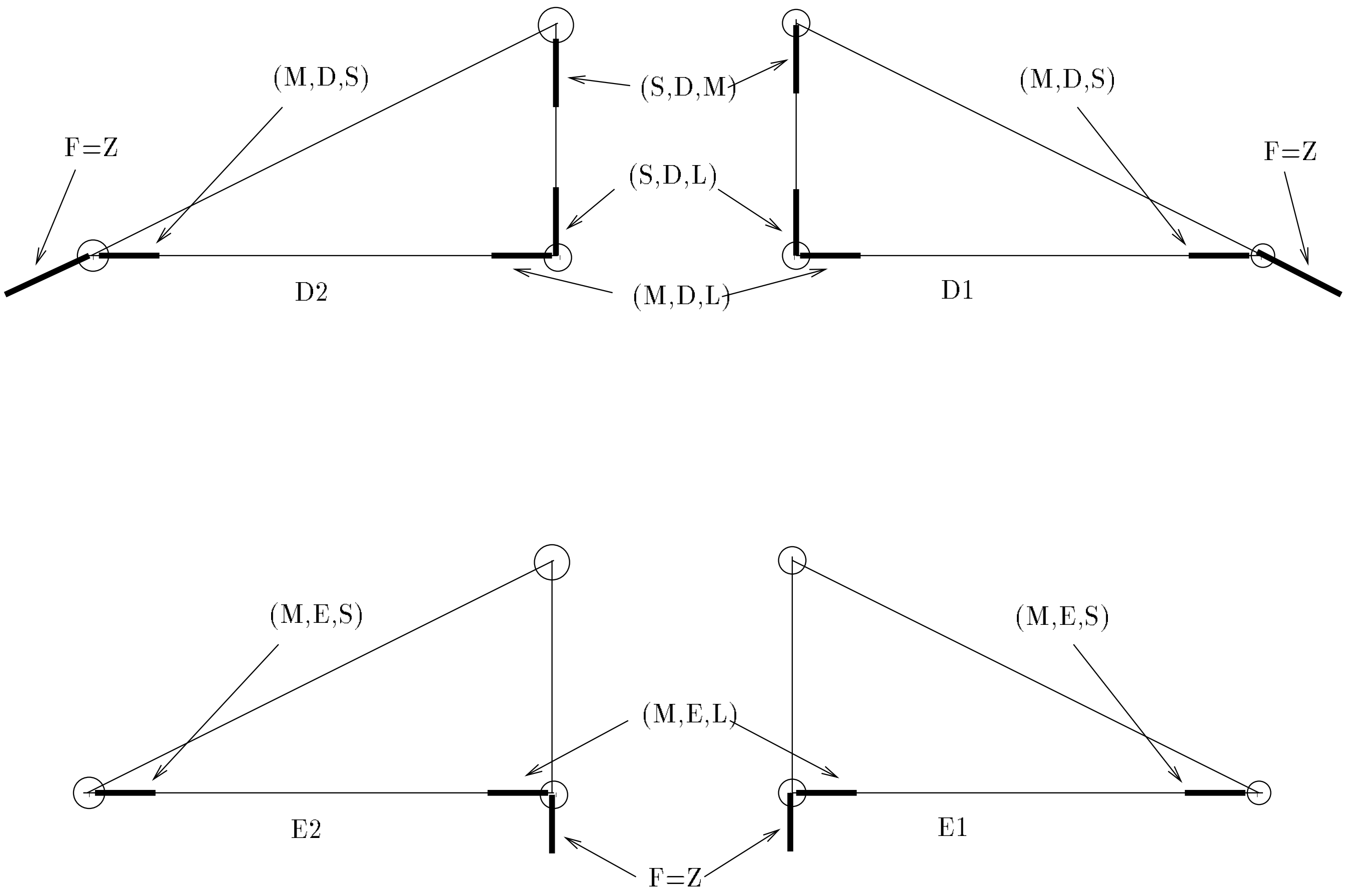


Figure 17b

A triangle of class Xj and level $n \geq 1$, defined above, is said to be of type X.

In summary, a triangle of level $n \geq 1$ is defined inductively as a collection of five triangles of level $n-1$ in one of two possible specified geometric relationships, satisfying one of ten (mutually exclusive) sets of conditions on values of $\mathcal{F}$, $\mathcal{N}$ and $\mathcal{S}$ values in those edge-marks referring to the edges of the collection; $\mathcal{F} = Z$ at all interior vertices, and $(\mathcal{F}, \mathcal{N}, \mathcal{S})$ exhibit one of ten specified patterns at the three vertices of the collection. Note (using V1 $\cdots$ V10) that triangles of level 0 (namely tiles) satisfy all the properties of triangles of level n, for $n = 0$.

III b. Results

Theorem 1. *If in the substitution tessellation of the plane we consider the Tiles as tiles, we obtain a tiling of the plane with all matching rules satisfied.*

Proof.

We need to prove that the markings on the Tiles satisfy the restrictions for markings of tiles, A1 $\cdots$ E2, V1 $\cdots$ V10, and R1 $\cdots$ R22, and that the neighbor rules are satisfied.

For the most part, the restrictions on tiles and the definition of "neighbor" are constructed so as to obviously conform with the tessellation. We first show that abutting Tiles occur in the tessellation only in ways allowed by the neighbor rules.

Our claim is obviously true for Level 1 Triangles. New types of neighbor combinations may appear as a result of some Triangle of a certain Level n being divided in five parts according to our procedure; two abutting Tiles in a Level n Triangle, when subdivided, may create along their common edge something new at Level $n+1$. Consequently, we just have to verify that any two Tiles, neighbors according to the rules, give rise only to such neighboring Tiles after being divided. We now examine four cases.

If the Tiles match (full) edge-to-edge, with equal angles at coinciding vertices, then they produce only edge-to-edge matchings after the subdivision.

If the Tiles match edge-to-edge, with unequal angles at coinciding vertices (this happens in the tessellation only for L edges, but this is irrelevant to our argument), then they could not produce edge-to-edge matchings unless they border along L edges. In this case they produce the A-to-A and A-to-B matchings on half the medium edge and the A-to-E small edge to half medium edge matching (see Figure 3).

All matchings of an S edge to half an M edge give rise to edge to edge matchings after a subdivision (see Figure 3).

All half medium edge to half medium edge matchings give rise to edge to edge matchings after a subdivision (see Figure 3).

This completes our argument concerning the neighbor rules. The only restrictions we consider are R9, R10, and parts of V1 and V6.

Consider a typical case of R9: assume $(\mathcal{S}, \mathcal{N}) = (M, D)$ in $\mathcal{E}_1^1$ of $\mathcal{V}_2$, and $G_D^{\epsilon 1 S}$ in $\mathcal{E}_e^2$ of $\mathcal{V}_1$ equals $G_D^{\alpha 1 S}$ in $\mathcal{E}_1^1$ of $\mathcal{V}_3$. We will show that this only occurs if the Tile is as shown in Figure 18, namely the Tile T_S of a Triangle of class D2 (which is of course consistent with the conclusion).

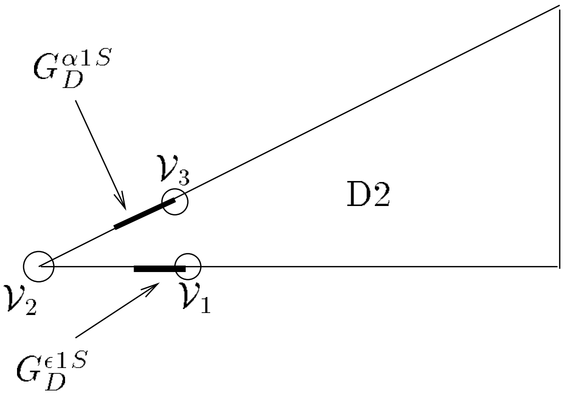


Figure 18

We know from the hypothesis there is a Triangle T of type D, with the medium edge of T containing vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ of the Tile. Our proof will be by contradiction. So assume that $\mathcal{F} \neq S$ in $\mathcal{E}_1^1$ of $\mathcal{V}_2$. It follows that $\mathcal{V}_2$ and $\mathcal{V}_3$ lie on a complete edge which ends at $\mathcal{V}_2$, and from the geometry this edge must be either: the medium edge of a type D or E Triangle, or the large edge of a type B Triangle. The three cases are similar, and we illustrate the argument with the case of the medium edge of a type E Triangle, T'. The geometry must then be as in Figure 19.

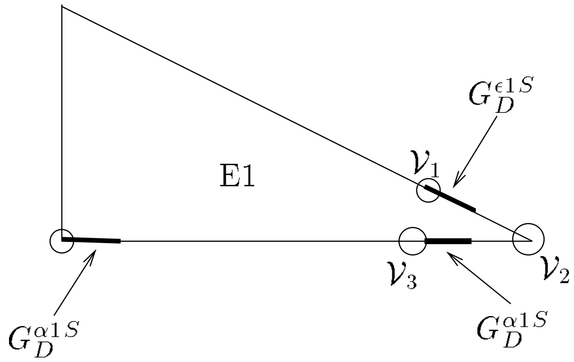


Figure 19

So consider M6 with $Y1 = E1$ and $X = D$. We will have proven our assertion (by contradiction) if we can show that $G_D^{\alpha 1S}$ in $\mathcal{E}_e^1[+\pi/2]$ of vertex V_{LL} of T' equals $G_D^{\alpha 1S}$ in $\mathcal{E}_1^1$ of vertex V_{SM} of T'. This lies at the heart of our technique. We have specified in M1 $\cdots$ M15 precisely how the different variables $\mathcal{G}$ and $\mathcal{H}$ vary, as one moves from vertex to vertex within a given complete edge. We now need to note a qualitative feature of the variation which follows from examination of the subtriangle structure in Figure 20, noting that any variation in G_m^{jkn} in the first triangle is repeated in the second.

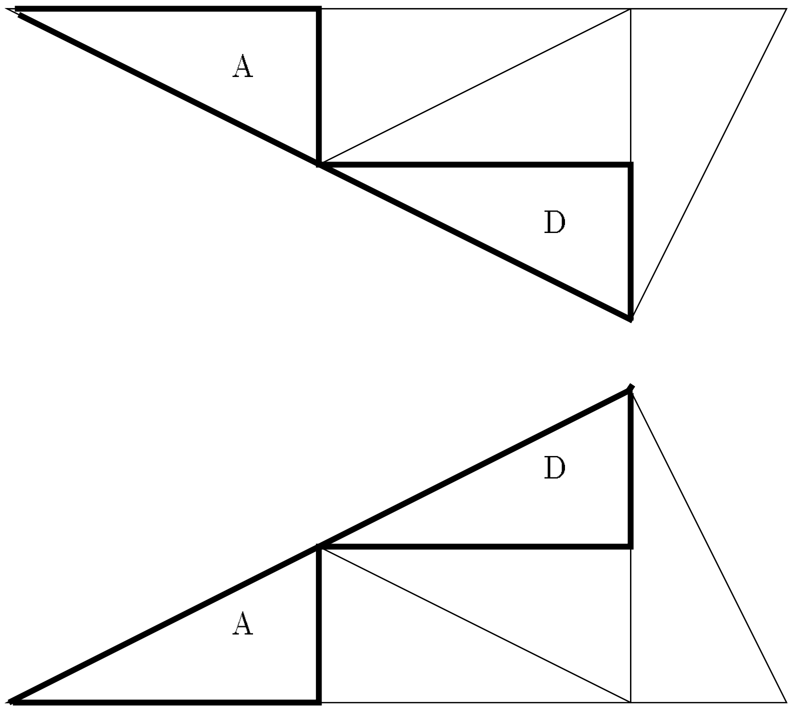


Figure 20

Lemma 1. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two vertices of a Tile, with $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ and $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$ both referring to the same physical edge. Assume $\mathcal{S} = M$, and $\mathcal{F} = S$ or L, in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of $\mathcal{V}_1$ (signifying that a medium edge ends at $\mathcal{V}_1$), and assume the other end of this edge occurs in edge-mark $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of some vertex $\mathcal{V}_4$, possibly of a different Tile. Then each variable G_m^{jkn} has the same value in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) $\mathcal{V}_4$ as it has in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of $\mathcal{V}_2$.

To continue our proof, we needed to show that a given variable did not change sign as it moved from one end of a medium edge to the other, and this follows from the Lemma. This completes our discussion of R9.

We now consider R10. More specifically assume $G_A^{\alpha 1M}$ in $\mathcal{E}_e^2$ of $\mathcal{V}_1$ equals $G_A^{\epsilon 1M}$ in $\mathcal{E}_1^1$ of $\mathcal{V}_3$, and that $(\mathcal{S}, \mathcal{N}) = (S, A)$ in $\mathcal{E}_e^2$ of $\mathcal{V}_2$; we must show that $\mathcal{F} = M$ in $\mathcal{E}_e^2$ of $\mathcal{V}_2$ as in Figure 21.

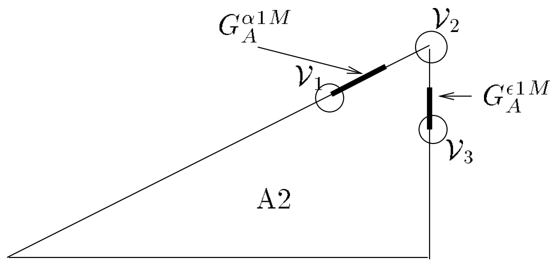


Figure 21

We know from the hypothesis there is a Triangle T of type A, with the short edge of T containing vertices $\mathcal{V}_3$ and $\mathcal{V}_2$ of the Tile. Our proof will be by contradiction. So assume $\mathcal{F} \neq M$ in $\mathcal{E}_e^2$ of $\mathcal{V}_2$. It follows that $\mathcal{V}_2$ and $\mathcal{V}_1$ lie on a complete edge which ends at $\mathcal{V}_2$, and from the geometry this edge must be the short edge of a type D or A Triangle. The two cases are similar, and we illustrate the argument with the case of a type D Triangle, T'. The geometry must then be as in Figure 22.

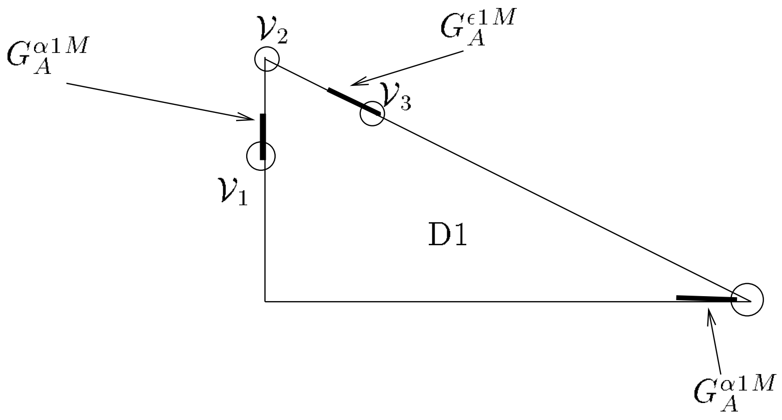


Figure 22

So consider M3 with $Y1 = D1$ and $X = A$. We will have proven our assertion (by contradiction) if we can show that the ratio of the value of $G_A^{\alpha 1M}$ in $\mathcal{E}_e^2$ of vertex V_{LL} of Tile T_L of T' to its value in $\mathcal{E}_e^2$ of $\mathcal{V}_1$, is equal to the ratio of the value of $G_A^{\epsilon 1M}$ in $\mathcal{E}_1^1$ of vertex V_{SS} of Tile T_S to its value in $\mathcal{E}_1^1$ of $\mathcal{V}_3$. This follows from Lemma 1 and the following lemma (which follows from examination of Figure 2).

Lemma 2. In any Triangle T'' of class E1 (resp. E2), some (and therefore any) variable G_m^{jkn} has the same value in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of vertex $\mathcal{V}_{LL}$ of Tile T_L of T'' as it has in $\mathcal{E}_e^2$ (resp. $\mathcal{E}_1^1$) of vertex $\mathcal{V}_{ML}$ of Tile T_M of T'', if and only if some (and therefore any) variable $G_{m'}^{j'k'n'}$ has the same value in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of vertex

$\mathcal{V}_{SS}$ of Tile T_S of T'' as it has in $\mathcal{E}_1^1$ (resp. $\mathcal{E}_e^2$) of vertex $\mathcal{V}_{MS}$ of Tile T_M of T''.
This completes our discussion of R10.

The nonobvious portions of V1 and V6 concern the values of the G_m^{jkn}. To take a specific case, we consider V1 and show that the quotient of H_A^{1M} by $G_A^{\epsilon 1M}$ in $\mathcal{E}_1^1$ of $\mathcal{V}_2$ equals $G_A^{\alpha 1M}$ in $\mathcal{E}_e^2[-\pi/2]$ of $\mathcal{V}_1$ if $\mathcal{P} = A$ in $\mathcal{E}_e^2$ of $\mathcal{V}_1$. From this hypothesis we know $\mathcal{E}_e^2[-\pi/2]$ of $\mathcal{V}_1$ lies on the large edge of a Triangle T of class A2. From M2, M13, M14 and M15 our problem is equivalent to showing that $G_A^{\alpha 1M}$ has the same value in $\mathcal{E}_e^2[-\pi/2]$ of $\mathcal{V}_1$ as in $\mathcal{E}_e^2$ of vertex V_{MS} of Triangle T. But this follows immediately from Lemmas 1 and 2.

This completes our proof of the theorem. ∎

Theorem 2. *In any tiling of the plane satisfying the matching rules, there is a unique decomposition into triangles of level $n \geq 0$.*

Proof.

Preview: The structure of this proof is the following. To prove by induction the existence of unique triangles of all levels one needs to show that each triangle of level N is contained in a unique "appropriate collection" at level $N+1$, which furthermore has all the distinguishing characteristics of some class, so that one can unambiguously form the level $N+2$ "appropriate collection". Some of this is easy to arrange using the "accessible" parts of the components of the collection, in particular the $\mathcal{F}$ values at the large vertex of the collection, which is accessible from the E component. (Accessible means known directly from the definition of the class of the component.) The harder step is to arrange for the needed characteristics of the collection at inaccessible parts of the components, in particular the $\mathcal{F}$ values at small and medium vertices of the collection. This is accomplished by encoding the information in the $\mathcal{G}$ and $\mathcal{H}$ variables, which can then carry it from accessible to inaccessible parts along pathways defined by the $\mathcal{F}$ variables.

To begin the proof we first note that for $n = 0$ the conclusion is satisfied by the assumption of a tiling. So to continue the induction, assume we have a tiling of the plane (with all matching rules for tiles satisfied of course), and that the tiling is uniquely decomposable into triangles of level n for $0 \leq n \leq N$. We must show that the tiling is uniquely decomposable into triangles of level $N+1$.

It is routine to check using R5 and V1 $\cdots$ V10 that given any triangle of level N in the tiling there is a unique way to extend it to an "appropriate collection" of five such triangles, and that there is a unique value of $\mathcal{P}$ in the nine complete edges of the five level N triangles, in particular the small edge of the A and the small and medium edges of the D. It then follows from R4, R8, V1, V5, V6 and V10 that this $\mathcal{P}$ value is converted into the $\mathcal{N}$ values in all those edges of the collection appropriate for the value of $\mathcal{P}$, where the $\mathcal{N}$ values are meant to help define a type, and thus class, for the collection, making it an $N+1$ triangle. It remains to show that the needed $\mathcal{F}$ values are automatically correct, thereby justifying these $\mathcal{N}$ values.

The needed $\mathcal{F}$ values (namely all Z) in edge-marks interior to the edges of the collection are automatically correct by properties of the five component triangles. The only possible difficulty is therefore at the three vertices of the collection. There are ten cases, depending on the ten possible classes, A1 $\cdots$ E2, of the collection. The proof is similar for all cases. Using the $\mathcal{P}$ value used to determine the class of the triangle of level $N+1$, we get the needed information as follows, the general method being illustrated in the interests of clarity for the cases of class B1 and A2.

We consider first the case where $\mathcal{P} = B$ in the edge-marks in, say, the small edge of the component triangle of type D of the collection. The needed information in V_{LL} of the collection follows from V5 using this value of $\mathcal{P}$ in the small edge of the triangle of type D, which shares its medium vertex with the small vertex of T_L. The needed information in V_{MM} of the collection follows from R11, applied to this tile, again using the above $\mathcal{P}$ value, but now using it with R4 and R8.

The needed information in V_{SS} of the collection follows from R9 applied to tile,T_S as follows. We know from V5 together with R19 $\cdots$ R22 that $G_B^{\alpha 1 S}$ in $\mathcal{E}_e^1[+\pi/2]$ of vertex T_{LL} of the component A triangle equals $G_B^{\epsilon 1 S}$ in $\mathcal{E}_1^2[-\pi/2]$ of vertex T_{LL} of the collection. To carry this information to vertex $\mathcal{V}_S$ of the collection, in order to use R9, we use Lemma 1, which holds for tiles since it depends only on the variation of the $\mathcal{G}$ and $\mathcal{H}$ variables, which are the same for Tiles and tiles.

We now consider another case of the theorem, class A2, and only the needed information at the medium vertex of the collection (the information at the large vertex following as above.)

The needed information in V_{MM} of the collection follows from R10 applied to this tile, as follows. We know from V5 together with R19 $\cdots$ R22 that $G_B^{\alpha 1 M}$ in $\mathcal{E}_e^1[-\pi/2]$ of vertex T_{LL} of the component A triangle equals $G_B^{\epsilon 1 M}$ in $\mathcal{E}_1^1$ of vertex T_{LL} of the collection. To carry this information to vertex V_{SS} of the collection, in order to use R10, we use Lemmas 1 and 2, the latter holding for tiles as discussed above for Lemma 1.

This completes our proof of the theorem. ∎

IV. Concluding Remarks

In the above we have constructed the first example of a finite set of shapes (prototiles) which can tile the Euclidean plane, but **only** using tiles in infinitely many orientations (and thus of course only nonperiodically.) This requires, essentially for the first time, the use of rotations in the study of forced tilings. We make the following supplementary remarks.

• If our restrictions for tile markings are replaced by the stronger restrictions that the only markings allowed are those which appear on Tiles in the tessellation, then of course Theorem 1 and Theorem 2 still hold. (The reason we did not do this is that it would then be more difficult to determine if a given marked prototile satisfied the restrictions.)

• It is common in the tiling literature to restrict attention to tilings of polygonal tiles which are "edge-to-edge", that is, in which the vertex of a tile can only intersect another tile at a vertex, as distinct from the way our tiles of type E meet tiles of type C for example. To accomodate this (mistaken) prejudice, we note that our results can be recaptured with this additional property by the doubling of the number of prototiles; each of our prototiles listed above could be decomposed into two by cutting on a straight line from the center of the medium edge to the medium vertex, each half retaining of course the information on the two vertices it inherits: see Figure 23.

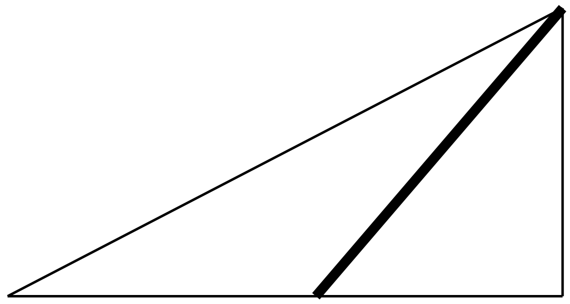


Figure 23

One would then add matching rules along these fresh edges, forcing only the original pairs to be able to meet along such an edge.

• It is convenient to insert a distinguished point within each triangle of type C, where the bisector of the large angle meets a straight line perpendicular to the center of the small edge, as in Figure 24.

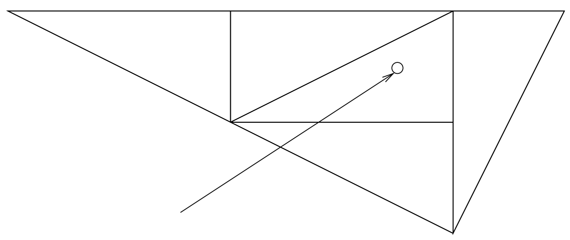


Figure 24

It is easy to check that for a triangle of type C of level $n \geq 1$ this point coincides with that associated with the triangle of type C of one lower level (and of course continuing down to level 0). These points are of interest because the triangles appear rotated about these points, in opposite directions for triangles of the same type but different class. This is the origin of the name we have given to this tiling system, with reference to the pinwheel toy which has blades, attached to a stick, which rotate in the wind.

• Daniel Berend has generalized the tessellation of Figure 4, using as prototile any right triangle with integer lengths for its small and medium edges, as is illustrated in Figure 25 for the case of legs of length 3 and 4.

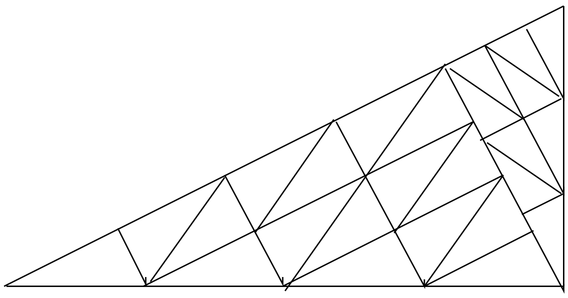


Figure 25

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