SPACE TILINGS AND SUBSTITUTIONS*

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Abstract

We generalize the study of symbolic dynamical systems of finite type and $\mathbf{Z}^2$ action, and the associated use of symbolic substitution dynamical systems, to dynamical systems with $\mathbf{R}^2$ action. The new systems are associated with tilings of the plane. We generalize the classical technique of the matrix of a substitution to include the geometrical information needed to study tilings, and we utilize rotation invariance to eliminate discrete spectrum. As an example we prove that Conway's pinwheel tilings have no discrete spectrum.

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1. INTRODUCTION.

Our main objective is to determine the limit of disorder which is possible in a certain class of dynamical systems with $\mathbf{R}^2$ action, a class, which we call "tiling dynamical systems", analogous to the symbolic systems of finite type. A subsidiary goal is to extend the classical tools of symbolic substitution dynamics to the geometrical needs of tilings of space.

We begin with a definition of tiling dynamical systems [10,1]. By a "prototile" p we mean a homeomorphic image of the closed unit disk in $\mathbf{R}^2$, with center of mass at the origin, which has small surface to volume ratio in the sense that:

$$\frac{area\ \{x \in tp\ :\ \|x-y\| \le 1,\ \text{for some } y \in \partial(tp)\}}{area\ \{tp\}} \longrightarrow 0 \qquad 1)$$

as the expansion factor $t \to \infty$. Assume given a finite set S of prototiles, and a subgroup $\mathbf{G}$ (called the "allowed isometries") of the connected subgroup $\mathbf{I}$ of the topological group of isometries of $\mathbf{R}^2$. $\mathbf{G}$ is assumed to be either $\mathbf{Z}^2$, $\mathbf{R}^2$ or $\mathbf{I}$, and the image of a prototile by an allowed isometry will be called a "tile". A final requirement: defining a "swatch" as a finite collection of tiles with pairwise disjoint interiors, we assume that any bounded set in the plane can only be covered by finitely many different swatches, up to isometry. We associate with given S and $\mathbf{G}$ the set $\mathbf{X}(S)$ of all infinite collections of tiles, called "tilings", that have union $\mathbf{R}^2$, and are such that each pair of tiles in a tiling has disjoint interiors. We assume $\mathbf{X}(S)$ is nonempty.

We next define a topology on $\mathbf{X}(S)$. There is an obvious action of $\mathbf{I}$ on the space $\mathbf{X}(S)$. Then a typical element of a base for the topology on $\mathbf{X}(S)$ is $C(P,\epsilon)$, where P is a swatch $\{\tau_j[\rho_j(p_j)]\ :\ 1 \le j \le n\}$, ($p_j$ being a prototile, ρ_j a rotation about the origin, and τ_j a translation), consisting of all tilings which contain a swatch $P' = \{\tau'_j[\rho'_j(p_j)]\ :\ 1 \le j \le n\}$ where $\|\tau_j - \tau'_j\| < \epsilon/2$ and $\|\rho_j - \rho'_j\| < \epsilon/2$. It is not hard to check that with this topology $\mathbf{X}(S)$ is metrizable and compact, and that $\mathbf{I}$ acts continuously on it [10].

Let $\mathbf{G}_T$ be the subgroup of translations in $\mathbf{G}$. The "tiling dynamical system" associated with given S and $\mathbf{G}$ is $(\mathbf{X}(S), \mathbf{G}_T)$. With this notation, our main objective is to determine the limits of disorder possible for uniquely ergodic tiling dynamical systems; in particular we wish to determine if such a system can have pure absolutely continuous spectrum. We do not solve this problem, but we do make a significant generalization of classical substitution dynamics and give evidence of its usefulness in our problem. Our basic problem has origins in several directions. For a review see [7], and for more recent developments see

[8,9,10,1].

One historical root of our problem was the search for tiling dynamical systems which have no closed orbits; the original examples (indeed all we know) were obtained by use of some form of hierarchical structure. We will follow this path. So given a tiling dynamical system, we assume further that we are given a "substitution function" F on the set of prototiles, which applied to prototile p produces a finite collection $F(p)$ of tiles, with pairwise disjoint interiors, such that the set theoretic union (also denoted $F(p)$) is geometrically similar to p. We call the collection, or any allowed isometry of it, an "LS–tile". LS–tiles $F(p)$ must have the same expansion factor $E > 1$ for all tiles p. One can iterate the substitution, producing larger and larger collections of tiles, each collection remaining geometrically similar to its predecessor. We will call any such collection, or an allowed isometry of such a collection, a "VLS–tile", and define its "order" as the number of substitutions used to produce it from a tile. (So an LS–tile is a VLS–tile of order 1.)

Given the substitution F we define the closed, $\mathbf{I}$-invariant subset $\mathbf{X}(S)_F$ of $\mathbf{X}(S)$ to consist of all tilings in which each swatch is a subset of a VLS–tile. (We call $(\mathbf{X}(S)_F, \mathbf{G}_T)$ a "substitution-tiling dynamical system".) It can easily be proven that every tiling in $\mathbf{X}(S)_F$ decomposes (not necessarily uniquely) into VLS–tiles of any fixed order with disjoint interiors. (Consider an expanding sequence of finite subcollections of a tiling, namely all the tiles inside an expanding disk. Using the definition we see then that every tiling is the union of an expanding sequence of VLS–tiles, and by diagonalization we can assume that in this expanding sequence the grouping of tiles into VLS–tiles of any one fixed order does not change.) We also note [10] that for all small enough ϵ, swatches P' associated with open sets $C(P,\epsilon)$ must be of the form $P' = g(P)$ for some $g \in \mathbf{I}$.

Previous examples, of tiling dynamical systems without closed orbits, were obtained by use of a substitution function F (or something similar) such that $\mathbf{X}(S)/\mathbf{X}(S)_F$ is either empty or at least has measure zero with respect to any invariant measure. In order that $(\mathbf{X}(S), \mathbf{G}_T)$ be uniquely ergodic clearly it cannot be the case that two tilings have disjoint orbit closures. This is the reason why $\mathbf{G}$ is taken to be $\mathbf{Z}^2$ rather than $\mathbf{R}^2$ for Wang dominoes [12,7], and $\mathbf{R}^2$ rather than $\mathbf{I}$ for Penrose tilings [4,7], together with a correspondingly restrictive replacement for $\mathbf{X}(S)$.

Let us consider further the special case of Wang dominoes, where the prototiles are small deformations of unit squares in the plane, all with edges parallel to Cartesian axes and with $\mathbf{G} = \mathbf{Z}^2$. Such tilings were considered by Mozes in

[5], where extensive use was made of the obvious alternative description of such systems as symbolic dynamical systems with $\mathbf{Z}^2$ action. One of the main results in that work was the construction of examples not just without closed orbits, but without discrete spectrum. This was generalized to examples with $\mathbf{G} = \mathbf{R}^2$ in [1]. In the present work we will generalize the approach further by allowing tiles with rather arbitrary shapes, and making use of a new and powerful mechanism to eliminate the discrete spectrum, namely rotational symmetry.

2. RESULTS.

Our first result gives simple hypotheses which force unique (uniform) orientational distribution.

Theorem. Assume given a set S of prototiles, a substitution function F, and the substitution dynamical system $(\mathbf{X}(S)_F, \mathbf{G}_T)$ with $\mathbf{G} = \mathbf{I}$. Assume there is some $r \geq 1$ such that:

a) for each prototile p, $F^r(p)$ contains tiles of every tile-type;

b) for some prototile p, $F^r(p)$ contains two tiles p', p'' of the same tile-type, whose relative rotation is irrational with respect to π.

Then $(\mathbf{X}(S)_F, \mathbf{G}_T)$ is uniquely ergodic.

Proof. For each open set $C(P, \epsilon)$ (defined by swatches P and P' as above), and with χ denoting the characteristic function for $C(P, \epsilon)$ and B_N denoting the disk in $\mathbf{R}^2$ of area N centered at the origin, we will prove that

$$\frac{1}{N} \int_{B_N} \chi[\tau'(T)]\ d\tau' \qquad 2)$$

has a limit as $N \to \infty$ independent of the tiling T, which, by Birkhoff's ergodic theorem, will prove our claim.

It is easy to show that, for large N, 2) can be replaced by

$$\frac{1}{N}[\epsilon^2 \times n(T, N, P, \epsilon)] \qquad 3)$$

where $n(T, N, P, \epsilon)$ is the number of times a translate of a swatch P' appears in the portion of T within B_N.

We will analyze the limit of 3) by keeping track of swatches strictly within VLS-tiles. For this we need the following generalization (to $m \neq 0$) of the matrix usually used in symbolic substitution dynamics. Labeling the tile-types by $j \in \{1, \cdots, q\}$, we define, for each integer m, the $q \times q$ matrix $A[m]$ for which the matrix element $A[m]_{jk}$ is the sum

$$A[m]_{jk} = \sum_n e^{i m a_n(j,k)} \qquad 4)$$

over all tiles of type j contained in the LS–tile $F(k)$, where $a_n(j,k)$ is the angle of rotation of the tile compared to the corresponding prototile. Let α be the spectral radius of $A[0]$. Using a) and a well known result of Perron-Frobenius [11], there are column vectors ξ and η, with strictly positive components, such that for all j, k

$$\frac{(A[0]^L)_{jk}}{\alpha^L} \longrightarrow \eta_k \xi_j \tag{5}$$

as $L \to \infty$. We interpret $(A[0]^{L-M})_{jk}$ as the number of VLS–tiles of order M and type j inside any VLS–tile of order L and type k. If we denote the area of the prototile of type j by vol_j, it follows that

Lemma. For fixed M, the relative area occupied by VLS–tiles of order M and tile-type j inside a VLS–tile of order L and tile-type k approaches $vol_j\, \xi_j / \sum_j vol_j\, \xi_j$ as $L \to \infty$, **independently of k and M.**

Next we will choose M so as to approximate $n(T, N, P, \epsilon)/N$ in the form

$$\frac{n(T,N,P,\epsilon)}{N} \simeq \sum_{j=1}^{q} X_j Y_j Z_j. \tag{6}$$

where (roughly): X_j is the number of (isometric) copies of the swatch P in an order M VLS–tile of type j, divided by $E^M\, vol_j$; Y_j is the relative area of the portion of T, inside B_N, taken up by order M VLS–tiles of type j; and Z_j is the fraction of the order M VLS–tiles of type j, in T inside B_N, which have orientation, compared to the prototile of type j, in the interval $(\rho - \epsilon/2, \rho + \epsilon/2)$. None of the terms X_j, Y_j, Z_j are precisely defined yet, so that is our next goal.

We note from 1) that for any given swatch P the ratio, of the number of copies of the swatch which are partly inside and partly outside an order M VLS–tile to the number of copies which are completely inside the VLS–tile, goes to 0 as M goes to infinity, uniformly in the VLS–tile and the tiling containing it. So we fix M such that this ratio is as small as desired. Then we define X_j in 6) as the number of copies of the swatch P completely inside an order M VLS–tile of type j, divided by $E^M\, vol_j$.

Using the Lemma we define Y_j as $vol_j\, \xi_j / \sum_j vol_j\, \xi_j$. To define Z_j we first prove that each tile-type of the order M VLS–tiles is uniformly distributed in orientation in any tiling. Now if one considers the set of VLS–tiles of order M and tile-type j, contained in a VLS–tile of order L of tile-type k, Weyl's criterion [2; p. 157] shows that their angles of orientation become uniformly distributed as L increases if and only if, for all $m \neq 0$

$$\lim_{L\to\infty} \frac{|(A[m]^{L-M})_{jk}|}{|(A[0])_{jk}^{L-M}|} = 0. \tag{7}$$

From 5) we know that the denominator in 7) behaves as α^L as $L \to \infty$, where α is the spectral radius of $A[0]$. The numerator is bounded above by a constant times β^L where β is the spectral radius of $A[m]$. Now from b) it follows that for each $m \neq 0$ some matrix element in $A[m]^r$ is smaller in absolute value than the corresponding element in $A[0]^r$, and so by a result of Wielandt [3; p. 57] the spectral radius of $A[m]^r$ is strictly less than that of $A[0]^r$. Therefore the spectral radius of $A[m]$ is strictly less than that of $A[0]$, and so 7) holds.

Getting back to 6), we therefore define $Z_j = \epsilon/2\pi$. And from the above analysis it is easy to see that 6) holds for large N, which completes the proof. ∎

Example. As an example we consider John Conway's "pinwheel system" [9] defined as follows. There are two prototiles, a right triangle with legs $1, 2, 5^{1/2}$ and its reflection; two iterations of the substitution rule are given in Figure 1, and a portion of a tiling is given in Figure 2.

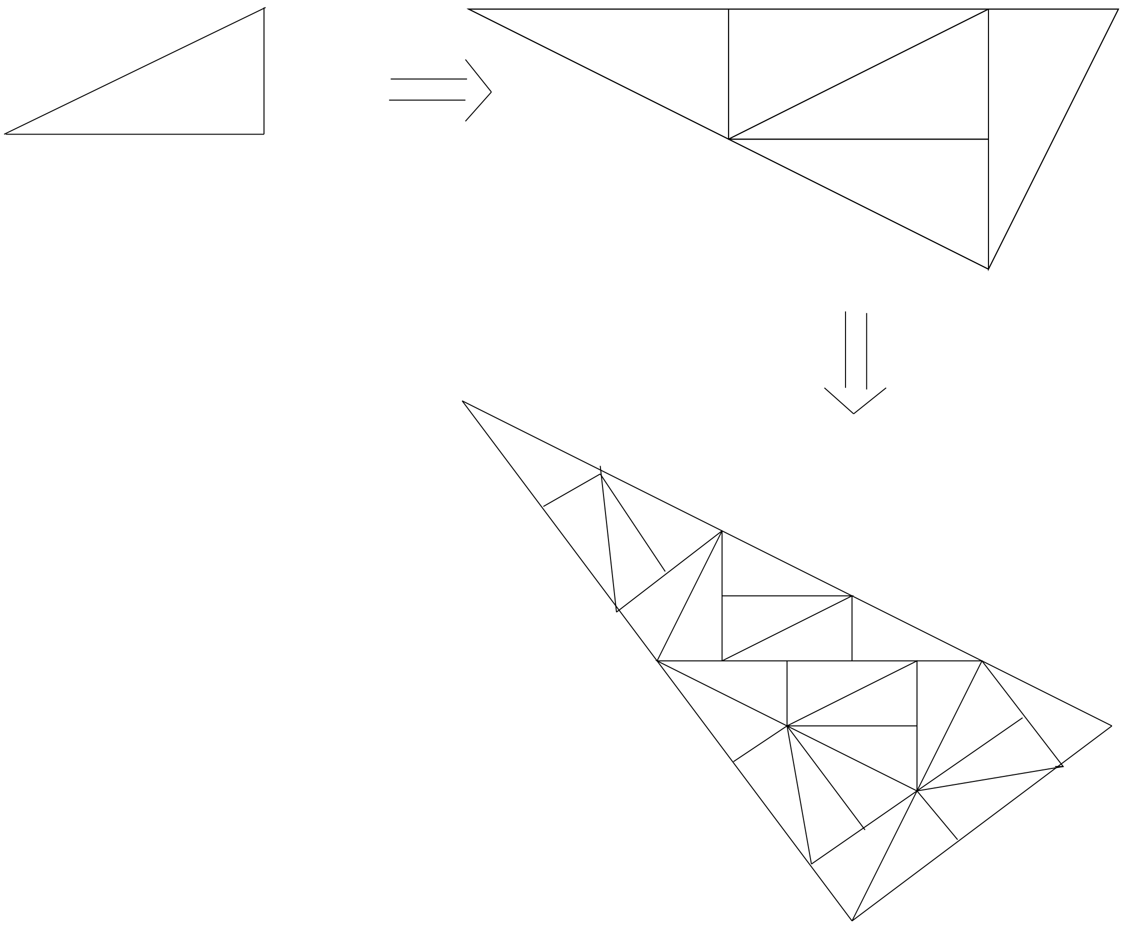

Figure 1

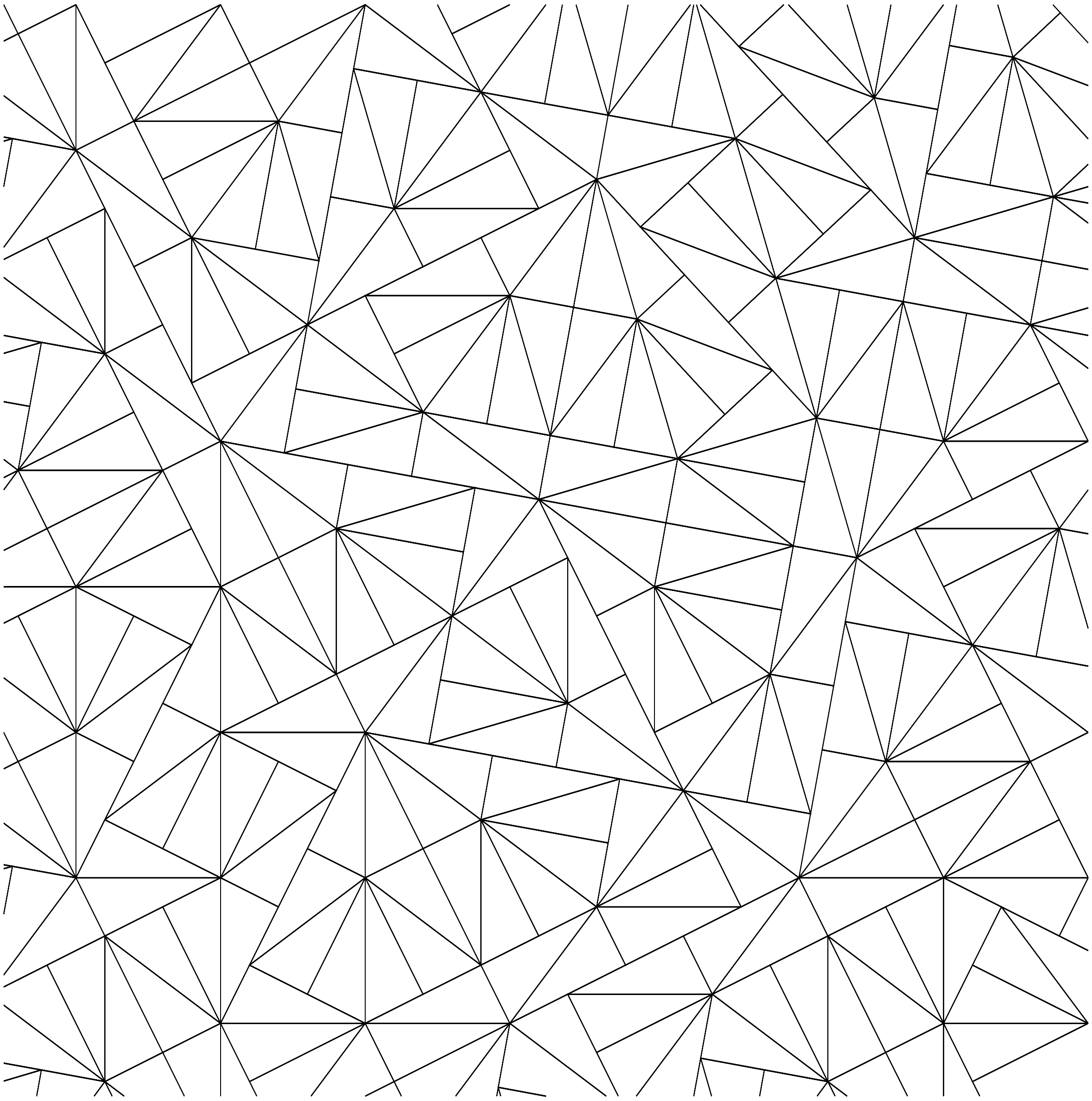

Figure 2

(Although this example does not satisfy the condition that the boundary of a prototile can only be covered in finitely many ways, this condition does hold in the tilings associated with the substitution and therefore the theorem does apply to this example.) The matrix $A[m]$ associated with this substitution is:

$$\begin{pmatrix} e^{ims} + e^{im(s+1/2)} & 2e^{-im(s+1/2)} + e^{-im(s+3/4)} \\ 2e^{im(s+1/2)} + e^{im(s+3/4)} & e^{ims} + e^{im(s+1/2)} \end{pmatrix}$$

where $s = \tan^{-1}(1/2)$ is irrational with respect to π [6; p. 143]. Since element $(A[m]^2)_{11}$ contains angles $2s$ and 0, the substitution satisfies the conditions of our Theorem.

This substitution system is known to be of finite type, that is, measure-theoretically isomorphic to a tiling dynamical system [9]. As for the spectrum of this substitution system, we now use a well known technique to show that since it is rotationally symmetric it cannot have any discrete spectrum.

Proposition. If $(\mathbf{X}(S)_F, \mathbf{G}_T)$ with $\mathbf{G} = \mathbf{I}$ is uniquely ergodic, then it has no discrete spectrum.

Proof. Since any rotation of $\mathbf{R}^2$ leaves invariant the set of translation-invariant probability measures on $\mathbf{X}(S)_F$, and since that set is assumed to be a singleton, which we will call μ, μ must be invariant under rotations, and in fact all of $\mathbf{I}$.

Let $U : g \longrightarrow U^g$ be the usual continuous unitary representation of $\mathbf{I}$ on the complex separable Hilbert space $\mathbf{H} = L^2(\mathbf{X}(S)_F, d\mu)$. Assume $f \neq 0$ is an eigenvector for the translations, corresponding to λ, that is: $U^t f = e^{i\lambda \cdot t} f$ for all translations t. The translations are a normal subgroup of $\mathbf{I}$, and in fact for any rotation R about the origin, and any translation t, $U^t U^R = U^R U^{R(t)}$ where $R(t)$ is the obvious translation. Therefore for all t, $U^t U^R f = U^R U^{R(t)} f = e^{i\lambda \cdot R(t)} U^R f = e^{iR^{-1}(\lambda)\cdot t} U^R f$, which implies that $R^{-1}(\lambda)$ is also in the discrete spectrum, for every rotation R. Since $\mathbf{H}$ is separable and eigenspaces of unitary operators are orthogonal, this implies $\lambda = 0$. ■

3. CONCLUSION.

The study of symbolic dynamical systems of finite type and with $\mathbf{Z}^n$ action often requires the use of symbolic substitution dynamical systems, as in [5].

We have considered here the generalization of such theory to dynamical systems with $\mathbf{R}^n$ action, associated with tilings of $\mathbf{R}^n$; the generalization of systems of finite type we call tiling dynamical systems, and the generalization of substitution systems we call substitution-tiling dynamical systems.

We discuss two results. The first is a natural generalization of the classical technique of the matrix of a substitution, including in an effective way the geometrical information needed to study tilings. Our second result is a new method for producing examples of tiling dynamical systems without discrete spectrum. Our method uses the new matrix to give sufficient conditions for a substitution-tiling system to be uniquely ergodic with respect to translations, and then uses the resulting rotation invariance to eliminate discrete spectrum. In particular our results show that the "pinwheel tilings" of the plane [9] have no discrete spectrum.

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