A continuation method to study periodic orbits of the Froeschlé map.

Arturo Olvera

Depto. de Matemáticas y Mecánica
IIMAS-UNAM
Apdo. Postal 20-726
01000 México, D.F.

Cristóbal Vargas

Departamento de Matemáticas
CINVESTAV-IPN
Apdo. Postal 14-740
07000 México, D.F.

Abstract.

The dynamics of many hamiltonian systems with three degrees of freedom is represented by the Froechlé map, which is symplectic and four dimensional. In this paper we study sequences of periodic orbits approaching the invariant tori in order to obtain information about its stability. A homotopic method is used to continue the branches of periodic orbits. We found that the existence of turning points is related to the linear stability of the periodic orbit and its rotation vector.

1. Introduction.

The structure of periodic orbits and invariant sets that belong to the symplectic twist maps in the plane is well known. The problem of finding these periodic orbits can be restated as a variational problem, where the minimum of an adequate functional corresponds to a periodic orbit [12]. For any rational rotation number there is a unique minimal periodic orbit, which is also a monotone orbit. This fact gives the possibility to approximate invariant circles by sequences of minimal periodic orbits. This result can be used to show that any invariant circle becomes a minimal cantorian set after it breaks down.

For higher dimensional symplectic maps the structure of the periodic orbits is more complex than the two dimensional case. However, periodic orbits can still be obtained by minimizing a certain functional, although the concept of rotation number is extended to that of rotation vector [13].

Given a rational rotation vector, there are many corresponding minimal periodic orbits. Therefore, it is not straightforward how to obtain a suitable approaching sequence of minimal periodic orbits to study the structure of the invariant tori. The existence of multiple periodic orbits, with the same rotation vector, opens many possibilities to obtain sequences of periodic orbits that converge to a given invariant torus. In this paper we show that the structure of the periodic orbits is strongly related to the form of the rational numbers that define the rotation vector.

There are many problems in mechanics where the dynamics is represented by the Froeschlé map; for example, two coupled perturbed penduli, the dynamics of stars in elliptic galaxies [1], among others. Therefore, many authors have chosen to study this map as a suitable model of four dimensional maps [1,8,13,14]. The Froeschlé map is a symplectic four dimensional map defined in $(\mathsf{S}^1 \times \mathrm{IR}) \times (\mathsf{S}^1 \times \mathrm{IR})$ by:

$$f(\mathbf{x}) = \begin{cases} \overline{y}_i = y_i - \frac{K_i}{2\pi}\sin(2\pi x_i) - \frac{\lambda}{2\pi}\sin(2\pi(x_1 + x_2)) \ , \\ \\ \overline{x}_i = x_i + \overline{y}_i \pmod 1 \ , \end{cases} \qquad i = 1, 2 \tag{1}$$

where $\mathbf{x} = (x_1, y_1, x_2, y_2)$, $x_i \in \mathsf{S}^1$ and $y_i \in \mathrm{IR}$, K_i and λ are positive parameters.

Due to its importance we decided to study the structure of the periodic orbits of this map. Since there exist approaching sequences of periodic orbits to the invariant tori it is possible to get information about the stability of the map and the existence of these tori. Therefore these sequences were computed for a set of values of one parameter of the map, and the branches were obtained by a continuation method. We found that there are some rotation vectors (with reducible rational components) whose corresponding solution branches have turning points, in this case the obtained sequences are not good approximants.

Froeschlé map is integrable when $K_i = 0$ and $\lambda = 0$, then the phase space is foliated by two dimensional invariant tori. For small perturbations of the parameters most of these tori breakdown , but KAM theory shows that some others still exist [2]. An important question is to know the evolution and persistence of the invariant tori for different values of K_i and λ.

One way to study the mentioned breakdown, is to consider the integrable case and to increase the value of the parameters until we reach the breakdown; but to obtain the right combination of the parameters could take a lot of effort and the method could also be too expensive. Therefore, what we do is to start with one situation where we know that we are close to the breakdown of the invariant tori.

Examining map (1), it is easy to find two cases which are fully understood and can be used for our purpose. One case is when $\lambda = 0$, here we get two uncoupled standard maps with parameters K_1 and K_2. The second case is when $K_i = 0$ and $\lambda \neq 0$, here we have a trivial twist map and a standard map which are not coupled. If we can move continuosly from one of the states to the other, we expect to have all of these different states of our system close to the breakdown, and from here to obtain the desired information.

2. Periodic orbits.

Since to study directly an invariant torus is quite difficult, the common technique is to approximate it by a sequence of periodic orbits [3]. Therefore, what we have to do is to characterize the periodic orbits and then to extract the approaching ones to the invariant torus, to this end it is introduced the concept of rotation vector.

Every periodic orbit, $\mathcal{X}$, is defined by its rotation vector, which is a couple of rational numbers $(\frac{p}{n}, \frac{q}{n})$, where n is the period of the orbit, such that, for any $\mathbf{x} \in \mathcal{X}$ then:

$$F^n(\mathbf{x}) = \mathbf{x} + (p, 0, q, 0) \ . \tag{2}$$

where $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ is the lift of map (1).

Every orbit on an invariant torus is quasiperiodic then its rotation vector is given by a couple of numbers (ω_1, ω_2) such that $\frac{\omega_1}{\omega_2}$ is irrational. This vector can be approached by a sequence of rational vectors whose entries have the same denominator. Every one of these rational vectors can be associated to a rotation vector of a periodic orbit, it is known that this sequence of periodic orbits approaches the invariant torus with rotation vector (ω_1, ω_2) [3,13]. Therefore, it is possible to study the properties of the torus (like continuity, differentiability, etc.) by studying the corresponding properties of the approaching sequence.

One important property of the Froeschlé map is that of being reversible, thus it is possible to write it as the product of two involution maps: $F = S \circ T$ where $S \circ S = T \circ T =$ Identity. As a consequence we have that for any rotation vector there exist symmetric periodic orbits, $\mathcal{X}$, this kind of orbits have the following property: if $\mathbf{x} \in \mathcal{X}$ then $S(\mathbf{x}) \in \mathcal{X}$ [8,13]. Posteriorly, we shall exploit this fact to find the mentioned periodic orbits.

It is known that for the map (1) there are eight symmetry planes [8], these two–dimensional planes are defined by :

$$\mathrm{P}_{m_1,m_2} = \left\{(x_1, y_1, x_2, y_2) : x_1 = \frac{m_1}{2}, x_2 = \frac{m_2}{2}\right\} \quad , \tag{3}$$

$$\mathrm{PS}_{m_1,m_2} = \left\{(x_1, y_1, x_2, y_2) : y_1 = 2(x_1 + \frac{m_1}{2}), y_2 = 2(x_2 + \frac{m_2}{2})\right\} \quad ,$$

with $m_1, m_2 = 0, 1$. Every symmetric orbit will have two different points, each one of them belongs to some of the above planes (different points in different planes). For any symmetric periodic orbit of period n we take one point in one symmetry plane, after l iterations of f, we have to arrive to the second symmetry plane, where $l = \frac{n}{2}$ if n is even and $l = \frac{n+1}{2}$ if n is odd. Therefore, looking for periodic orbits is reduced to look for points lying in one symmetry plane, such that after applying f^l the image is in the other symmetry plane. Since any invariant torus always cuts P_{m_1,m_2} and PS_{m_1,m_2}, then we can approximate this torus by a sequence of symmetric periodic orbits because any orbit in the torus is a symmetric orbit [8].

The procedure to find such a sequence of symmetric periodic orbits is the following: First, we should have a sequence of rational numbers $(\frac{p_i}{n_i}, \frac{q_i}{n_i})$ converging to the rotation vector (ω_1, ω_2) of the torus. The next step is to find symmetric periodic orbits with rotation vector $(\frac{p_i}{n_i}, \frac{q_i}{n_i})$. This problem is equivalent to find zeros of the function:

$$F^{n_i}(\mathbf{x}) - \mathbf{x} - (p_i, 0, q_i, 0) = \mathbf{0} \ , \tag{4}$$

where $\mathbf{x} \in \mathrm{I\!R}^4$. It can be reduced to a two-dimensional problem, because symmetric orbits always have points in the symmetry planes. Then, we only need to look for points in these planes such that their images, after some iterations, belong to other symmetry planes. Because the planes are two dimensional we require a two-dimensional root finder.

For $\lambda = 0$ or $K_1 = K_2 = 0$ we know exactly the structure of the symmetric periodic orbits and the invariant tori of the Froeschlé map. This will allow us to study the evolution of any invariant torus beginning with the case $\lambda = 0$, $K_1 = K_1^0$, $K_2 = K_2^0$ until we reach the state $\lambda = \lambda^f$ and $K_1 = K_2 = 0$. Therefore, we want the solution of (4), with rotation vector $(\frac{p_i}{n_i}, \frac{q_i}{n_i})$, to be continued from the initial value of the parameters to their final value. Homotopic methods can be used to continue the solution of the periodic orbits. We choose λ as our homotopic parameter, the other parameters being functions of λ.

Begining with $\lambda = 0$, map (1) becomes a set of two uncouple standard maps:

$$f(\mathbf{x}) = \begin{cases} \overline{y}_i = y_i - \frac{K_i}{2\pi}\sin(2\pi x_i) \ , \\ \overline{x}_i = x_i + \overline{y}_i \pmod 1 \ , \end{cases} \qquad i = 1, 2 \tag{5}$$

then for our procedure, the initial periodic orbit is obtained in the following way: the first two coordinates correspond to a symmetric periodic orbit of the standard map with rotation number $\frac{p_i}{n_i}$ [6], where the value of the corresponding parameter is K_1^0. And the second coordinates are obtained similarly, but with rotation number $\frac{q_i}{n_i}$ and corresponding parameter K_2^0.

In the standard map, for any rational rotation number there are two symmetric periodic orbits and these orbits have two points on the symmetry lines of the map. Then, for any rational rotation vector we have eight initial periodic orbits for the map (5). The homotopic method continues the solution branches starting from $\lambda = 0$. These branches have different behaviour, some of them connect the initial and the final states of the parameters, and others have turning points such that they return to the initial state ($\lambda = 0$).

The behaviour of the branches, depends upon the rotation vector and the linear stability of the periodic orbits in the standard map. We are interested in finding conditions on the rotation vectors, such that all the branches connect the initial and final states of the parameters. This will give the possibility to obtain a suitable sequence of periodic orbits which approach the invariant torus for any value of λ.

3. Numerical Method.

The problem to find periodic orbits of map (1) can be solved by numerical methods. In the last section, we showed a method to find symmetric periodic orbits in $\mathbb{R}^4$ given a rational rotation vector $(\frac{p}{n}, \frac{q}{n})$. The idea is to use the symmetries of map (1) to reduce problem (4) such that it becomes a two dimensional one.

For our purpose we need to introduce some projection maps, these maps will help to reduce our problem to a two dimensional one. The idea of the $\mathcal{T}$ maps is that they transform the local coordinates of the symmetry planes (3) to coordinates in $\mathbb{R}^4$, and the $\mathcal{P}$ maps give a measure of the closeness of a point in $\mathbb{R}^4$ to a symmetry plane:

$$\begin{array}{ll} \mathcal{T}_{1,m_1,m_2} : P_{m_1,m_2} \to \mathbb{R}^4 \ , & \mathcal{T}_{1,m_1,m_2}(x,y) = (\frac{m_1}{2}, x, \frac{m_2}{2}, y) \ ; \\ \mathcal{T}_{2,m_1,m_2} : PS_{m_1,m_2} \to \mathbb{R}^4 \ , & \mathcal{T}_{2,m_1,m_2}(x,y) = (x, 2(x+\frac{m_1}{2}), y, 2(y+\frac{m_2}{2})) \ ; \end{array}$$

$$\begin{array}{ll} \mathcal{P}_{1,m_1,m_2} : \mathbb{R}^4 \to \mathbb{R}^2 \ , & \mathcal{P}_{1,m_1,m_2}(x_1,y_1,x_2,y_2) = (x_1-\frac{m_1}{2}, x_2-\frac{m_2}{2}) \ ; \\ \mathcal{P}_{2,m_1,m_2} : \mathbb{R}^4 \to \mathbb{R}^2 \ , & \mathcal{P}_{2,m_1,m_2}(x_1,y_1,x_2,y_2) = (y_1-2(x_1+\frac{m_1}{2}), y_2-2(x_2+\frac{m_2}{2})) \ ; \end{array}$$

here m_1 and m_2 are integers which can take only values 1 and 2.

To get a periodic orbit of period n we start with a point (x, y) in the symmetry plane, and after l iterations of the map F we want to be sure that its image is in the other symmetry plane, this would mean that symmetric periodic orbits are the solution of the following equation:

$$\mathcal{P}_{i,m_1',m_2'} \circ F^l \circ \mathcal{T}_{j,m_1,m_2}(x,y) = 0 \ . \qquad (6)$$

In this case, F^l is the l-th iteration of map (1), with $l = \frac{n}{2}$ if n is even and $l = \frac{n+1}{2}$ if n is odd. The numbers i, j, m_1, m_2, m_1' and m_2' are integers chosen in function of the parity of p, q, n and the linear stability of the initial point, we refer the reader to [8].

Now, our problem is to find the roots of equation (6), and this can be done using a two dimensional root finder. Newton's methods could be rather expensive, since if the period of the orbit is n, the evaluation of the jacobian matrix F^l is obtained by the product of l jacobian matrices of F, all of them evaluated at different points. Therefore we selected Broyden's method [5], which belongs to the class of quasi–newton methods. The advantage of Broyden's method is that we do not need to evaluate the jacobian matrix at each step.

It is know that any Newton's type method does not converge unless the initial guess is close to the solution. Nevertheless, if we have a root of equation (6), for some value of the parameters K_1, K_2 and λ, then we can get a good guess of the root for small changes of the parameters. The homotopic methods help us to obtain a new guess for small perturbations of the parameters. The idea of these methods is to introduce an ordinary differential equation for the root as function of the variation of the parameter [7].

It is rather common for homotopic methods to vary only one parameter in order to continue the solution of the function. Since we have three parameters and we want to have a one-parameter homotopic procedure, we select K_1 and K_2 to depend upon λ by means of the relation:

$$K_i(\lambda) = K_i^0 \left(\frac{\lambda^f - \lambda}{\lambda^f} \right)^{\frac{1}{2}} \quad \text{for } i = 1,2 \text{ and } \lambda \in [0, \lambda^f] \quad . \tag{7}$$

For $\lambda = 0$ we have the initial value of the parameter $K_i(0) = K_i^0$, for $i = 1, 2$. The final value of λ is λ^f and at this point $K_i(\lambda^f) = 0$. The choice of the square root in this homotopy was taken to make the important part of the dynamics of the branches to come out, and this was the most succesfull among several that we tried.

We showed in the previous section that for $\lambda = 0$ and any rotation vector it is not difficult to find periodic orbits. Based upon the initial stability conditions, one of these solutions is taken as our initial point, then we want to continue the solution through all the possible values of λ.

The set of points obtained by the continuation method defines a solution branch, every point of the branch is a solution of equation (6) for some value of λ. If we represent this branch in the (λ, y_i) plane, where y_i is one of the coordinates of a point that belongs to the periodic orbit, for $i = 1, 2$, the geometric form of any branch, in the (λ, y_i) plane, is not necessarily the graph of some function, since it is possible to have two or more solutions for a single value of λ. Because the solution branch is a continuous curve in the (λ, y_i) plane, the existence of multiple solutions for a single value of λ implies the existence of turning points. The direction of the branch at these points changes in the sense of λ, and any simple homotopic method will find troubles at the turning points. To solve this problem it is introduced a new variable and a new equation involving the new variable, the parameter and the variables of (6).

This procedure increases the dimension of (6), and the new variable becomes the homotopic parameter of the new equations. To solve this new system we used the pseudo–arclenght continuation method given by Keller [7]. The homotopic parameter is a kind of arc–lenght of the branch and the augmented equation system is now well defined in the turning points. The parameter λ is now a function of the new homotopic parameter and the pseudo–arclenght increases its value while we follow the branch. The augmented equation system is given by:

$$\begin{aligned} \mathcal{P}_{i,m'_1,m'_2} \circ F^l \circ \mathcal{T}_{j,m_1,m_2}(x,y) &= 0 \ , \\ (x-x_0)^2 + (y-y_0)^2 + |\lambda - \lambda_0|^2 - |s - s_0|^2 &= 0 \ , \end{aligned}$$

where s is the pseudo–arclenght parameter and s_0, λ_0, x_0, y_0 are initial values of the corresponding variables.

It is common that the homotopic method includes an Euler method in the predictor step. In order to improve the efficiency of this step, we replaced the Euler method by a multistep method, we used the Adams–Bashfort method, in this case the new guess is obtained by the extrapolation of the previous solutions.

Our numerical experience in computing solution branches, shows that there is not a unique way to implement the continuation method optimizing the computing time. We found two kinds of problems which had to be solved by a hybrid method, alternating each version depending upon the situation, we discused all of this below.

The first problem is to continue the solution branch close to the turning points, where the simple homotopic methods become singular. This problem is due to the fact that a turning point represents a bifurcation point, where two periodic orbits emanate for some value of λ, this means that at least two eigenvalues of the periodic orbit start from 1. Keller's pseudo–arclenght method [7] is a common way to continue the solution branch, since it is well behaved at the turning points. We must remark that this procedure does not remove all the difficulties, since new points appear where the determinant of the jacobian of the augmented system is zero. We can remove this new problem because the augmented system has some free parameters, which can be used to re–scale the pseudo–arclenght in such a way that we can avoid the zero value of the determinant.

The second problem is when the periodic orbits become unstable, this reduces the attracting basin of the root finder, therefore the step size of the homotopic method should be reduced in order to insure the convergence of the quasi–newton method. For any section of the branch, where the periodic orbit has two very large eigenvalues there are not turning points, therefore it is not necessary to use a pseudo–arclenght procedure, a simple continuation method can do the job. Not only that, but we found thus this simpler method insures a better convergence, because the dimension of the system is smaller and the attracting basin of the Newton's type method increases its size. An improving of the simpler continuation method, can also be obtained by introducing the Adams–Bashfort multistep method, as already was done for the pseudo–arclength method.

All of the above concerns only to one branch, in the case of multiple branches which have not common points, the homotopic method must jump from one branch to another. Close to the turning points we can get this jump to happen, if desired, by increasing sufficiently the step size of Euler's method.

4. Turning points.

There are many approximating rational sequences to any irrational rotation vector (ω_1, ω_2). And each one of these generates an approximating sequence of periodic orbits to the torus, each one having its own characteristics. In order to study the evolution of the torus for a range of values of λ, we want to have those sequences of approaching periodic orbits such that for all this range of λ there exists a solution which as a function of λ is continuous.

Solution branches with turning points do not satisfy the previous requirement, because for one value of λ there are multiple solutions. Therefore, we want to characterize which kind of sequences approaching (ω_1, ω_2) gives birth to branches with turning points, in order to avoid them.

There are some algorithms to obtain a sequence of rational vectors $(\frac{p_i}{n_i}, \frac{q_i}{n_i})$ converging to an irrational vector (ω_1, ω_2). Kim and Ostlund [4] gave an algorithm to obtain these sequences, by using a modified Farey tree method. For any irrational rotation vector they constructed a triangle in a recursive form, every node of the triangle being defined by three integers (a, b, c) such that this node represents the vector $(\frac{a}{c}, \frac{b}{c})$. Any node of the level $n+1$ is obtained from the sum of two neighbour nodes of level n using the following rule:

$$(a_1, b_1, c_1) \oplus (a_2, b_2, c_2) \rightarrow (a_1 + a_2, b_1 + b_2, c_1 + c_2) \quad .$$

In this form, any irrational vector is approximated by a sequence of nodes of level n, where $n = 1, 2, 3, \ldots$.

For example, let τ be an irrational number which is the real root of $\tau^3 - \tau - 1 = 0$, and let us form the irrational rotation vector (τ^{-2}, τ^{-1}), then the first terms of the approaching sequence are:

$$(\frac{1}{2}, \frac{1}{2}) \ , \ (\frac{1}{2}, \frac{2}{2}) \ , \ (\frac{2}{3}, \frac{2}{3}) \ , \ (\frac{2}{4}, \frac{3}{4}) \ , \ (\frac{3}{5}, \frac{4}{5}) \ , \ (\frac{4}{7}, \frac{5}{7}) \ , \ (\frac{5}{9}, \frac{7}{9}) \ , \ (\frac{7}{12}, \frac{9}{12}) \ ,$$
$$(\frac{9}{16}, \frac{12}{16}) \ , \ (\frac{12}{21}, \frac{16}{21}) \ , \ (\frac{16}{28}, \frac{21}{28}) \ , \ (\frac{21}{37}, \frac{28}{37}) \ , \ \ldots \quad .$$

We can see that some of these terms are not irreducible rational fractions, take for example the tenth element of the sequence, $(\frac{16}{28}, \frac{21}{28})$. We found that the solution branches with this kind of rotation vector have a geometry which is related to the linear stability of the initial point, and it is possible for the branch to have turning points. The importance of having turning points is that the branch can take again the initial value of the parameters, this would imply that the homotopic parameter not necesarily runs over all its domain of definition. Thus, it would be impossible to connect the initial and final states, as was our original purpose. We show an example of the behaviour of this kind of orbits.

Let the rotation vector be $(\frac{16}{28}, \frac{21}{28})$. The initial point of the required branch is obtained by looking for monotone periodic orbits, with rotation numbers $\frac{16}{28}$ and $\frac{21}{28}$ in the standard map, the definition of monotone orbit can be found in the appendix. A usefull property of the standard map is that any monotone orbit is also a symmetric orbit [10]. The theory of minimal orbits given by Mather [9], shows that there are only two monotone periodic orbits with rotation number $\frac{p}{q}$, where p and q are relative primes. As a consequence all the monotone periodic orbits with rotation number $\frac{m \cdot p}{m \cdot q}$, $m = 1, 2, \ldots$, coincide with one of the previous two orbits. Therefore the monotone periodic orbits with rotation number $\frac{16}{28}$ and $\frac{21}{28}$ coincide with one of the monotone orbits with rotation numbers $\frac{4}{7}$ and $\frac{3}{4}$ respectively.

In our example, the initial periodic orbit is given by the cartesian product of two monotone periodic orbits of the standard map with rotation numbers $\frac{4}{7}$ and $\frac{3}{4}$. If we denote the coordinates for the first orbit by (x_1, y_1) and of the second one by (x_2, y_2), then the initial point of map (1) has coordinates (x_1, y_1, x_2, y_2). The linear stability of this initial point is given by the stability of the monotone periodic orbits and we have the following combinations: E–E, E–H, H–E, and H–H (where E means elliptic stability and H means hyperbolic stability).

The continuation method gave a solution branch, the shape of this branch was found to be strongly related to the linear stability and the rotation vector of the initial orbit.

Since our rotation vector was not a pair of irreducible rational numbers, the branches for which the initial periodic orbit had one of the following stabilities E–E, E–H or H–E, had turning points such that the branches returned to a point corresponding to the initial value of the parameters ($\lambda = 0$, $K_1 = K_1^0$, $K_2 = K_2^0$).

For the branches mentioned above, we found that there exists a relation between the coordinates of the initial and final points of the branch. Let (x_1, y_1, x_2, y_2) be the initial coordinates and (x_1', y_1', x_2', y_2') be the final coordinates of one of these branches with turning points, for our rotation vector we found that these coordinates obey the relations shown in table 1.

Initial orbit	Coordinates	
H–E	$(x_1, y_1) = (x_1', y_1')$	$(x_2, y_2) \neq (x_2', y_2')$
E–H	$(x_1, y_1) \neq (x_1', y_1')$	$(x_2, y_2) = (x_2', y_2')$
E–E	$(x_1, y_1) \neq (x_1', y_1')$	$(x_2, y_2) \neq (x_2', y_2')$

Table 1:

Consider an initial periodic orbit whose coordinates (x_1, y_1, x_2, y_2) are given by two periodic orbits of the standard map, with rotation numbers $\frac{4}{7}$ and $\frac{3}{4}$. If one of these orbits in the standard map has elliptic stability, the solution branch will return to a different periodic point in the standard map, qualitative picture is shown in figure 1. The final point of the branch will belong to a non–monotone periodic orbit of period 28, this is because there are not more than two monotone periodic orbits with rotation vector of the form $\frac{m \cdot 4}{m \cdot 7}$ for $m = 1, 2, \ldots$ (or $\frac{m' \cdot 3}{m' \cdot 4}$ for $m' = 1, 2, \ldots$ respectively).

In order to understand the behaviour of the branches mentioned above, we continue our example. We took the periodic orbit of type H–E with rotation vector $(\frac{16}{28}, \frac{21}{28})$, this orbit has period 28. We studied the evolution of the coordinates (x_1, y_1, x_2, y_2) of one point of the orbit, and our goal was to continue the periodic orbit for all posible values of λ starting from zero, but we found that the continuation method could not reach the value $\lambda = \lambda^f$, the branch obtained returned to the point corresponding to the value of $\lambda = 0$. In this case, the constants in the homotopy (7) where: $K_1^0 = 0.93$, $K_2^0 = 0.95$ and $\lambda^f = 0.6$.

Let us study in more detail what is happening in this example. As we said before, the initial periodic point is given by two monotone periodic orbits of the standard map with rotation numbers $\frac{4}{7}$ and $\frac{3}{4}$ with H–E stability. We shall study in two stages the evolution of the coordinates (x_1, y_1) and (x_2, y_2) of the branch:

First stage: The periodic orbit with rotation number $\frac{4}{7}$ has hyperbolic stability, this orbit is composed by seven hyperbolic points. Let (x_1, y_1) be the coordinates of one of these points. Figure 2 shows the evolution of the coordinate y_1 in the (λ, y_1) plane, we can see that the branch has a turning point and we verified that it returned exactly to the initial value of the coordinates (x_1, y_1). We conclude from this fact that the solution branch, which comes out from the hyperbolic point, returns to this initial point.

When λ takes positive values a bifurcation phenomena occours, four branches raise from each hyperbolic point, figure 3 shows this hyperbolic point with rotation number $\frac{4}{7}$ in the (x_1, y_1) plane. We can see four branches that start from that point and how they return. These branches are four images of the branch showed in figure 2 but now in the (x_1, y_1) plane. Each branch corresponds to an image of the periodic point with period 28 (they are the coordinates (x_1, y_1) of a point belonging to a periodic orbit with rotation vector $(\frac{16}{28}, \frac{21}{28})$). These branches return to the initial hyperbolic point when the solution branch reaches the initial value of the parameter.

Second stage: We choose a monotone periodic orbit with rotation number $\frac{3}{4}$ and with elliptic stability, like in the first stage the orbit is composed of four elliptic points. Let (x_2, y_2) be the coordinates of one of these points, figure 4 shows the projection of the solution branch in the (λ, y_2) plane.

We can see that this branch does not return to the initial coordinates (x_2, y_2). For positive values of λ we have again a bifurcation phenomenon, figure 5 shows one of the elliptic points in the (x_2, y_2) plane, where seven branches come out from this point. Each branch is one of the images of the periodic orbit of period 28. We must remark that these branches do not return to the initial point, that means that there are seven final points intead of one. When the parameter λ returns to the zero value, these seven points are placed around each elliptic point, and these points belong to an orbit which is also of period 28 in the standard map.

Finally, figure 1 shows a qualitative picture of the evolution of these branches. They start from the elliptic points and return to a periodic orbit whose image points are located, like islands, around the initial elliptic points, and they lie in the space between the separatrices of the hyperbolic points.

We have already seen, in the last example, that there are branches that connect an elliptic point with its islands, and this connections can be extended from islands to islands. We want to explain the structure of these connections using another example. Suppose, that we have a periodic orbit with rotation vector $(\frac{12}{42}, \frac{14}{42})$, the entries of this rotation vector are reducible rational numbers, they can be written in the following form: $\frac{12}{42} = (\frac{2}{2}) \cdot (\frac{3}{3}) \cdot (\frac{2}{7})$ and $\frac{14}{42} = (\frac{2}{2}) \cdot (\frac{7}{7}) \cdot (\frac{1}{3})$. When $\lambda = 0$, the initial periodic orbit is given by the direct product of two periodic orbits of the standard map, whose rotation numbers are $\frac{2}{7}$ and $\frac{1}{3}$. Let us assume that the linear stability of the periodic orbit with rotation number $\frac{2}{7}$ is elliptic, we are going to show all the possible connections of the branches between this elliptic point and its islands.

The elliptic periodic orbit with rotation number $\frac{2}{7}$ has seven elliptic points in the (x_1, y_1) plane. From the above decomposition of $\frac{12}{42}$, and our previous experience of the last example, we can expect that the orbits with rotation number $\frac{2}{2}\frac{2}{7} = \frac{4}{14}$, $\frac{3}{3}\frac{2}{7} = \frac{6}{21}$ and $\frac{6}{6}\frac{2}{7} = \frac{12}{42}$, are the only ones which can be connected to the elliptic points, and each one of these orbits has respectivelly 2, 3 and 6 images around any of the seven elliptic points. Figure 6 shows one of this elliptic points and three sets of islands which have 2, 3 and 6 elements. We denote the elliptic point by the letter (e) and the set of islands around (e) by the numbers (2), (3) and (6). There are many possibilities of connection between (e) and the sets of islands by solution branches, which start from one point and return to another one.

One of this possible connections is shown in figure 6: from the elliptic point (e) emanate six branches which end in the two islands of (2); from each island of (2) emanate three branches ending in the six islands of (6); finally, from each island of (6) rises one branch that ends in one of the three islands of (3). In a short form we can write the previous connection as follow: $(e) \to (2) \to (6) \to (3)$. All the possibles connections are written in the next table.

$$
\begin{array}{c}
(e) \to (2) \to (3) \to (6) \\
(e) \to (2) \to (6) \to (3) \\
(e) \to (3) \to (2) \to (6) \\
(e) \to (3) \to (6) \to (2) \\
(e) \to (6) \to (2) \to (3) \\
(e) \to (6) \to (3) \to (2)
\end{array}
$$

The set of all possible connections is similar to all the permutations of four elements taking off the cyclic permutations.

We have a similar situation for the periodic orbit of rotation number $\frac{1}{3}$ if we also assume that it has elliptic stability. The possible connections are given by:

$$
\begin{array}{c}
(e) \to (2) \to (7) \to (14) \\
(e) \to (2) \to (14) \to (7) \\
(e) \to (7) \to (2) \to (14) \\
(e) \to (7) \to (14) \to (2) \\
(e) \to (14) \to (2) \to (7) \\
(e) \to (14) \to (7) \to (2)
\end{array}
$$

Around the elliptic point of rotation number $\frac{2}{7}$ there could be several sets of islands of types (2), (3) and (6), therefore in our diagrams the connection of type $\ldots \to (3) \to \ldots$, for example, means that one branch ends in an island of type (3) and the next branch rises from a set of islands of type (3) although not necessarily these sets of islands are the same.

We must remember that any point that belongs to the connecting branches is a point of an orbit of period 42, for this reason we can not have, for any connection, more than 42 branches in the (x_1, y_1) plane (six branches for any of the seven elliptic points).

In the case of hyperbolic points, there is only one branch which rises from this point and returns to it.

From the last examples together with others that were studied, not shown here, we conclude the following conjecture: For a rotation vector $(\frac{p}{n}, \frac{q}{n})$, where $\frac{p}{n}$ (or $\frac{q}{n}$) can be decomposed in the form:

$$\frac{m_1 \cdot m_2 \cdot \ldots \cdot m_k \cdot a}{m_1 \cdot m_2 \cdot \ldots \cdot m_k \cdot b}$$

where a and b are relative primes, the number of possible connections of the branches between the elliptic point and its sets of islands is bounded. This would mean that the number of possible branches for a fix value of λ and fix rotation vector, is know to be bounded in advance.

How to calculate this bound?. It is given by the number of sets of islands, around the elliptic point, which take part in the connection. How do we know this number?. This is given by the total of possible combinations of the set $\{m_1, m_2, \ldots, m_k\}$ taken one at a time, two at a time, until k at a time. Which gives $2^k - 1$ as a result.

Also, we would conjecture that the numbers of islands in each set is given by the product of the elements in each one of the above combinations.

An additional information that we could obtain from this conjecture would be the number of different connections between the sets of islands. And this would be given by the number of permutations that can be obtained from the number of sets of islands.

5. Study of the solution branches.

The main interest of this work is to understand the evolution of the invariant torus for different values of the parameters, in particular, how the torus can loose its continuity. There are, at least, two possible ways in which this can happen.

We can observe readily that one of these possibilities is happen when $\lambda = 0$, then map (1) is given by two uncoupled standard maps. Any invariant torus, with irrational rotation vector (ω_1, ω_2), is given by the direct product of two invariant circles of the standard map whose rotation numbers are ω_1 and ω_2. There is a critical value, K_{ω_i}, for the parameter K_i, $i = 1, 2$, such that if $K_i > K_{\omega_i}$ then the corresponding invariant circle becomes a cantorian set. In that case, if K_1 and K_2 are greater than their critical values, the invariant set is defined by the direct product of two cantorian sets (also called cantorus) [9]. Figure 7 shows all this possibilities.

A second possibility is when the parameters K_1 and K_2 are zero but $\lambda \neq 0$. To obtain more information in this case, we introduce the following coordinate transformation:

$$v_1 = \tfrac{y_1+y_2}{2} \quad , \quad z_1 = \tfrac{x_1+x_2}{2} \quad ,$$

$$v_2 = \tfrac{y_1-y_2}{2} \quad , \quad z_2 = \tfrac{x_1-x_2}{2} \quad ,$$

which applied to equation (1) gives a set of two uncoupled twist maps:

$$\begin{aligned} \overline{v}_1 &= v_1 - \tfrac{2\lambda}{2\pi}\sin(2\pi z_1) \ , \\ \overline{z}_1 &= z_1 + \overline{v_1} \ , \\ \overline{v}_2 &= v_2 \ , \\ \overline{z}_2 &= z_2 + \overline{v}_2 \ . \end{aligned} \tag{8}$$

The first two equations above represent a standard map with parameter value 2λ. The second ones represent an integrable twist map. Any rotation vector (a, b) is transformed into the vector $(\frac{a+b}{2}, \frac{a-b}{2})$ in the new coordinates. In this case, any invariant torus of rotation vector (ω_1, ω_2) is changed to a torus with rotation vector $(\frac{\omega_1+\omega_2}{2}, \frac{\omega_1-\omega_2}{2})$. This new torus is the direct product of the invariant circle with rotation number $\frac{\omega_1+\omega_2}{2}$, given by the standard map, with the invariant circle whose rotation number is $\frac{\omega_1-\omega_2}{2}$ of the integrable twist map. There is a critical value of λ for which the invariant circle of the standard map becomes a cantorian set, under these conditions the invariant torus becomes the product of a cantorian set with a circle.

Summarizing, we can say that for any irrational rotation vector there is an invariant set. In the case $\lambda = 0$ and $K_i > K_{\omega_i}$ it is given by the product of two cantorian sets; and for $K_1 = K_2 = 0$ and $\lambda \neq 0$, the mentioned invariant set is given by the product of a cantorian set with a circle, see figure 7.

We can not compute directly the invariant torus, nevertheless we can approximate it by a sequence of periodic orbits whose rational rotation vectors converge to (ω_1, ω_2). In order to do this we can use the continuation method, described in the previous section, to understand the evolution of this invariant torus from the initial case $\lambda = 0$ and K_1 , $K_2 \neq 0$, to the final state $\lambda \neq 0$, $K_1 = K_2 = 0$. Therefore we shall look for periodic orbits for all the range of λ.

The first step is to find a suitable approaching sequence, of rational rotation vectors, converging to (ω_1, ω_2), in order to find a sequence of periodic orbits which approach the invariant torus for all λ. Therefore, for any periodic orbit of these sequence, we want to have a unique branch that connects the initial and final values of λ. In order to avoid branches that return to the initial value of the parameter, the sequence of rational vectors should be of the form $(\frac{p_i}{n_i}, \frac{q_i}{n_i})$ where the entries are irreducible fractions. Unfortunaly, we do not have a general method to produce this kind of sequence, the method of Kim and Ostlund does not have this property because some of the elements of the produced sequence can be reducible.

Nevertheless there are some examples where it is possible to obtain sequences with the required properties. We show the case of an invariant torus of resonant type such that $\omega_1 + \omega_2 = 1$. Let the rotation vector be $(\frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2})$, the following sequence is an approximant to that rotation vector:

$$\left\{ \left(\frac{H_{i-2}}{H_i}, \frac{H_{i-1}}{H_i} \right) \right\}_{i=2}^{\infty} ,$$

where H_i are the Fibonacci numbers. We use this example to show how to approximate an invariant torus by means of periodic orbits.

Our interest is to study the behaviour of the invariant torus with rotation vector $(\frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}+1}{2})$ when the value of the parameter λ goes from 0 to λ^f. In the examples that we show in this paper we used the homotopy (7) with constants $K_1^0 = 0.93$, $K_2^0 = 0.95$ and λ^f taking diferent values for different rotation vectors. We choose these values because the invariant circles of the standard map, with rotation numbers $\frac{3-\sqrt{5}}{2}$ and $\frac{\sqrt{5}-1}{2}$, still exist for them, but these value of the parameters are close to the break down of the corresponding invariant circles.

We study the first approximants to the invariant torus starting with $i = 7$, in table 2 we show some of the corresponding rational vectors, also the value of λ^f and the linear stability of the initial periodic point. There are many possibilities for the stability of the initial conditions and we found that periodic orbits that start with hyperbolic linear stability give an adequate approximation to the invariant tori, because the solution branch is really a continuous function of λ in the (x_i, y_i, λ) space. This means that for every λ in the domain, there exists only one periodic orbit. Thus studying the behaviour of this sequence of orbits we can obtain in the limit the desired information about the structure of the invariant tori.

However we are still interested in the orbits with elliptic stability, because for the two dimensional twist map, Greene's method [15] is used to determine the break down of the invariant tori, and we want to know if it is still possible to use a similar method to analyse the change of stability of the elliptic periodic orbits which converge to the invariant torus. But here in dimension four, we found that the structure of the solution branches is much more complex because we obtain turning points. For this reason we studied more throughly this particular example to understand the structure of the solution branches, and in a near future try to decide if it is possible to give a similar method to Greene's one.

i	rotation vector	λ^f	stability
7	$\left(\frac{8}{21}, \frac{13}{21}\right)$	0.4	E–E
8	$\left(\frac{13}{34}, \frac{21}{34}\right)$	0.4	E–E
9	$\left(\frac{21}{55}, \frac{34}{55}\right)$	0.1	E–E
10	$\left(\frac{34}{89}, \frac{55}{89}\right)$	0.05	E–E

Table 2:

For the rotation vectors of table 2, using the homotopic method, we obtained several branches which are shown in figures 8 through 11. Because the rotation vectors are irreducible, there is a unique branch that departs from the initial condition. But, it is interesting to note that most of the branches leave from the state $K_1 = K_2 = 0$ and $\lambda = \lambda^f$ and then return to it. This means that there are turning points on the branches.

It is important to understand the structure of the branches in the bifurcation diagram. We could ask ourselves why there are many branches that start from $K_1 = K_2 = 0$ and $\lambda = \lambda^f$ and return to this initial situation?. In order to answer this question we must examine the behaviour of the periodic orbit close to λ^f. Using the new coordinates given in (3), we obtain a periodic orbit of period $2H_i$ whose rotation vector is given by:

$$\left(\frac{H_{i-1} + H_{i-2}}{2H_i}, \frac{H_{i-1} - H_{i-2}}{2H_i}\right) \quad \text{for } i = 3, 4, \ldots \quad .$$

Since $H_{i-1} + H_{i-2} = H_i$ and $H_{i-1} - Hi - 2 = H_{i-3}$, then it becomes:

$$\left(\frac{H_i}{2H_i}, \frac{H_{i-3}}{2H_i}\right) \quad ,$$

it is clear that the first component of this rotation vector is given by reducible rational numbers since it is equal to $\frac{1}{2}$. Because at least one of these entries is reducible then there exists the posibility to have multiple branches which have turning points. From the previous section we know that this phenomenon means that there exists a connection between the elliptic points of period $\frac{1}{2}$ and their islands as shown in figure 1. The number of branches is related to the decomposition of the Fibonacci numbers as described in section 4. Let us examine the branches given by the periodic orbit with rotation vector $(\frac{8}{21}, \frac{13}{21})$. Writting the rotation vector in the new coordinates we have $(\frac{21}{42}, \frac{5}{42})$, the rational number $\frac{21}{42}$ can be decomposed in the form $\frac{3 \cdot 7 \cdot 1}{3 \cdot 7 \cdot 2}$. From this decomposition we can expect the existence of an elliptic periodic orbit with rotation number $\frac{1}{2}$ which has islands of rotation numbers $\frac{3}{6}$, $\frac{7}{14}$ and $\frac{21}{42}$ when $\lambda = \lambda^f$. Figure 12 shows a qualitative picture of these orbits in the (v_1, z_1) plane of the phase space. Our conjeture says that it is possible to find branches connecting the elliptic point and its islands, therefore we expect to have at most four of these branches.

Examining figure 8, when λ is close to λ^f, we can see that there are four branches as expected. But we want to stablish the relationship of the branches with the elliptic points and their islands. In order to do this, we had to follow the branches in the (v_1, z_1) plane and we observed that one branch started from the elliptic point of rotation number $\frac{1}{2}$ and arrived to the island with rotation number $\frac{7}{14}$. We also observed that the others branches connected the islands of rotation numbers $\frac{7}{14}$ and $\frac{21}{42}$, also $\frac{21}{42}$ and $\frac{21}{42}$. Finally there was a branch starting from an island of rotation number $\frac{21}{42}$ which did not return to $\lambda = \lambda^f$.

Figure 13 shows the relationship between figures 8 and 12, where (a) represents the elliptic point, (b) and (c) are two different images of the periodic orbit of rotation number $\frac{7}{14}$. Points (d) and (e) belong to the orbit of rotation number $\frac{21}{42}$ and finally (f) and (g) are in another periodic orbit of rotation number $\frac{21}{42}$. We could not find any branch emanating from the island with rotation number $\frac{3}{6}$.

From other picture in the (v_1, z_1) plane not shown here, we noticed that (d) and (e) belong to an orbit which is not the same to the one to which (f) and (g) belong. These two periodic orbits have the same rotation number, but looking carefully at the orbit to which (f) and (g) belong, we found that most of the points of this orbit are cluster around the hyperbolic points with rotation number $\frac{1}{2}$. Furthermore this orbit does not behave as a rotation around the elliptic point, since does not follow the dynamics close to the elliptic point. Therefore, it is possible that this orbit belongs to the set of periodic orbits associated to the hyperbolic periodic point, with rotation number $\frac{1}{2}$. This set of orbits is created by the mechanism of the horseshoe map, due to the the dynamics of the invariant manifolds of the hyperbolic point [11].

From figures 8 through 11 we observe that for a fixed value of λ there are many periodic orbits that have the same rotation vector. The question is, which one of them are close to the invariant torus?. Since the branches are separated among themselves, it is clear that it is impossible that all of these orbits are close to the invariant torus. In order to find which ones are the right approximation to the invariant torus, we have to compare the dynamics of the orbits, in the invariant torus, with the dynamics of the approaching periodic orbits. In order to do this we need to introduce a new concept, which is the analogous to monotone orbits for two–dimensional maps, this is the concept of well ordered orbit (we refer the reader to the appendix).

If all the orbits in the invariant torus are well ordered orbits, then any periodic orbit that approaches an invariant torus must be a well ordered orbit. Therefore we must identify the right approximations to the invariant tori with those branches which are composed of well ordered orbits. We marked with a bold line in figures 8 through 11 those orbits which have this property. We can see that there is not a continous branch such that is made out completely of well ordered orbits, however all the branches have a continous section of well ordered orbits. Also we can see that for any value of λ there is at least one orbit with the property of well order. All of this tells us that we can define a discontinous branch of well order orbits, this branch is made out of the sections of the branches which have the well ordered orbits.

This discontinous branch of well ordered orbits is not a function of λ, because close to the turning point, there exists an open set of values of λ, such that there are two well ordered periodic orbits with the same rotation vector. In these open sets of λ, there exist two different periodic orbits with the same kind of stability and the same rotation vector that approach in the right way the invariant torus. This shows a difference with the two dimensional case where the rotation number and stability define only one approximant to the torus.

One could think that the discontinuities of the well ordered branches, that we can observe in figures 8 through 11, are due to numerical errors and that actually we should have transcritical bifurcations and the branches should form a continuous curve. In order to prove that the discontinuities do not come from numerical error, we made some numerical experiments with different kinds of perturbations at the points close to the discontinuities, to try to reconnect the branches. If these connections were possible, the discontinuities should really be transcritical bifurcations, but for all of them the branches never came together.

6. Conclusions.

From the above sections we can conclude the following:

1. Periodic orbits with reducible rational rotation vector are inadequate to approximate the invariant tori, because the solution branches come back to the initial state.

2. The number of solution branches, that appears for periodic orbits with reducible rational rotation vector, can be estimated in function of the initial linear stability.

3. A necessary condition to have a good sequence of approximating periodic orbits, to the invariant tori, is that the orbits have irreducible rational rotation vector.

4. If a sequence as described in 3) also has initial hyperbolic linear stability, it will be an adequate approximating sequence to the invariant tori.

5. Those sequences as described in 3) but with initial elliptic linear stability, give raise to multiple solution branches, making them inadequate to approximate the invariant tori.

6. For sequences as in 5) no one of the branches is made out only of well ordered periodic orbits. But it is possible to define a discontinuos branch, which consists of well ordered periodic orbits.

7. A natural way to apply Greene's method is to follow solution branches with the property of well ordered. However the well ordered solution branches are discontinuous and its is not clear at all, that the method still works through the discontinuities.

7. Appendix.

In this appendix we define some properties of the angular coordinates of the orbits of two and four dimensional maps.

Monotone orbits: An orbit, of a map $f : \mathsf{S}^1 \times \mathsf{IR} \to \mathsf{S}^1 \times \mathsf{IR}$, is a sequence $\{(x_i, y_i)\}$ such that $(x_{i+1}, y_{i+1}) = f(x_i, y_i)$. The variables x_i belong to S^1 and are called angular variables. From an orbit we can obtain a sequence of angular variables, namely $\{x_i\}$, where $i \in \mathbb{Z}$. The elements of the sequence belong to S^1, then we can get a map $g : \mathsf{S}^1 \to \mathsf{S}^1$ following the elements of the sequence, $x_{i+1} = g(x_i)$.

We can classify the maps on the circle in two clases, monotone maps which are those maps that follow a similar order of the iteration like a simple rotation of the circle and those that do not follow that order. We would say that any orbit is a monotone orbit if the corresponding map of its angular variable is a monotone map.

Another way to write the above geometrical idea for monotone orbits is the following: Let f be a two dimensional map where $f : \mathsf{S}^1 \times \mathsf{IR} \to \mathsf{S}^1 \times \mathsf{IR}$, the map F is the lift of f if:

1. $F : \mathsf{IR} \times \mathsf{IR} \to \mathsf{IR} \times \mathsf{IR}$.
2. $F \circ S_a = S_a \circ f$, where $S_a(x, y) = (x - a, y)$ and $a \in \mathbb{Z}$.

Let (x_i, y_i) a point that belongs to an orbit $\mathcal{X}$ of the map f, and $(X_i, Y_i) = S_a(x_i, y_i)$ the corresponding point under map F. If the sequence $\{X_i\}$ has the property that $X_i < X_j$ implies $X_{i+1} < X_{j+1}$ we say that $\mathcal{X}$ is a monotone orbit of f [12].

Well ordered orbits: Let f be a four dimensional map:

$$f : (\mathsf{S}^1 \times \mathsf{IR}) \times (\mathsf{S}^1 \times \mathsf{IR}) \to (\mathsf{S}^1 \times \mathsf{IR}) \times (\mathsf{S}^1 \times \mathsf{IR}) \quad ,$$

such that $(x_{i+1}, y_{i+1}, z_{i+1}, w_{i+1}) = f(x_i, y_i, z_i, w_i)$. Since the variables, x_i and z_i, belong to S^1, are called angular variables and they define the sequences $\{x_i\}$ and $\{z_i\}$, where $i = 1, 2, \ldots, n$. These sequences define two maps g_1 and g_2 on the circle $x_{i+1} = g_1(x_i)$, $z_{i+1} = g_2(z_i)$. We say that the orbit $\{x_i, y_i, z_i, w_i\}$ is well ordered if the corresponding maps g_1, g_2, are monotone.

8. Acknowledgements.

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9. References.

[1] Froeschlé, C. *Astrophys. Space Sci.*, **14**, (1971) 110.

[2] Arnold, V. I. Mathematical Methods of Classical Mechanics, Springer–Verlag, New York (1978).

[3] Bernstein, D. and Katok, A. *Invent. Math.*, **88**, (1987) 225.

[4] Kim, S. and Ostlund, S. *Phys. Rev.* **A34**, (1986) 3426.

[5] Broyden, C. G. *Math. Comp.* **19**, (1965) 577.

[6] Olvera, A. and Simó, C. *Physica 26D*, (1987) 181.

[7] Keller, H. B. Lectures of Numerical Methods in Bifurcation Problems, Springer–Verlag, New York (1987).

[8] Kook, H. and Meiss J. Periodic Orbits for Reversible, Symplectic Mapping, Inst. for Fusion Studies, Univ. of Texas–Austin, ♯ 336 (1988).

[9] Mather, J. *Topology* **21**, (1982) 457.

[10] Aubry, S. *Physica 7D*, (1983) 240.

[11] Olvera, A. and Simó, C. Elliptic non-Birkhoff periodic orbits, in the twist maps., to appear in Bifurcation and Chaos (1991).

[12] Katok, A. *Lecture Notes in Physics* **179** Ed. Garrido E., Springer–Verlag, (1982) 47.

[13] Muldoon, M. Ghosts of Order on the Frontier of Chaos Ph. D. Thesis, California Institute of Technology (1989).

[14] Chen, Q., Mackay, R. S. and Meiss, J.D. Cantori for Symplectic Maps, Inst. for Fusion Studies, Univ. of Texas–Austin (1989).

[15] Greene J.M. *J. Math. Phys.* **20**, (1979) 1183.

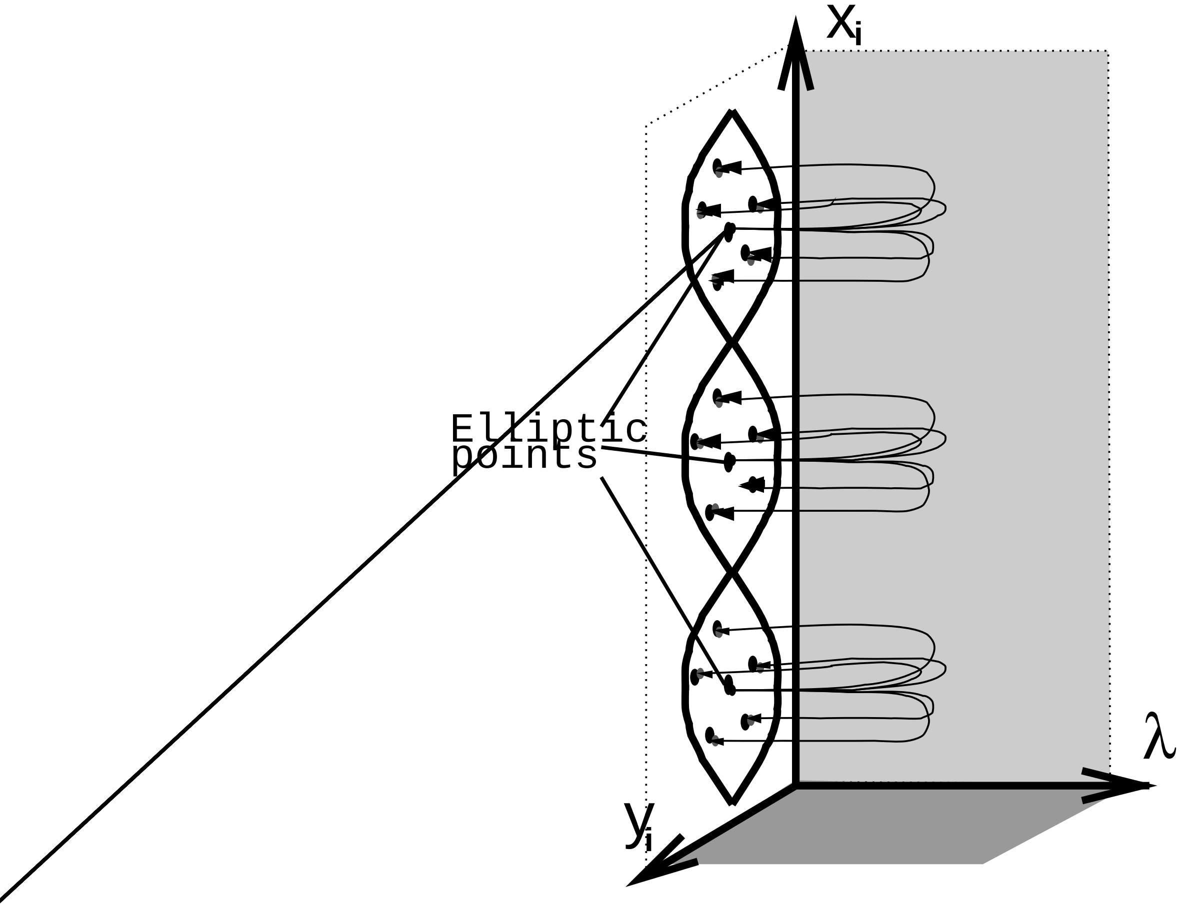

Figure 1: Qualitative behaviour of the solution branch in the (x_2, y_2, λ) space. The thin lines show the evolution of the coordinates (x_2, y_2) of the image points that belong to the periodic orbit.

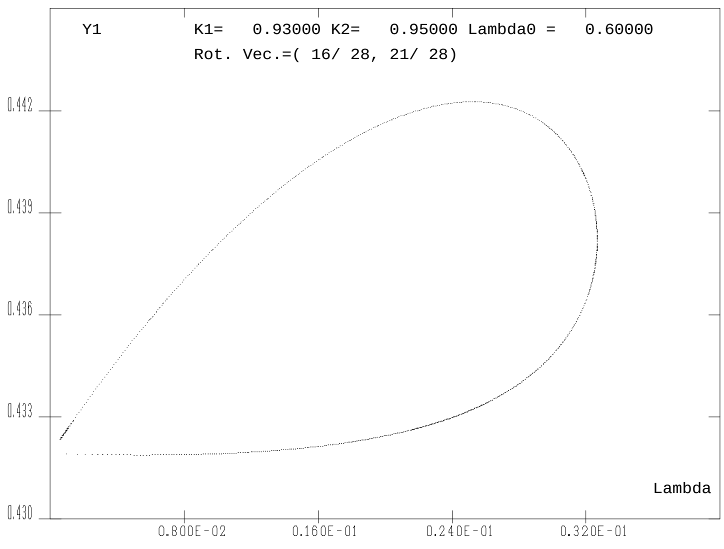

Figure 2: Solution branch of the periodic orbit with rotation vector $(\frac{16}{28}, \frac{21}{28})$ in the (λ, y_1) plane. The initial periodic orbit with rotation number $\frac{4}{7}$ has hyperbolic stability.

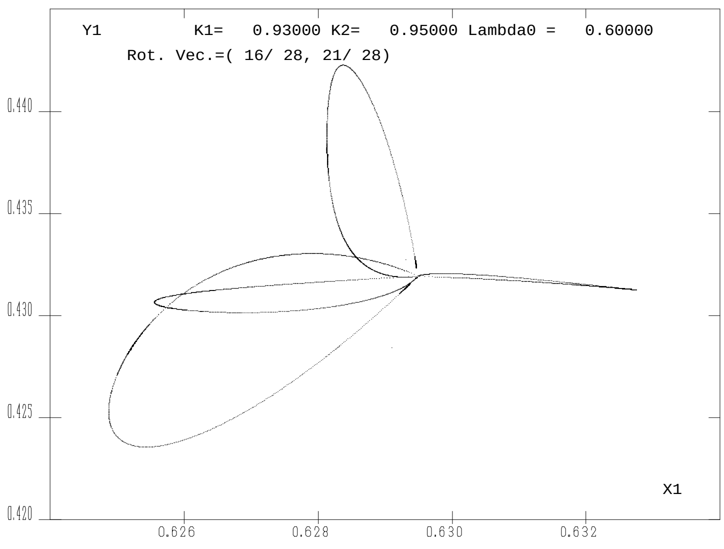

Figure 3: Evolution of the coordinates (x_1, y_1) of the solution branch given in figure 2. The point $(\bullet)$ is the hyperbolic point.

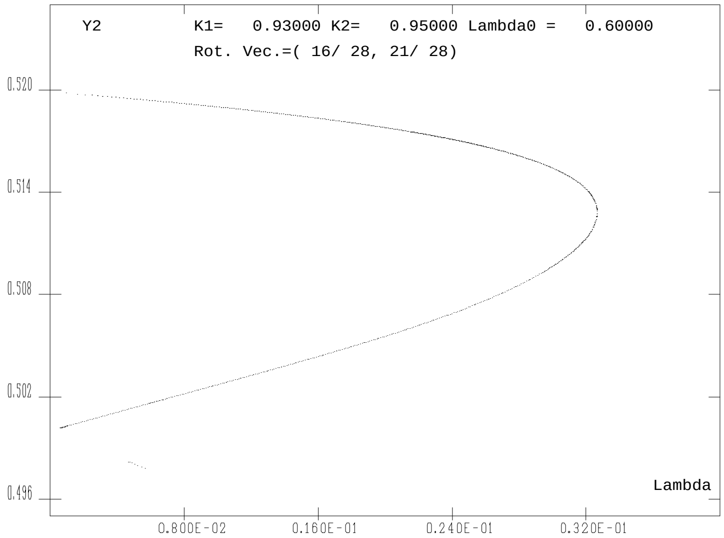

Figure 4: Solution branch of the periodic orbit with rotation vector $(\frac{16}{28}, \frac{21}{28})$ in the (λ, y_2) plane. The initial periodic orbit with rotation number $\frac{3}{4}$ has elliptic stability.

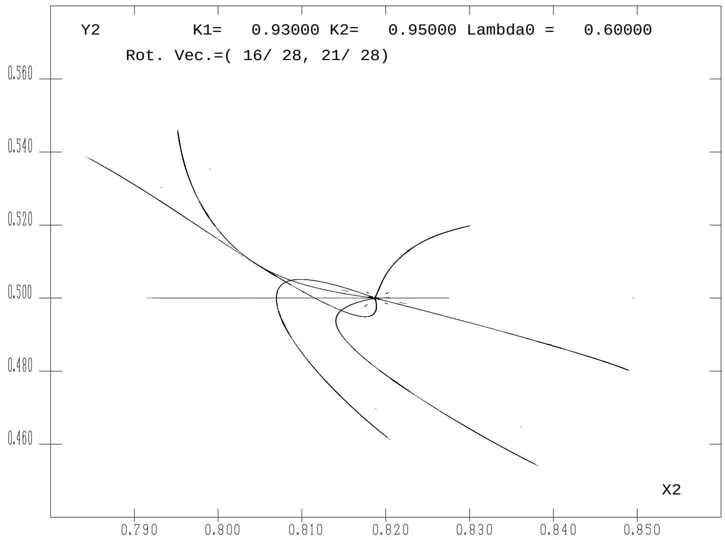

Figure 5: Evolution of the coordinates (x_2, y_2) of the solution branch given in figure 4. The point $(\bullet)$ represents the initial elliptic point. The final points $(\times)$ belong to the orbit of period 28.

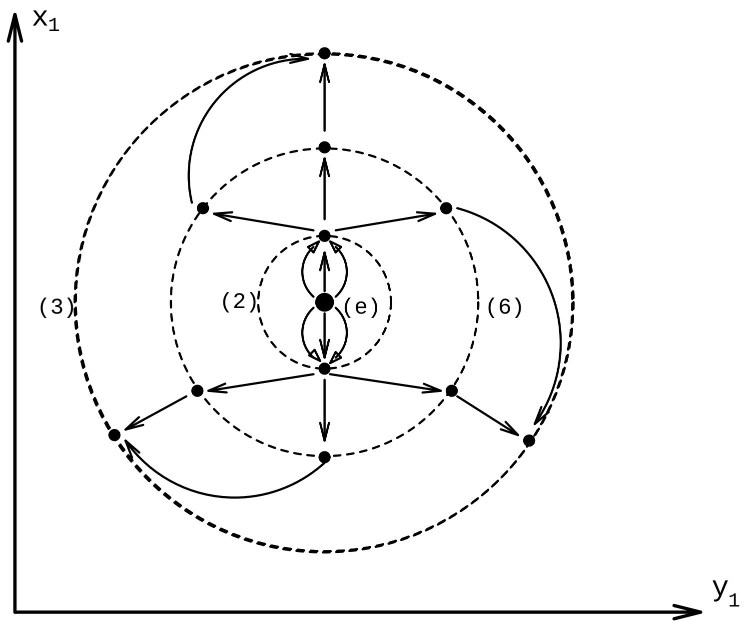

Figure 6: Schematic picture of the connection of the elliptic point (e) and its set of islands: $(e) \rightarrow (2) \rightarrow (6) \rightarrow (3)$.

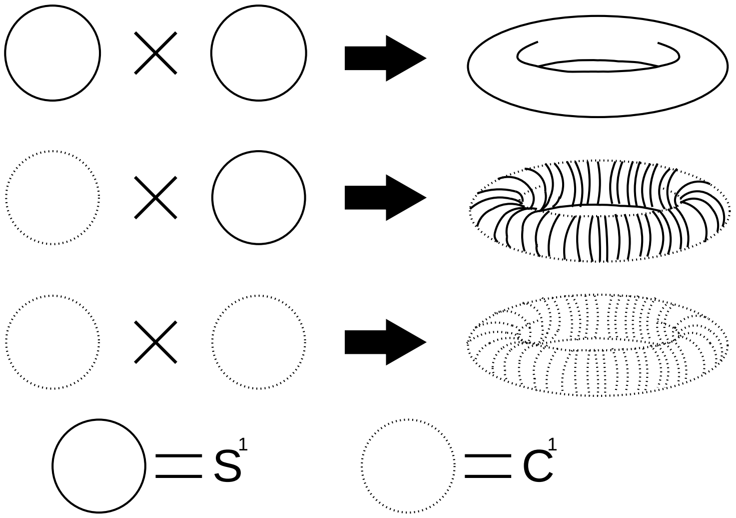

Figure 7: Evolution of the invariant tori. Where S^1 represents a circle and C^1 represents a cantorian set in S^1.

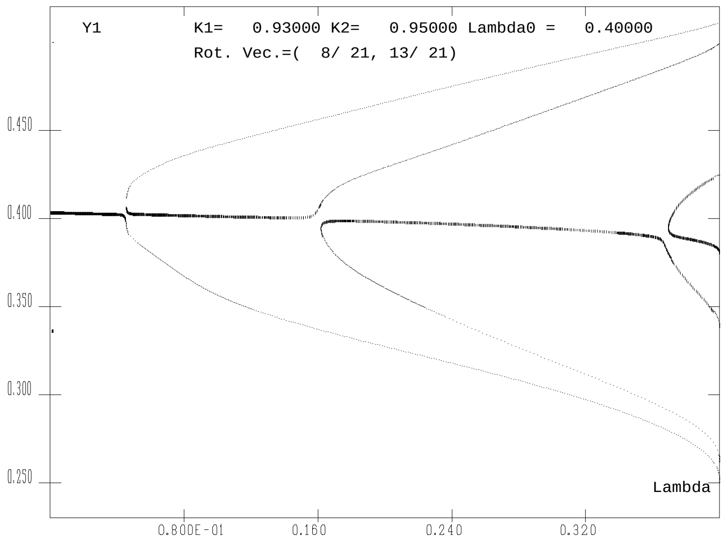

Figure 8: $\lambda^f = 0.4$ and $(\frac{p}{n}, \frac{q}{n}) = (\frac{8}{21}, \frac{13}{21})$

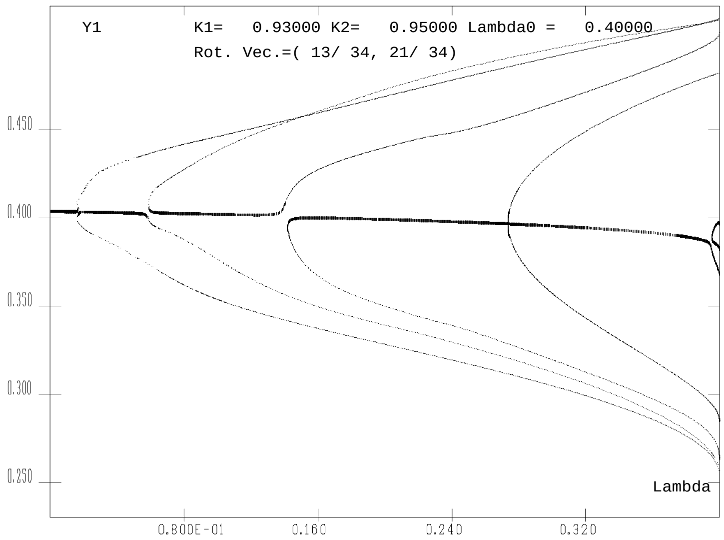

Figure 9: $\lambda^f = 0.4$ and $(\frac{p}{n}, \frac{q}{n}) = (\frac{13}{34}, \frac{21}{34})$

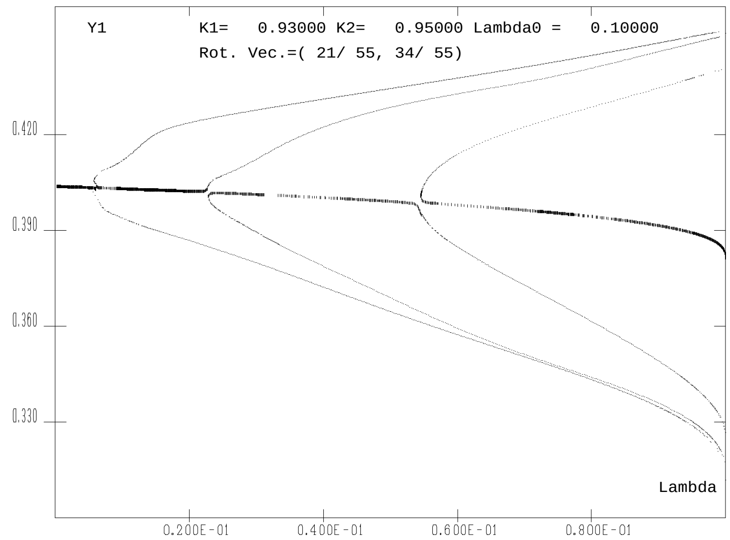

Figure 10: $\lambda^f = 0.1$ and $(\frac{p}{n}, \frac{q}{n}) = (\frac{21}{55}, \frac{34}{55})$

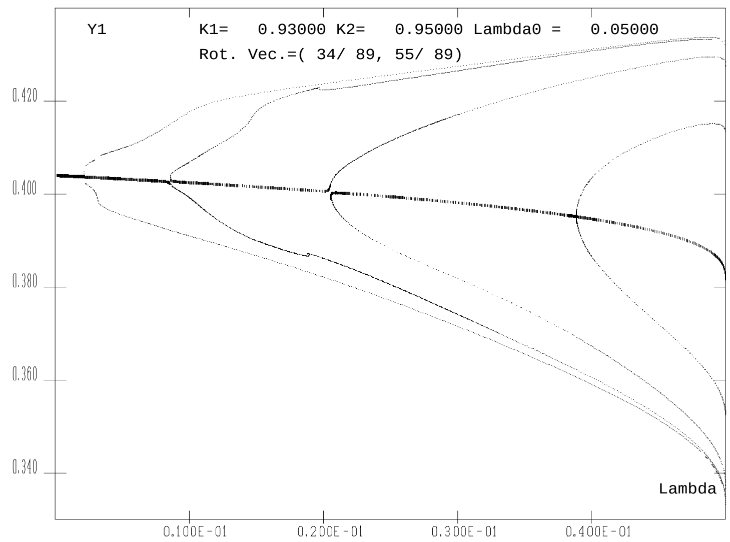

Figure 11: $\lambda^f = 0.05$ and $(\frac{p}{n}, \frac{q}{n}) = (\frac{34}{89}, \frac{55}{89})$

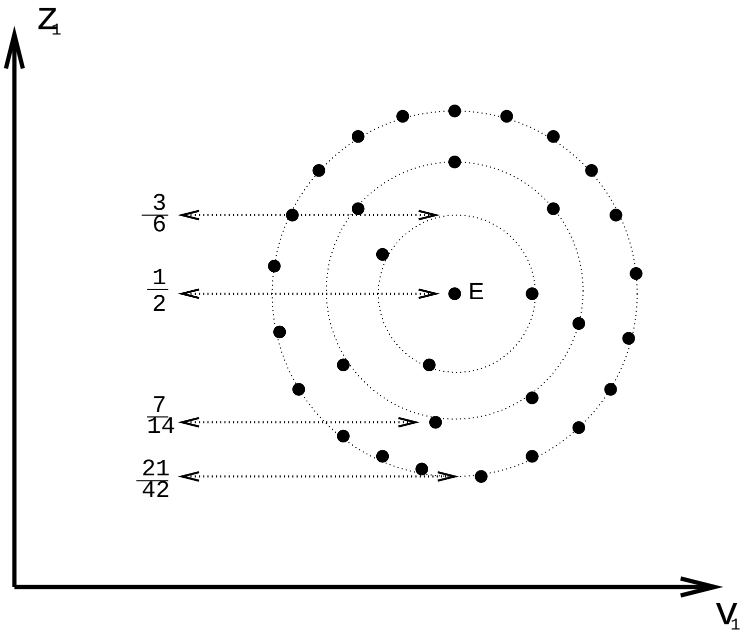

Figure 12: Qualitative picture of elliptic point of rotation number $\frac{1}{2}$ and its islands of rotation number $\frac{3}{6}$, $\frac{7}{14}$ and $\frac{21}{42}$.

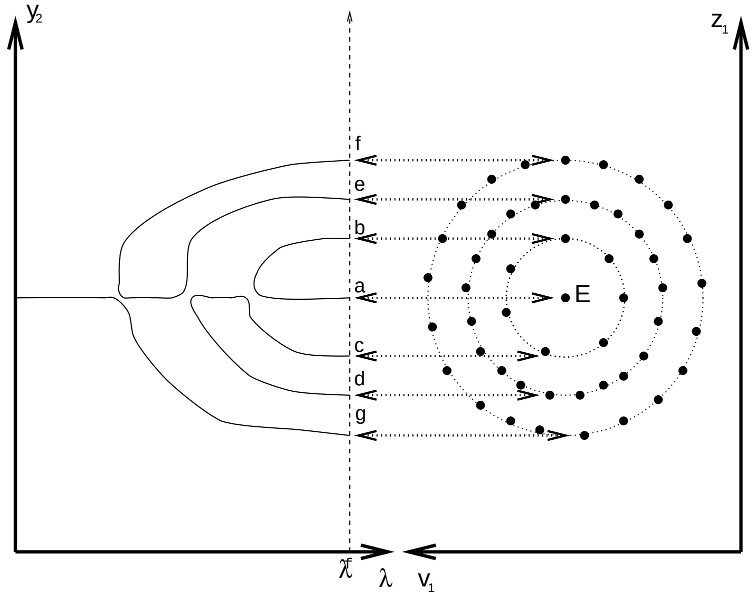

Figure 13: Branching connection of the elliptic point and its islands