A Galilean dance

1:2:4 resonant periodic motions and their librations of Jupiter and his Galilean moons

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Abstract

The four Galilean moons of Jupiter were discovered by Galileo in the early 17th century, and their motion was first seen as a miniature solar system. Around 1800 Laplace discovered that the Galilean motion is subjected to an orbital 1:2:4–resonance of the inner three moons Io, Europa and Ganymedes. In the early 20th century De Sitter gave a mathematical explanation for this in a Newtonian framework. In fact, he found a family of stable periodic solutions by using the seminal work of Poincaré, which at the time was quite new. In this paper we review and summarize recent results of Broer, Hanßmann and Zhao on the motion of the entire Galilean system, so including the fourth moon Callisto. To this purpose we use a version of parametrised Kolmogorov–Arnol'd–Moser theory where a family of multi-periodic isotropic invariant three-dimensional tori is found that combines the periodic motions of De Sitter and Callisto. The 3–tori are normally elliptic and excite a family of invariant Lagrangean 8–tori that project down to librational motions. Both the 3– and the 8–tori occur for an almost full Hausdorff measure set in the product of corresponding dimension in phase space and a parameter space, where the external parameters are given by the masses of the moons.

1 Setting of the problem in a historical background

In 1610 Galileo Galilei published his *Sidereus Nuncius* [21], where he reports of the existence of the four moons Io, Europa, Ganymedes and Callisto that move around the planet Jupiter, see Figure 1, a kind of miniature solar system. We shall not enter into the discussion that followed on the heliocentric world view, but just mention that this 5 body system has played a role in Newton's hypothesis of universal gravity.[1]

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[1]In fact, he had the astronomer royal Flamsteed check that Kepler's 2/3-rd law holds for this system, implying that Jupiter attracts his moons by inverse square gravitation [44] and thus giving an additional example for the inverse square law.

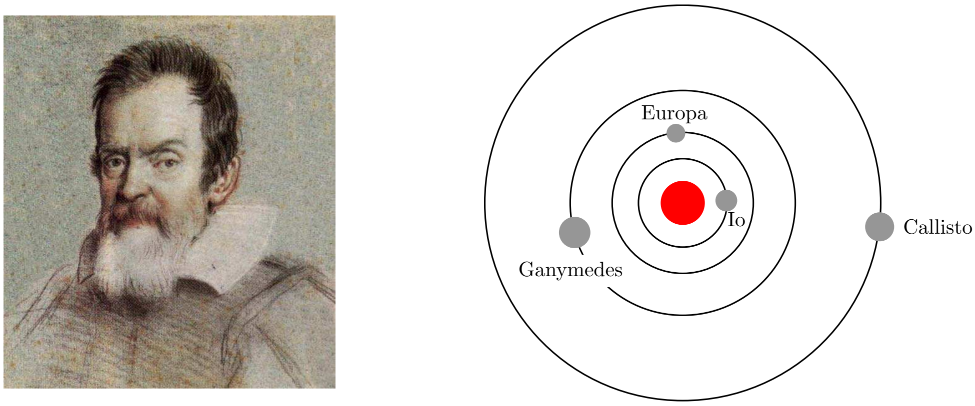

Figure 1: Left: Galileo Galilei 1564-1642. Right: The Galilean moons of Jupiter.

In the centuries after the discovery it became clear that the three inner moons do not move independently, in particular they never seem to be in conjuction, *i.e.* collinear and at the same side of Jupiter. In 1799 this led Pierre Simon de Laplace to postulate a 1:2:4 orbital resonance between the inner three Galilean moons [30], also see [43]. Indeed, it turns out that Europa has an approximately twice as long period of revolution as Io and the same holds for Ganymedes with respect to Europa. Since then this phenomenon has been subject of study within celestial mechanics, where the aim is to give a theoretical explanation of this resonant phenomenon in terms of Newton's laws. Our story starts with Willem de Sitter [40, 41, 42], who established the existence of such a stable 1:2:4–resonant periodic motion, based on the work of Poincaré [35]. For historical details see [23, 28].

The present paper describes possible motions of the 5–body system Jupiter, Io, Europa, Ganymedes, also including the outer Galilean moon Callisto, for details referring to [8, 18]. Here the theoretical background is extended with Kolmogorov–Arnol'd–Moser theory, in particular with parametrised KAM theory as developed in [14] and overcoming the problem of different scales in the various frequencies (so-called proper degeneracy); for the latter see also [1, 20, 24].

Remarks.

- The solar system contains a lot of resonances [28]. First of all, apart from the 1:2:4–resonance at hand, we mention the following orbital (period) resonances: a 2:5 resonance between Jupiter and Saturnus, a 1:3 resonance between Saturnus and Uranus, a 1:2 resonance between Uranus and Neptunus, and finally a 3:8 resonance between the Earth and Venus and between Callisto and Ganymedes.

- Also the following spin-orbit (period) resonances have been observed: the 1:1 resonance of the Moon versus the Earth and of Io versus Jupiter, and the mutual 1:1 resonance of Charon and Pluto. Finally we mention the 2:3 spin-orbit resonance of Mercurius versus the sun.

This paper is organized as follows. In the next section we introduce some notation, present the result on the motion of Jupiter's inner moons and give a sketch of the proof. Section 3 contains

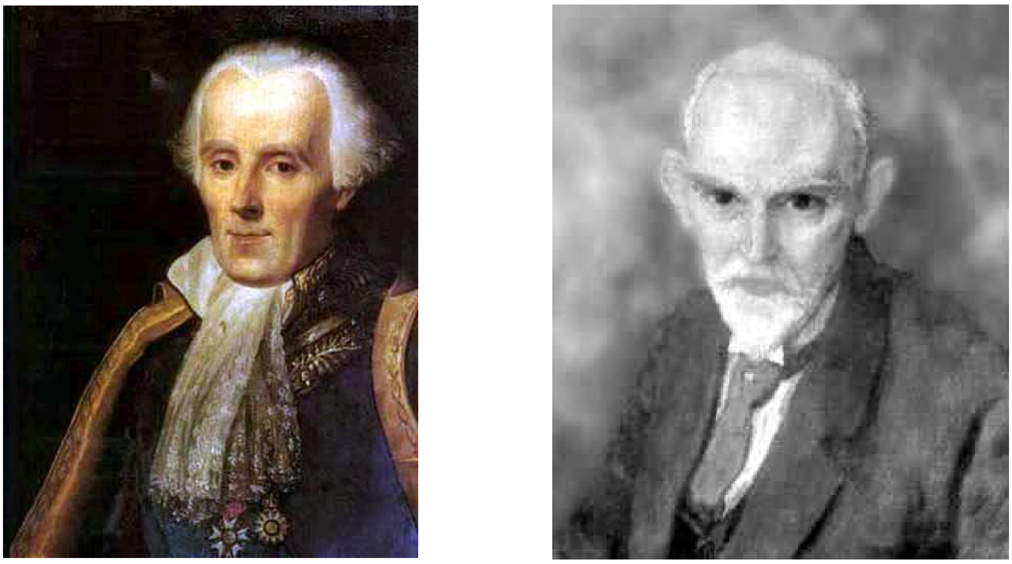

Figure 2: Left: Pierre Simon de Laplace 1749-1827. Right: Willem de Sitter 1872-1934.

the KAM theorem on which our result is based, preceded by a reformulation of the problem on a suitable covering space. In the final section 4 KAM theory is used to prove Theorem 1.

2 Set-up and sketch of results

Combining the periodic motion of De Sitter with the nearly circular periodic motion of Callisto we established the persistent existence of multi-periodic motion of 3 frequencies, which moreover generate multi-periodic librations of 8 frequencies. All of this occurs for an almost full measure set of parameters, consisting of masses and certain action variables. A librational motion was already foreseen by Moser and Zehnder, see [33], p. 120.

In our set-up the Galilean moons are all considered point masses where the oblateness of Jupiter is accounted for in a special potential. Moreover, the influences of Saturn, the sun etc. are neglected, while all motions are assumed to take place in one (ecliptic) plane. The 5–body system thus has 10 degrees of freedom. Taking into account the translational symmetry belonging to the conservation of linear momentum reduces the system to 8 degrees of freedom, moving through the ecliptic plane with its center of gravity.

The 3–frequency multi-periodic motion is found as the projection on the ecliptic plane of a multi-periodic motion in a 3–dimensional isotropic torus. Similarly the 8–frequency librational motion is the projection on this plane of a multi-periodic motion in an invariant 8–dimensional (Lagrangean) torus. One special aspect of the present approach is the need for and use of equivariant KAM theory on a covering space, related to the use of co-ordinates that co-rotate with the 1:2:4–resonance.

2.1 Specifications

The masses of Jupiter, Io, Europa, Ganymedes and Callisto are denoted m_0, m_1, m_2, m_3, m_4 and the eccentricities of their frozen ellipses by e_1, e_2, e_3, e_4. Since m_0 is much larger than m_j, for $1 \le j \le 4$, a scaling (small) parameter $\mu \ge 0$ is introduced such that

$$\mu \sim \frac{m_j}{m_0}, \quad j = 1, 2, 3, 4\,,$$

where we put

$$m_1 = \mu\bar{m}_1, \quad m_2 = \mu\bar{m}_2, \quad m_3 = \mu\bar{m}_3, \quad m_4 = \mu\bar{m}_4\,.$$

Also the eccentricities are assumed small:

$$e_1, e_2, e_3, e_4 \le e; \quad \text{where } e \text{ is a small constant.}$$

A multi-parameter λ is introduced that combines masses of the Galilean moons and certain action variables. The multi-periodic motions of 3 and 8 frequencies occur persistently for a nowhere dense set of positive measure in a specified part of λ–space, where the measure tends to full as $\mu \downarrow 0$. Note that this set sits in the product of phase space and the space of external parameters. For more details see below.

Remarks.

- We briefly explain the terminology concerning multi- and quasi-periodicity. An invariant n–torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ is multi-periodic if its motion is smoothly conjugate to the flow of a constant vector field
$$\sum_{j=1}^{n} \omega_j \frac{\partial}{\partial x_j}\,.$$
- This multi-periodicity is even quasi-periodic when $\omega_1, \omega_2, \ldots, \omega_1$ are non-resonant, *i.e.*, that for $k \in \mathbb{Z}^n$
$$\langle k, \omega\rangle = 0 \quad \text{implies that} \quad k = 0\,,$$
where $\langle k, \omega\rangle = \sum_{j=1}^{n} k_j\omega_j$.

2.2 Result

The 5–body system of Jupiter, Io, Europa, Ganymedes and Callisto in the plane has the symmetry-group $\mathbb{R}^2 \times \mathbb{S}^1$ of translation in the plane (with the linear momentum as conserved quantity) and rotation about the origin of the plane (with the angular momentum as conserved quantity). After reduction of these symmetries the reduced system has 7 degrees of freedom. The De Sitter periodic orbit of Io, Europa and Ganymedes in 1:2:4 resonance together with Callisto moving close to a circular orbit gives rise to a normally elliptic, isotropic invariant 2–torus. The normal modes of the isotropic 2–tori excite Lagrangean tori that carry librational motions.

Theorem 1 (Invariant 2–tori and their librations). *Let $N \geq 2$ be a given order of normalization. In the Diophantine conditions* (11) *we take $n = 2$, $r = 5$, $\tau > 6$ and the gap-parameter $\gamma > 0$ sufficiently small. Then there exists a bound $e_0 > 0$ on the eccentricities such that for $\mu \ll e^3$ and $e \leq e_0$ the following holds true.*

1. *Persistence of the invariant 2–tori: there exists a symplectic, equivariant C^∞–diffeomorphism*

$$\Phi : \mathbb{T}^2 \times U \times V \times \Gamma \subseteq \tilde{N} \longrightarrow \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}^{10} \times \Gamma$$

   onto its image, where U is a neighbourhood of 0 in $\mathbb{R}^2$ and V is a neighbourhood of 0 in $\mathbb{R}^{10}$, with the properties of Theorem 5.

2. *The 5 elliptic normal modes of these 2–tori give rise to a Cantor family of invariant Lagrangean 7–tori perturbed from $\mathbb{T}^7 \times \{0\} \times \Gamma \subseteq \tilde{N} \subseteq \tilde{Q} \subseteq \mathbb{T}^7 \times \mathbb{R}^7 \times \Gamma$ where the Cantorisation results from the Diophantine conditions*

$$|\langle k, \omega \rangle + \langle \ell, \beta \rangle| \geq \frac{\gamma}{(|k| + |\ell|)^\tau},$$

   for all $k \in \mathbb{Z}^2$ and all $\ell \in \mathbb{Z}^5$.

3. *The measure of the gaps in the union of surviving* KAM *tori can be estimated by* const. μe^N.

De-reducing the axial symmetry (generated by the angular momentum) yields invariant 3–tori surrounded by Lagrangean tori in 8 degrees of freedom. In fact one has a whole hierarchy of isotropic tori, exciting the normal modes one by one. Also, note that the fast frequencies of the periodic motions of the three inner moons and the periodic motion of Callisto do not satisfy the strong non-resonance condition of being Diophantine but instead are in 3:8 resonance.

2.3 Strategy of the proof

We briefly sketch how the proof runs. Following De Sitter, we start considering the 4 body problem of Jupiter, Io, Europa and Ganymedes, which lives in 8 degrees of freedom. Reduction of the translational symmetry ($\sim$ linear momentum) then leaves us with 6 degrees of freedom. For $\mu = 0$ there are no mutual interactions between Io, Europa and Ganymedes, which leads to three independent (Keplerian) elliptic motions, that foliate an open part of the phase space by (resonant) Lagrangean invariant 6–tori. The idea is to perturb away from this situation by taking $\mu > 0$ 'sufficiently small'. A further reduction of the rotational symmetry ($\sim$ angular momentum) leaves us in 5 degrees of freedom.

In 5 degrees of freedom we normalize by averaging out the 'fast' angle of 1:2:4 resonant motion by completely standard methods, going back to Poincaré and Birkhoff, and take a symmetric truncation that closely approximates the original system. Reduction of this 'resonant' symmetry takes us to 4 degrees of freedom. Following De Sitter and Poincaré, we study 'near collinear' equilibria of this system by their Hessian 8×8–matrix and thus find one stable equilibrium. Returning to 5 degrees of freedom, with help of the Implicit Mapping Theorem this equilibrium leads to the De Sitter family of stable (elliptic) periodic orbits [40, 41, 42].

Adding the periodic motion of Callisto to our considerations brings us to 7 degrees of freedom. Applying parametrised KAM theory we then obtain for large λ–measure, among other things accounting for Diophantic non-resonance between the periodic motions of the moon Callisto and of the three inner moons (i.e. of the periodic motion found by De Sitter), a family of quasi-periodic normally elliptic isotropic invariant 2–tori. These 2–tori project in 5 degrees of freedom down to (nearly) De Sitter periodic orbits. The normally elliptic eigenvalues excite quasi-periodic Lagrangean 7–tori. As said earlier this approach is based on [14, 24]. De-reducing the rotational symmetry ($\sim$ angular momentum) then provides us with the multi-periodic 3– and 8–tori as mentioned before.

3 On the proof

In this section we follow the main elements of the proof in [8], for details also referring to [18].

3.1 Delaunay variables

We start with the unperturbed 4–body system as this occurs for $\mu = 0$, which in 6 degrees of freedom takes place on three independent Keplerian ellipses. Here we introduce the Delaunay canonical variables (L_j, ℓ_j, G_j, g_j) , $j = 1, 2, 3$. For the angles ℓ_j and g_j see Figure 3; L_j, G_j are corresponding angular momenta. In these variables the Hamiltonian reads as

$$F_{\text{Kep}} = 4\nu L_1 + 2\nu L_2 + \nu L_3\,. \tag{1}$$

The explicit formulas are

$$\begin{cases} L_i = \mu_i \sqrt{M_i}\,\sqrt{a_i} & \text{circular angular momentum} \\ \ell_i & \text{mean anomaly} \\ G_i = L_i \sqrt{1 - e_i^2} & \text{angular momentum} \\ g_i & \text{argument of pericentre}\,, \end{cases}$$

where a_j are the semi-major axes and

$$\mu \sim \frac{m_j}{m_0}, \quad j = 1, 2, 3$$

with

$$m_1 = \mu\bar{m}_1, \quad m_2 = \mu\bar{m}_2, \quad m_3 = \mu\bar{m}_3$$

as before, and

$$\mu_j = \frac{m_0\bar{m}_j}{m_0 + \mu\bar{m}_j}, \quad M_j = m_0 + \mu\bar{m}_j\,.$$

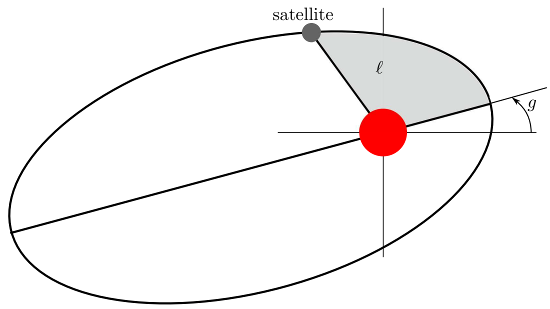

Figure 3: Delaunay angles: 'mean anomaly' ℓ and 'argument' g of the pericenter of a Keplerian motion.

Moving to co-rotating variables we introduce

$$
\begin{aligned}
D_1 &= L_1 & \delta_1 &= \ell_1 - 2\ell_2 \\
D_2 &= 2L_1 + L_2 & \delta_2 &= \ell_2 - 2\ell_3 \\
D_3 &= 4L_1 + 2L_2 + L_3 & \delta_3 &= \ell_3 \quad \text{(fast angle)} \\
Z_1 &= G_1 & \eta_1 &= g_1 - g_2 \\
Z_2 &= G_1 + G_2 & \eta_2 &= g_2 - g_3 \\
Z_3 &= G_1 + G_2 + G_3 & \eta_3 &= g_3
\end{aligned}
\tag{2}
$$

which leads to new canonical variables (D_j, δ_j) and (Z_j, η_j), $j = 1, 2, 3$, in which the unperturbed Hamiltonian system (1) obtains the form

$$\text{the Hamiltonian} \quad F_{\text{Kep}} = \nu D_3 \quad \text{generates the vector field} \quad X_{\text{kep}} = \nu\, \partial_{\delta_3} \,.$$

Note that Z_3 is the total angular momentum, the conservation of which is reflected by the fact that the angle η_3 is cyclic, compare with [4]. Fixing $Z_3 = \kappa_0 \neq 0$ reduces the system to 5 degrees of freedom.

3.2 Covering and deck group

Co-rotating co-ordinates often can be best understood in terms of covering mappings and deck transformations [10, 16, 17, 25]. This also holds for the transformation $(L, \ell, G, g) \mapsto (D, \delta, Z, \eta)$ and the covering mapping $\Pi : \mathbb{T}^6 \times \mathbb{R}^6 \longrightarrow \mathbb{T}^6 \times \mathbb{R}^6$, $\Pi = (\Pi_1 \times \Pi_2) \times (\Pi_3 \times \Pi_4)$, where

$$
\begin{aligned}
\Pi_1 : (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/4\pi\mathbb{Z}) \times (\mathbb{R}/8\pi\mathbb{Z}) &\longrightarrow (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \\
(\delta_3, \delta_2, \delta_1) &\mapsto (\ell_3, \ell_2, \ell_1) := (\delta_3, \delta_2 + 2\delta_3, \delta_1 + 2\delta_2 + 4\delta_3) \\
&\ \operatorname{mod}(2\pi\mathbb{Z}) \,,
\end{aligned}
$$

which is multiple to one, with deck transformations

$$\Delta_{1,2} : (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/4\pi\mathbb{Z}) \times (\mathbb{R}/8\pi\mathbb{Z}) \longrightarrow (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/4\pi\mathbb{Z}) \times (\mathbb{R}/8\pi\mathbb{Z})$$

defined as

$$\Delta_1(\delta_3,\delta_2,\delta_1) = (\delta_3,\delta_2 - 2\pi,\delta_1) \quad \text{and} \quad \Delta_2(\delta_3,\delta_2,\delta_1) = (\delta_3,\delta_2,\delta_1 - 2\pi)$$

generating the deck group

$$\Lambda = \langle \Delta_1, \Delta_2 \mid \Delta_1^2 = \Delta_2^4 = \mathrm{Id}, \Delta_1 \circ \Delta_2 = \Delta_2 \circ \Delta_1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \,.$$

In the other directions

$$\begin{aligned} \Pi_2 : (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \quad &\longrightarrow \quad (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \\ (\eta_3,\eta_2,\eta_1) \quad &\mapsto \quad (g_3,g_2,g_1) := (\eta_3, \eta_2+\eta_3, \eta_1+\eta_2+\eta_3) \\ &\qquad \bmod(2\pi\mathbb{Z}) \end{aligned}$$

$$\begin{aligned} \Pi_3 : \mathbb{R}^3 \quad &\longrightarrow \quad \mathbb{R}^3 \\ (D_1,D_2,D_3) \quad &\mapsto \quad (L_1,L_2,L_3) := (D_1, D_2 - 2D_1, D_3 - 2D_2) \end{aligned}$$

$$\begin{aligned} \Pi_4 : \mathbb{R}^3 \quad &\longrightarrow \quad \mathbb{R}^3 \\ (Z_1,Z_2,Z_3) \quad &\mapsto \quad (G_1,G_2,G_3) := (Z_1, Z_2 - Z_1, Z_3 - Z_2) \end{aligned}$$

which are all automorphisms. Hence in these directions there are only trivial deck transformations.

When working with the (D,δ,Z,η)–variables we are situated on the covering space, while the (L,ℓ,G,g)–variables live on the base space. Objects on the covering space are only well-defined, when they project nicely to the base space, which means that they have to respect the deck group Λ. To be more definite, below we transform to the normal form generated by the 'fast' Keplerian vector field

$$X_{\mathrm{kep}} = \nu\, \partial_{\delta_3} \,,$$

which is equivalent to averaging over the 'fast' angle δ_3 (or over the 1:2:4-resonance), and the normalizing transformations have to be equivariant with respect to Λ. The resulting normal form function or vector field on the covering space is symmetric with respect to Λ so that it projects to a well-defined function or vector field on the base space, expressed in the (L,ℓ,G,g)–variables.

This is already in order when looking for the De Sitter periodic motions in 5 degrees of freedom. To include the 4th moon Callisto we augment the Delaunay variables by

$$\begin{cases} L_4 = \mu_4 \sqrt{M_4}\,\sqrt{a_4} & \text{circular angular momentum} \\ \ell_4 & \text{mean anomaly} \\ G_4 = L_4 \sqrt{1 - e_4^2} & \text{angular momentum} \\ g_4 & \text{argument of pericentre}\,, \end{cases}$$

where a_4 is the semi-major axis of Callisto and $m_4 = \mu \bar{m}_4$ as already stated, and

$$\mu_4 = \frac{m_0 \bar{m}_4}{m_0 + \mu \bar{m}_4}, \qquad M_4 = m_0 + \mu \bar{m}_4 \,.$$

In the co-rotating variables on our covering space we replace Z_3 by

$$Z_3' = G_1 + G_2 + G_3 + G_4$$

and furthermore add

$$\begin{aligned} D_4 &= L_4 \\ \delta_4 &= \ell_4 + g_4 - g_3 \end{aligned}$$

together with

$$\xi_4 + \mathrm{i}\eta_4 = \sqrt{2(L_4 - G_4)}\mathrm{e}^{-\mathrm{i}(g_4 - g_3)}.$$

Note that the latter choice (going back to Poincaré) allows the orbit of Callisto to be circular while the other orbits have to be ellipses with non-zero eccentricity. Fixing $Z_3' = \kappa_0 \neq G_4 \neq 0$ and neglecting the cyclic angle η_3 now reduces the system to 7 degrees of freedom. When applying KAM theory in 7 degrees of freedom in order to find the invariant 2– and 7–tori, transformations are being applied that also have to be equivariant with respect to Λ. And, as before, the resulting function or vector field again projects nicely to the base space.

These problems however are easily overcome by using only Lie algebraic methods, compare with [7, 14]. In fact, the structure preservation like a symplectic form or invariance under a group can be neatly formulated in terms of Lie algebras and groups. The rest of the proof applies perturbation theory on the real analytic family of Hamiltonian functions

$$F = F_{\mathrm{Kep}} + F_{\mathrm{pert}}, \quad \text{where} \quad F_{\mathrm{pert}} = O(\mu)\,. \tag{3}$$

3.3 Normal Form

We start with recalling the well-known normal form or averaging theorem, the origins of which go back to Poincaré [34], e.g., compare with [7, 37] and references therein. As this does not concern Callisto we formulate this in 6 degrees of freedom. Recall that we work on the covering space $\mathbb{T}^6 \times \mathbb{R}^6$ with co-ordinates (D, δ, Z, η).

Theorem 2 (Normal Form). *For an appropriate domain $\mathcal{N} \subseteq \mathbb{T}^6 \times \mathbb{R}^6$ (avoiding collisions), there exists a near-identity, Λ–equivariant, real analytic, symplectic transformation $\Phi : \mathcal{N} \longrightarrow \mathcal{N}$, such that*

$$F \circ \Phi = F_{\mathrm{Kep}} + F^2_{\mathrm{res}} + F_{\mathrm{rem}} \tag{4}$$

and $|\Phi - \mathrm{Id}| = O(|\mu|)$ on complex extension of $\mathcal{N}$, where

$$F^2_{\mathrm{res}} = \frac{1}{2\pi}\int_0^{2\pi} F_{\mathrm{pert}}\, \mathrm{d}\delta_3$$

and $F_{\mathrm{rem}} = O(\mu e^2) + O(\mu^2)$. When extending the normalization up to order $N \geq 2$ we obtain $F_{\mathrm{Kep}} + F^N_{\mathrm{res}} + F^N_{\mathrm{rem}}$, with F^N_{res} again independent of δ_3. In that case we obtain the estimates

$$\begin{aligned} F^N_{\mathrm{res}} - F^2_{\mathrm{res}} &= O(\mu e^2 + \mu^2) \\ F^N_{\mathrm{rem}} &= O(\mu e^N) + O(\mu^N)\,. \end{aligned}$$

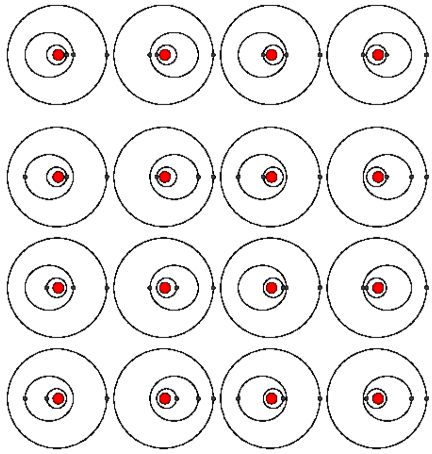

Figure 4: Poincaré's Ansatz: look near the 16 possible collinearities of Jupiter–Io–Europa–Ganymedes. It turns out that the one in the first column in the second row from above corresponds to the stable situation, where Io and Ganymedes are in their perijoves and Europa in her apojove.

Remarks.

- We note that the Hamiltonian function $F \circ \Phi$, transformed from (3) thereby gets the form

$$F_{\mathrm{Kep}}(D_1, D_2, D_3) + F^N_{\mathrm{res}}(D_1, D_2, \delta_1, \delta_2, Z_1, Z_2, \eta_1, \eta_2; D_3, Z_3) + O(\mu e^N) + O(\mu^N)\,.$$

- In the 5–body system, with Callisto included, the same transformation holds for the variables $D_1, D_2, D_3, \delta_1, \delta_2, \delta_3, Z_1, Z_2, Z_3, \eta_1, \eta_2, \eta_3$, while in the Callisto co-ordinates $D_4, \delta_4, \xi_4, \eta_4$ it acts as the identity mapping.

3.4 Retrieving the De Sitter periodic motions

The truncated normal form $F_{\mathrm{Kep}} + F_{\mathrm{res}}$ is invariant under the flow of ∂_{δ_3}, *i.e.,* it has D_3 as an integral. Reducing this symmetry leads to a 4–degree-of-freedom Hamiltonian

$$F_{\mathrm{res}}(D_1, D_2, \delta_1, \delta_2, Z_1, Z_2, \eta_1, \eta_2; D_3, Z_3)\,,$$

where the conserved quantities, that act as (distinguished) parameters, are indicated behind the semicolon. Equilibria of F_{res} then de-reduce to periodic orbits of $F_{\mathrm{Kep}} + F_{\mathrm{res}}$ in 5 degrees of freedom, which by an application of the Implicit Function Theorem on an appropriate Poincaré mapping give families of periodic orbits of $F_{\mathrm{Kep}} + F_{\mathrm{res}} + F_{\mathrm{rem}}$ in 5 degrees of freedom.

Starting point is Poincaré's Ansatz, asserting that the periodic motion should be looked for when 'starting' at a collinearity of the three moons. For the reduced F_{res} in 4 degrees of freedom we know the following. Let

$$\nu_1 = \frac{\partial F_{\mathrm{res}}}{\partial Z_1} \quad \text{and} \quad \nu_2 = \frac{\partial F_{\mathrm{res}}}{\partial Z_2}$$

be the (relative) frequencies of $\eta_1 = g_1 - g_2$ and $\eta_2 = g_2 - g_3$, then the equations for equilibria of F_{res} read

$$\nu_1 = \nu_2 = 0\,,$$

which have 16 possible parametrised solutions as indicated in Figure 4. It turns out that only one of these is elliptic (stable), which then yields De Sitter's periodic orbits. This result follows after considerable computations on the (normal) linear part in $\mathrm{sp}(2 \times 4, \mathbb{R})$. For details we refer to [18] and to [40, 41, 42].

3.5 Kolmogorov–Arnol'd–Moser Theory

For the description of the 5 body (Galilean) problem Jupiter–Io–Europa–Ganymedes–Callisto we invoke Kolmogorov–Arnol'd–Moser (KAM) theory as developed by Moser [32] in the formulation of Pöschel [36], as elaborated in Broer, Huitema and Takens [14]. For a slightly more popular version see [13]. We shall formulate a transparent version of the KAM theorem that covers all the possible applications of the present Galilean problem. For more or less similar applications of KAM theory in celestial mechanics see [1, 20, 45], also compare with [18]. For a general treatment of KAM theory also see [6].

General starting point is a Hamiltonian vector field family X of the form

$$\begin{aligned} \dot{x} &= \omega(\lambda) + f(x, y, z, \lambda) \\ \dot{y} &= g(x, y, z, \lambda) \\ \dot{z} &= \Omega(\lambda) z + h(x, y, z, \lambda) \end{aligned} \tag{5}$$

with $x \in \mathbb{T}^n, y \in \mathbb{R}^n, z \in \mathbb{R}^{2p}$ and where $\lambda \in \mathbb{R}^s$ is a multi-parameter. The symplectic form is given by

$$\sigma = dx \wedge dy + dz^2 = \sum_{j=1}^{n} dx_j \wedge dy_j + \sum_{j=1}^{p} dz_{2j-1} \wedge dz_{2_j}\,. \tag{6}$$

In our applications we choose x among the angles (δ, η) and we choose y among the conjugate actions (D, Z) depending on the kind of tori under consideration.

Remarks.

- In the Lagrangean case we have $p = 0$.
- The most interesting case $n = 2$, $p = 5$ is isotropic and detailed in (12) of § 4 below.

In all cases f, g and h are given by partial derivative of F as determined by

$$X \lrcorner\, \sigma = dF\,. \tag{7}$$

The family X is approximated by a family of integrable vector fields

$$\tilde{X} = \omega(\lambda)\frac{\partial}{\partial x} + \Omega(\lambda) z \frac{\partial}{\partial z} \tag{8}$$

and the central problem is the persistence of the *isotropic* n–torus $y = y_0$, $\lambda = \lambda_0$ and of *Lagrangean* tori excited by normal frequencies / modes of this.

In fact, following [14] we introduce a parameter $\kappa \in \mathbb{R}^n$, studying the isotropic torus $y_0 = \kappa$, also introducing the localized variable

$$y_{\text{loc}} = y - \kappa\,.$$

In order not to burden notation we include the (distinguished) parameter κ in λ and from now on simply again write y instead of y_{loc}, remembering that we study the tori given by the equation $y_0 = \kappa$. Again for simplicity we set $\lambda_0 = 0$.

An important tool in our formulation is the amended frequency mapping

$$\mathcal{F} : \lambda \in \mathbb{R}^s \quad \mapsto \quad (\omega(\lambda), \Omega(\lambda)) \in \mathbb{R}^n \times \mathrm{sp}(2p, \mathbb{R})\,,$$

in terms of which we shall consider two conditions, namely nondegeneracy and Diophanticity. Here $\mathrm{sp}(2p, \mathbb{R})$ denotes the Lie algebra of symplectic $2p\times 2p$–matrices. For the following concepts we refer to [14] and to [9, 12, 15], largely following Arnol'd [3]. Also see [13] and [11, 25].

3.5.1 Nondegeneracy and scaling

A proper adaptation of the classical Kolmogorov nondegeneracy condition for the persistence of Lagrangean tori in Hamiltonian systems in the present setting is the following, where the terminology was introduced by Sevryuk [38, 39].

Definition 3 (BHT–nondegeneracy). *The system* (8) *is* Broer-Huitema-Takens (BHT)–nondegenerate *at $\lambda = 0$, if the mapping*

$$\mathcal{F} : \lambda \in \mathbb{R}^s \quad \mapsto \quad \mathbb{R}^n \times \mathrm{sp}(2p, \mathbb{R})$$

is transverse to

$$\{\omega(0)\} \times \mathrm{SP}(2p, \mathbb{R})(\Omega(0)) \quad \subseteq \quad \mathbb{R}^n \times \mathrm{sp}(2p, \mathbb{R})\,,$$

where $\mathrm{SP}(2p, \mathbb{R})(\Omega(0))$ *is the orbit of* $\Omega(0)$ *under the adjoint action of* $\mathrm{SP}(2p, \mathbb{R})$.

Let us explain what this means in the present context. Assume the matrix $\Omega(\lambda)$ to be in Williamson diagonal form [22] with eigenvalues of type

$$\pm \mathrm{i}\beta, \quad \pm\alpha \pm \mathrm{i}\beta \quad \text{and} \quad \pm\alpha$$

of respective numbers n_E, n_C and n_R, where $n_E + 2n_C + n_R = p$. We put $r = n_E + n_C$. If $\beta \in \mathbb{R}^r$ is the vector of normal frequencies where all entries are non-zero, then BHT–nondegeneracy implies that the extended frequency mapping

$$\tilde{\mathcal{F}} : \lambda \in \mathbb{R}^s \quad \mapsto \quad (\omega(\lambda), \beta(\lambda)) \in \mathbb{R}^n \times \mathbb{R}^r \tag{9}$$

is a submersion [14]. In the present cases of elliptic tori, where $n_C = n_R = 0$ and $r = p = n_E$, the two statements are even equivalent.

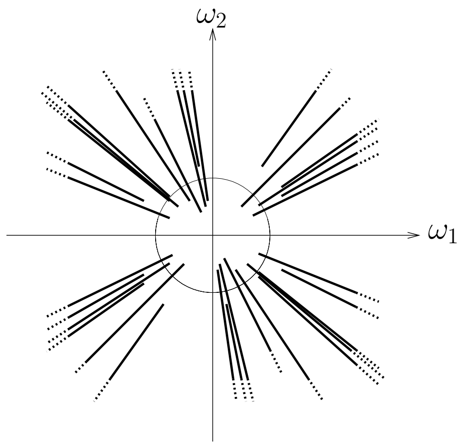

Figure 5: Sketch of the set $(\mathbb{R}^2)_{\tau,\gamma}$.

Remarks.

- Since we have n internal frequencies, the problem needs $s = n + n_E = n + p$ parameters for a versal unfolding. This implies that even if we use the full n–dimensional distinguished parameter $\kappa = y$, the parameter λ still needs p further components.
- Note that for Lagrangean tori $p = 0$ and we can take $s = 0$ whence BHT–nondegeneracy reduces to (local) Kolmogorov nondegeneracy.
- It so happens that all different cases of the present Galilean problem can be captured by taking various values of n and r.

In the present case of multiple time scales the extended frequency mapping (9) is split into

$$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m) : \mathbb{R}^s \quad \longrightarrow \quad \Pi_{j=1}^m \mathbb{R}^{n_j} \times \mathbb{R}^{r_j}, \tag{10}$$

where the values of $\mathcal{F}$ turn into

$$(\omega, \beta) = \left((\omega^1, \beta^1), (\omega^2, \beta^2), \ldots, (\omega^m, \beta^m)\right).$$

Here we have n_j frequencies ω^j of order ε_j and r_j normal frequencies β^j of the same order ε_j. The orders of magnitude of the time scales are

$$1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_m > 0.$$

Considering the BHT-condition and the splitting (10) we note the following. First of all, when $\mathcal{F}$ is submersive, so are all its components $\mathcal{F}_j$. To ensure that the latter also implies the former requires a further splitting of the parameters space.

Remarks.

- For the 4–body problem without Callisto we have $m = 3$, with $n = (1, 0, 0)$ and $r = (0, 2, 2)$, while for the 5–body problem including Callisto we have $m = 4$ with $n = (2, 0, 0, 0)$ and $r = (0, 2, 2, 1)$.
- In particular the fast frequencies are the internal frequencies of the 1:2:4–resonant motion of Io–Europa–Ganymedes and of the Keplerian motion of Callisto while the various magnitudes of small frequencies concern the normal frequencies.

3.5.2 Diophantine conditions

In the present perturbation problem in the space $\mathbb{R}^{n+r}$ of internal and normal frequencies we have to deal with the dense set of resonances given by the equations

$$\langle k, \omega\rangle + \langle \ell, \beta\rangle = 0$$

for $k \in \mathbb{Z}^n, \ell \in \mathbb{Z}^r$. Even when excluding these resonances the notorious small divisor problem shows up. This problem can be overcome by the following Diophantine non-resonance condition.

Definition 4 (Diophanticity). *Let $\tau > n - 1$ and $\gamma > 0$ be given. Then the pair $(\omega, \beta) \in \mathbb{R}^n \times \mathbb{R}^r$ is (τ, γ)–Diophantine if for all $k \in \mathbb{Z}^n \setminus \{0\}$ and all $\ell \in \mathbb{Z}^r$ with $|\ell| \leq 2$ one has*

$$|\langle k, \omega\rangle + \langle \ell, \beta\rangle| \geq \frac{\gamma}{|k|^\tau}\,. \tag{11}$$

The set of all Diophantine frequencies is denoted $(\mathbb{R}^n \times \mathbb{R}^r)_{\tau,\gamma}$.

We give a few properties of the set $(\mathbb{R}^n \times \mathbb{R}^r)_{\tau,\gamma} \subseteq \mathbb{R}^n \times \mathbb{R}^r$. First of all the set is closed. Moreover it is a nowhere dense union of closed halflines intersecting the unit sphere $\mathbb{S}^{n+r-1}$ in a Cantor set, where

$$\text{measure}\,(\mathbb{S}^{n+r-1} \setminus (\mathbb{R}^n \times \mathbb{R}^r)_{\tau,\gamma}) = O(\gamma)$$

as $\gamma \downarrow 0$. We refer to γ as the gap-parameter.

3.5.3 Persistence

The Hamiltonian systems of our interest are defined on

$$\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2p} \times \Gamma,$$

with $\Gamma \subseteq \mathbb{R}^s$ a compact box with $\lambda = 0$ in its interior. We also introduce the box

$$\Gamma^\gamma = \{\lambda \in \Gamma \mid \text{dist}\,(\lambda, \partial\Gamma) > \gamma\} \quad \text{as well as} \quad \Gamma^\gamma_{\tau,\gamma} = \Gamma^\gamma \cap \tilde{\mathcal{F}}^{-1}(\mathbb{R}^n \times \mathbb{R}^r)_{\tau,\gamma}\,,$$

observing that the measure of $\Gamma^\gamma_{\tau,\gamma}$ again is full up to order $O(\gamma)$ as $\gamma \downarrow 0$.

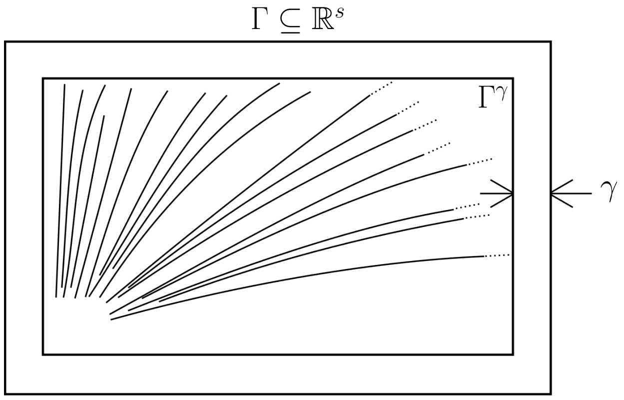

Figure 6: Sketch of $\Gamma^\gamma_{\tau,\gamma} \subseteq \Gamma^\gamma \subseteq \Gamma$.

In the case of multiple scales we furthermore split the parameter space

$$\Gamma^\gamma_{\tau,\gamma} = \Gamma^{\gamma,1}_{\tau,\gamma} \times \Gamma^{\gamma,2}_{\tau,\gamma} \times \cdots \times \Gamma^{\gamma,m}_{\tau,\gamma}$$

where we require that the component $\mathcal{F}_j : \Gamma^\gamma \longrightarrow \mathbb{R}^{n_j} \times \mathbb{R}^{r_j}$ not only is a submersion, but that its derivative already has maximal rank with respect to the variables in $\Gamma^{\gamma,j}$. If this property holds for all $j = 1, 2, \ldots, m$ we say that the unperturbed vector field $\tilde{X}$ satisfies the *scaled* BHT nondegeneracy condition. This condition is sufficient for $\mathcal{F}$ being submersive. Moreover, the condition is easy to check in our applications: the multiple time scales actually facilitate the solution of our problems.

Theorem 5 (Parametrised KAM with time scales [8]). *Let two real analytic, Λ–invariant vector fields X and $\tilde{X}$ as in* (5) *and* (8) *be given, both Hamiltonian with respect to the symplectic form $dx \wedge dy + dz^2$. Assume that $\tilde{X}$ is scaled* BHT *nondegenerate. Fix constants $\tau > n + n_E - 1$ and $\gamma > 0$ 'sufficiently small' and assume that $X - \tilde{X} = O(\gamma)$ in the compact-open topology on complex extensions. Then there exists a compact box Γ, with $\lambda = 0$ in its interior, and a C^∞–diffeomorphism, onto its image,*

$$\Phi : \mathbb{T}^n \times U \times V \times \Gamma \longrightarrow \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2p} \times \Gamma,$$

for neighbourhoods U of $0 \in \mathbb{R}^n$ and V of $0 \in \mathbb{R}^{2p}$ such that the mapping Φ preserves fibres in the sense that the diagram

$$\begin{array}{ccc}
\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2p} \times \Gamma & \stackrel{\Phi}{\longrightarrow} & \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2p} \times \Gamma \\
\downarrow & & \downarrow \\
\mathbb{T}^n \times \mathbb{R}^n \times \Gamma & \longrightarrow & \mathbb{T}^n \times \mathbb{R}^n \times \Gamma \\
\downarrow & & \downarrow \\
\mathbb{T}^n \times \Gamma & \longrightarrow & \mathbb{T}^n \times \Gamma \\
\downarrow & & \downarrow \\
\Gamma & \longrightarrow & \Gamma
\end{array}$$

commutes, where vertical arrows denote natural projections and where the horizontal arrows indicate the relevant components of Φ. The mapping Φ is real analytic in x and affine in the y– and z–direction. Moreover

- *Φ is near the identity mapping in the C^∞–topology and Φ preserves the symplectic form $dx \wedge dy + dz^2$. Moreover Φ is equivariant with respect to the deck group Λ.*

- *Restricted to $\mathbb{T}^n \times \{0\} \times \{0\} \times \Gamma^\gamma_{\tau,\gamma}$ the mapping Φ conjugates $\tilde{X}$ to X, i.e.,*

$$\Phi_*(\tilde{X}) = X .$$

  The restriction $\Phi_*|_{\mathbb{T}^n\times\{0\}\times\{0\}\times\Gamma^\gamma_{\tau,\gamma}}$ preserves the normal linear dynamics of these invariant tori.

- *The n_E elliptic normal frequencies give rise to a smooth Cantor family of invariant $(n+n_E)$–tori excited from*

$$\mathbb{T}^{n+n_E} \times \{0\} \times \{0\} \times \Gamma \subseteq \mathbb{T}^{n+n_E} \times \mathbb{R}^{n+n_E} \times \mathbb{R}^{2(p-n_E)} \times \Gamma$$

  under Diophantine conditions slightly adapted from (11)*, where the Cantorisation results from Diophantine conditions of the form* (11) *with $\ell \in \mathbb{Z}^r$ restricted to $|\ell_{n_E+1}| + \cdots + |\ell_r| \leq 2$ and $|k|^\tau$ in the denominator of the right hand side replaced by* $(|k| + |\ell_1| + \cdots + |\ell_{n_E}|)^\tau$.

In [8] the proof of Theorem 5 is based on [14].

Remarks.

- Note that the mapping Φ conjugates the unperturbed restriction $\tilde{X}_{\mathbb{T}^n\times\{0\}\times\{0\}\times\Gamma^\gamma_{\tau,\gamma}}$ to a quasi-periodic subsystem of the perturbation X. Therefore the perturbed tori have the same frequencies as the unperturbed ones. Moreover, perturbed and unperturbed tori are close to each other since Φ is a near-the-identity mapping.

- For the case where $m = 1$ and $\Lambda = \{\mathrm{Id}\}$ Theorem 5 reduces to Theorem 6.1 in [14]. Note that the extension to Λ–equivariance of Φ does not form an extra complication as this already follows from Theorem 8.1 of [14], which uses the Lie algebra set-up as this goes back to Moser [32].

- Excitation of normal modes behaves well with respect to different time scales as normal frequencies β_j of time scale ε_j are merely turned into internal frequencies of that same time scale.

- Excitation of normal modes not simultaneously, but one by one yields tori of all intermediate dimensions. This can be proven along the same lines as the final point of Theorem 5, compare with [27].

4 Applications of Theorem 5 to the Galilean system

We return to the 5–body Galilean system, with Callisto included as this lives in 7 degrees of freedom, *i.e.*, where the rotational symmetry related to the conserved angular momentum Z_3' has been reduced.

The interest then is with the persistence of quasi-periodic invariant 2–tori as generated from the 1:2:4–resonant De Sitter periodic motion, and the almost circular motion of Callisto. We therefore apply Theorem 5 with $n = 2$ and $p = 5$. From the ellipticity of the De Sitter periodic motion it follows that also $r = 5$ the quasi-periodic 2–tori are normally elliptic. The equations of motion can be expressed in the co-ordinates

$$\begin{array}{rclcrcl} x_1 & = & \delta_3 & \qquad & y_1 & = & D_3 \\ x_2 & = & \delta_4 & & y_2 & = & D_4 \\ z_1 & = & \delta_1 & & z_2 & = & D_1 - D_1^0 \\ z_3 & = & \delta_2 & & z_4 & = & D_2 - D_2^0 \\ z_5 & = & \eta_1 - \pi & & z_6 & = & Z_1 - Z_1^0 \\ z_7 & = & \eta_2 - \pi & & z_8 & = & Z_2 - Z_2^0 \\ z_9 & = & \xi_4 & & z_{10} & = & \eta_4\,, \end{array} \tag{12}$$

which are the co-ordinates on the covering space obtained after normalization by Theorem 2 around the De Sitter periodic orbit. The symplectic form is given by (6) and the Hamiltonian (4) through (7) yields the equations of motion in the format (5). The normally elliptic part of these 2–tori excites librational quasi-periodic invariant Lagrangean 7–tori. The frequencies of these tori are close to the normally linear ones, compare with [13, 27].

It turns out [8, 18] that the 7–dimensional frequency vector $(\nu, \nu_4, \nu_{n,1}, \nu_{n,2}, \nu_{n,3}, \nu_{n,4}, \nu_{n,5})$ has scaling properties

$$\begin{array}{rcl} \text{internal frequencies } \nu \text{ of } \delta_3 \text{ and } \nu_4 \text{ of } \delta_4 & \sim & \text{order } \varepsilon_1 = 1 \\ \text{normal frequencies } \nu_{n,1} \text{ and } \nu_{n,2} & \sim & \text{order } \varepsilon_2 = \sqrt{\mu e} \\ \text{normal frequencies } \nu_{n,3} \text{ and } \nu_{n,4} & \sim & \text{order } \varepsilon_3 = \mu/e \\ \text{normal frequency } \nu_{n,5} & \sim & \text{order } \varepsilon_4 = \mu e^2. \end{array}$$

Moreover it follows that for $0 < \mu \ll e^3 \ll 1$ one has

$$1 = \varepsilon_1 \gg \varepsilon_2 \gg \varepsilon_3 \gg \varepsilon_4 > 0\,.$$

which, in terms of (5), leads to corresponding smallness conditions of the form

$$|f_1, g_1, h_1| \sim \varepsilon_1\gamma, \quad |f_2, g_2, h_2| \sim \varepsilon_2\gamma, \quad |f_3, g_3, h_3| \sim \varepsilon_3\gamma, \quad |f_4, g_4, h_4| \sim \varepsilon_4\gamma\,,$$

which somewhat relaxes the condition $X - \tilde{X} = O(\gamma)$ from Theorem 5. Compare with [24].

Our multi-parameter λ is introduced as

$$\lambda = (\bar{m}_1, \bar{m}_2, \bar{m}_3, \bar{m}_4, e_2, D_3, D_4),$$

where the triple (e_2, D_3, D_4) is distinguished, *i.e.*, depends on the multi-parameter $\kappa = (\kappa_0, \kappa_1, \kappa_2)$ related to the angular momentum Z_3' and the action variables y. The precise choice of parameters is somewhat proof generated and for mathematical convenience.

If N is the order of normalization in Theorem 2, then, if we take $N \geq 2$ 'sufficiently large' and choose the gap parameter $\gamma = e^N$, the total gap measure is of order $O(\mu e^N)$. However, it turns out that here $N = 2$ already works.

De-reduction of the rotational symmetry related to the angular momentum amounts to re-introduction of the angle η_3, which adds one degree of freedom. This also augments the toral dimensions by one, leading to normally elliptic, multi-periodic, isotropic invariant 3–tori with librational motions in multi-periodic, Lagrangian invariant 8–tori. De-reduction of the translational symmetry related to the linear momentum then lets the 5–body Galilean system move through the ecliptic plane, circling around the common centre of mass.

Remarks.

- In the 5–degree-of-freedom 4–body system Jupiter–Io–Europa–Ganymedes the elliptic De Sitter periodic motions excite quasi-periodic Lagrangean invariant 5–tori for large measure in product of phase space and parameter space. Here we can take $n = 1$ and $r = 4$ in the application of Theorem 5. De-reducing the rotational symmetry by re-introducing the angle η_3 then leads to a smooth family of multiperiodic elliptic 2–tori surrounded by 6–dimensional Lagrangean tori.

  However, Celletti *et al.* [19] use observational data to argue that the real Laplace resonant orbits lie outside a certain small neighbourhood of the De Sitter periodic orbits. It may well be of interest to adapt the present 5–body approach combining this Laplace motion with that of Callisto.

- The 'true' motion of Callisto is in 3:8 resonance with Ganymedes which puts an upper bound on N if we want to obtain a normal form in which the superposed motion of the De Sitter periodic orbit and Callisto's periodic motion form a 2–periodic invariant torus. On the other hand, it puts a lower bound on the normalization order N if we become interested in the fine details of the dynamics as influenced by the additional 3:8 resonance.

- The present approach still is highly simplified and, for instance, a 3–dimensional description is lacking. The real motion is even more complicated. One question may be what is its past. The Galilean moons probably were not always in their present resonance and other resonances may have preceeded the present one, see [31] and references therein. A related question is whether Arnol'd diffusion [2, 6] has taken place in its long past.

- One of the purposes of mathematical modeling is the possibility to predict ephemerides. For a more modern approach that directly attacks the equations of motion with numerics, we refer to [29], which contains relativistic corrections by De Sitter.

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