Average consensus in multi-agent systems with uncertain topologies and multiple time-varying delays

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Abstract

In this paper, we investigate average consensus problem in continuous-time multi-agent systems with uncertain topologies as well as multiple time-varying communication delays. The network of dynamic agents is directed and both fixed and switching topologies are considered. By using the linear matrix inequality method, we prove that all the nodes in the network can achieve average consensus asymptotically for appropriate delays and uncertainties if the network topology is weakly connected and balanced. Feasible linear matrix inequalities are established to determine the maximal allowable upper bounds of multiple delays. Numerical examples are presented to illustrate the availability of the theoretical results.

Key words: Consensus, linear matrix inequality, directed network, uncertainty, time-varying delay.

1 Introduction

In the past few years, decentralized coordinated control of multi-agent systems has attracted great research attention. Multi-agent systems have been widely used in the areas of formation control [1], flocking or swarming behaviors [2, 3, 4, 5], path planning [6] and congestion control in communication networks [7, 8], etc. In coordinated control of multi-agent systems, a critical problem is to design appropriate protocols and algorithms such that multiple agents can achieve a common consensus value on the shared information in the presence of limited and unreliable information exchange as well as communication delays.

In many applications involving multi-agent systems, time delays may frequently occur due to agents moving, asymmetric interactions, communication congestion, or finite transmission speed. It is well-known that introducing a delay generally leads to a reduce of the performance or to instability [10]. Thus, investigating the effect of time delays in the consensus problem is an important issue. Consensus behaviors are first analyzed in [9] for a continuous-time multi-agent system with switching topology and a common constant communication delay. Necessary and sufficient consensus conditions are established under varied assumptions on presence or absence of time delays, and directed or undirected information flow. Based on Lyapunov-Krasovskii techniques, stability criteria are provided in [11] for systems with non-uniform constant communication delay. In [12, 13], average consensus problem of multi-agent networks with time-dependent delays is addressed by using frequency approach and linear matrix inequalities (LMIs).

So far, to the best of our knowledge, just a few works deal with consensus problems in the case of multiple time-varying delay. Theoretical analysis for this case is more challenging. By means of LMI method, average consensus problem is first studied in undirected networks of dynamic agents with multiple time-varying communication delays in [15]. Since the close-loop system matrix is singular, a special matrix is introduced to overcome this singularity. Alternatively, the authors in [16, 17] employ orthogonal transformations to change the original singular system into a reduced dimensional system. By constructing Lyapunov-Krasovskii functions, the maximal allowed upper bounds of multiple time-varying delays are expressed in terms of feasible LMIs. A common assumption in these works is that the network topology is undirected. This assumption is too rigorous in practice and dramatically limits the applicability of the results. One goal of this paper is to lift this constraint by considering balanced digraphs (directed graphs). Note that average consensus problem and leader-following consensus problem of high-order multi-agent systems with multiple time-varying delay are also treated in [18] and [14], respectively.

The above mentioned works [15, 16, 17, 18, 14] implicitly assume the interaction strengths or link weights can be exactly measured. In other words, external disturbances and communication uncertainties have been ignored. However, this is not realistic in most of real-world complex networks, because measurement error and environmental uncertainties cannot be avoided in real systems (see e.g. [19, 20, 21, 22]). For instance, in regulatory networks, which are directed and weighted, no tools have been found yet to estimate the link weights.

Motivated by these considerations, in this paper we study average consensus problems in directed multi-agent systems with uncertain topologies as well as multiple time-varying delays. Both fixed and switching network topologies are considered. With the aid of LMI approach, we show that the group of first-order dynamic agents can reach average consensus asymptotically for suitable multiple time-varying delays and uncertainties if the network topology is weakly connected and balanced. Our main results are expressed in terms of feasible LMIs, from which the maximal admissible upper bounds of multiple time-varying delays can be easily obtained by using Matlab's LMI Toolbox [23]. Simulation examples are given to demonstrate the effectiveness of the theoretical results.

The rest of the paper is organized as follows. Section 2 contains the problem formulation. Sections 3 and 4 are devoted to the consensus analysis for fixed and switching topologies, respectively. Numerical results are presented in Section 5. The concluding remarks are made in Section 6.

2 Problem statement

Let $G = (V, E, A)$ denote a weighted digraph (directed graph) of order n with the set of nodes $V = \{1, 2, \cdots, n\}$, edges $E \subseteq V \times V$, and the weighted adjacency matrix $A = (a_{ij})$. A directed edge (j, i) from j to i exists if and only if $a_{ij} > 0$. Moreover, we assume that $a_{ii} = 0$ for all $i \in V$. The set of neighbors of node i is denoted by $N_i = \{j \in V : (j, i) \in E\}$. The graph Laplacian (or Laplacian matrix) $L = (l_{ij})$ induced by the digraph G is defined by (see e.g. [24])

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{k=1}^{n} a_{ik}, & i = j. \end{cases} \tag{1}$$

A digraph G is called balanced [9] if $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$ for all $i \in V$. In other words, a digraph is balanced if and only if the total weight of edges entering a vertex and leaving the same vertex are equal for all vertices. By definition, any undirected graph is balanced. An important property of balanced digraphs is that $\mathbf{1} = (1, \cdots, 1)^T \in \mathbb{R}^n$ is a left eigenvector of their Laplacian, i.e., $\mathbf{1}^T L = 0$.

We consider a continuous-time system consisting of n agents, which is depicted by a digraph G, with dynamics given by

$$\dot{x}_i(t) = u_i(t) \tag{2}$$

for $i = 1, \cdots, n$, where $x_i(t) \in \mathbb{R}$ is the state of agent (or node) i at time t, and $u_i(t) \in \mathbb{R}$

is the control input (or protocol) at time t. Let $x = (x_1, \cdots, x_n)^T$. We say system (2) achieves average consensus asymptotically if for any initial value $x(0) \in \mathbb{R}^n$, the states of agents satisfy $x_i(t) \to \mathrm{Ave}(x(0)) := (\sum_{i=1}^n x_i(0))/n$ as $t \to \infty$ for each $i \in V$.

In a network with uncertain topology and multiple time-varying delay, we consider the following protocol

$$u_i(t) = \sum_{m=1}^{M} \sum_{j \in N_i} (a_{ij} + \Delta a_{ij}^{(m)}(t))(x_j(t - \tau_m(t)) - x_i(t - \tau_m(t))), \tag{3}$$

where $\Delta a_{ij}^{(m)}(t)$ with $\Delta a_{ii}^{(m)}(t) = 0$ represent system uncertainties, and $\tau_m(t)$ $(m = 1, \cdots, M)$ are time-varying communication delays. Let $\Delta L_m(t) = (\Delta l_{ij}^{(m)}(t))$ be the uncertain Laplacian matrix with

$$\Delta l_{ij}^{(m)}(t) = \begin{cases} -\Delta a_{ij}^{(m)}(t), & i \neq j, \\ \sum_{k=1}^n \Delta a_{ik}^{(m)}(t), & i = j. \end{cases}$$

Thus we can rewrite the system (2) with (3) in a compact form as

$$\dot{x}(t) = -\sum_{m=1}^{M} (L + \Delta L_m(t)) x(t - \tau_m(t)) \tag{4}$$

for $t \geq 0$. We assume the time delays $\tau_m(t)$ satisfy

(A1) $0 \leq \tau_m(t) \leq \delta_m$, $\dot{\tau}_m(t) \leq \gamma_m < 1$ for $t \geq 0$ and $m = 1, \cdots, M$;

or

(A2) $0 \leq \tau_m(t) \leq \delta_m$ for $t \geq 0$ and $m = 1, \cdots, M$.

In a network of mobile agents, the following switched system is considered

$$\dot{x}(t) = -\sum_{m=1}^{M} (L^{(k)} + \Delta L_m^{(k)}(t)) x(t - \tau_m(t)), \quad k = \sigma(t), \tag{5}$$

where $L^{(k)}$ is the Laplacian matrix of the digraph $G^{(k)}$ (sited on the same node set V) with $k = 1, 2, \cdots, \Xi$, and $\Delta L_m^{(k)}(t)$ is the corresponding uncertain Laplacian matrix defined as above. The map $\sigma(t) : [0, \infty) \to \{1, 2, \cdots, \Xi\}$ is a switching signal that determines the network topology at time t from the collection $\{G^{(1)}, \cdots, G^{(\Xi)}\}$.

Recall that a digraph is strongly connected if, between every pair of distinct vertices, there is a directed path. On the other hand, a digraph is referred to as weakly connected if it is connected when viewed as a graph (replacing directed edges by undirected ones). An interesting result is that a balanced digraph is weakly connected if and only if it is strongly connected [25]. Let $I_n \in \mathbb{R}^{n\times n}$ be the identity matrix. For two symmetric matrices X and Y with the same dimension, we say $X > Y$ if $X - Y$ is positive definite. To simplify the notations, we use $*$ to represent the elements below the main diagonal of a symmetric matrix. The following lemmas will be useful in our later derivations.

Lemma 1. *Assume that G with the Laplacian L is a weakly connected and balanced digraph, then we have*

$$U^T(L+L^T)U > 0,$$

where

$$U = \begin{bmatrix} I_{n-1} \\ U_0 \end{bmatrix} \in \mathbb{R}^{n\times(n-1)}, \quad U_0 = (-1, -1, \cdots, -1).$$

By our above comments and the fact that the digraph with the Laplacian matrix $L+L^T$ is strongly connected and balanced, Lemma 1 can be proved with the same reasoning as in [15, Lemma 2]. Here, we omit the proof.

Lemma 2. (Schur complement, [26]) *Let R, P, Q be given matrices such that $Q > 0$. Then*

$$\begin{bmatrix} P & R \\ R^T & -Q \end{bmatrix} < 0$$

if and only if $P + RQ^{-1}R^T < 0$.

The next lemma can be proved by using Schur complement.

Lemma 3. ([27]) *For any real differentiable vector function $x(t) \in \mathbb{R}^n$ and any $n \times n$ constant matrix $W = W^T > 0$, we have the following inequality*

$$(x(t) - x(t-\tau(t)))^T W (x(t) - x(t-\tau(t))) \leq \delta \int_{t-\tau(t)}^{t} \dot{x}(s)^T W \dot{x}(s) ds,$$

for $t \geq 0$ and $0 \leq \tau(t) \leq \delta$.

Lemma 4. ([28]) *Let P, Q, F be real matrices of appropriate dimensions with $F^T F \leq I$. Then for any $h > 0$, we have*

$$PFQ + Q^T F^T P^T \leq h^{-1} P P^T + h Q^T Q.$$

3 Consensus with fixed topology

This section focuses on the convergence analysis of system (4) with fixed interconnection topology G. Some feasible linear matrix inequalities will be established to determine the allowable upper bounds of multiple delays and uncertainties that guarantee the average consensus of the system.

We assume that the digraph $G(t)$ with Laplacian matrix $L_{m,t} := L + \Delta L_m(t)$ is balanced for $t \geq 0$ and $m = 1, \cdots, M$. Notice that $\mathbf{1}^T L_{m,t} = 0$ and hence $\mathbf{1}^T \dot{x}(t) = 0$ by (4). Therefore, $\bar{x}_0 := \mathrm{Ave}(x(0))$ is an invariant quantity, which allows the following decomposition of x

$$x = \bar{x}_0 \mathbf{1} + \varepsilon, \tag{6}$$

where $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)^T$ satisfies $\mathbf{1}^T \varepsilon = 0$. The vector ε is called the disagreement (or error) vector, and its time evolution follows

$$\dot{\varepsilon}(t) = -\sum_{m=1}^{M} L_{m,t} \varepsilon(t - \tau_m(t)). \tag{7}$$

Theorem 1. *Assume that (A1) holds and $G = (V, E, A)$ is a weakly connected and balanced digraph. If*

$$\left(\sum_{m=1}^{M} \Delta L_m(t)\right)\left(\sum_{m=1}^{M} \Delta L_m(t)\right)^T + \sum_{m=1}^{M} \Delta L_m(t) \Delta L_m(t)^T \leq \alpha^2 I_n, \tag{8}$$

then for any $0 \leq \gamma_m < 1$ $(m = 1, \cdots, M)$ and appropriate $\alpha > 0$, there exist $\delta_1, \cdots, \delta_M > 0$ such that system (4) achieves average consensus asymptotically. In particular, the allowable δ_m $(m = 1, \cdots, M)$ can be obtained from the following feasible LMI

$$\Psi = \begin{bmatrix} \Psi_{1,1} + \Omega & \Psi_{1,2} & \Psi_{1,3} & \cdots & \Psi_{1,M+1} & \Psi_{1,M+2} & U^T \\ * & \Psi_{2,2} + \Omega & 0 & \cdots & 0 & \Psi_{2,M+2} & 0 \\ * & * & \Psi_{3,3} + \Omega & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots \\ * & * & \cdots & * & \Psi_{M+1,M+1} + \Omega & \Psi_{M+1,M+2} & 0 \\ * & * & \cdots & \cdots & * & -(\sum_{m=1}^{M} \delta_m)^{-1} Q & Q \\ * & * & \cdots & \cdots & * & * & -hI_n \end{bmatrix} < 0, \tag{9}$$

where

$$\Psi_{1,1} = U^T \left(\sum_{m=1}^{M} (\gamma_m P - L - L^T)\right) U, \quad \Omega = \alpha^2 h U^T U,$$

$$\Psi_{1,m+1} = U^T(L + (1-\gamma_m)P)U, \quad \Psi_{m+1,m+1} = U^T((\gamma_m - 1)P - \delta_m^{-1}Q)U,$$

$$\Psi_{1,M+2} = -MU^TL^TQ, \quad \Psi_{m+1,M+2} = U^TL^TQ, \quad m = 1, \cdots, M,$$

U is defined as in Lemma 1, $P \geq 0$ and $Q > 0$ are $n \times n$ matrices to be determined, and $h > 0$ is a constant.

Proof. First, we show that (9) is always feasible for any $0 \leq \gamma_m < 1$ $(m = 1, \cdots, M)$ and suitable $\alpha > 0$ under the assumptions of Theorem 1. In other words, there exist some matrices $P \geq 0$, $Q > 0$, constants $h > 0$, $\alpha > 0$, and $\delta_m > 0$ $(m = 1, \cdots, M)$ such that (9) holds for any $0 \leq \gamma_m < 1$ $(m = 1, \cdots, M)$. Since G is weakly connected and balanced, we have $U^T(L + L^T)U > 0$ by Lemma 1. Take $P = 0$ and $Q = I_n$. Lemma 2 implies that (9) is tantamount to

$$\begin{bmatrix} MU^T(\alpha^2hM^{-1}I_n - L - L^T)U & U^TLU & \cdots & U^TLU & -MU^TL^T \\ * & (\alpha^2h - \delta_1^{-1})U^TU & 0 & 0 & U^TL^T \\ \vdots & * & \ddots & 0 & \vdots \\ * & * & * & (\alpha^2h - \delta_M^{-1})U^TU & U^TL^T \\ * & * & \cdots & * & (\sum_{m=1}^M \delta_m)^{-1}I_n \end{bmatrix}$$

$$+h^{-1}\begin{bmatrix} U^T \\ 0 \\ \vdots \\ 0 \\ I_n \end{bmatrix}[U \ 0 \ \cdots \ 0 \ I_n] < 0. \tag{10}$$

Since $U^T(L + L^T)U > 0$ and $U^TU > 0$, by choosing suitable h, α, δ_m and repeatedly employing Lemma 2, we can easily see that (10) holds. Therefore, (9) is always feasible for any $0 \leq \gamma_m < 1$ and suitable $\alpha > 0$.

Next, we will show that for any $0 \leq \gamma_m < 1$ and appropriate $\alpha > 0$, the multi-agent system (4) achieves average consensus asymptotically for $\tau_m \leq \delta_m$ for $m = 1, \cdots, M$, when (9) holds. It follows from (6) that it suffices to show the asymptotical stability of the zero solution of error system (7). We define the Lyapunov function as follows

$$V(t) = \varepsilon(t)^T\varepsilon(t) + \sum_{m=1}^M \int_{t-\tau_m(t)}^t \varepsilon(s)^TP\varepsilon(s)\mathrm{d}s + \sum_{m=1}^M \int_{t-\delta_m}^t (s - t + \delta_m)\dot{\varepsilon}(s)^TQ\dot{\varepsilon}(s)\mathrm{d}s. \tag{11}$$

Let $\eta_m(t) := \varepsilon(t) - \varepsilon(t - \tau_m(t))$. We rewrite the system (7) as

$$\dot{\varepsilon}(t) = -\sum_{m=1}^M L_{m,t}\varepsilon(t) + \sum_{m=1}^M L_{m,t}\eta_m(t).$$

Hence, by using the assumption (A1) and $\varepsilon(t-\tau_m(t)) = \varepsilon(t) - \eta_m(t)$, we obtain

$$\begin{aligned}
\dot{V}(t) \;\le\;& -\sum_{m=1}^{M} \varepsilon(t)^T (L_{m,t} + L_{m,t}^T)\varepsilon(t) + 2\sum_{m=1}^{M} \varepsilon(t)^T L_{m,t}\eta_m(t) + \sum_{m=1}^{M} \varepsilon(t)^T P \varepsilon(t) \\
& -\sum_{m=1}^{M} (1-\gamma_m)\varepsilon(t-\tau_m(t))^T P \varepsilon(t-\tau_m(t)) + \sum_{m=1}^{M} \delta_m \dot{\varepsilon}(t)^T Q \dot{\varepsilon}(t) \\
& -\sum_{m=1}^{M} \int_{t-\tau_m(t)}^{t} \dot{\varepsilon}(s)^T Q \dot{\varepsilon}(s) \mathrm{d}s \\
=\;& \sum_{m=1}^{M} \varepsilon(t)^T (\gamma_m P - L_{m,t} - L_{m,t}^T)\varepsilon(t) + \sum_{m=1}^{M} (\gamma_m - 1)\eta_m(t)^T P \eta_m(t) \\
& +2\sum_{m=1}^{M} \varepsilon(t)^T (L_{m,t} + (1-\gamma_m)P)\eta_m(t) + \left(\sum_{m=1}^{M} \delta_m\right) y(t)^T \\
& \cdot \left[-\sum_{m=1}^{M} L_{m,t} \; L_{1,t} \; L_{2,t} \; \cdots \; L_{M,t}\right]^T Q \left[-\sum_{m=1}^{M} L_{m,t} \; L_{1,t} \; L_{2,t} \; \cdots \; L_{M,t}\right] y(t) \\
& -\sum_{m=1}^{M} \int_{t-\tau_m(t)}^{t} \dot{\varepsilon}(s)^T Q \dot{\varepsilon}(s) \mathrm{d}s, \qquad (12)
\end{aligned}$$

where $y(t)^T = (\varepsilon(t)^T, \eta_1(t)^T, \eta_2(t)^T, \cdots, \eta_M(t)^T) \in \mathbb{R}^{1\times(M+1)n}$. Since $Q > 0$, Lemma 3 implies that

$$-\sum_{m=1}^{M} \int_{t-\tau_m(t)}^{t} \dot{\varepsilon}(s)^T Q \dot{\varepsilon}(s) \mathrm{d}s \le -\sum_{m=1}^{M} \delta_m^{-1} \eta_m(t)^T Q \eta_m(t).$$

Inserting this into (12), we obtain

$$\dot{V}(t) \le y(t)^T \Phi(t) y(t), \qquad (13)$$

where $\Phi(t) \in \mathbb{R}^{(M+1)n\times(M+1)n}$ is given by

$$\Phi(t) = \begin{bmatrix} \sum_{m=1}^{M}(\gamma_m P - L_{m,t} - L_{m,t}^T) & L_{1,t} + (1-\gamma_1)P & \cdots & L_{M,t} + (1-\gamma_M)P \\ * & (\gamma_1 - 1)P - \delta_1^{-1}Q & 0 & 0 \\ \vdots & * & \ddots & 0 \\ * & * & * & (\gamma_M - 1)P - \delta_M^{-1}Q \end{bmatrix}$$
$$+ \left(\sum_{m=1}^{M} \delta_m\right) \left[-\sum_{m=1}^{M} L_{m,t} \; L_{1,t} \; L_{2,t} \; \cdots \; L_{M,t}\right]^T Q \left[-\sum_{m=1}^{M} L_{m,t} \; L_{1,t} \; L_{2,t} \; \cdots \; L_{M,t}\right].$$

Define $\hat{\varepsilon} := (\varepsilon_1, \cdots, \varepsilon_{n-1})^T$. We see that $\varepsilon = U\hat{\varepsilon}$, where U is defined as in Lemma 1. By using (13) we obtain

$$\dot{V}(t) \le \hat{y}(t)^T W^T \Phi(t) W \hat{y}(t), \qquad (14)$$

where $\hat{y}(t)^T = (\hat{\varepsilon}(t)^T, \hat{\eta}_1(t)^T, \cdots, \hat{\eta}_M(t)^T) \in \mathbb{R}^{1\times(M+1)(n-1)}$, $\hat{\eta}_m(t) = \hat{\varepsilon}(t) - \hat{\varepsilon}(t-\tau_m(t))$, for $m = 1, \cdots, M$, and $W = \begin{bmatrix} U & 0 & 0 \\ * & \ddots & 0 \\ * & * & U \end{bmatrix} \in \mathbb{R}^{(M+1)n\times(M+1)(n-1)}$.

By straightforward calculation, we derive that

$$W^T\Phi(t)W =$$

$$\begin{bmatrix} \Psi_{1,1} - U^T\left(\sum_{m=1}^M \Delta L_m(t) + \Delta L_m(t)^T\right)U & \Psi_{1,2} + U^T\Delta L_1(t)U & \cdots & \Psi_{1,M+1} + U^T\Delta L_M(t)U \\ * & \Psi_{2,2} & 0 & 0 \\ \vdots & * & \ddots & 0 \\ * & * & * & \Psi_{M+1,M+1} \end{bmatrix}$$

$$+\left(\sum_{m=1}^M \delta_m\right)\begin{bmatrix} \Psi_{1,M+2} - U^T(\sum_{m=1}^M \Delta L_m(t)^T)Q \\ \Psi_{2,M+2} + U^T\Delta L_1(t)^TQ \\ \vdots \\ \Psi_{M+1,M+2} + U^T\Delta L_M(t)^TQ \end{bmatrix} Q^{-1} \begin{bmatrix} \Psi_{1,M+2} - U^T(\sum_{m=1}^M \Delta L_m(t)^T)Q \\ \Psi_{2,M+2} + U^T\Delta L_1(t)^TQ \\ \vdots \\ \Psi_{M+1,M+2} + U^T\Delta L_M(t)^TQ \end{bmatrix}^T.$$

On the other hand, set

$$\hat{\Psi} = \begin{bmatrix} \Psi_{1,1} & \Psi_{1,2} & \cdots & \Psi_{1,M+1} & \Psi_{1,M+2} \\ * & \Psi_{22} & 0 & 0 & \Psi_{2,M+2} \\ \vdots & * & \ddots & 0 & \vdots \\ * & * & * & \Psi_{M+1,M+1} & \Psi_{M+1,M+2} \\ * & * & \cdots & * & -(\sum_{m=1}^M \delta_m)^{-1}Q \end{bmatrix} \in \mathbb{R}^{((M+1)(n-1)+n)\times((M+1)(n-1)+n)}.$$

Since

$$\begin{bmatrix} -U^T(\sum_{m=1}^M \Delta L_m(t)^T)U & 0 & -U^T(\sum_{m=1}^M \Delta L_m(t)^T)Q \\ U^T\Delta L_1(t)^TU & 0 & U^T\Delta_1(t)^TQ \\ \vdots & \vdots & \vdots \\ U^T\Delta L_M(t)^TU & 0 & U^T\Delta_M(t)^TQ \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -U^T & 0 & \cdots & \cdots & 0 \\ * & U^T & 0 & 0 & \vdots \\ \vdots & * & \ddots & 0 & \vdots \\ \vdots & * & * & U^T & 0 \\ * & \cdots & \cdots & * & 0 \end{bmatrix} \begin{bmatrix} \sum_{m=1}^M \Delta L_m(t)^T \\ \Delta L_1(t)^T \\ \vdots \\ \Delta L_M(t)^T \\ 0 \end{bmatrix} [U\ 0\ \cdots\ 0\ Q],$$

by Lemma 2 we obtain that $W^T\Phi(t)W < 0$ if and only if

$$\hat{\Psi} + \alpha \begin{bmatrix} -U^T & 0 & \cdots & \cdots & 0 \\ * & U^T & 0 & 0 & \vdots \\ \vdots & * & \ddots & 0 & \vdots \\ \vdots & * & * & U^T & 0 \\ * & \cdots & \cdots & * & 0 \end{bmatrix} F(t)^T [U\ 0\ \cdots\ 0\ Q]$$

$$+\alpha \begin{bmatrix} U^T \\ 0 \\ \vdots \\ 0 \\ Q \end{bmatrix} F(t) \begin{bmatrix} -U & 0 & \cdots & \cdots & 0 \\ * & U & 0 & 0 & \vdots \\ \vdots & * & \ddots & 0 & \vdots \\ \vdots & * & * & U & 0 \\ * & \cdots & \cdots & * & 0 \end{bmatrix} < 0, \qquad (15)$$

where $F(t) = \alpha^{-1}[\sum_{m=1}^{M} \Delta L_m(t)\ \Delta L_1(t)\ \cdots\ \Delta L_M(t)\ 0] \in \mathbb{R}^{n\times(M+2)n}$.

By virtue of Lemma 4 and (8), we see that

$$\alpha \begin{bmatrix} -U^T & 0 & \cdots & \cdots & 0 \\ * & U^T & 0 & 0 & \vdots \\ \vdots & * & \ddots & 0 & \vdots \\ \vdots & * & * & U^T & 0 \\ * & \cdots & \cdots & * & 0 \end{bmatrix} F(t)^T [U\ 0\ \cdots\ 0\ Q]$$

$$+\alpha \begin{bmatrix} U^T \\ 0 \\ \vdots \\ 0 \\ Q \end{bmatrix} F(t) \begin{bmatrix} -U & 0 & \cdots & \cdots & 0 \\ * & U & 0 & 0 & \vdots \\ \vdots & * & \ddots & 0 & \vdots \\ \vdots & * & * & U & 0 \\ * & \cdots & \cdots & * & 0 \end{bmatrix}$$

$$\leq \alpha^2 h \begin{bmatrix} U^TU & 0 & \cdots & \cdots & 0 \\ * & U^TU & 0 & 0 & \vdots \\ \vdots & * & \ddots & 0 & \vdots \\ \vdots & * & * & U^TU & 0 \\ * & \cdots & \cdots & * & 0 \end{bmatrix} + h^{-1} \begin{bmatrix} U^T \\ 0 \\ \vdots \\ 0 \\ Q \end{bmatrix} [U\ 0\ \cdots\ 0\ Q].$$

Therefore, by using Lemma 2, we obtain that (9) yields (15), and hence $\dot{V}(t) < 0$ for $t \geq 0$. There exists a constant $\beta > 0$ such that $\dot{V}(t) \leq -\beta\hat{\varepsilon}(t)^T\hat{\varepsilon}(t)$. Moreover, by the

Cr-inequality, we arrive at $\hat{\varepsilon}(t)^T\hat{\varepsilon}(t) \geq \varepsilon(t)^T\varepsilon(t)/n$, and thus,

$$\dot{V}(t) \leq -\frac{\beta}{n}\varepsilon(t)^T\varepsilon(t).$$

This implies that the trivial solution of error system (7) is asymptotically stable (see e.g. [10, p. 132]). The proof of Theorem 1 is complete. $\square$

Note that the above analysis can be easily adapted to the general system

$$\dot{x}(t) = \sum_{m=1}^{M} a_m(L + \Delta L_m(t))x(t - \tau_m(t)),$$

where $a_m > 0$ $(m = 1, \cdots, M)$.

If some $\gamma_m \geq 1$ or nothing is known about the derivative of the multiple time delays, instead of (11), we may construct the Lyapunov function as

$$V(t) = \varepsilon(t)^T\varepsilon(t) + \sum_{m=1}^{M}\int_{t-\delta_m}^{t}(s - t + \delta_m)\dot{\varepsilon}(s)^T Q\dot{\varepsilon}(s)\mathrm{d}s.$$

The following corollary can be obtained similarly.

Corollary 1. *Assume that (A2) holds and $G = (V, E, A)$ is a weakly connected and balanced digraph. If*

$$\left(\sum_{m=1}^{M}\Delta L_m(t)\right)\left(\sum_{m=1}^{M}\Delta L_m(t)\right)^T + \sum_{m=1}^{M}\Delta L_m(t)\Delta L_m(t)^T \leq \alpha^2 I_n,$$

then for appropriate $\alpha > 0$, there exist $\delta_1, \cdots, \delta_M > 0$ such that system (4) achieves average consensus asymptotically. In particular, the allowable δ_m $(m = 1, \cdots, M)$ can be obtained from the following feasible LMI

$$\begin{bmatrix} U^T(\alpha^2 hI_n - ML - ML^T)U & U^TLU & \cdots & U^TLU & -MU^TL^TQ & U^T \\ * & U^T(\alpha^2 hI_n - \delta_1^{-1}Q)U & 0 & 0 & U^TL^TQ & 0 \\ \vdots & * & \ddots & 0 & \vdots & \vdots \\ * & * & * & U^T(\alpha^2 hI_n - \delta_M^{-1}Q)U & U^TL^TQ & 0 \\ * & * & \cdots & * & -(\sum_{m=1}^{M}\delta_m)^{-1}Q & Q \\ * & * & \cdots & * & * & -hI_n \end{bmatrix} \quad (16)$$

where U is defined as in Lemma 1, $Q > 0$ is an $n \times n$ matrix to be determined, and $h > 0$ is a constant.

4 Consensus with switching topology

Consider the switching system (5). We assume that the digraphs $G^{(k)}(t)$ with Laplacian matrices $L^{(k)}_{m,t} := L^{(k)} + \Delta L^{(k)}_m(t)$ are balanced for $t \geq 0$, $m = 1, \cdots, M$ and $k = 1, \cdots, \Xi$. Therefore, $\bar{x}_0 = \mathrm{Ave}(x(0))$ is also an invariant quantity, which allows the decomposition (6). Thus, the error dynamics follows

$$\dot{\varepsilon}(t) = -\sum_{m=1}^{M} L^{(k)}_{m,t}\varepsilon(t - \tau_m(t)), \quad k = \sigma(t) \in \{1, 2, \cdots, \Xi\}. \tag{17}$$

Theorem 2. *Assume that (A1) holds and $G^{(k)}$ are weakly connected and balanced digraphs. If*

$$\left(\sum_{m=1}^{M} \Delta L^{(k)}_m(t)\right)\left(\sum_{m=1}^{M} \Delta L^{(k)}_m(t)\right)^T + \sum_{m=1}^{M} \Delta L^{(k)}_m(t)\Delta L^{(k)}_m(t)^T \leq \alpha_k^2 I_n, \quad k = 1, \cdots, \Xi,$$

then for any $0 \leq \gamma_m < 1$ ($m = 1, \cdots, M$) and appropriate constants $\alpha_k > 0$, there exist $\delta_1, \cdots, \delta_M > 0$ such that system (5) achieves average consensus asymptotically under any switching signal. In particular, the allowable δ_m ($m = 1, \cdots, M$) can be obtained from the following feasible LMIs

$$\begin{bmatrix} \Psi^{(k)}_{1,1} + \Omega^{(k)} & \Psi^{(k)}_{1,2} & \cdots & \Psi^{(k)}_{1,M+1} & \Psi^{(k)}_{1,M+2} & U^T \\ * & \Psi^{(k)}_{2,2} + \Omega^{(k)} & 0 & 0 & \Psi^{(k)}_{2,M+2} & 0 \\ \vdots & * & \ddots & 0 & \vdots & \vdots \\ * & * & * & \Psi^{(k)}_{M+1,M+1} + \Omega^{(k)} & \Psi^{(k)}_{M+1,M+2} & 0 \\ * & * & \cdots & * & -(\sum_{m=1}^{M}\delta_m)^{-1}Q & Q \\ * & * & \cdots & * & * & -h_k I_n \end{bmatrix} < 0, \tag{18}$$

for $k = 1, \cdots, \Xi$, where $\Psi^{(k)}_{i,j}$ and $\Omega^{(k)}$ are taken from Theorem 1 by replacing L, α and h with $L^{(k)}$, α_k and h_k, respectively, $P \geq 0$ and $Q > 0$ are $n \times n$ matrices to be determined.

Proof. Analogous to Theorem 1, we can show that (18) is always feasible for any $k = 1 \cdots, \Xi$.

Next, we show that, for any $0 \leq \gamma_m < 1$ and appropriate $\alpha_k > 0$, the multi-agent system (5) achieves average consensus asymptotically for any switching signal $\sigma(t)$ and communication delay $\tau_m \leq \delta_m$ for $m = 1, \cdots, M$, where δ_m's are determined by (18). It suffices to prove that the zero solution of switching system (17) is asymptotically stable for any switching signal $\sigma(t)$ if (18) holds. Let $V(t)$ defined in (11) be the common

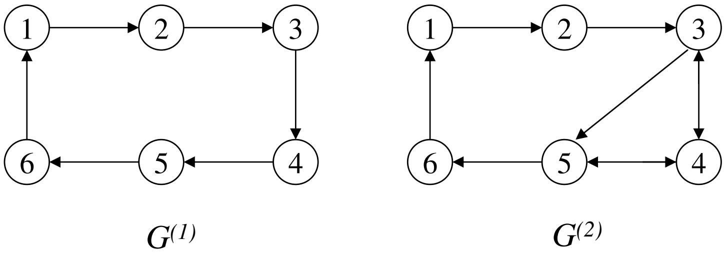


Figure 1: Two weakly connected and balanced digraphs.

Lyapunov function and assume that the kth subsystem is activated at time t, i.e., $\sigma(t) = k$. Consider the right-hand side derivative of $V(t)$, denoted by $D_+V(t)$, along the trajectories of system (17). Similar to the argument in Theorem 1, (18) implies that there exists $\beta_k > 0$ such that $D_+V(t) \leq -(\beta_k/n)\varepsilon(t)^T\varepsilon(t)$. Set $\beta := \min_{k=1,\cdots,\Xi}\beta_k$. We have $D_+V(t) \leq -(\beta/n)\varepsilon(t)^T\varepsilon(t)$ for any switching signal $\sigma(t)$. Hence, we have that the zero solution of (17) is asymptotically stable for any switching signal. The proof is then complete. □

Similar to Corollary 1, we obtain the following corollary for switching system.

Corollary 2. *Assume that (A2) holds and $G^{(k)}$ are weakly connected and balanced digraphs. If*

$$\left(\sum_{m=1}^{M}\Delta L_m^{(k)}(t)\right)\left(\sum_{m=1}^{M}\Delta L_m^{(k)}(t)\right)^T + \sum_{m=1}^{M}\Delta L_m^{(k)}(t)\Delta L_m^{(k)}(t)^T \leq \alpha_k^2 I_n, \quad k = 1, \cdots, \Xi,$$

then for appropriate $\alpha_k > 0$, there exist $\delta_1, \cdots, \delta_M > 0$ such that system (5) achieves average consensus asymptotically under any switching signal. In particular, the allowable δ_m $(m = 1, \cdots, M)$ can be obtained from the feasible LMIs (18) with $P = 0$.

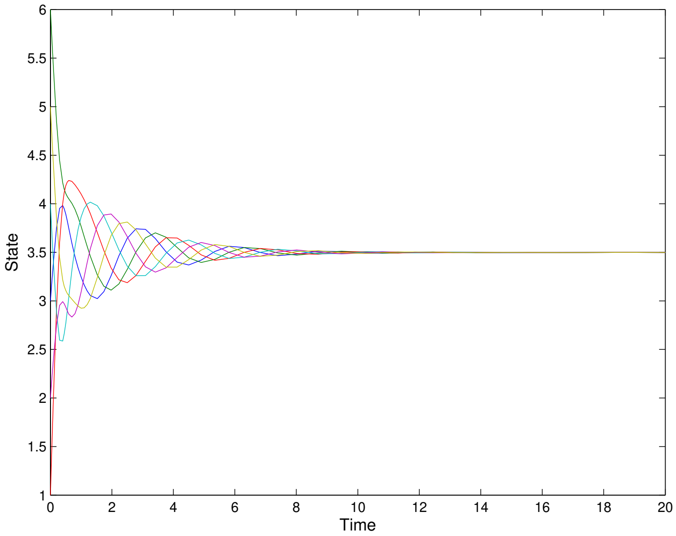
5 Numerical examples

We present two examples in this section.

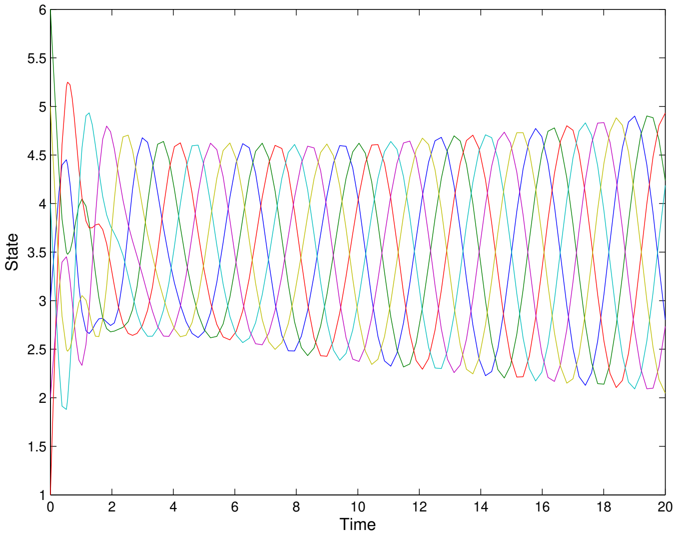
Example 1. Consider a digraph $G := G^{(1)}$ with 0-1 weights in Fig. 1. We set $\alpha = 0.1$, $M = 2$ and $\gamma_m = 0$ for $m = 1, 2$. Using the Matlab's LMI Toolbox [23] to solve (9) and (16) we obtain the estimates of allowable time delay as: $\delta_1 + \delta_2 \leq 0.39$ and $\delta_1 + \delta_2 \leq 0.31$ (when nothing has been known about the derivatives of the delays $\tau_1(t)$ or $\tau_2(t)$), respectively.

On the other hand, let $\Delta L_1(t) = 0.1\alpha\sin(t)L$, $\Delta L_2(t) = 0.1\alpha\sin(2t)L$, and the initial condition be $x(0) = (3, 6, 1, 4, 2, 5)^T$. Let the time delays be $\tau_1(t) = 0.1$ and $\tau_2(t) = 0.2$.

From Theorem 1, we obtain that system (4) achieves average consensus asymptotically. This agrees with the simulation result shown in Fig. 2(a). In Fig. 2(b), we set the time delays as $\tau_1(t) = 0.26$ and $\tau_2(t) = 0.28$. It is seen that the system is divergent in this case.



(a) $\tau_1(t) = 0.1, \tau_2(t) = 0.2$



(b) $\tau_1(t) = 0.26, \tau_2(t) = 0.28$

Figure 2: States of all nodes in the fixed network topology $G^{(1)}$.

Example 2. Consider two digraphs $G^{(1)}$ and $G^{(2)}$ with 0-1 weights shown in Fig. 1. We set $\alpha_1 = 0.2$, $\alpha_2 = 0.1$, $M = 2$ and $\gamma_m = 0$ for $m = 1, 2$. Using the Matlab's LMI Toolbox to solve (18), we obtain the estimates of allowable time delay as: $\delta_1 + \delta_2 \leq 0.26$ and $\delta_1 + \delta_2 \leq 0.17$ (when nothing has been known about the derivatives of the delays $\tau_1(t)$

or $\tau_2(t)$), respectively.

Let $\Delta L_1^{(1)}(t) = \Delta L_2^{(1)}(t) = 0.1\alpha_1 \sin(t)L$, $\Delta L_1^{(2)}(t) = \Delta L_2^{(2)}(t) = 0.05\alpha_2 \sin(2t)L$, and the initial condition be $x(0) = (3, 6, 1, 4, 2, 5)^T$. Set the time delays as $\tau_1(t) = 0.08$ and $\tau_2(t) = 0.12$. From Theorem 2, we obtain that system (5) achieves average consensus asymptotically for arbitrary switching signal $\sigma(t)$. In Fig. 3, we present the corresponding simulation result under random switching signal, which matches our theoretical prediction.

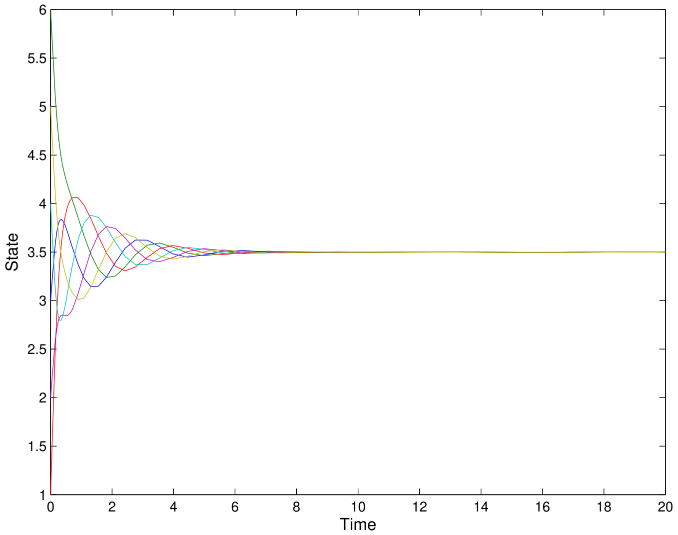


Figure 3: States of all nodes in the switching network topology $\{G^{(1)}, G^{(2)}\}$.

6 Conclusion

This paper has addressed the average consensus problems in continuous-time multi-agent systems with uncertain topologies as well as multiple time-varying communication delays. Directed networks with both fixed and switching topologies are considered. Based on LMI approach, we showed that all the nodes in the network can reach average consensus asymptotically if the network topology is weakly connected and balanced. Some feasible LMIs have also been derived to obtain the maximal allowable upper bounds of the delays and uncertainties. Numerical examples and simulations showed the effectiveness of our theoretical results.

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