Consensus in a class of nonlinear multi-agent systems with time-varying topologies

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Abstract

Most consensus protocols for multi-agent systems presented in the past do not consider nonlinear factors in the exchange of information between the agents. In this paper, we study a discrete-time nonlinear protocol for time-varying multi-agent systems described by stochastic matrices. Based on stochastic matrix theory, we provide convergence conditions (i.e., consensus criteria) for the systems subject to exponential decaying disturbances as well as general disturbances. Under the proposed framework, different convergence speeds, such as exponential convergence, can be achieved.

MSC 2000: 93C10, 15A60.

Keywords: consensus problem; multi-agent system; nonlinear protocol; stochastic matrix; disturbance.

1 Introduction

Distributed decision-making for coordination of networks of dynamic agents involving information flow enabling enhanced operational effectiveness via cooperative control in networked autonomous systems has attracted a considerable research effort in recent years. Vicsek et. al. [24] introduces a simple and popular model for self-driven particles alignment problem in which each agent updates its heading based on the average of its own heading and its neighbors'. By simulation, all agents are observed to move in the same direction eventually. To explain the above phenomenon, a discrete-time consensus protocol is proposed in [6] based on stochastic matrix theory. A simple continuous-time consensus protocol is given in [13]. It is shown that, for balanced communication topologies, this protocol solves the average consensus problem, i.e., the states of all agents converge to the average of the initial values exponentially fast. Moreover, consensus problems under general connected directed graphs are discussed in [15, 16, 21]. Recent reviews on consensus and coordination can be found in [2, 12] and [22].

Consensus problems in multi-agent systems cover a very broad spectrum of applications including motion coordination [4], sensor fusion [14], rendezvous problems [8], congestion control [1], clock synchronization [17] and distributed Kalman filters [11], to name just a few. In many applications involving multi-agent systems, individual agents need to share information of the system objectives and are required to agree on certain quantities of interest. Distributed consensus algorithms over dynamic neighbor-to-neighbor interaction topologies guarantee agreement between agents for a given coordination task. In reality agents update their information state usually in a nonlinear manner based on the information states of the neighboring agents. However, most papers until now in the literature mainly concern the consensus problem under linear protocols, with the connection topologies time-varying, state-dependent (see e.g. [6, 13, 15, 16, 20]). Even in those papers investigating nonlinear protocols, like [3, 5, 7, 10, 19, 23], communication disturbances are typically not considered. However, disturbances and noises are inevitable in real systems.

In this paper, we will look at the consensus problem in time-varying systems under communication disturbances. Based on a novel time synchronization linear protocol [17], we propose a nonlinear consensus protocol which is shown to converge exponentially fast under some mild conditions. Our proof will largely rely on the properties of stochastic matrices [18].

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries and state our consensus result. In Section 3, we give the proof and in Section 4, we discuss the convergence speed of the system and conclude the paper.

2 Preliminaries and main result

Communication network is usually modeled by a directed graph $G = (V, E)$, where $V = \{1, 2, \cdots, n\}$ represents the vertices in the network and the edge set E represents the available directed communication links, i.e., $(i, j) \in E$ if vertex j sends information to vertex i. Let $N_i = \{j | (i, j) \in E, i \neq j\}$ be the set of neighbors of i. A matrix $P = (P_{ij}) \in \mathbb{R}^{n\times n}$ is said stochastic if $P_{ij} \geq 0$ and $\sum_{j=1}^{n} P_{ij} = 1$ for any $i \in V$. For convenience, we denote the previous constraints by $P \geq 0$ and $P\mathbf{1} = \mathbf{1}$ where $\mathbf{1} = (1, \cdots, 1)^T \in \mathbb{R}^n$. Given a stochastic matrix P, its associated graph is defined as $G_P = (V, E_P)$ where $(i, j) \in E_P$ if and only if $P_{ij} > 0$. It is easy to see that G_P is a complete graph if and only if $P > 0$. For a vector $x \in \mathbb{R}^n$ and a matrix $X \in \mathbb{R}^{n\times n}$, we denote by $\|x\|$ and $\|X\|$ their infinity norm and induced infinity norm, respectively.

We will need the following technical lemma, which introduces a Lyapunov candidate and its property in the context of systems whose dynamics is described by a stochastic matrix.

Lemma 1.([18]) *Let $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ and $P = (P_{ij}) \in \mathbb{R}^{n\times n}$ be a stochastic matrix. Let $V(x) = \max(x) - \min(x)$, then we have*

$$V(Px) \leq \big(1 - \max_{j=1}^{n} \min_{i=1}^{n} \{P_{ij}\}\big)V(x), \tag{1}$$

where $\max(x) = \max_{i=1}^{n}\{x_i\}$ *and* $\min(x) = \min_{i=1}^{n}\{x_i\}$.

Note that $V(x)$ is nonnegative and has the property that $V(x) = 0$ if and only if $x = \alpha\mathbf{1}$ for some $\alpha \in \mathbb{R}$. Moreover, it is obvious that $\max_{j=1}^{n} \min_{i=1}^{n}\{P_{ij}\} > 0$ if and only if there is at least one column of P whose elements are all positive. In other words, there is at least one vertex which is directly connected to all the other vertices.

Our main result (Theorem 1) is the convergence of nonlinear iterative stochastic matrices in time-varying systems subject to exponential decaying disturbances, which extends a discrete-time linear clock synchronization protocol in [17].

Theorem 1. *Consider the following system*

$$x(k+1) = P(k)x(k) + \triangle(k)f_k(x(k)) + v(k), \tag{2}$$

where $x(k) \in \mathbb{R}^n$, $P(k) \in \mathbb{R}^{n\times n}$ are stochastic matrices, $f_k : \mathbb{R}^n \to \mathbb{R}^n$, and $\triangle(k) \in \mathbb{R}^{n\times n}$ and $v(k) \in \mathbb{R}^n$ are unknown. Suppose that $\|\triangle(k)\| \leq a\rho^k$, $\|v(k)\| \leq a\rho^k$ and $\|f_k(x)\| \leq b\|x\|$ for some $a, b > 0$ and $0 \leq \rho < 1$. If there exists an integer K such that $P((l+1)K - 1) \cdots P(lK + 1)P(lK) \geq \varepsilon > 0$ for all $l = 0, 1, \cdots$, then there is some $\alpha \in \mathbb{R}$ such that

$$\lim_{k\to\infty} x(k) = \alpha\mathbf{1} \tag{3}$$

exponentially fast.

We remark that the strictly positiveness of products of time-varying stochastic matrices $P(k)$ in Theorem 1 is the necessary and sufficient condition for a connected communication graph over an arbitrary but finite time window of length H, i.e., $\cup_{s=h_l}^{h_{l+1}} G_{P(s)}$ for all $l = 0, 1, \cdots$, where $0 = h_0 < h_1 < \cdots < h_l < \cdots$ and $h_{l+1} - h_l < H$ (c.f. [9, 18]).

In additon, we note that the uniform boundedness of operators $\{f_k\}$ is not a necessary condition for convergence to consensus, i.e., $\lim_{k\to\infty} x(k) = \alpha\mathbf{1}$. More general conditions together with disturbances will be discussed in Section 4.

3 Proof of Theorem 1

We will first to show that $x(k)$ is bounded, i.e., $\|x\| \leq M$ for some $M > 0$. Note that $\|P(k)\| = 1$ for any $k \geq 0$. Therefore, we have

$$\begin{aligned}
\|x(k+1)\| &= \|P(k)x(k) + \triangle(k)f_k(x(k)) + v(k)\| \\
&\leq \|P(k)x(k)\| + \|\triangle(k)f_k(x(k))\| + \|v(k)\| \\
&\leq (1 + ab\rho^k)\|x(k)\| + a\rho^k \\
&\leq \gamma_{k,0}\|x(0)\| + \sum_{m=1}^{k} \gamma_{k,m} a\rho^{m-1} + a\rho^k \qquad (4)
\end{aligned}$$

where $\gamma_{k,m} = (1 + ab\rho^k)(1 + ab\rho^{k-1}) \cdots (1 + ab\rho^m)$ for $k \geq m$. The inequality (4) follows by induction. Since $\ln(1 + x) \leq x$ for any $x \geq 0$, we obtain

$$\begin{aligned}
1 \leq \gamma_{k,m} &\leq \gamma_{k,0} = e^{\ln \gamma_{k,0}} = e^{\sum_{m=0}^{k} \ln(1+ab\rho^m)} \\
&\leq e^{\sum_{m=0}^{k} ab\rho^m} \leq e^{\frac{ab}{1-\rho}} := \bar{\gamma}. \qquad (5)
\end{aligned}$$

Combining (4) and (5), we get

$$\begin{aligned}
\|x(k+1)\| &\leq \bar{\gamma}\|x(0)\| + \sum_{m=1}^{k} \bar{\gamma} a\rho^{m-1} + \bar{\gamma} a\rho^k \\
&\leq \bar{\gamma}\|x(0)\| + \sum_{m=0}^{\infty} \bar{\gamma} a\rho^m \\
&\leq \bar{\gamma}\|x(0)\| + \bar{\gamma}\frac{a}{1-\rho} := M, \qquad (6)
\end{aligned}$$

which implies that $\|x(k)\|$ is bounded for all k.

Next, we want to prove that $\lim_{k\to\infty} V(x(k)) = 0$ exponentially fast. Let $w(k) = \triangle(k)f_k(x(k)) + v(k)$ and $Q(k+h, h) = P(k+h) \cdots P(h+1)P(h)$, then (2) can be rewritten as

$$x(k+1) = P(k)x(k) + w(k), \qquad (7)$$

and more generally

$$\begin{aligned} x(k+h+1) &= Q(k+h,k)x(k) + Q(k+h,k+1)w(k) + \cdots \\ &\quad + Q(k+h,k+h)w(k+h-1) + w(k+h) \\ &:= Q(k+h,k)x(k) + \tilde{w}(k+h,k). \end{aligned} \tag{8}$$

Since $\|x\| \leq M$, we have $\|w(k)\| \leq \|\triangle(k)\|\|f_k(x(k))\| + \|v(k)\| \leq a(bM+1)\rho^k$. As $Q(k+h,k)$ is still a stochastic matrix, $\|Q(k+h,k)\| = 1$. Hence we get

$$\begin{aligned} \|\tilde{w}(k+h,k)\| &\leq \|Q(k+h,k+1)\|\|w(k)\| + \cdots + \|w(k+h)\| \\ &\leq a(bM+1)\rho^k \sum_{m=0}^{h} \rho^m \leq \frac{a(bM+1)\rho^k}{1-\rho}. \end{aligned} \tag{9}$$

Assume that there exists some integer K such that $\lim_{l\to\infty} x(lK) = \alpha\mathbf{1}$ exponentially fast. By letting $lK = k$ and $h+1 = K$ in (8) we have

$$x((l+1)K) = Q((l+1)K-1, lK)x(lK) + \tilde{w}((l+1)K-1, lK). \tag{10}$$

Therefore, by using (8), (9), (10) and the fact $P\mathbf{1} = \mathbf{1}$ for any stochastic matrix P, the above assumption yields that $\lim_{k\to\infty} \|x(k+h+1) - x(k)\| = 0$ exponentially fast for all $0 \leq h \leq K$. Hence the study of the convergence can be confined to the subsequence (10).

For the notation simplicity, we denote $x_l = x(lK)$, $Q_l = Q((l+1)K-1, lK)$ and $\tilde{w}_l = \tilde{w}((l+1)K-1, lK)$. By the hypotheses of the theorem, we get $Q_l \geq \varepsilon > 0$. Also, it follows from (9) that

$$\|\tilde{w}_l\| \leq \frac{a(bM+1)\rho^{lK}}{1-\rho} \leq \frac{a(bM+1)}{1-\rho}\rho^l. \tag{11}$$

Consequently, we have

$$\begin{aligned} V(x_{l+1}) &= \max(x_{l+1}) - \min(x_{l+1}) \\ &= \max(Q_l x_l + \tilde{w}_l) - \min(Q_l x_l + \tilde{w}_l) \\ &\leq \max(Q_l x_l) + \max(\tilde{w}_l) - \min(Q_l x_l) - \min(\tilde{w}_l) \\ &\leq (1-\varepsilon)V(x_l) + \frac{2a(bM+1)}{1-\rho}\rho^l, \end{aligned} \tag{12}$$

where we use the fact $V(Q_l x_l) \leq (1-\varepsilon)V(x_l)$ by Lemma 1, $\max(x) \leq \|x\|$ and $\min(x) \geq -\|x\|$. Let us define

$$z_{l+1} = (1-\varepsilon)z_l + \frac{2a(bM+1)}{1-\rho}\rho^l \tag{13}$$

with initial value $z_0 = V(x_0)$, then we are ready to get $V(x_l) \leq z_l$ for any $l \geq 0$ by induction. On the other hand, we can solve (13) by iteration to get

$$z_l = (1-\varepsilon)^l z_0 + \frac{2a(bM+1)}{1-\rho} \frac{(1-\varepsilon)^l - \rho^l}{1 - \frac{\rho}{1-\varepsilon}}, \tag{14}$$

which implies that $\lim_{l\to\infty} z_l = 0$ exponentially fast since $\varepsilon \in (0,1)$ and $\rho \in [0,1)$. Accordingly, we have $\lim_{l\to\infty} V(x_l) = 0$. By the above comments, we also have $\lim_{k\to\infty} V(x(k)) = 0$, which implies that

$$\lim_{k\to\infty} x(k) - \alpha(k)\mathbf{1} := \lim_{k\to\infty} u(k) = 0 \tag{15}$$

exponentially fast, for some $\alpha(k) \in \mathbb{R}$.

Now we need to show that $\alpha(k) = \alpha$ for all k. By previous analysis, we get $\|u(k)\| \leq c\gamma^k$ for some $c > 0$ and $\gamma \in (0,1)$. From (7), we have

$$\begin{aligned} \alpha(k+1)\mathbf{1} + u(k+1) &= x(k+1) = P(k)(\alpha(k)\mathbf{1} + u(k)) + w(k) \\ &= \alpha(k)\mathbf{1} + P(k)u(k) + w(k). \end{aligned} \tag{16}$$

Rearranging the terms in (16) we may derive that

$$\begin{aligned} |\alpha(k+1) - \alpha(k)| &= \|\alpha(k+1)\mathbf{1} - \alpha(k)\mathbf{1}\| = \|P(k)u(k) + w(k) - u(k+1)\| \\ &\leq \|u(k)\| + \|w(k)\| + \|u(k+1)\| \leq 2c\gamma^k + a(bM+1)\rho^k. \end{aligned} \tag{17}$$

It is easy to see that the sequence $\{\alpha(k)\}$ satisfies the Cauchy convergence test. Therefore, we have $\lim_{k\to\infty} \alpha(k) = \alpha$ exponentially fast, which readily concludes our proof.

4 Concluding remarks

We have proved the convergence of a class of discrete-time nonlinear time-varying multi-agent systems. For the convergence speed, as alluded at the end of Section 2, our result can be generalized to the following situation by a bit more efforts. If we replace the corresponding assumptions in Theorem 1 by

$$\|\triangle(k)\| \leq a_k, \quad \|v(k)\| \leq c_k, \quad \|f_k(x(k))\| \leq b_k\|x(k)\| \tag{18}$$

and

$$\sum_{k=0}^{\infty} a_k b_k < \infty, \quad \sum_{k=0}^{\infty} c_k < \infty, \tag{19}$$

then the system (2) will converge in a speed according to $\max\{a_k b_k, c_k, (1-\varepsilon)^{k/K}\}$.

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