On singularly perturbed q-difference-differential equations with irregular singularity

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Abstract

We study a q-analog of a singularly perturbed Cauchy problem with irregular singularity in the complex domain. We construct solutions of this problem which are holomorphic on open half q-spirals. Using a version of a q-analog of the Malgrange-Sibuya theorem obtained by J-P. Ramis, J. Sauloy and C. Zhang, we show the existence of a formal power series solution in the perturbation parameter which is the q-asymptotic expansion of these holomorphic solutions.

Key words: q-Laplace transform, Whitney C^{∞} functions, formal power series, Poincaré asymptotic expansions. 2000 MSC: 35C10, 35C20.

1 Introduction

In the paper [7], we have studied singular perturbations of the following form

$$\epsilon t^2 \partial_t \partial_z^S u(\epsilon,t,z) = F(\epsilon,z,\partial_t,\partial_z)u(\epsilon,t,z) \tag{1}$$

where F is some differential operator with polynomial coefficients. These equations (1) belong to a class of partial differential equations with so-called irregular singularity at $t=0$, for $\epsilon \neq 0$. We constructed formal power series $\hat{u}(\epsilon,t,z) = \sum_{k \geq 0} h_k(t,z)\epsilon^k$ with coefficients $h_k(t,z)$ which are holomorphic functions with respect to t in some sector centered at 0 in $\mathbb{C}$ and z uniformly on a small disc centered at 0. We showed that there exist actual holomorphic solutions $u_i(\epsilon,t,z)$ of (1), having $\hat{u}$ as Gevrey asymptotic of order 1 on some sectors S_i with respect to ϵ, where S_i have opening larger than π and whose union covers a punctured neighborhood of the origin in $\mathbb{C}$.

In this paper, we study a q-analog of the equation (1) with respect to the variable t, namely we will consider the following q-difference-differential equations

$$\epsilon t \partial_z^S X(\epsilon,qt,z) + \partial_z^S X(\epsilon,t,z) = \sum_{k=0}^{S-1} p_k(\epsilon,z)(t\sigma_q)^{m_{0,k}}(\partial_z^k X)(\epsilon,t,zq^{-m_{1,k}}) \tag{2}$$

where $q \in \mathbb{C}$ such that $|q| > 1$, $m_{0,k}, m_{1,k}$ are positive integers, $p_k(\epsilon, z)$ are polynomials in z with holomorphic coefficients in ϵ on some neighborhood of 0 in $\mathbb{C}$ and σ_q is the dilation operator defined by $(\sigma_q X)(\epsilon, t, z) = X(\epsilon, qt, z)$. As in previous works, see [5], [6], the maps $(t, z) \mapsto (q^{m_{0,k}} t, zq^{-m_{1,k}})$ are assumed to be volume shrinking maps, meaning that the modulus of the Jacobian determinant $|q|^{m_{0,k}-m_{1,k}}$ is less than 1, for all $0 \leq k \leq S-1$.

We will construct actual holomorphic solutions $X(\epsilon, t, z)$ of (2) in the variable ϵ on some open q-spirals with holomorphic coefficients on a domain in $\mathbb{C}^2$ which is a product of a half q-spiral and a disc, for given initial conditions $(\partial_z^j X)(\epsilon, t, 0)$, $0 \leq j \leq S-1$. Our goal is to investigate the asymptotic of these solutions with respect to ϵ near the origin on the given open q-spirals.

The methodology of proof will be the same as the one introduced by M. Canalis-Durand, J. Mozo-Fernandez and R. Schäfke in the paper [1]. The idea consists in transforming the singularly perturbed equation (2) using the linear map $t \mapsto t/\epsilon$ into an auxiliary regularly perturbed q-difference differential equation which has a so-called irregular singularity at $t = 0$, see (47). The approach we will follow is based on a resummation process of formal power series

$$\hat{Y}(\epsilon, t, z) = \sum_{l \geq 0} W_l(\epsilon, z) t^l$$

with respect to the variable t, which are solutions of the constructed auxiliary problem. This resummation method was introduced by J-P. Ramis and C. Zhang in the papers [11], [12], and is a q-analog of the classical Borel-Laplace summability procedure. This method has been used with success to study the asymptotic properties of analytic solutions to q-difference equations with irregular singularity, see [2], [3].

We study a q-Borel transform of $\hat{Y}(\epsilon, t, z)$ with respect to t,

$$\hat{W}(\epsilon, \tau, z) = \sum_{l \geq 0} W_l(\epsilon, z) \frac{\tau^l}{q^{l(l-1)/2}}$$

which satisfies a regular differential equation with contraction in the variable z, rational coefficients in τ, and holomorphic in (ϵ, z) near the origin in $\mathbb{C}^2$, see (20). Under suitable initial data, we show that $\hat{W}$ converges near the origin in $\mathbb{C}^3$ (Theorem 3) and can be analytically extended to functions $W_i(\epsilon, \tau, z)$ defined on products $U_i q^{-\mathbb{N}} \times V_i q^{\mathbb{N}} \times D(0, \rho)$ where $\{U_i q^{-\mathbb{N}}\}_{i \in I}$ is a finite set of open half q-spirals whose union forms a so-called good covering (in the terminology of [10]) and V_i, $i \in I$, are well chosen open sets of $\mathbb{C}^*$. The functions W_i, $i \in I$, have also a q-exponential growth with respect to τ, meaning that there exist constants $C, K > 0$ such that

$$\sup_{x \in V_i, z \in D(0,\rho)} |W_i(\epsilon, xq^l, z)| \leq C|q|^{l^2/2} K^l$$

for all $l \geq 0$, all $\epsilon \in U_i q^{-\mathbb{N}}$, $i \in I$ (Theorem 2). To get these estimates, we reduce the problem to the study of the growth estimates of the coefficients of a formal series $\hat{V}(\xi, x) = \sum_{l,h \geq 0} v_{l,h} \xi^l x^h / h! \in \mathbb{C}[[\xi, x]]$, solution of a differential equation with contraction in the variable x, dilation in ξ and polynomial coefficients in x, see (17). We construct the formal series solution in a Banach space of formal series $\hat{U}(\xi, x) = \sum_{l,h \geq 0} u_{l,h} \xi^l x^h / h!$ for which there exist constants $C, K_1, K_2 > 0$ such that

$$|u_{l,h}| \leq C|q|^{\frac{1}{2}l^2 - \frac{1}{4}h^2} K_1^l K_2^h h!$$

for all $l, h \geq 0$ (Theorem 1).

We construct actual solutions $Y_i(\epsilon, t, z)$, $i \in I$, of the auxiliary equation (47) as a discret q-Laplace transform of the functions $W_i(\epsilon, \tau, z)$, with respect to τ, in directions $\lambda_i q^{\mathbb{Z}}$ where $\lambda_i \in V_i$.

These functions Y_i, $i \in I$, define holomorphic functions on domains $U_i q^{-\mathbb{N}} \times T_i \times D(0,\rho)$, where T_i are complements of closed q-spirals in some neighborhood of 0 (Theorem 4).

We observe that the functions $X_i(\epsilon,t,z) := Y_i(\epsilon,\epsilon t,z)$, $i \in I$, are actual holomorphic solutions of the initial singularly perturbed equation (2) on domains of the form $U_i q^{-\mathbb{N}} \times \mathcal{T} \times D(0,\rho)$, where $\mathcal{T}$ is some well chosen open half q-spiral in a small neighborhood of 0. From the fact that the functions Y_i, $i \in I$, all have the formal series $\hat{Y}$ as q-Gevrey asymptotic expansion with respect to t (Theorem 4), we get that the functions $G_{i,j}(\epsilon) := X_j(\epsilon,t,z) - X_i(\epsilon,t,z)$ have the formal series $\hat{0}$ as q-asymptotic expansion (in the Poincaré sense) on the intersections $U_j q^{-\mathbb{N}} \cap U_i q^{-\mathbb{N}}$ as $\mathbb{H}_{\mathcal{T},\rho}$-valued functions, where $\mathbb{H}_{\mathcal{T},\rho}$ denotes the Banach space of holomorphic bounded functions on $\mathcal{T} \times D(0,\rho)$ equipped with the supremum norm. The tuple of functions $(G_{i,j}(\epsilon))_{(i,j)\in I^2}$ forms what is called a $\mathbb{H}_{\mathcal{T},\rho}$-valued cocyle (Proposition 4). Following the approach of J.P. Ramis, J. Sauloy and C. Zhang in [10], we show that this cocycle splits in the space of C^∞ Whitney functions on closed sets. More precisely, using the Whitney extension theorem (see [8], [9]), we construct a family of C^∞ functions f_i in the sense of Whitney on closed sets $K_i q^{-\mathbb{N}} \cup \{0\} \subset \mathbb{R}^2$, where K_i are compact sets of U_i such that $G_{i,j}(\epsilon) = f_j(\epsilon) - f_i(\epsilon)$, for all $\epsilon \in (K_i q^{-\mathbb{N}} \cap K_j q^{-\mathbb{N}}) \cup \{0\}$, all $(i,j) \in I^2$ (Proposition 5).

Solving a $\bar{\partial}$-problem for $\mathbb{H}_{\mathcal{T},\rho}$-valued C^∞ functions on open sets in $\mathbb{R}^2$, we show that there exists a C^∞ function u on a neighborhood of the origin in $\mathbb{R}^2$ such that $X_i = f_i + u$ on $K_i q^{-\mathbb{N}}$, $i \in I$ (Proposition 6). Finally, we deduce the main result of this paper which is the existence of a formal series

$$\hat{X}(\epsilon) = \sum_{k\geq 0} X_k \epsilon^k / k!$$

where $X_k \in \mathbb{H}_{\mathcal{T},\rho}$, $k \geq 0$, which is the q-asymptotic expansion of X_i on $K_i q^{-\mathbb{N}}$, $i \in I$, and solves the equation (2) (Theorem 5).

The paper is organized as follows.
In section 2, we introduce Banach spaces of formal power series and we solve a contraction/dilation differential equation with polynomial coefficients in these spaces.
In section 3, we solve a parameter depending differential equation with rational coefficients in analytic spaces of q-exponential growing functions on open half q-spirals.
In section 4.1, we recall the definition and properties of the q-Laplace transform and the definition of the q-asymptotic expansion.
In section 4.2, we construct actual solutions Y_i, $i \in I$, of an auxiliary q-difference differential equation with irregular singularity (47) and we give their formal asymptotic expansion $\hat{Y}$.
In section 4.3, we construct actual solutions X_i, $i \in I$, of our initial equation (2).
In section 5.1, we build a $\mathbb{H}_{\mathcal{T},\rho}$-valued cocyle with the help of the functions X_i, $i \in I$.
In section 5.2, we show that this latter cocyle splits in the space of Whitney C^∞ functions.
In section 5.3, we state the main result of this paper, namely the existence of a formal series $\hat{X}(\epsilon) \in \mathbb{H}_{\mathcal{T},\rho}[[\epsilon]]$ solving (2) which is the q-asymptotic expansion of the functions X_i, $i \in I$.

2 Weighted Banach space of formal power series

Definition 1 *We consider the vector space $\mathbb{E}_{q,(T,X)}$ of formal power series*

$$V(\xi,x) = \sum_{l\geq 0, h\geq 0} v_{l,h} \xi^l \frac{x^h}{h!} \in \mathbb{C}[[\xi,x]] \tag{3}$$

such that

$$||V(\xi,x)||_{(T,X)} := \sum_{l\geq 0,h\geq 0} \frac{|v_{l,h}|}{q^{P(l,h)}} T^l \frac{X^h}{h!} < \infty,$$

where $T,X > 0$, $q > 1$ are positive real numbers and where

$$P(l,h) = \frac{1}{4}l^2 + \frac{1}{2}lh - \frac{1}{2}h^2$$

for all $l,h \geq 0$. The space $(\mathbb{E}_{q,(T,X)}, ||.||_{(T,X)})$ is a Banach space.

Remark: Notice that we have a continuous inclusion $(\mathbb{E}_{q,(T,X)}, ||.||_{(T,X)}) \hookrightarrow (\mathbb{E}_{q,(T',X')}, ||.||_{(T',X')})$ when $0 < X' \leq X$, $0 < T' \leq T$.

We consider the integration operator ∂_x^{-1} defined on $\mathbb{C}[[\xi,x]]$ by

$$\partial_x^{-1}(V(\xi,x)) := \sum_{l\geq 0,h\geq 1} v_{l,h-1}\xi^l \frac{x^h}{h!} \in \mathbb{C}[[\xi,x]]$$

Lemma 1 *Let $s,h_1,h_2 \geq 0$ be positive integers. Let $T,X > 0$. Assume that the inequalities hold*

$$s + h_2 \geq 2h_1 \quad , \quad m_1 \geq s + h_2. \tag{4}$$

Then, there exists $C > 0$ (depending q, s,h_1,h_2,m_1) such that

$$||x^s(\partial_x^{-h_2}V)(q^{h_1}\xi, \frac{x}{q^{m_1}})||_{(T,X)} \leq CX^{(s+h_2)}||V(\xi,x)||_{(T,X)} \tag{5}$$

for all $V(\xi,x) \in \mathbb{E}_{q,(T,X)}$.

Proof Let $V(\xi,x) \in \mathbb{C}[[\xi,x]]$. We have that

$$x^s(\partial_x^{-h_2}V)(q^{h_1}\xi, \frac{x}{q^{m_1}}) = \sum_{l\geq 0,h\geq h_2+s} v_{l,h-(s+h_2)} \frac{q^{h_1 l}h!}{q^{m_1(h-s)}(h-s)!}\xi^l \frac{x^h}{h!}$$

From the definition of the norm $||.||_{(T,X)}$, we get that

$$||x^s(\partial_x^{-h_2}V)(q^{h_1}\xi, \frac{x}{q^{m_1}})||_{(T,X)} = \sum_{l\geq 0,h\geq h_2+s} \frac{|v_{l,h-(s+h_2)}|}{q^{P(l,h-(s+h_2))}} T^l \frac{X^{h-(s+h_2)}}{(h-(s+h_2))!} \times \left\{ \frac{1}{q^{P(l,h)-P(l,h-(s+h_2))-h_1 l+m_1(h-s)}} \frac{(h-(s+h_2))!}{(h-s)!} X^{s+h_2} \right\} \tag{6}$$

In the rest of the proof, we will show that there exists a constant $C > 0$ (depending on q, s,h_1,h_2,m_1) such that

$$\frac{1}{q^{P(l,h)-P(l,h-(s+h_2))-h_1 l+m_1(h-s)}} \leq C \tag{7}$$

for all $l \geq 0$, $h \geq 0$. Indeed,

$$P(l,h) - P(l,h-(s+h_2)) - h_1 l + m_1(h-s) = l(\frac{s+h_2}{2} - h_1) + h(m_1 - (s+h_2)) - m_1 s + \frac{(s+h_2)^2}{2}$$

for all $l,h \geq 0$. From the assumption (4), we deduce that the inequalities (7) hold for all $l \geq 0$, $h \geq 0$. Finally, the inequality (5) follows from the expression (6) and the estimates (7). □

Lemma 2 *Let $s, h_1 \geq 0$ and $T_0, X_0 > 0$. Then, for all $T_1, X_1 > 0$ satisfying*

$$0 < X_1 \leq X_0 q^{-s} \quad , \quad 0 < T_1 \leq T_0 q^{\frac{s}{2} - h_1}, \tag{8}$$

there exists a constant $C_1 > 0$ (depending on q, s, h_1, T_0, X_0) such that

$$||x^s V(q^{h_1}\xi, x)||_{(T_1,X_1)} \leq C_1 ||V(\xi, x)||_{(T_0,X_0)} \tag{9}$$

for all $V(\xi, x) \in \mathbb{E}_{q,(T_0,X_0)}$.

Proof Let $V(\xi, x) \in \mathbb{C}[[\xi, x]]$. From the definition of the norm $||.||_{(T,X)}$, one can write

$$||x^s V(q^{h_1}\xi, x)||_{(T_1,X_1)} = \sum_{l \geq 0, h \geq s} \frac{|v_{l,h-s}|}{q^{P(l,h-s)}} T_0^l \frac{X_0^{(h-s)}}{(h-s)!} \times \\ \left\{ \frac{1}{q^{P(l,h) - P(l,h-s) - h_1 l}} (\frac{T_1}{T_0})^l (\frac{X_1}{X_0})^h X_0^s \right\} \tag{10}$$

In the rest of the proof, we will show that there exists a constant $C_1 > 0$ (depending on q, s, h_1) such that

$$\frac{1}{q^{P(l,h) - P(l,h-s) - h_1 l}} (\frac{T_1}{T_0})^l (\frac{X_1}{X_0})^h \leq C_1 \tag{11}$$

for all $l \geq 0$, all $h \geq 0$.

We have

$$P(l,h) - P(l,h-s) - h_1 l = l(\frac{s}{2} - h_1) + h(-s) + \frac{s^2}{2}$$

for all $h \geq 0$. For all T_1, X_1 satisfying (8), we get that (11) holds for all $l \geq 0$, all $h \geq 0$. Finally, the inequality (9) follows from the expression (10) and the estimates (11). □

Lemma 3 *Let $h_2 \geq 0$ and $T_0, X_0 > 0$. Then, for all $T_1, X_1 > 0$ satisfying*

$$0 < X_1 \leq X_0 q^{-h_2} \quad , \quad 0 < T_1 \leq T_0 q^{h_2/2}, \tag{12}$$

there exists a constant $C_2 > 0$ (depending on q, h_2, T_0, X_0) such that

$$||\partial_x^{-h_2} V(\xi, x)||_{(T_1,X_1)} \leq C_2 ||V(\xi, x)||_{(T_0,X_0)} \tag{13}$$

for all $V(\xi, x) \in \mathbb{E}_{q,(T_0,X_0)}$.

Proof Let $V(\xi, x) \in \mathbb{C}[[\xi, x]]$. From the definition of the norm $||.||_{(T,X)}$, one can write

$$||\partial_x^{-h_2} V(\xi, x)||_{(T_1,X_1)} = \sum_{l \geq 0, h \geq h_2} \frac{|v_{l,h-h_2}|}{|q|^{P(l-h_2)}} T_0^l \frac{X_0^{(h-h_2)}}{(h-h_2)!} \times \\ \left\{ \frac{1}{q^{P(l,h) - P(l,h-h_2)}} (\frac{T_1}{T_0})^l (\frac{X_1}{X_0})^h \frac{(h-h_2)!}{h!} X_0^{h_2} \right\} \tag{14}$$

In the rest of the proof, we will show that there exists a constant $C_2 > 0$ (depending on q, h_2) such that

$$\frac{1}{q^{P(l,h)-P(l,h-h_2)}}(\frac{T_1}{T_0})^l(\frac{X_1}{X_0})^h \leq C_2 \tag{15}$$

for all $l \geq 0$, all $h \geq 0$.

We have

$$P(l,h) - P(l,h-h_2) = l(\frac{h_2}{2}) + h(-h_2) + \frac{h_2^2}{2},$$

for all $h \geq 0$. For all T_1, X_1 satisfying (12), we get that (15) holds for all $l \geq 0$, all $h \geq 0$. Finally, the inequality (13) follows from the expression (14) and the estimates (15). □

Let $S, m_{0,k}, m_{1,k}$, $0 \leq k \leq S-1$ be positive integers. Let $\mathcal{D}$ the linear operator from $\mathbb{C}[[\xi, x]]$ into $\mathbb{C}[[\xi, x]]$ defined by

$$\mathcal{D}(V(\xi,x)) := \partial_x^S V(\xi,x) - \sum_{k=0}^{S-1} a_k(x)(\partial_x^k V)(q^{m_{0,k}}\xi, x/q^{m_{1,k}}),$$

for all $V \in \mathbb{C}[[\xi, x]]$, where $a_k(x) = \sum_{s \in I_k} a_{ks}x^s \in \mathbb{C}[x]$, with I_k be a finite subset of $\mathbb{N}$, for $0 \leq k \leq S-1$.

We make the following hypothesis.

Assumption (A) For all $0 \leq k \leq S-1$, for all $s \in I_k$, we have

$$s + S - k \geq 2m_{0,k} \quad , \quad m_{1,k} \geq s + S - k.$$

We consider the operator $\mathcal{A}$ from $\mathbb{C}[[\xi, x]]$ into $\mathbb{C}[[\xi, x]]$ defined by

$$\mathcal{A}(V(\xi,x)) = V(\xi,x) - \mathcal{D}(\partial_x^{-S}V(\xi,x)) = \sum_{k=0}^{S-1} a_k(x)(\partial_x^{k-S}V)(q^{m_{0,k}}\xi, x/q^{m_{1,k}})$$

for all $V \in \mathbb{C}[[\xi, x]]$.

From the lemma 1, we deduce the following

Lemma 4 *Let $T > 0$. Then, there exists $X > 0$ such that $\mathcal{A}$ is a linear bounded operator from $(\mathbb{E}_{q,(T,X)}, ||.||_{(T,X)})$ into itself, with the following estimates*

$$||\mathcal{A}(V(\xi,x))||_{(T,X)} \leq \frac{1}{2}||V(\xi,x)||_{(T,X)},$$

for all $V \in \mathbb{E}_{q,(T,X)}$.

From the lemma 4, we deduce the next

Corollary 1 *Let $T > 0$. Then, there exists $X > 0$ such that $\mathcal{D} \circ \partial_x^{-S}$ is an invertible linear operator from $(\mathbb{E}_{q,(T,X)}, ||.||_{(T,X)})$ into itself. In particular, there exists $C > 0$ such that*

$$||\mathcal{D}(\partial_x^{-S}b(\xi,x))||_{(T,X)} \leq C||b(\xi,x)||_{(T,X)} \tag{16}$$

for all $b(\xi,x) \in \mathbb{E}_{q,(T,X)}$.

Theorem 1 *Let $S \geq 1$ be an integer. For all $0 \leq k \leq S-1$, let $m_{0,k}, m_{1,k}$ be positive integers and $a_k(x) = \sum_{s \in I_k} a_{ks} x^s \in \mathbb{C}[x]$. We make the hypothesis that the assumption* **(A)** *holds.*

We consider the following functional equation

$$\partial_x^S V(\xi, x) = \sum_{k=0}^{S-1} a_k(x)(\partial_x^k V)(q^{m_{0,k}} \xi, x/q^{m_{1,k}}) \tag{17}$$

with initial conditions

$$(\partial_x^j V)(\xi, 0) = \phi_j(\xi) \quad , \quad 0 \leq j \leq S-1. \tag{18}$$

We assume that $\phi_j(\xi) \in \mathbb{E}_{q,(T_0,X_0)}$, for $0 \leq j \leq S-1$, where $T_0, X_0 > 0$ are given positive real numbers. Then, there exist $T, X > 0$ with $0 < T \leq T_0$, $0 < X \leq X_0$ such that the problem (17), (18) has a unique solution $V(\xi, x) \in \mathbb{E}_{q,(T,X)}$.

Proof A formal series $V(\xi, x) \in \mathbb{C}[[\xi, x]]$ which satisfies (18) can be uniquely written in the form $V(\xi, x) = \partial_x^{-S} U(\xi, x) + I(\xi, x)$ where

$$I(\xi, x) = \sum_{j=0}^{S-1} \phi_j(\xi) \frac{x^j}{j!}$$

and $U(\xi, x) \in \mathbb{C}[[\xi, x]]$. A formal series $V(\xi, x) \in \mathbb{C}[[\xi, x]]$ is a solution of the problem (17), (18) if and only if $U(\xi, x)$ satisfies the equation

$$\mathcal{D}(\partial_x^{-S} U(\xi, x)) = -\mathcal{D}(I(\xi, x)). \tag{19}$$

By construction, we have that

$$-\mathcal{D}(I(\xi, x)) = \sum_{k=0}^{S-1} \sum_{j=k}^{S-1} \sum_{s \in I_k} \frac{a_{ks}}{q^{m_{1,k}(j-k)}(j-k)!} x^{s+j-k} \phi_j(q^{m_{0,k}} \xi)$$

From the lemma 2, there exist $T_1, X_1 > 0$ such that

$$x^{s+j-k} \phi_j(q^{m_{0,k}} \xi) \in \mathbb{E}_{q,(T_1,X_1)}$$

for all $0 \leq k \leq S-1$, all $k \leq j \leq S-1$, all $s \in I_k$. We deduce that $\mathcal{D}(I(\xi, x)) \in \mathbb{E}_{q,(T_1,X_1)}$. From the corollary 1, we deduce that the equation (19) has a unique solution $U(\xi, x) \in \mathbb{E}_{q,(T_1,X_1)}$. From the lemma 3, there exist $0 < T_2 < T_1$ and $0 < X_2 < X_1$ such that $\partial_x^{-S} U(\xi, x) \in \mathbb{E}_{q,(T_2,X_2)}$. From the lemma 2, there exist $0 < T_3 < T_2$, $0 < X_3 < X_2$ such that $I(\xi, x) \in \mathbb{E}_{q,(T_3,X_3)}$. Finally, the formal series $V(\xi, x) = \partial_x^{-S} U(\xi, x) + I(\xi, x)$, unique solution of the problem (17), (18), belongs to $\mathbb{E}_{q,(T_3,X_3)}$. □

3 A parameter depending Cauchy problem in analytic spaces of q-exponential growth

Let U, V be open bounded sets in $\mathbb{C}^*$ and let $q \in \mathbb{C}^*$ such that $|q| > 1$. We define

$$Uq^{-\mathbb{N}} = \{\epsilon_u q^{-n} \in \mathbb{C} / \epsilon_u \in U, n \in \mathbb{N}\} \quad , \quad Vq^{\mathbb{N}} = \{\tau_v q^l \in \mathbb{C} / \tau_v \in V, l \in \mathbb{N}\}.$$

We assume that the following condition is satisfied: there exists $\delta > 0$ such that $|\tau + 1| > \delta$, for all $\tau \in Vq^{\mathbb{N}}$.

Let $S \geq 1$ be an integer. For all $0 \leq k \leq S-1$, let $m_{0,k}$,$m_{1,k}$ be positive integers and $b_k(\epsilon, z) = \sum_{s \in I_k} b_{ks}(\epsilon) z^s$ be a polynomial in z, where I_k is a subset of $\mathbb{N}$ and $b_{ks}(\epsilon)$ are holomorphic bounded functions on some open disc $D(0, r_0)$ with $r_0 > 0$ such that $Uq^{-\mathbb{N}} \subset D(0, r_0)$.
We consider functions $(\epsilon, \tau) \mapsto W_j(\epsilon, \tau)$, for $0 \leq j \leq S-1$, which are holomorphic on $Uq^{-\mathbb{N}} \times Vq^{\mathbb{N}}$. We consider the following Cauchy problem

$$\partial_z^S W(\epsilon, \tau, z) = \sum_{k=0}^{S-1} \frac{b_k(\epsilon, z)}{\tau + 1} \tau^{m_{0,k}} (\partial_z^k W)(\epsilon, \tau, zq^{-m_{1,k}}) \tag{20}$$

with initial conditions

$$(\partial_z^j W)(\epsilon, \tau, 0) = W_j(\epsilon, \tau) \quad , \quad 0 \leq j \leq S-1. \tag{21}$$

Theorem 2 *Let the assumption* **(A)** *(of section 2) be fulfilled by the sets I_k and the integers $m_{0,k}$,$m_{1,k}$, for $0 \leq k \leq S-1$. We make the following assumptions on the initial conditions. There exist constants $K, T_0 > 0$ such that*

$$\sup_{x \in V} |W_j(\epsilon, xq^l)| \leq K |q|^{\frac{1}{4} l^2} (\frac{1}{T_0})^l \frac{1}{1 + l^2} \tag{22}$$

for all $0 \leq j \leq S-1$, all $l \geq 0$, all $\epsilon \in Uq^{-\mathbb{N}}$. Then, there exists a unique solution of (20), (21)

$$(\epsilon, \tau, z) \mapsto W(\epsilon, \tau, z) = \sum_{h \geq 0} W_h(\epsilon, \tau) \frac{z^h}{h!}$$

which is holomorphic on $Uq^{-\mathbb{N}} \times Vq^{\mathbb{N}} \times D(0, \rho)$, for every discs $D(0, \rho)$ centered at 0, with radius $\rho > 0$. Moreover, there exist two constants $C, T > 0$ (depending on ρ,S,$|q|$,$b_k(\epsilon, z)$,$m_{0,k}$,$m_{1,k}$, for $0 \leq k \leq S-1$ and T_0) such that

$$\sup_{x \in V, z \in D(0, \rho)} |W(\epsilon, xq^l, z)| \leq C |q|^{\frac{1}{2} l^2} (\frac{1}{T})^l \tag{23}$$

for all $l \geq 0$, all $\epsilon \in Uq^{-\mathbb{N}}$.

Proof We consider the sequence of functions $W_h(\epsilon, \tau)$, $h \geq S$, defined as follows

$$\frac{W_{h+S}(\epsilon, xq^l)}{h!} = \sum_{k=0}^{S-1} \sum_{h_1 + h_2 = h, h_1 \in I_k} \frac{b_{kh_1}(\epsilon) x^{m_{0,k}} q^{m_{0,k} l}}{(xq^l + 1)} \frac{W_{h_2+k}(\epsilon, xq^l)}{h_2! q^{m_{1,k} h_2}} \tag{24}$$

for all $h \geq 0$, all $l \in \mathbb{N}$, all $x \in V$ and all $\epsilon \in Uq^{-\mathbb{N}}$. One checks that the sequence $W_h(\epsilon, \tau)$, $h \geq 0$, of holomorphic functions on $Uq^{-\mathbb{N}} \times Vq^{\mathbb{N}}$, satisfies the recursion (24) if and only if the formal series

$$W(\epsilon, \tau, z) = \sum_{h \geq 0} W_h(\epsilon, \tau) \frac{z^h}{h!}$$

in the z variable, satisfies the problem (20), (21).

We consider the sequence of positive real numbers defined by

$$w_{l,h}(\epsilon) = \sup_{x \in V} |W_h(\epsilon, xq^l)|,$$

for all $l \geq 0$, all $h \geq 0$, all $\epsilon \in Uq^{-\mathbb{N}}$. The sequence $(w_{l,h}(\epsilon))_{l \geq 0, h \geq 0}$ satisfies the following inequalities: there exists $r > 0$ (depending on $m_{0,k}$,V) such that

$$\frac{w_{l,h+S}(\epsilon)}{h!} \leq \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1 \in I_k} \frac{|b_{kh_1}(\epsilon)| r |q|^{m_{0,k} l}}{\delta} \frac{w_{l,h_2+k}(\epsilon)}{h_2! |q|^{(m_{1,k} h_2)}} \tag{25}$$

for all $l \geq 0$, all $h \geq 0$, all $\epsilon \in Uq^{-\mathbb{N}}$.
We consider the sequence of real numbers $(v_{l,h}(\epsilon))_{l \geq 0, h \geq 0}$ defined by the following recursion

$$\frac{v_{l,h+S}(\epsilon)}{h!} = \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1 \in I_k} \frac{|b_{kh_1}(\epsilon)| r |q|^{m_{0,k} l}}{\delta} \frac{v_{l,h_2+k}(\epsilon)}{h_2! |q|^{(m_{1,k} h_2)}} \tag{26}$$

with initial conditions $v_{l,j}(\epsilon) = w_{l,j}(\epsilon)$, for $0 \leq j \leq S-1$, all $l \geq 0$ and all $\epsilon \in Uq^{-\mathbb{N}}$. By construction, we have that

$$w_{l,h}(\epsilon) \leq v_{l,h}(\epsilon) \tag{27}$$

for all $l \geq 0$, all $h \geq 0$, all $\epsilon \in Uq^{-\mathbb{N}}$.

In the following, we put $a_k(\epsilon, x) = \sum_{s \in I_k} (|b_{ks}(\epsilon)| r / \delta) x^s$ for $0 \leq k \leq S-1$ and we consider the formal power series

$$V(\epsilon, \xi, x) = \sum_{l \geq 0, h \geq 0} v_{l,h}(\epsilon) \xi^l \frac{x^h}{h!}.$$

From the recursion (26), we get that $V(\epsilon, \xi, x)$ satisfies the following Cauchy problem

$$\partial_x^S V(\epsilon, \xi, x) = \sum_{k=0}^{S-1} a_k(\epsilon, x) (\partial_x^k V)(\epsilon, q^{m_{0,k}} \xi, x/|q|^{m_{1,k}}) \tag{28}$$

with initial conditions

$$(\partial_x^j V)(\epsilon, \xi, 0) = \phi_j(\epsilon, \xi) := \sum_{l \geq 0} w_{l,j}(\epsilon) \xi^l \quad , \quad 0 \leq j \leq S-1. \tag{29}$$

From the hypothesis (22), we get that $\phi_j(\epsilon, \xi)$ belongs to $\mathbb{E}_{|q|,(T_0,X_0)}$, for all $X_0 > 0$, all $\epsilon \in Uq^{-\mathbb{N}}$. By hypothesis, the assumption **(A)** holds for the sets I_k and the numbers $m_{0,k}$,$m_{1,k}$. From Theorem 1, we deduce that the unique solution $V(\epsilon, \xi, x)$ of the problem (28), (29) satisfies $V(\epsilon, \xi, x) \in E_{|q|,(T,X)}$ for real numbers $T, X > 0$ with $0 < T < T_0$, $0 < X < X_0$. From the inequality $P(l,h) \leq \frac{l^2}{2} - \frac{h^2}{4}$, for all $l \geq 0$, $h \geq 0$, we get the estimates : there exists $C' > 0$ such that

$$|v_{l,h}(\epsilon)| \leq C' |q|^{\frac{l^2}{2}} |q|^{\frac{-h^2}{4}} h! (\frac{1}{T})^l (\frac{1}{X})^h \tag{30}$$

for all $l \geq 0$, all $h \geq 0$, all $\epsilon \in Uq^{-\mathbb{N}}$. From the inequalities (27) and (30), we get that

$$\sup_{x \in V, z \in D(0,\rho)} |W(\epsilon, xq^l, z)| \leq C' |q|^{\frac{l^2}{2}} (\frac{1}{T})^l (\sum_{h \geq 0} |q|^{-h^2/4} (\frac{\rho}{X})^h) \tag{31}$$

for all $l \geq 0$, all $\rho > 0$, all $\epsilon \in Uq^{-\mathbb{N}}$. So that the estimates (23) hold. □

Theorem 3 *Let the assumption* **(A)** *(of section 2) be fulfilled by the sets* I_k *and the integers* $m_{0,k}$,$m_{1,k}$*, for* $0 \leq k \leq S-1$*. Assume that the initial conditions* $(\epsilon,\tau) \mapsto W_j(\epsilon,\tau)$*, for* $0 \leq j \leq S-1$*, are holomorphic and bounded on* $D(0,r_0) \times D(0,\rho_0)$*, for some* $0 < \rho_0 < 1$*. Then, there exists a unique solution of (20), (21)*

$$(\epsilon,\tau,z) \mapsto W(\epsilon,\tau,z) = \sum_{h \geq 0} W_h(\epsilon,\tau)\frac{z^h}{h!}$$

which is bounded holomorphic on $D(0,r_0) \times D(0,\rho_0) \times D(0,\rho)$*, for every discs* $D(0,\rho)$ *centered at 0, with radius* $\rho > 0$*.*

Proof We consider the sequence of functions $W_h(\epsilon,\tau)$, $h \geq S$, defined as follows

$$\frac{W_{h+S}(\epsilon,\tau)}{h!} = \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1 \in I_k} \frac{b_{kh_1}(\epsilon)\tau^{m_{0,k}}}{(\tau+1)} \frac{W_{h_2+k}(\epsilon,\tau)}{h_2! q^{m_{1,k}h_2}} \tag{32}$$

for all $h \geq 0$, all $\tau \in D(0,\rho_0)$ and all $\epsilon \in D(0,r_0)$. One checks that the sequence $W_h(\epsilon,\tau)$, $h \geq 0$, of holomorphic functions on $D(0,r_0) \times D(0,\rho_0)$, satisfies the recursion (32) if and only if the formal series

$$W(\epsilon,\tau,z) = \sum_{h \geq 0} W_h(\epsilon,\tau)\frac{z^h}{h!}$$

in the z variable, satisfies the problem (20), (21).
We consider the sequence of positive real numbers defined by

$$w_h(\epsilon) = \sup_{\tau \in D(0,\rho_0)} |W_h(\epsilon,\tau)|,$$

for all $h \geq 0$, all $\epsilon \in D(0,r_0)$. The sequence $(w_h(\epsilon))_{h \geq 0}$ satisfies the following inequalities

$$\frac{w_{h+S}(\epsilon)}{h!} \leq \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1 \in I_k} \frac{|b_{kh_1}(\epsilon)|{\rho_0}^{m_{0,k}}}{1-\rho_0} \frac{w_{h_2+k}(\epsilon)}{h_2!|q|^{(m_{1,k}h_2)}} \tag{33}$$

for all $h \geq 0$, all $\epsilon \in D(0,r_0)$.
We consider the sequence of real numbers $(v_h(\epsilon))_{h \geq 0}$ defined by the following recursion

$$\frac{v_{h+S}(\epsilon)}{h!} = \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1 \in I_k} \frac{|b_{kh_1}(\epsilon)|{\rho_0}^{m_{0,k}}}{1-\rho_0} \frac{v_{h_2+k}(\epsilon)}{h_2!|q|^{(m_{1,k}h_2)}} \tag{34}$$

with initial conditions $v_j(\epsilon) = w_j(\epsilon)$, for $0 \leq j \leq S-1$ and all $\epsilon \in D(0,r_0)$. By construction, we have that

$$w_h(\epsilon) \leq v_h(\epsilon) \tag{35}$$

for all $h \geq 0$, all $\epsilon \in D(0,r_0)$.

In the following, we put $c_k(\epsilon,x) = \sum_{s \in I_k}(|b_{ks}(\epsilon)|{\rho_0}^{m_{0,k}}/(1-\rho_0))x^s$ for $0 \leq k \leq S-1$ and we consider the formal power series

$$V(\epsilon,x) = \sum_{h \geq 0} v_h(\epsilon)\frac{x^h}{h!}.$$

From the recursion (34), we get that $V(\epsilon,x)$ satisfies the following Cauchy problem

$$\partial_x^S V(\epsilon,x) = \sum_{k=0}^{S-1} c_k(\epsilon,x)(\partial_x^k V)(\epsilon, x/|q|^{m_{1,k}}) \tag{36}$$

with initial conditions

$$(\partial_x^j V)(\epsilon,0) = \phi_j(\epsilon) := w_j(\epsilon) \quad , \quad 0 \leq j \leq S-1. \tag{37}$$

We see that $\phi_j(\epsilon) \in E_{|q|,(T_0,X_0)}$, for all $T_0, X_0 > 0$, all $\epsilon \in D(0,r_0)$. From our hypotheses, we also get that the assumptions of Theorem 1 are fulfilled for the problem (36), (37). From Theorem 1, we get that there exists a solution $V(\epsilon,x)$ of (36), (37) such that $V(\epsilon,x) \in E_{|q|,(T,X)}$ for some $0 < T < T_0$, $0 < X < X_0$, for all $\epsilon \in D(0,r_0)$. In particular, we get the estimates : there exists $C > 0$ such that

$$v_h(\epsilon) \leq C(\frac{1}{X})^h h! |q|^{-\frac{h^2}{2}} \tag{38}$$

for all $h \geq 0$, all $\epsilon \in D(0,r_0)$. From the estimates (35) and (38), we get that the formal series $W(\epsilon,\tau,z)$ defines a bounded holomorphic function on $D(0,r_0) \times D(0,\rho_0) \times D(0,\rho)$, for every discs $D(0,\rho)$ centered at 0, with radius $\rho > 0$. □

4 Analytic solutions in a small parameter of a singularly perturbed Cauchy problem

4.1 A q-analog of the Laplace transform and q-asymptotic expansion

In this section, we recall the definition of a q-analog of the Laplace transform introduced by J-P. Ramis and C. Zhang in the papers [11], [12] and some properties that will be usefull in the sequel.

In the next proposition, which is a Banachic version of Theorem 1.3.2 in [12], we give the definition of the q-Laplace transform and we recall estimates for its asymptotic expansion. For the proof, we refer to [12].

Proposition 1 *Let $q \in \mathbb{C}$ such that $|q| > 1$. Let V be an open and bounded set in $\mathbb{C}^*$ and $D(0,\rho_0)$ a disc such that $V \cap D(0,\rho_0) \neq \emptyset$. Let $(\mathbb{F}, ||.||_{\mathbb{F}})$ be a complex Banach space. Let $\phi : Vq^{\mathbb{N}} \cup D(0,\rho_0) \rightarrow \mathbb{F}$ be a holomorphic function which satisfies the following estimates : there exist $C, M > 0$ such that*

$$||\phi(xq^m)||_{\mathbb{F}} \leq M|q|^{m^2/2}C^m \tag{39}$$

for all $m \geq 0$, all $x \in V$. Let Θ be the Theta Jacobi function defined in $\mathbb{C}^*$ by

$$\Theta(x) = \sum_{n \in \mathbb{Z}} q^{-n(n-1)/2} x^n.$$

Let $\delta > 0$ and $\lambda \in V \cap D(0,\rho_0)$. We denote by

$$\mathcal{T}_{\lambda,q,\delta,r_1} = \{t \in \mathbb{C}^*/|1 + \frac{\lambda}{tq^k}| > \delta, \forall k \in \mathbb{Z}, |t| < r_1\}.$$

The q-Laplace transform of ϕ *in the direction* $\lambda q^{\mathbb{Z}}$ *is defined by*

$$\mathcal{L}_q^{\lambda}(\phi)(t) := \sum_{m \in \mathbb{Z}} \phi(q^m \lambda)/\Theta(\frac{q^m \lambda}{t})$$

for all $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$*, if* $r_1 < |\lambda q^{1/2}|/C$*. Moreover,* $\mathcal{L}_q^{\lambda}(\phi)(t)$ *defines a bounded holomorphic function on* $\mathcal{T}_{\lambda,q,\delta,r_1}$ *with values in* $\mathbb{F}$ *when* $r_1 < |\lambda q^{1/2}|/C$*. Assume that the function* ϕ *has the following Taylor expansion*

$$\phi(\tau) = \sum_{n \geq 0} \frac{f_n}{q^{n(n-1)/2}} \tau^n \tag{40}$$

on $D(0,\rho_0)$*, where* $f_n \in \mathbb{F}$*,* $n \geq 0$*. Then, there exist two constants* $D, B > 0$ *such that*

$$||\mathcal{L}_q^{\lambda}(\phi)(t) - \sum_{m=0}^{n-1} f_m t^m||_{\mathbb{F}} \leq D B^n |q|^{n(n-1)/2} |t|^n \tag{41}$$

for all $n \geq 1$*, for all* $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$*.*

In the following proposition, we recall a commutation formula for the q-Laplace transform and the multiplication by a monomial.

Proposition 2 *Let* V *be an open and bounded set in* $\mathbb{C}^*$ *and* $D(0,\rho_0)$ *be a disc such that* $V \cap D(0,\rho_0) \neq \emptyset$*. Let* ϕ *be a holomorphic function on* $Vq^{\mathbb{N}} \cup D(0,\rho_0)$ *with values in* $(\mathbb{F}, ||.||_{\mathbb{F}})$ *which satisfies the estimates : there exist* $C, K > 0$*, such that*

$$||\phi(xq^m)||_{\mathbb{F}} \leq K |q|^{m^2/2} C^m \tag{42}$$

for all $m \geq 0$*, all* $x \in V$*. Then, the function* $m\phi(\tau) = \tau\phi(\tau)$ *is holomorphic on* $Vq^{\mathbb{N}} \cup D(0,\rho_0)$ *and satisfies estimates of the form (39). Let* $\lambda \in V \cap D(0,\rho_0)$*. We have the following equality*

$$\mathcal{L}_q^{\lambda}(m\phi)(t) = t\mathcal{L}_q^{\lambda}(\phi)(qt) \tag{43}$$

for all $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$*, if* $r_1 < |\lambda q^{1/2}|/(C|q|)$*.*

Proof From the estimates (42), we get a constant $r > 0$ such that

$$||(m\phi)(xq^m)||_{\mathbb{F}} \leq rK|q|^{m^2/2}(|q|C)^m$$

for all $m \geq 0$, all $x \in V$. From Proposition 1, we deduce that $\mathcal{L}_q^{\lambda}(m\phi)(t)$ defines a holomorphic function on $\mathcal{T}_{\lambda,q,\delta,r_1}$, if $r_1 < |\lambda q^{1/2}|/(C|q|)$. On the other hand,

$$t\mathcal{L}_q^{\lambda}(\phi)(qt) = \sum_{m \in \mathbb{Z}} \frac{t\phi(q^m\lambda)}{\Theta(q^m\lambda/(qt))} \tag{44}$$

But we have that

$$\frac{t}{\Theta(q^m\lambda/(qt))} = \frac{q^m\lambda}{\Theta(q^m\lambda/t)}$$

for all $m \in \mathbb{Z}$. Indeed, put $y = q^m\lambda/(qt)$ in the equation $\Theta(qy) = qy\Theta(y)$ which holds for all $y \in \mathbb{C}^*$. From (44), we get that

$$t\mathcal{L}_q^{\lambda}(\phi)(qt) = \sum_{m\in\mathbb{Z}} \frac{q^m\lambda\phi(q^m\lambda)}{\Theta(q^m\lambda/t)} = \mathcal{L}_q^{\lambda}(m\phi)(t)$$

for all $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$. □

In the following, we recall the definition of q-asymptotic expansion as introduced in the paper [10].

Definition 2 *Let U be an open set in $\mathbb{C}^*$. Let $(\mathbb{H}, ||.||_{\mathbb{H}})$ be a complex Banach space. Let $f : Uq^{-\mathbb{N}} \rightarrow \mathbb{H}$ be a holomorphic function and let $\hat{f}(\epsilon) = \sum_{n\geq 0} a_n\epsilon^n \in \mathbb{H}[[\epsilon]]$ be a formal power series with coefficients in $\mathbb{H}$. The formal series $\hat{f}$ is the q-asymptotic expansion of f on $Uq^{-\mathbb{N}}$ if for all $n \geq 1$, all compact set $K \subset U$, there exists a constant $C_{K,n} > 0$ such that*

$$||f(\epsilon) - \sum_{k=0}^{n-1} a_k\epsilon^k||_{\mathbb{H}} \leq C_{K,n}|\epsilon|^n, \tag{45}$$

for all $\epsilon \in Kq^{-\mathbb{N}}$.

Remark: If f admits a q-asymptotic expansion $\hat{f}$, then it is unique. Moreover, if we denote by f' the complex derivative of f on $Uq^{-\mathbb{N}}$ and $\hat{f}'$ the formal derivative of $\hat{f}$, we have that $\hat{f}'$ is the q-asymptotic expansion of f' on $Uq^{-\mathbb{N}}$, see [10] p.31.

4.2 Analytic solutions in a parameter of a singular Cauchy problem

Let $S \geq 1$ be an integer. For all $0 \leq k \leq S-1$, let $m_{0,k}$,$m_{1,k}$ be positive integers and $b_k(\epsilon, z) = \sum_{s\in I_k} b_{ks}(\epsilon)z^s$ be a polynomial in z, where I_k is a subset of $\mathbb{N}$ and $b_{ks}(\epsilon)$ are bounded holomorphic functions on some disc $D(0, r_0)$ with $r_0 > 0$. Let U, V be open sets in $\mathbb{C}^*$ and let $q \in \mathbb{C}$ such that $|q| > 1$. We assume that there exists $\delta > 0$ such that $|\tau + 1| > \delta$ for all $\tau \in Vq^{\mathbb{Z}}$ and that $Uq^{-\mathbb{N}} \subset D(0, r_0)$.

Definition 3 *Let $T_0 > 0$, $0 < \rho_0 < 1$ be real numbers such that $V \cap D(0, \rho_0) \neq \emptyset$. Let $(\epsilon, \tau) \mapsto W(\epsilon, \tau)$ be a bounded holomorphic function on $D(0, r_0) \times D(0, \rho_0)$. Assume that $W(\epsilon, \tau)$ can be extended to an analytic function $(\epsilon, \tau) \mapsto W_{UV}(\epsilon, \tau)$ on $Uq^{-\mathbb{N}} \times (Vq^{\mathbb{N}} \cup D(0, \rho_0))$. Assume moreover that there exists $K > 0$ such that*

$$\sup_{x\in V} |W_{UV}(\epsilon, xq^l)| \leq K|q|^{\frac{1}{4}l^2}(\frac{1}{T_0})^l \frac{1}{1+l^2} \tag{46}$$

for all $l \geq 0$, all $\epsilon \in Uq^{-\mathbb{N}}$. We say that the set $\{W, W_{UV}, T_0, \rho_0\}$ is admissible.

We consider the following singular Cauchy problem

$$t\partial_z^S Y_{UV}(\epsilon, qt, z) + \partial_z^S Y_{UV}(\epsilon, t, z) = \sum_{k=0}^{S-1} b_k(\epsilon, z)(t\sigma_q)^{m_{0,k}}(\partial_z^k Y_{UV})(\epsilon, t, zq^{-m_{1,k}}) \tag{47}$$

for given initial conditions

$$(\partial_z^j Y_{UV})(\epsilon, t, 0) = \varphi_{UV,j}(\epsilon, t) \quad , \quad 0 \leq j \leq S-1, \tag{48}$$

where σ_q is the dilation operator defined by $(\sigma_q Y)(\epsilon,t,z) = Y(\epsilon,qt,z)$. The initial conditions $\varphi_{UV,j}(\epsilon,t)$ are constructed as follows. Let $T_0, \rho_0 > 0$. For all $0 \leq j \leq S-1$, let $\{W_j, W_{UV,j}, T_0, \rho_0\}$ be admissible sets.

Let $\lambda_V \in V \cap D(0,\rho_0)$ be a complex number. We put

$$\varphi_{UV,j}(\epsilon,t) = \mathcal{L}_q^{\lambda_V}(\tau \mapsto W_{UV,j}(\tau,\epsilon))(t)$$

for all $0 \leq j \leq S-1$. From the proposition 1 and the estimates (46), we deduce that, for all $0 \leq j \leq S-1$, $\varphi_{UV,j}(\epsilon,t)$ is holomorphic and bounded on $Uq^{-\mathbb{N}} \times \mathcal{T}_{\lambda_V,q,\delta,r_1}$ for $r_1 > 0$ small enough. We have the following result.

Theorem 4 *Let the assumption* **(A)** *(of section 2) be fulfilled by the sets I_k and the integers $m_{0,k}$,$m_{1,k}$, for $0 \leq k \leq S-1$. Let the initial data (48) constructed as above.*
Then, the problem (47),(48) has a solution $Y_{UV}(\epsilon,t,z)$ which is holomorphic and bounded on $Uq^{-\mathbb{N}} \times \mathcal{T}_{\lambda_V,q,\delta,r_2} \times D(0,\rho)$, for $r_2 \leq r_1$ and all $\rho > 0$.
Moreover, there exists a sequence $(W_n(\epsilon,z))_{n\geq 0}$ of functions which are independent of U,V, that belongs to the Banach space of bounded holomorphic functions on $D(0,r_0) \times D(0,\rho)$ and satisfies the following property: there exist $C,A > 0$ such that

$$\sup_{z\in D(0,\rho)} |Y_{UV}(\epsilon,t,z) - \sum_{m=0}^{n-1} W_m(\epsilon,z)t^m| \leq CA^n|q|^{n(n-1)/2}|t|^n \tag{49}$$

for all $n \geq 1$, for all $\epsilon \in Uq^{-\mathbb{N}}$, all $t \in \mathcal{T}_{\lambda_V,q,\delta,r_2}$.

Proof We consider the Cauchy problem (20),(21) for the initial conditions $W_{UV,j}(\epsilon,\tau)$, for all $0 \leq j \leq S-1$. From the fact that the initial conditions are holomorphic functions on $D(0,r_0) \times D(0,\rho_0)$, we deduce from Theorem 3 that the problem (20),(21) has a solution $W(\epsilon,\tau,z)$ which is holomorphic and bounded on $D(0,r_0) \times D(0,\rho_0) \times D(0,\rho)$, for all $\rho > 0$. Let

$$W(\epsilon,\tau,z) = \sum_{n\geq 0} \frac{W_n(\epsilon,z)}{q^{n(n-1)/2}}\tau^n$$

it's Taylor expansion with respect to τ on $D(0,\rho_0)$.

From the estimates (46) for the initial conditions, we deduce from Theorem 2 that the function $W(\epsilon,\tau,z)$ can be extended to a solution $W_{UV}(\epsilon,\tau,z)$ of the problem (20),(21) which is a holomorphic function on $Uq^{-\mathbb{N}} \times Vq^{\mathbb{N}} \times D(0,\rho)$, for all $\rho > 0$. Moreover, the function W_{UV} satisfies the following estimates: there exist two constants $C,T > 0$, such that

$$\sup_{x\in V, z\in D(0,\rho)} |W_{UV}(\epsilon,xq^l,z)| \leq C|q|^{\frac{1}{2}l^2}(\frac{1}{T})^l \tag{50}$$

for all $l \geq 0$, all $\epsilon \in U_I q^{-\mathbb{N}}$. From the proposition 1 and the estimates (50), we deduce that the function

$$Y_{UV}(\epsilon,t,z) := \mathcal{L}_q^{\lambda_V}(\tau \mapsto W_{UV}(\epsilon,\tau,z))(t)$$

is a holomorphic and bounded function on $Uq^{-\mathbb{N}} \times \mathcal{T}_{\lambda_V,q,\delta,r_2} \times D(0,\rho)$, for some $r_2 \leq r_1$ and all $\rho > 0$. Moreover, the estimates (49) hold. Finally, from the proposition 2, we get that $Y_{UV}(\epsilon,t,z)$ is a solution of (47) and by construction of the initial conditions $\varphi_{UV,j}(\epsilon,t)$, $0 \leq j \leq S-1$, $Y_{UV}(\epsilon,t,z)$ satisfies also (48). □

4.3 Analytic solutions in a parameter of a singularly perturbed Cauchy problem

We recall the definition of a good covering as defined in [10], p.36.

Definition 4 *For any tuple $I = (A, B)$, where A, B are open intervals of $\mathbb{R}$ with length smaller than $1/4$, let U_I be the bounded open sets in $\mathbb{C}^*$ defined by*

$$U_I = \{\exp(2\sqrt{-1}\pi u)q^{\nu} \in \mathbb{C}^*/\forall u \in A, \forall \nu \in B\}.$$

We choose a finite family $\mathcal{I}$ of tuple I as above, such that

1) $\cup_{I\in\mathcal{I}}(U_I q^{-\mathbb{N}}) = \mathcal{V} \setminus \{0\}$, where $\mathcal{V}$ is a neighborhood of 0 in $\mathbb{C}$.
2) The open sets $U_I q^{-\mathbb{N}}$, $I \in \mathcal{I}$, are four by four disjoint.

Then, the family $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$ is called a good covering.

Definition 5 *Let $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$ be a good covering. Let $\delta, r_2 > 0$. We consider a family of open bounded sets $(V_I)_{I\in\mathcal{I}}$, $\mathcal{T}$ in $\mathbb{C}^*$ satisfying the following properties:*
1) There exists $0 < \rho_0 < 1$ such that $V_I \cap D(0, \rho_0) \neq \emptyset$, for all $I \in \mathcal{I}$.
2) For all $I \in \mathcal{I}$, for all $\tau \in V_I q^{\mathbb{Z}}$, $|\tau + 1| > \delta$.
3) For all $I \in \mathcal{I}$, for all $t \in \mathcal{T}$, all $\epsilon_u \in U_I$, all $\lambda_v \in V_I \cap D(0, \rho_0)$, we have

$$|1 + \frac{\lambda_v}{\epsilon_u t q^k}| > \delta \quad , \quad |\epsilon_u t| < r_2,$$

for all $k \in \mathbb{Z}$. We say that the family $\{(V_I)_{I\in\mathcal{I}}, \mathcal{T}\}$ is associated to the good covering $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$.

Let $S \geq 1$ be an integer. For all $0 \leq k \leq S - 1$, let $m_{0,k}$,$m_{1,k}$ be positive integers and $b_k(\epsilon, z) = \sum_{s\in I_k} b_{ks}(\epsilon)x^s$ be a polynomial in x, where I_k is a subset of $\mathbb{N}$ and $b_{ks}(\epsilon)$ are bounded holomorphic functions on some disc $D(0, r_0)$ in $\mathbb{C}$. Let $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$ be a good covering such that $U_I q^{-\mathbb{N}} \subset D(0, r_0)$ for all $I \in \mathcal{I}$.

For all $I \in \mathcal{I}$, we consider the following singularly perturbed Cauchy problem

$$\epsilon t \partial_z^S X_I(\epsilon, qt, z) + \partial_z^S X_I(\epsilon, t, z) = \sum_{k=0}^{S-1} b_k(\epsilon, z)(\epsilon t \sigma_q)^{m_{0,k}} (\partial_z^k X_I)(\epsilon, t, zq^{-m_{1,k}}) \tag{51}$$

for given initial conditions

$$(\partial_z^j Y_I)(\epsilon, t, 0) = \varphi_{I,j}(\epsilon, t) \quad , \quad 0 \leq j \leq S - 1, \tag{52}$$

where the functions $\varphi_{I,j}(\epsilon, t)$ are constructed as follows. Let $\{(V_I)_{I\in\mathcal{I}}, \mathcal{T}\}$ be a family of open sets associated to the good covering $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$. For all $0 \leq j \leq S - 1$, for all $I \in \mathcal{I}$, let $\{W_j, W_{U_I V_I, j}, T_0, \rho_0\}$ be an admissible set. Let λ_I be a complex number in $V_I \cap D(0, \rho_0)$. We put

$$\varphi_{I,j}(\epsilon, t) := \mathcal{L}_q^{\lambda_I}(\tau \mapsto W_{U_I V_I, j}(\tau, \epsilon))(\epsilon t)$$

From the proposition 1 and the estimates (46), we deduce that $\varphi_{I,j}(\epsilon, t)$ is holomorphic and bounded on $U_I q^{-\mathbb{N}} \times \mathcal{T}$, for all $I \in \mathcal{I}$, $0 \leq j \leq S - 1$, if r_2 is small enough.

Proposition 3 *Let the assumption* **(A)** *(of section 2) be fulfilled by the sets* I_k *and the integers* $m_{0,k}$*,*$m_{1,k}$*, for* $0 \leq k \leq S-1$*. Let the initial data (52) constructed as above.*
Then, for all $I \in \mathcal{I}$*, the problem (51), (52) has a solution* $X_I(\epsilon,t,z)$ *which is holomorphic and bounded on* $U_I q^{-\mathbb{N}} \times \mathcal{T} \times D(0,\rho)$*, for all* $\rho > 0$*.*
Moreover, there exist two constants $C, A > 0$ *such that*

$$\sup_{t\in\mathcal{T}} \sup_{z\in D(0,\rho)} |X_I(\epsilon,t,z) - X_{I'}(\epsilon,t,z)| \leq CA^n |q|^{n(n-1)/2} |\epsilon|^n \tag{53}$$

for all $I, I' \in \mathcal{I}$*,* $I \neq I'$ *such that* $U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}} \neq \emptyset$*, for all* $\epsilon \in U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}}$*, for all* $n \geq 1$*.*
In other words, if we denote by $\mathbb{H}_{\mathcal{T},\rho}$ *the Banach space of holomorphic and bounded functions on* $\mathcal{T} \times D(0,\rho)$ *with the supremum norm, the holomorphic and bounded function*

$$g_{I,I'}(\epsilon) := (t,z) \mapsto X_I(\epsilon,t,z) - X_{I'}(\epsilon,t,z)$$

with values in $\mathbb{H}_{\mathcal{T},\rho}$ *has the formal series* $\hat{0}$ *as* q*-asymptotic expansion on* $U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}}$*.*

Proof For all $I \in \mathcal{I}$, we consider the Cauchy problem (47),(48) for the initial conditions $\varphi_{U_I V_I,j}(\epsilon,t) := \mathcal{L}_q^{\lambda_I}(\tau \mapsto W_{U_I V_I,j}(\tau,\epsilon))(t)$, for $0 \leq j \leq S-1$. From Theorem 4, this problem has a solution $Y_{U_I V_I}(\epsilon,t,z)$ which is holomorphic and bounded on $U_I q^{-\mathbb{N}} \times \mathcal{T}_{\lambda_I,q,\delta,r_2} \times D(0,\rho)$, for $r_2 \leq r_1$ and all $\rho > 0$. Now, we put

$$X_I(\epsilon,t,z) := Y_{U_I V_I}(\epsilon,\epsilon t,z)$$

for all $(\epsilon,t,z) \in U_I q^{-\mathbb{N}} \times \mathcal{T} \times D(0,\rho)$. By construction, $X_I(\epsilon,t,z)$ is a solution of the Cauchy problem (51), (52) which is holomorphic and bounded on $U_I q^{-\mathbb{N}} \times \mathcal{T} \times D(0,\rho)$.

In the rest of the proof, we show the estimates (53). From the estimates (49) there exists a sequence $(W_n(\epsilon,z))_{n\geq 0}$ of holomorphic and bounded functions on $D(0,r_0) \times D(0,\rho)$ and two constants $C, A > 0$ such that

$$\sup_{z\in D(0,\rho)} |X_I(\epsilon,t,z) - \sum_{m=0}^{n-1} W_m(\epsilon,z)(\epsilon t)^m| \leq CA^n |q|^{n(n-1)/2} |\epsilon t|^n \tag{54}$$

for all $\epsilon \in U_I q^{-\mathbb{N}}$, all $t \in \mathcal{T}$, for all $n \geq 1$, for all $I \in \mathcal{I}$. From the estimates (54), we get a constant $B > 0$ such that

$$\sup_{t\in\mathcal{T}, z\in D(0,\rho)} |X_I(\epsilon,t,z) - X_{I'}(\epsilon,t,z)| \leq 2C(AB)^n |q|^{n(n-1)/2} |\epsilon|^n \tag{55}$$

for all $I, I' \in \mathcal{I}$ such that $U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}} \neq \emptyset$, for all $\epsilon \in U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}}$, for all $n \geq 1$. □

5 Existence of formal series solutions and q-asymptotic expansion in the parameter ϵ

In this section, we will establish the existence of a formal series solution of the problem (51), (52), $\hat{X}(\epsilon) \in \mathbb{H}_{\mathcal{T},\rho}[[\epsilon]]$ with coefficients in $\mathbb{H}_{\mathcal{T},\rho}$, the Banach spaces of bounded holomorphic functions on $\mathcal{T} \times D(0,\rho)$ with the supremum norm, such that the solutions $X_I(\epsilon,t,z)$ of the problem (51),(52) constructed above, has $\hat{X}$ as q-asymptotic expansion on $U_I q^{-\mathbb{N}}$, for all $I \in \mathcal{I}$. The proof is based on a q-analog of the Malgrange-Sibuya theorem obtained in [10].

5.1 Construction of Banach valued cocyles on a good covering.

We denote by $\mathbb{H}_{\mathcal{T},\rho}$ the Banach space of holomorphic and bounded functions on the open set $\mathcal{T} \times D(0,\rho)$ with the supremum norm.

Definition 6 *A set of functions* $\{g_{I,I'}(\epsilon)\}_{(I,I') \in \mathcal{I}^2}$ *is called a* $\mathbb{H}_{\mathcal{T},\rho}$*-cocyle attached to a good covering* $(U_I q^{-\mathbb{N}})_{I \in \mathcal{I}}$ *if the following properties are satisfied :*

1) For all $(I,I') \in \mathcal{I}^2$*, the function* $g_{I,I'}$ *is a* $\mathbb{H}_{\mathcal{T},\rho}$*-valued holomorphic function on* $U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}}$ *having the formal series* $\hat{0}$ *as q-asymptotic expansion on* $U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}}$*.*
2) For all $I, I' \in \mathcal{I}$*, we have* $g_{I,I'}(\epsilon) = -g_{I',I}(\epsilon)$ *for all* $\epsilon \in U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}}$*.*
3) For all $I, I', I'' \in \mathcal{I}$*, we have* $g_{I,I''}(\epsilon) = g_{I,I'}(\epsilon) + g_{I',I''}(\epsilon)$*, for all* $\epsilon \in U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}}$*.*

Proposition 4 *We consider the family of holomorphic functions* $\{X_I(\epsilon,t,z)\}_{I \in \mathcal{I}}$ *constructed in the proposition 3. Then, the set of functions* $\{g_{I,I'}(\epsilon)\}_{(I,I') \in \mathcal{I}^2}$ *defined by*

$$g_{I,I'}(\epsilon) := (t,z) \mapsto X_{I'}(\epsilon,t,z) - X_I(\epsilon,t,z)$$

for all $I, I' \in \mathcal{I}$ *is a* $\mathbb{H}_{\mathcal{T},\rho}$*-cocyle attached to the good covering* $(U_I q^{-\mathbb{N}})_{I \in \mathcal{I}}$*, for* $\mathcal{T} \subset \mathbb{C}^*$*,*$\rho > 0$ *given in the proposition 3.*

Proof The property 1) in the definition 6 is a consequence of the proposition 3. The properties 2) 3) are fulfilled by construction. □

5.2 Splitting of the cocyles in spaces of Banach valued Whitney C^∞ functions

We recall the definition of a C^∞ function the the sense of Whitney, see [8], [9], p.67.

Definition 7 *Let* $f : A \to \mathbb{H}$ *be a continuous function defined on a compact set* $A \subset \mathbb{R}^2$ *with values in a Banach space* $(\mathbb{H}, ||.||_{\mathbb{H}})$*. The function* f *is said to be* C^∞ *in the sense of Whitney on* A *if there exists a family of continuous functions* $f^k : A \to \mathbb{H}$*, for* $k \in \mathbb{N}^2$*, where* $f^{(0,0)} = f$ *satisfying the following property* **(W)** *: For all* $m \geq 0$*, for all* $k \in \mathbb{N}^2$ *with* $|k| \leq m$*, for all* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$(56) \quad ||f^k(x_1,y_1) - \sum_{h=(h_1,h_2) \in \mathbb{N}^2, |h| \leq m-|k|} \frac{f^{(k+h)}(x_2,y_2)}{h_1! h_2!} (x_1-x_2)^{h_1} (y_1-y_2)^{h_2} ||_{\mathbb{H}} \\ < \epsilon ||(x_1-x_2, y_1-y_2)||_{\mathbb{R}^2}^{m-|k|}$$

for all $(x_1,y_1),(x_2,y_2) \in A$*,* $(x_1,y_1) \neq (x_2,y_2)$ *with* $||(x_1-x_2,y_1-y_2)||_{\mathbb{R}^2} < \delta$*.*

Remark: If a continuous function f is C^∞ in the sense of Whitney on a compact set A, then f is C^∞ in the usual sense on the interior $\mathrm{Int}(A)$ of A. Moreover, we have that

$$f^k(x,y) = \partial_x^{k_1} \partial_y^{k_2} f(x,y),$$

if $k = (k_1,k_2) \in \mathbb{N}^2$, for all $(x,y) \in \mathrm{Int}(A)$.

In the following we will denote by $\mathcal{E}(A)$ the vector space of C^∞ functions in the sense of Whitney on A. We recall the lemma 3.2 from [10].

Lemma 5 *Let U be an open set in $\mathbb{C}^*$. Let $f : Uq^{-\mathbb{N}} \to \mathbb{H}$ be a holomorphic function having a formal series $\hat{f}(\epsilon) = \sum_{n\geq 0} a_n\epsilon^n \in \mathbb{H}[[\epsilon]]$ as q-asymptotic expansion on $Uq^{-\mathbb{N}}$. Then, the family $\partial_\epsilon^n f(\epsilon)$ of $n-th$ complex derivatives of f, for $n \geq 0$, satisfy the following property $(\mathbf{W_H})$: Let $K \subset U$ be a compact set. Then, for all $k, m \geq 0$, there exists a constant $C_{K,m,k}$ such that for all $\epsilon_a, \epsilon_b \in Kq^{-\mathbb{N}} \cup \{0\}$, we have*

$$||\partial_\epsilon^k f(\epsilon_a) - \sum_{h=0}^{m} \frac{\partial_\epsilon^{k+h} f(\epsilon_b)}{h!}(\epsilon_a - \epsilon_b)^h||_{\mathbb{H}} \leq C_{K,m,k}|\epsilon_a - \epsilon_b|^{m+1} \tag{57}$$

where, by definition, we put $\partial_\epsilon^n f(0) = n!a_n$, for all $n \geq 0$.

The next lemma shows that all holomorphic functions having q-asymptotic expansion must satisfy the property $(\mathbf{W})$.

Lemma 6 *Let $f : Uq^{-\mathbb{N}} \to \mathbb{H}$ be a holomorphic function having a formal series $\hat{f}(\epsilon) = \sum_{n\geq 0} a_n\epsilon^n \in \mathbb{H}[[\epsilon]]$ as q-asymptotic expansion on $Uq^{-\mathbb{N}}$. Let $K \subset U$ be compact set. Then, the function $(\epsilon_1, \epsilon_2) \mapsto \phi(\epsilon_1, \epsilon_2) = f(\epsilon_1 + i\epsilon_2)$ is a C^∞ function in the Whitney sense on the compact set*

$$A = \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 / \epsilon_1 + i\epsilon_2 \in Kq^{-\mathbb{N}} \cup \{0\}\}$$

Proof For all $\epsilon \in Kq^{-\mathbb{N}}$, we have that

$$(\partial_{\epsilon_1}\phi)(\epsilon_1, \epsilon_2) = \partial_\epsilon f(\epsilon) \quad , \quad (\partial_{\epsilon_2}\phi)(\epsilon_1, \epsilon_2) = i\partial_\epsilon f(\epsilon).$$

Then, we define a set of functions $\phi^{(k_1,k_2)}(\epsilon_1, \epsilon_2)$, $(k_1, k_2) \in \mathbb{N}^2$ as follows :

$$\phi^{(k_1,k_2)}(\epsilon_1, \epsilon_2) := i^{k_2}\partial_\epsilon^{k_1+k_2} f(\epsilon)$$

for all $(k_1, k_2) \in \mathbb{N}^2$, all $(\epsilon_1, \epsilon_2) \in A$.

In the next step, we show that the family $\{\phi^{(k_1,k_2)}\}_{(k_1,k_2)\in\mathbb{N}^2}$ satisfies the property $(\mathbf{W})$. From the lemma 5, the function f satisfies the inequality (57) which can be rewritten in terms of the functions $\phi^{(k_1,k_2)}$ as follows. Let $K \subset U$ be a compact set. For all $(k_1, k_2) \in \mathbb{N}^2$, all $m \geq 0$, there exists a constant $C_{K,m,k_1,k_2} > 0$ such that

$$||\frac{1}{i^{k_2}}\phi^{(k_1,k_2)}(x_1, y_1) - \sum_{p=0}^{m} \sum_{h_1+h_2=p} \frac{\phi^{(k_1+h_1,k_2+h_2)}(x_2, y_2)/(i^{k_2+h_2})}{p!}$$
$$\times \frac{p!}{h_1!h_2!}(x_1 - x_2)^{h_1} i^{h_2}(y_1 - y_2)^{h_2}||_{\mathbb{H}} \leq C_{K,m,k_1,k_2}||(x_1 - x_2, y_1 - y_2)||_{\mathbb{R}^2}^{m+1}$$

for all $(x_1, y_1), (x_2, y_2) \in A$. In particular, we get that the inequalities (56) hold. □

In the next proposition, we will see that the cocycle constructed in the proposition 4 splits in the space of Whitney $\mathcal{C}^\infty$ functions.

Proposition 5 *Let $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$ be a good covering and let $\{g_{I,I'}(\epsilon)\}_{(I,I')\in\mathcal{I}^2}$ be the $\mathbb{H}_{\mathcal{T},\rho}$-cocyle constructed in the proposition 4. We choose a family of compact sets $K_I \subset U_I$, for all $I \in \mathcal{I}$, with $\mathrm{Int}(K_I) \neq \emptyset$, in such a way that $\cup_{I\in\mathcal{I}}(K_I q^{-\mathbb{N}})$ is of the form $\mathcal{U} \setminus \{0\}$, where $\mathcal{U}$ is a neighborhood of 0 in $\mathbb{C}$.*
Then, for all $I \in \mathcal{I}$, there exists a C^∞ function $f_I(\epsilon_1, \epsilon_2)$ in the sense of Whitney on the compact set

$$A_I = \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 / \epsilon_1 + i\epsilon_2 \in K_I q^{-\mathbb{N}} \cup \{0\}\},$$

with values in the Banach space $\mathbb{H}_{\mathcal{T},\rho}$*, such that*

$$g_{I,I'}(\epsilon_1 + i\epsilon_2) = f_{I'}(\epsilon_1,\epsilon_2) - f_I(\epsilon_1,\epsilon_2) \tag{58}$$

for all $I, I' \in \mathcal{I}$ *such that* $A_I \cap A_{I'} \neq \emptyset$*, for all* $(\epsilon_1,\epsilon_2) \in A_I \cap A_{I'}$*.*

Proof The proof follows the arguments given in Lemma 3.12 from [10]. First of all, for every $I, I' \in \mathcal{I}$ such that $A_I \cap A_{I'} \neq \emptyset$, the function $(\epsilon_1,\epsilon_2) \mapsto g_{I,I'}(\epsilon_1 + i\epsilon_2)$ is C^∞ in the sense of Whitney on $A_I \cap A_{I'}$. Indeed, by construction, $g_{I,I'}(\epsilon)$ has the $\hat{0}$ formal series as q-asymptotic expansion on $K_I q^{-\mathbb{N}} \cap K_{I'} q^{-\mathbb{N}}$, so that, by the lemma 6, we get the result.

Let one $I \in \mathcal{I}$. Consider a C^∞ function f_I in the sense of Whitney on A_I. We will see how to construct the functions $f_{I'}$ on all the I' such that $A_{I'} \cap A_I \neq \emptyset$.

Case 1. There is at least one $I' \in \mathcal{I}$, $I' \neq I$, such that $A_{I'} \cap A_I \neq \emptyset$ and there is no $I'' \in \mathcal{I}$ with $I'' \neq I' \neq I$ such that $A_I \cap A_{I'} \cap A_{I''} \neq \emptyset$. We put $e_{I,I'}(\epsilon_1,\epsilon_2) = f_I(\epsilon_1,\epsilon_2) + g_{I,I'}(\epsilon_1 + i\epsilon_2)$ for all $(\epsilon_1,\epsilon_2) \in A_I \cap A_{I'}$. From the Whitney extension theorem, see [8], [9], p.67–68, we can extend the C^∞ function $e_{I,I'}$ as a C^∞ function $f_{I'}(\epsilon_1,\epsilon_2)$ in the sense of Whitney on $A_{I'}$. So that

$$g_{I,I'}(\epsilon_1 + i\epsilon_2) = f_{I'}(\epsilon_1,\epsilon_2) - f_I(\epsilon_1,\epsilon_2)$$

for all $(\epsilon_1,\epsilon_2) \in A_I \cap A_{I'}$.

Case 2. There are two $I', I'' \in \mathcal{I}$ with $I' \neq I'' \neq I$, such that $A_I \cap A_{I'} \cap A_{I''} \neq \emptyset$. First, as in case 1, we construct a C^∞ function $f_{I'}(\epsilon_1,\epsilon_2)$ in the sense of Whitney on $A_{I'}$ such that

$$g_{I,I'}(\epsilon_1 + i\epsilon_2) = f_{I'}(\epsilon_1,\epsilon_2) - f_I(\epsilon_1,\epsilon_2) \tag{59}$$

for all $(\epsilon_1,\epsilon_2) \in A_I \cap A_{I'}$. Then, we put

$$e_{I,I''}(\epsilon_1,\epsilon_2) = f_I(\epsilon_1,\epsilon_2) + g_{I,I''}(\epsilon_1 + i\epsilon_2) \tag{60}$$

for all $(\epsilon_1,\epsilon_2) \in A_I \cap A_{I''} \subset A_{I''}$ and

$$e_{I',I''}(\epsilon_1,\epsilon_2) = f_{I'}(\epsilon_1,\epsilon_2) + g_{I',I''}(\epsilon_1 + i\epsilon_2) \tag{61}$$

for all $(\epsilon_1,\epsilon_2) \in A_{I'} \cap A_{I''} \subset A_{I''}$. Now, we show that

$$e_{I,I''}(\epsilon_1,\epsilon_2) = e_{I',I''}(\epsilon_1,\epsilon_2) \tag{62}$$

for all $(\epsilon_1,\epsilon_2) \in A_I \cap A_{I'} \cap A_{I''}$. Indeed, from the relation (59), we get that

$$e_{I',I''}(\epsilon_1,\epsilon_2) = f_I(\epsilon_1,\epsilon_2) + g_{I,I'}(\epsilon_1 + i\epsilon_2) + g_{I',I''}(\epsilon_1 + i\epsilon_2)$$

for all $(\epsilon_1,\epsilon_2) \in A_I \cap A_{I'} \cap A_{I''}$ But, from the the cocycle relation $g_{I,I'} + g_{I',I''} = g_{I,I''}$ on $A_I \cap A_{I'} \cap A_{I''}$, we deduce (62). Now, we put

$$e_{I''}(\epsilon_1,\epsilon_2) = \begin{cases} e_{I,I''}(\epsilon_1,\epsilon_2) & \text{if } (\epsilon_1,\epsilon_2) \in A_I \cap A_{I''}, \\ e_{I',I''}(\epsilon_1,\epsilon_2) & \text{if } (\epsilon_1,\epsilon_2) \in A_I \cap A_{I''}. \end{cases}$$

From Lemma 3.10 in [10], we know that the sets $A_I \cap A_{I''}$ and $A_{I'} \cap A_{I''}$ are regularly separated, so that from the Lojasiewicz theorem, [8], [9] p.83, the function $e_{I''}(\epsilon_1,\epsilon_2)$ is C^∞ in the sense of Whitney on $(A_I \cap A_{I''}) \cup (A_{I'} \cap A_{I''}) \subset A_{I''}$. From the Whitney extension theorem, we deduce that the function $e_{I''}(\epsilon_1, \epsilon_2)$ extends to a C^∞ function in the sense of Whitney on $A_{I''}$ called $f_{I''}(\epsilon_1, \epsilon_2)$. Finally, we have the relation (59), from (60) we get

$$f_{I''}(\epsilon_1, \epsilon_2) = f_I(\epsilon_1, \epsilon_2) + g_{I,I''}(\epsilon_1 + i\epsilon_2)$$

for all $(\epsilon_1, \epsilon_2) \in A_I \cap A_{I''}$ and from (61) we get

$$f_{I''}(\epsilon_1, \epsilon_2) = f_{I'}(\epsilon_1, \epsilon_2) + g_{I',I''}(\epsilon_1 + i\epsilon_2)$$

for all $(\epsilon_1, \epsilon_2) \in A_{I'} \cap A_{I''}$.

By construction of the good covering, we cannot have $A_I \cap A_{I'} \cap A_{I''} \cap A_{I'''} \neq \emptyset$, if $I \neq I' \neq I'' \neq I'''$. So only the case 1 or 2 can appear. Starting from one function f_I on A_I and using the procedures described in cases 1 and 2, we can construct a set of functions f_J, for all $J \in \mathcal{I} \setminus \{I\}$, C^∞ in the sense of Whitney on A_J, which satisfy the relations (58). □

5.3 Existence of a formal series and q-asymptotic expansions

In the next proposition, we will give a useful decomposition of the functions X_I constructed in Proposition 3 in terms of the functions f_I, constructed in Proposition 5, C^∞ and analytic functions on some neighborhood of the origin in $\mathbb{R}^2$.

Proposition 6 *For all $I \in \mathcal{I}$, we denote by $X_I(\epsilon)$, the holomorphic function from $U_I q^{-\mathbb{N}}$ into $\mathbb{H}_{\mathcal{T},\rho}$ defined by*

$$X_I(\epsilon) := (t, z) \mapsto X_I(\epsilon, t, z).$$

Then, there exist a C^∞ function $u(\epsilon_1, \epsilon_2)$ and an analytic function $a(\epsilon_1 + i\epsilon_2)$ on the neighborhood $\mathrm{Int}(\cup_{I \in \mathcal{I}} A_I)$ of the origin in $\mathbb{R}^2$ such that

$$X_I(\epsilon_1 + i\epsilon_2) = f_I(\epsilon_1, \epsilon_2) + u(\epsilon_1, \epsilon_2) + a(\epsilon_1 + i\epsilon_2) \tag{63}$$

for all $I \in \mathcal{I}$, for all $(\epsilon_1, \epsilon_2) \in \mathrm{Int}(A_I)$.

Proof From the definition of the $\mathbb{H}_{\mathcal{T},\rho}$-cocyle $g_{I,I'}(\epsilon_1 + i\epsilon_2)$ and the proposition 5, we get that

$$X_I(\epsilon_1 + i\epsilon_2) - f_I(\epsilon_1, \epsilon_2) = X_{I'}(\epsilon_1 + i\epsilon_2) - f_{I'}(\epsilon_1, \epsilon_2) \tag{64}$$

for all $I, I' \in \mathcal{I}$ such that $A_I \cap A_{I'} \neq \emptyset$, for all $(\epsilon_1, \epsilon_2) \in A_I \cap A_{I'} \setminus \{(0,0)\}$. From the relations (64), we see that the following function defined by

$$(X - f)(\epsilon_1, \epsilon_2) := X_I(\epsilon_1 + i\epsilon_2) - f_I(\epsilon_1, \epsilon_2)$$

for all $(\epsilon_1, \epsilon_2) \in A_I \setminus \{(0,0)\}$, for all $I \in \mathcal{I}$, is well defined on $W \setminus \{(0,0)\}$, where $W = \cup_{I \in \mathcal{I}} A_I$ is a closed neighborhood of $(0,0)$.

We will denote by $\partial_{\bar{\epsilon}}$ the operator defined by

$$\partial_{\bar{\epsilon}} = \frac{1}{2}(\partial_{\epsilon_1} + i\partial_{\epsilon_2}).$$

From the fact that X_I is a holomorphic function on $U_I q^{-\mathbb{N}}$, the Cauchy-Riemann equations yield $\partial_{\bar{\epsilon}} X_I(\epsilon_1 + i\epsilon_2) = 0$ for all $(\epsilon_1, \epsilon_2) \in A_I \setminus \{(0,0)\}$, for all $I \in \mathcal{I}$. We deduce that

$$\partial_{\bar{\epsilon}}(X - f)(\epsilon_1, \epsilon_2) = -\partial_{\bar{\epsilon}} f_I(\epsilon_1, \epsilon_2) \tag{65}$$

for all $(\epsilon_1,\epsilon_2) \in \mathrm{Int}(A_I)$. From the fact that f_I is C^∞ in the sense of Whitney on A_I, we get that $-\partial_{\bar{\epsilon}} f_I(\epsilon_1,\epsilon_2)$ can be extended to a C^∞ function in the sense of Whitney on A_I. We deduce that

$$\partial_{\bar{\epsilon}}(X - f) \in \mathcal{E}(A_I) \tag{66}$$

for all $I \in \mathcal{I}$. On the other hand, from the fact that $g_{I,I'}(\epsilon)$ is holomorphic on $U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}}$, we get from (58) that

$$\partial_{\bar{\epsilon}} f_I(\epsilon_1,\epsilon_2) = \partial_{\bar{\epsilon}} f_{I'}(\epsilon_1,\epsilon_2) \tag{67}$$

for all $(\epsilon_1,\epsilon_2) \in \mathrm{Int}(A_I \cap A_{I'})$, for all $I, I' \in \mathcal{I}$. From the fact that f_I is C^∞ in the sense of Whitney on A_I, we get that the relation (67) can be extended to $A_I \cap A_{I'}$.

From Lemma 3.10 in [10], we know that the sets A_I and $A_{I'}$ are regularly separated. From (66) and (67), we deduce with the Lojasiewicz extension theorem ([9] p.83), that

$$\partial_{\bar{\epsilon}}(X - f) \in \mathcal{E}(\cup_{I \in \mathcal{I}} A_I).$$

In particular, the function $\partial_{\bar{\epsilon}}(X - f)$ defines a C^∞ function in the usual sense on $\mathrm{Int}(W)$.

Now, from [4], theorem 14.4, we know that there exists a C^∞ function in the usual sense $u(\epsilon_1,\epsilon_2)$ on $\mathrm{Int}(W)$ such that

$$\partial_{\bar{\epsilon}} u(\epsilon_1,\epsilon_2) = \partial_{\bar{\epsilon}}(X - f)(\epsilon_1,\epsilon_2) \tag{68}$$

for all $(\epsilon_1,\epsilon_2) \in \mathrm{Int}(W)$. From this latter equality, we deduce that $u(\epsilon_1,\epsilon_2) - (X - f)(\epsilon_1,\epsilon_2)$ defines a holomorphic function on $\mathrm{Int}(W) \setminus \{(0,0)\}$. From the fact that for all $I \in \mathcal{I}$, the functions X_I are bounded (as $\mathbb{H}_{\mathcal{T},\rho}$ valued functions), we deduce that the holomorphic function $u(\epsilon_1,\epsilon_2) - (X - f)(\epsilon_1,\epsilon_2)$ is bounded on $\mathrm{Int}(W) \setminus \{(0,0)\}$. Finally, from the classical result on removable singularities, we deduce that $u(\epsilon_1,\epsilon_2) - (X - f)(\epsilon_1,\epsilon_2)$ can be extended as a holomorphic function on $\mathrm{Int}(W)$. This yields the decomposition (63). □

In the following, we state the main result of the paper.

Theorem 5 *Under the same hypotheses as in Proposition 3, there exists a formal series*

$$\hat{X}(\epsilon,t,z) = \sum_{k \geq 0} \frac{X_k(t,z)}{k!} \epsilon^k \in \mathbb{H}_{\mathcal{T},\rho}[[\epsilon]]$$

which satisfies the singularly perturbed problem

$$\epsilon t \partial_z^S \hat{X}(\epsilon,qt,z) + \partial_z^S \hat{X}(\epsilon,t,z) = \sum_{k=0}^{S-1} b_k(\epsilon,z)(\epsilon t \sigma_q)^{m_{0,k}}(\partial_z^k \hat{X})(\epsilon,t,zq^{-m_{1,k}}) \tag{69}$$

and which is the q-asymptotic expansion on $\tilde{K}_I q^{-\mathbb{N}}$ of the holomorphic functions $X_I(\epsilon,t,z)$ constructed in the proposition 3, where $\tilde{K}_I$ is any compact subset of $\mathrm{Int}(K_I)$, *for all $I \in \mathcal{I}$.*

Proof From the decomposition (63), we get that for all $I \in \mathcal{I}$, the function $X_I(\epsilon_1 + i\epsilon_2)$ can be extended to a C^∞ function in the sense of Whitney on every compact set of the form

$$\tilde{A}_I = \{(\epsilon_1,\epsilon_2) \in \mathbb{R}^2 / \epsilon_1 + i\epsilon_2 \in \tilde{K}_I q^{-\mathbb{N}} \cup \{0\}\} \subset \mathrm{Int}(A_I) \cup \{(0,0)\}$$

where $\tilde{K}_I$ is a compact set in $\mathrm{Int}(K_I)$. Let $I \in \mathcal{I}$. We consider the family of continuous functions $X_I^{(h_1,h_2)}(\epsilon_1,\epsilon_2)$ on $\tilde{A}_I$ associated to X_I from the definition 7, for all $(h_1,h_2) \in \mathbb{N}^2$. From the fact that $X_I(\epsilon)$ is holomorphic on $\mathrm{Int}(K_I)q^{-\mathbb{N}}$, we have the relations

$$X_I^{(h_1,h_2)}(\epsilon_1,\epsilon_2) = \partial_{\epsilon_1}^{h_1}\partial_{\epsilon_2}^{h_2}X_I(\epsilon_1 + i\epsilon_2) = i^{h_2}\partial_{\epsilon}^{h_1+h_2}X_I(\epsilon)$$

for all $\tilde{A}_I \setminus \{(0,0)\}$. By continuity of the functions $X_I^{(h_1,h_2)}$ at $(0,0)$, we deduce that there exists a sequence $X_{k,I} \in \mathbb{H}_{\mathcal{T},\rho}$, $k \geq 0$, such that

$$\frac{X_I^{(h_1,h_2)}(0,0)}{i^{h_2}} = X_{k,I} \tag{70}$$

for all $(h_1,h_2) \in \mathbb{N}^2$ such that $h_1 + h_2 = k$.

We now apply the estimates (56) satisfied by the family X_I^h for $k = (0,0)$ and $(x_2,y_2) = (0,0)$. Using the relations (70), we get that for all $m \geq 0$, all $\epsilon > 0$, there exists $\delta > 0$ such that

$$||X_I(\epsilon_1 + i\epsilon_2) - \sum_{k=0}^{m+1}\frac{X_{k,I}}{k!}(\epsilon_1 + i\epsilon_2)^k||_{\mathbb{H}_{\mathcal{T},\rho}} < \epsilon|\epsilon_1 + i\epsilon_2|^{m+1} \tag{71}$$

for all $\epsilon_1 + i\epsilon_2 \in \tilde{K}_I q^{-\mathbb{N}}$ with $|\epsilon_1 + i\epsilon_2| < \delta$. This implies, in particular, estimates of the form (45). This shows that the formal series $\hat{X}_I(\epsilon) = \sum_{k\geq 0} X_{k,I}\epsilon^k/k!$ is the q-asymptotic expansion of X_I on $\tilde{K}_I q^{-\mathbb{N}}$, for every $I \in \mathcal{I}$.

Now, we show that $\hat{X}_I$ is independent of I. Indeed, let $I, I' \in \mathcal{I}$ such that $\tilde{K}_I q^{-\mathbb{N}} \cap \tilde{K}_{I'} q^{-\mathbb{N}} \neq \emptyset$. We get that the formal series $\hat{X}_{I'}(\epsilon) - \hat{X}_I(\epsilon)$ is the q-asymptotic expansion of $X_{I'}(\epsilon) - X_I(\epsilon) = g_{I',I}(\epsilon)$ on $\tilde{K}_I q^{-\mathbb{N}} \cap \tilde{K}_{I'} q^{-\mathbb{N}}$. But, from the proposition 3, we know that $g_{I',I}(\epsilon)$ has $\hat{0}$ as q-asymptotic expansion on $\tilde{K}_I q^{-\mathbb{N}} \cap \tilde{K}_{I'} q^{-\mathbb{N}}$. From the unicity of the q-asymptotic expansion, we deduce that $\hat{X}_{I'}(\epsilon) = \hat{X}_I(\epsilon)$.

In conclusion, there exist a formal series $\hat{X}(\epsilon) = \sum_{k\geq 0} X_k\epsilon^k/k!$ which is the q-asymptotic expansion of X_I on $\tilde{K}_I q^{-\mathbb{N}}$, for every $I \in \mathcal{I}$.

In the last step of the proof, we will check that $\hat{X}(\epsilon,t,z)$ satisfies the equation (51). From the remark after the definition 2, we get that

$$\lim_{\epsilon\to 0, \epsilon \in \tilde{K}_I q^{-\mathbb{N}}} \partial_\epsilon^l X_I(\epsilon,t,z) = X_l(t,z) \in \mathbb{H}_{\mathcal{T},\rho}, \tag{72}$$

for all $l \geq 0$, all $I \in \mathcal{I}$. Let $I \in \mathcal{I}$. By construction, the function $X_I(\epsilon,t,z)$ satisfies the equation (51). Taking the $l-$derivative of the right and left handsides of the equation (51) with respect to ϵ, we deduce that $\partial_\epsilon^l X_I(\epsilon,t,z)$ satisfies the following equation

$$\begin{aligned}\epsilon t\partial_z^S\partial_\epsilon^l X_I(\epsilon,qt,z) + t\partial_z^S l\partial_\epsilon^{l-1}X_I(\epsilon,qt,z) + \partial_z^S\partial_\epsilon^l X_I(\epsilon,t,z)&\\ = \sum_{k=0}^{S-1}\sum_{l_1+l_2=l}\frac{l!}{l_1!l_2!}\partial_\epsilon^{l_1}(b_k(\epsilon,z)\epsilon^{m_{0,k}})\partial_\epsilon^{l_2}((t\sigma_q)^{m_{0,k}}\partial_z^k X_I)(\epsilon,t,zq^{-m_{1,k}}))&\end{aligned} \tag{73}$$

for all $l \geq 1$, all $(t,z) \in \mathcal{T} \times D(0,\rho)$, all $\epsilon \in \tilde{K}_I q^{-\mathbb{N}}$. Letting ϵ tend to 0 in (73) yields the recursion

$$t\partial_z^S\frac{X_{l-1}(qt,z)}{(l-1)!} + \partial_z^S\frac{X_l(t,z)}{l!} = \sum_{k=0}^{S-1}\sum_{l_1+l_2=l}\frac{\partial_\epsilon^{l_1}(\epsilon^{m_{0,k}}b_k(\epsilon,z))|_{\epsilon=0}}{l_1!}\frac{((t\sigma_q)^{m_{0,k}}\partial_z^k X_{l_2})(t,zq^{-m_{1,k}})}{l_2!} \tag{74}$$

for all $l \geq 1$, all $(t,z) \in \mathcal{T} \times D(0,\rho)$. From the fact that $b_k(\epsilon,z)$ is analytic with respect to ϵ at 0, we have that

$$\epsilon^{m_{0,k}} b_k(\epsilon,z) = \sum_{l \geq 0} \frac{\partial_\epsilon^l (\epsilon^{m_{0,k}} b_k(\epsilon,z))|_{\epsilon=0}}{l!} \epsilon^l \tag{75}$$

for all ϵ on a neighborhood of 0, all $z \in \mathbb{C}$. Finally, one checks from the recursion (74) and (75), that the formal series $\hat{X}(\epsilon,t,z) = \sum_{k \geq 0} X_k(t,z)\epsilon^k/k!$ satisfies the equation (51). □

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