CONNECTING ORBITS OF AUTONOMOUS LAGRANGIAN SYSTEMS

XIA LI AND CHONG-QING CHENG

Abstract. Different from time-dependent case, the time-t-section of Mañé set of autonomous Lagrangian systems is independent of time, thus, it is nowhere disconnected. It induces some difference in the study of dynamics, for instance, Mather's c-equivalence can not exist among those cohomology classes if they are not in a flat of the α function (cf. [Be1]). In this paper, we show how to construct connecting orbits in autonomous systems, and propose a modified c-equivalence. We also apply the result to construct diffusion orbits in an energy surface.

1. Introduction

In this paper, we study autonomous Lagrangian systems of Tonelli type, i.e. let $L : TM \to \mathbb{R}$ ($M = \mathbb{T}^n$) be a C^r-function ($r = 3, 4, \cdots, m, \cdots, \infty$) which satisfies the following conditions:

Positive definiteness. For each $x \in M$, the Lagrangian function is strictly convex in velocity: the Hessian $L_{\dot{x}\dot{x}}$ is positive definite.

Super-linear growth. L has fiber-wise superlinear growth: for each $x \in M$, we have $L/\|\dot{x}\| \to \infty$ as $\|\dot{x}\| \to \infty$.

For autonomous systems, in the Hamiltonian formalism, each orbit entirely stays in some energy surface which is compact because of the superlinearity. Therefore, all solutions of the Lagrangian equations are well defined for all $t \in \mathbb{R}$, i.e. the completeness is automatically satisfied.

Let ϕ_L^t denote the Lagrangian flow on TM, determined by the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

The solution of the equation corresponds to the critical point of the functional

$$A_c(\gamma) = \int (L - \eta_c)(\gamma, \dot{\gamma})dt,$$

where $\eta_c = \langle \eta_c(x), \dot{x}\rangle$, $\eta_c(x)$ denotes a closed 1-form $\langle \eta_c(x), dx\rangle$ evaluated at x, and its de-Rham cohomology $[\langle \eta_c(x), dx\rangle] = c \in H^1(M, \mathbb{R})$. For convenience and without danger of confusion, we call η_c closed 1-form also.

For Lagrangians of Tonelli type, Mather proved the existence of minimal measures and introduced the concept of Aubry set, Mañé set and barrier function (see [Ma1], [Ma2]). One can refer [It] for Lagrangians with more complex dependence on time.

Let $\mathfrak{M}$ be the set of Borel probability measures on TM, invariant for ϕ_L^t. Given an invariant measure μ such that $\int L d\mu < \infty$, the rotation vector $\omega(\mu) \in H_1(M, \mathbb{R})$ is the unique vector such that

$$\langle c, \omega(\mu)\rangle = \int \eta_c d\mu$$

holds for every closed 1-form η_c whose de Rham cohomology class $[\eta_c] = c$. For each $\nu \in \mathfrak{M}$ and $\int L d\nu < \infty$, the action $A_c(\nu)$ is defined as the following:

$$A_c(\nu) = \int (L - \eta_c) d\nu.$$

It is shown in [Ma1] that for each first de Rham cohomology class c there exists at least one invariant probability measure μ which minimizes the actions over $\mathfrak{M}$

$$A_c(\mu) = \inf_{\nu \in \mathfrak{M}} \int (L - \eta_c) d\nu.$$

From the super-linear growth of L in $\dot{x}$, one can easily see that the support of each c-minimal measure is compact. We use $\tilde{\mathcal{M}}(c)$ to denote the support of all c-minimal measures and call it Mather set. We use $-\alpha(c)$ to denote the average action of $L - \eta_c$: $\alpha(c) = -A_c(\mu)$ where μ is a c-minimal measure. It is usually called the α-function, its Legendre transformation β: $H_1(M, \mathbb{R}) \to \mathbb{R}$ is called β-function which is a function of rotation vectors.

Let $\gamma : \mathbb{R} \to M$ be an absolutely continuous curve. We call it a minimizer of $L - \eta_c$ if for any $a < b$ and any absolutely continuous curve $\zeta : [a, b] \to M$ with $\zeta(a) = \gamma(a)$, $\zeta(b) = \gamma(b)$ we have

$$\int_a^b (L - \eta_c)(d\gamma(t))dt \le \int_a^b (L - \eta_c)(d\zeta(t))dt,$$

where $d\gamma = (\gamma, \dot{\gamma})$. The minimizer is actually C^1 because of the completeness and $d\gamma(t)$ is a trajectory of the Euler-Lagrange flow [Ma1]. We define

$$h_c^t(m, m') = \min_{\substack{\gamma \in C^1([-t,t],M) \\ \gamma(-t)=m, \gamma(t)=m'}} \int_{-t}^{t} (L - \eta_c)(d\gamma(s))ds + 2t\alpha(c),$$

$$F_c(m, m') = \inf_{t>0} h_c^t(m, m'),$$

$$h_c^\infty(m, m') = \liminf_{t \to \infty} h_c^t(m, m').$$

Note that h_c, F_c and h_c^∞ depend on the choice of η_c. If $\eta_c' = \eta_c + df$, where f is a smooth function on M, then $h_c'(m,m') = h_c(m,m') + f(m') - f(m)$, but this would not change the class of curves which reach the minimum. According to a result in [Fa1], this limit infimum is actually a limit as $t \to \infty$. For non-autonomous systems, this limit may not exist (cf. [FM]). Given an absolutely continuous curve γ: $[a,b] \to M$, we use the notation

$$[A_c(\gamma|_{[a,b]})] = \int_a^b (L - \eta_c)(d\gamma(t))dt + (b-a)\alpha(c).$$

A curve $\gamma \in C^1(\mathbb{R}, M)$ is called c-semi-static if

$$[A_c(\gamma|_{[a,b]})] = F_c(\gamma(a), \gamma(b))$$

for each $[a,b] \subset \mathbb{R}$. In the same way, c-semi-static can be defined for one sided curve $(-\infty, 0] \to M$ and $[0,\infty) \to M$, called backward and forward c-semi-static curve respectively.

A curve $\gamma \in C^1(\mathbb{R}, M)$ is called c-static if, in addition

$$[A_c(\gamma|_{[a,b]})] = -F_c(\gamma(b), \gamma(a))$$

for each $[a,b] \subset \mathbb{R}$. An orbit $X(t) = d\gamma(t)$ is called c-static (semi-static) if γ is c-static (semi-static). We call the union of global c-semi-static orbits Mañé set $\tilde{\mathcal{N}}(c)$, and call the union of c-static orbits Aubry set $\tilde{\mathcal{A}}(c)$.

We use $\mathcal{M}(c)$, $\mathcal{A}(c)$ and $\mathcal{N}(c)$ to denote the standard projection of $\tilde{\mathcal{M}}(c)$, $\tilde{\mathcal{A}}(c)$ and $\tilde{\mathcal{N}}(c)$ from TM to M respectively. The following inclusions hold

$$\tilde{\mathcal{M}}(c) \subseteq \tilde{\mathcal{A}}(c) \subseteq \tilde{\mathcal{N}}(c).$$

The inverse of the projection is Lipschitz when it is restricted to $\mathcal{A}(c)$ and $\mathcal{M}(c)$, the Mañé set is the same as the Aubry set if the minimal measure is uniquely ergodic (cf. [Ma2]). We mention that the definition of Mañé set depends on what configuration manifold we choose. If we choose some finite covering manifold, the Mañé set could be larger.

An interesting problem of Lagrangian dynamics is to find connecting orbits from an Aubry set $\tilde{\mathcal{A}}(c)$ to another one $\tilde{\mathcal{A}}(c')$. We call a connecting orbit *global* if the corresponding cohomology classes c and c' are not close to each other, i.e. $|c - c'| = O(1)$. In autonomous case, it is possible only when $\alpha(c) = \alpha(c')$. In the language of Hamiltonian dynamics, it amounts to say that the orbit has to stay entirely in a surface of energy level $\{H = \alpha(c)\}$. Due to Carneiro ([Ca]), along c-minimal orbit the system has energy $\alpha(c)$. Throughout this paper, we assume that $\alpha(c) = 0$ for all cohomology classes under consideration. This can be realized by adding a constant $\alpha(c)$ to the Lagrangian.

In this paper, global connecting orbits are constructed along a sequence of finitely many successive *local* connecting orbits. Toward this goal, we find a path in the cohomology group connecting from c to c', Γ: $[0,1] \to H^1(M,\mathbb{R})$ with $\Gamma(0) = c$, $\Gamma(1) = c'$ and each $\Gamma(s)$-minimal measure has the same average action. For each $s \in [0,1]$ $\exists$ $\delta > 0$ such that there exists local connecting orbit from $\tilde{\mathcal{A}}(\Gamma(s))$ to $\tilde{\mathcal{A}}(\Gamma(s'))$ whenever $s' \in [0,1]$ and $|s - s'| < \delta$. The existence of a sequence of local connecting orbits is an immediate consequence. In this paper, two types of local connecting orbits are constructed for autonomous systems, we call them c-type and h-type respectively.

To construct a global connecting orbit shadowing this sequence of local connecting orbits, some local minimality is required to imposed on the curves which generate h-type of local connecting orbits. This local minimality for c-type of local connecting orbits appears to be naturally given. Roughly speaking, we call it locally minimal if the action along this curve is smaller than the action along any other curve staying in a small neighborhood of this curve. This minimal property appears to be certain variational version of the transversal intersection of the "stable set" of an Aubry set with the "unstable set" of another Aubry set. These sets are regarded as certain generalization of stable and unstable manifolds, and are generated by so-called weak KAM solutions to the Hamilton-Jacobi equation (cf. [Fa2]).

The sequence of successive local minimal connecting orbits induces a *generalized* transition chain, the construction of global connecting orbit appears to be a combination (cf. [CY2]) of the variational version of the argument of Arnold in [Ar1] and the technique of c-equivalence [Ma2].

In [CY1] and [CY2], local and global connecting orbits are constructed for time-periodic systems, by assuming the disconnectedness of the time-1-section of the whole or part of the Mañé set. In these works, the disconnectedness is also proved to be a generic property in *a priori* unstable systems. Different from time-periodic case, the time-t-section of Mañé set of autonomous systems is independent of time, thus, it is nowhere disconnected. It induces some difference in the implementation of the strategy as above, the arguments in [CY1] and [CY2] can not be directly applied. For autonomous systems, we need to find some co-dimension one sections in configuration space which are transversal to those local as well as global connecting curves. In this paper, we are able to construct global connecting orbits by assuming some kind of disconnectedness on these sections, and show it is a generic phenomenon in *a priori* unstable systems.

This paper is organized as follows. In section 2, we construct two types of local connecting orbits specifically for autonomous systems. With a sequence of these local connecting orbits, some global connecting orbits are constructed in section 3. In section 4, we show the existence of Arnold diffusion in autonomous systems of *a priori* unstable type, it is an application of the result obtained in the section 3.

2. Local Connecting Orbits

In this section, two types of local connecting orbits in autonomous systems are constructed, which are the building blocks for the construction of global connecting orbits. For convenience of notation, we call them c-type and h-type of local connecting orbits respectively. In our previous paper [CY2], these types of local connecting orbits have been constructed for time-periodic case. However, the conditions required there can not be satisfied in autonomous case.

To construct c-type of local connecting orbits, let us recall the concept of c-equivalence (cf. [Ma2]). There Mather introduced an equivalent relation in the group of first cohomology classes, sometimes called c-equivalence (where c refers to the fact that it is an equivalence relation between cohomology classes). With this definition, Mather pointed out that the two Aubry sets can be connected by an orbit if the corresponding cohomology classes are c-equivalent. However, this theorem is of no interest for autonomous case, as it was shown in [Be1] that c is equivalent to c' if and only if c and c' are contained in a flat of the α-function. From [Mas] it follows that if two cohomology classes are in the interior of the same flat of the α-function then the corresponding Aubry sets coincide. Therefore, connecting orbits are trivial.

To formulate a modified version of c-equivalence, we do not intend to deal with the general case. Instead, we consider a special but typical case. Let $\mathfrak{C} \subset H^1(\mathbb{T}^n,\mathbb{R})$ be a connected set where we are going to define c-equivalence. For each cohomology class $c \in \mathfrak{C}$, we assume that there exists a non-degenerate embedded $(n-1)$-dimensional torus $\Sigma_c \subset \mathbb{T}^n$ such that, restricted on the Mañé set $\mathcal{N}(c)$, the flow $\pi_x\phi_L^t$ is transversal to Σ_c. Here we use π_x to denote the standard projection $\pi_x(x,\dot{x}) = x$. Let

$$V_c = \bigcap_U \{i_{U*}H_1(U,\mathbb{R}) : U \text{ is a neighborhood of} \mathcal{N}(c)\cap\Sigma_c\}.$$

Here, we use i_U: $U \to M$ to denote the inclusion map and use i_{U*} and i_U^* to denote the maps induced on the homology and cohomology groups respectively. $V_c^{\perp}$ is defined to be the annihilator of V_c, i.e. if $c' \in H^1(\mathbb{T}^n,\mathbb{R})$, then $c' \in V_c^{\perp}$ if and only if $\langle c', h\rangle = 0$ for all $h \in V_c$. Clearly,

$$V_c^{\perp} = \bigcup_U \{\ker i_U^* : U \text{ is a neighborhood of} \mathcal{N}(c)\cap\Sigma_c\}.$$

Note that there exists a neighborhood U of $\mathcal{N}(c)\cap\Sigma_c$ such that $V_c = i_{U*}H_1(U,\mathbb{R})$ and $V_c^{\perp} = \ker i_U^*$ (see [Ma2]).

We say that $c, c' \in H^1(M,\mathbb{R})$ are c-equivalent if there exists a continuous curve Γ: $[0,1] \to \mathfrak{C}$ such that $\Gamma(0) = c$, $\Gamma(1) = c'$, $\alpha(\Gamma(s))$ keeps constant for all $s \in [0,1]$, and for each $s_0 \in [0,1]$ there exists $\delta > 0$ such that $\Gamma(s) - \Gamma(s_0) \in V^{\perp}_{\Gamma(s_0)}$ whenever $s \in [0,1]$ and $|s - s_0| < \delta$.

In this paper, we call Σ_c non-degenerate embedded $(n-1)$-dimensional torus by assuming a smooth injection φ: $\mathbb{T}^{n-1} \to \mathbb{T}^n$ such that Σ_c is the image of φ, and the induced map φ_*: $H_1(\mathbb{T}^{n-1}, \mathbb{Z}) \to H_1(\mathbb{T}^n, \mathbb{Z})$ is an injection. Let $\{e_i\}_{1\le i\le n-1}$ be the standard basis of $H_1(\mathbb{T}^{n-1}, \mathbb{Z})$, $g_{i+1} = \varphi_*(e_i) \in H_1(\mathbb{T}^n, \mathbb{Z})$. Obviously, g_i is an n-dimensional integer vector for $i = 2, 3, \cdots, n$. Because φ is injection, there exists $g_1 \in \mathbb{Z}^n$ such that the $n \times n$ matrix $G_c = (g_1, g_2, \cdots, g_n)$ is uni-module, i.e. $\det G_c = \pm 1$. (see the appendix for the existence proof of G_c.)

Let $q = G_c^{-1}x$, the Lagrangian $\tilde{L}(q, \dot{q}) = L(G_c q, G_c \dot{q})$ is also 2π-periodic in q. In this sense, we call $x \to q = G_c^{-1}x$ coordinate transformation on torus. In q-coordinates, let $\bar{M} = \mathbb{R} \times \mathbb{T}^{n-1} = \{q_1 \in \mathbb{R}, (q_2, \cdots, q_n) \in \mathbb{T}^{n-1}\}$ be a covering of $\mathbb{T}^n$, $\pi : \bar{M} \to M = \mathbb{T}^n$. The lift of Σ_c, $\pi^{-1}(\Sigma_c)$ has infinitely many compact components $\{\Sigma_c^i\}_{i\in\mathbb{Z}}$. If φ is linear, $\Sigma_c^i = \{q_1 = 2i\pi\}$.

Σ_c is said admissible for some ergodic component μ_c if $\exists$ $q = G_c^{-1}x$-coordinates such that the first component of the rotation vector of μ_c, $\omega_1(\mu_c) > 0$, and the lift of Σ_c in $\mathbb{R} \times \mathbb{T}^{n-1}$ has infinitely many compact components. Σ_c is said admissible for some c-minimal measure if it is admissible for each ergodic component of the measure.

Assume Σ_c is admissible for all c-minimal measures. Let $\bar{\gamma}_1$ denote the first component of $\bar{\gamma} = (\bar{\gamma}_1, \cdots, \bar{\gamma}_n)$ which is the lift of a c-semi-static curve γ, Clearly, in the q-coordinate system, $|\bar{\gamma}_1(t)| \to \infty$ as $t \to \pm\infty$. If $\bar{\gamma}_1(t) \to \infty$ $(\to -\infty)$ as $t \to \infty$, then $\bar{\gamma}_1(t) \to -\infty$ $(\to \infty)$ as $t \to -\infty$. We assume that $\bar{\gamma}_1(t) \to \pm\infty$ as $t \to \pm\infty$ which can be realized by choosing g_1 or $-g_1$ in the matrix G_c.

In the following, for each $c \in \mathfrak{C}$, we introduce such a coordinate transformation on torus $q = G_c^{-1}x$, work in q-coordinate system, and $\tilde{L}$ is still denoted by L. When we use the covering space $\bar{M} = \mathbb{R} \times \mathbb{T}^{n-1}$, it implies that this coordinate transformation has been done. Note that the c'-semi-static curves also intersects Σ_c transversally if $|c' - c|$ is sufficiently small, because of the upper semi-continuity of $\tilde{\mathcal{N}}(c)$ with respect to c ([Ma2]).

The codimension one hypersurface Σ_c^0 separates $\bar{M}$ into two parts, the upper part $\bar{M}^+$ and the lower part $\bar{M}^-$. Let $\Sigma_c^0 + \delta$ denotes the δ-neighborhood of Σ_c^0 in $\bar{M}$, we introduce a smooth function $\varrho \in C^r(\bar{M}, [0,1])$ such that $\varrho = 0$ if $q \in \bar{M}^- \backslash (\Sigma_c^0 + \delta)$, $\varrho = 1$ if $q \in \bar{M}^+ \backslash (\Sigma_c^0 + \delta)$. Let η and $\bar{\mu}$ be closed 1-forms on M such that $[\eta] = c$ and $[\eta + \bar{\mu}] = c'$. These forms have natural lifts on $\bar{M}$. Let $\mu = \varrho\bar{\mu}$ and

$$L_{\eta,\mu} = L - \eta - \mu.$$

Let $\bar m, \bar m'$ be two points in $\bar M$. Since $\alpha(c) = \alpha(c') = 0$ is assumed, we define

$$h^T_{\eta,\mu}(\bar m, \bar m') = \inf_{\substack{\gamma(-T)=\bar m\\ \gamma(T)=\bar m'}} \int_{-T}^{T} L_{\eta,\mu}(d\gamma(t))dt.$$

Lemma 2.1. *If the rotation vector of each ergodic component of a c-minimal measure has positive first component $\omega_1(\mu_c) > 0$, $|\bar m - \bar m'| > 0$ and $|c - c'|$ is suitably small, then*

$$\lim_{\substack{T\to 0,\text{or}\\ T\to\infty}} h^T_{\eta,\mu}(\bar m, \bar m') = \infty.$$

Proof. The case for $T \to 0$ is an immediate consequence of the super-linear growth of L in $\dot q$.

Let $\bar\gamma^T_{\eta,\mu}$: $[-T, T] \to \bar M$ be the minimizer of $h^T_{\eta,\mu}(\bar m, \bar m')$, and $d\gamma^T_{\eta,\mu}$ denote the projection of $d\bar\gamma^T_{\eta,\mu}$ to TM. Choosing $\mu = 0$, let ν_T denote the probability measure evenly distributed along $\gamma^T_{\eta,0}$ and let ν be a point of accumulation of ν_T, we have:

$$A_c(\nu) = \int L_{\eta,0}d\nu \le \liminf_{T\to\infty} \int L_{\eta,0}d\nu_T = \liminf_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} L_{\eta,0}(d\gamma(t))dt.$$

Obviously, there exists C such that $\int L_{\eta,0}d\nu_T \le C$ for all $T \ge 1$. Given any 1-form λ on M, the mapping $\mu \to \int \lambda d\mu$ is continuous on the set of probability measure μ such that $\int L d\mu \le C$ (cf. [Ma1]). Because $\bar m, \bar m'$ are fixed, $\omega_1(\nu_T) = (\bar m'_1 - \bar m_1)/2 \to 0$ as $T \to \infty$, where $\bar m'_1, \bar m_1$ denote the first coordinate of $\bar m', \bar m$ respectively. Consequently, the first component of rotation vector of ν is zero. Because of $\omega_1(\mu_c) > 0$ for each c-minimal measure μ_c, we obtain that the average action of ν is positive $A_c(\nu) > 0$.

As we assume the condition of super-linear growth, there exists $D > 0$ such that $|\dot{\bar\gamma}^T_{\eta,\mu}| \le D$ for each $t \in [-T, T]$ and all $T > 1$. By letting $|c - c'|$ be suitably small, we can choose μ such that $|L_{\eta,\mu} - L_{\eta,0}| < \frac{1}{2}A(\nu)$ if we restrict them in the region $\{|\dot q| \le D\} \times \bar M$. Thus, we obtain

$$\begin{aligned} h^T_{\eta,\mu}(\bar m, \bar m') &= \int_{-T}^{T} L_{\eta,\mu}(d\bar\gamma^T_{\eta,\mu}(t))dt \\ &> \int_{-T}^{T} L_{\eta,0}(d\bar\gamma^T_{\eta,\mu}(t))dt - A(\nu)T \\ &\ge \int_{-T}^{T} L_{\eta,0}(d\bar\gamma^T_{\eta,0}(t))dt - A(\nu)T \\ &\ge A(\nu)T \to \infty \quad \text{as } T \to \infty, \end{aligned} \tag{2.1}$$

this completes the proof. □

Definition 2.1. *The pseudo connecting orbit set is defined as:*

$$\mathscr{C}_{\eta,\mu}=\Big\{\bar\gamma:\mathbb{R}\to\bar M,\ \text{for all } T>0 \text{ the following holds}\\ \int_{-T}^{T}L_{\eta,\mu}(d\bar\gamma(t))dt=\inf_{T'>0}h^{T'}_{\eta,\mu}(\bar\gamma(-T),\bar\gamma(T))\Big\}.$$

Proposition 2.1. *If $\alpha([\eta])=\alpha([\eta+\bar\mu])=0$, the rotation vector of each ergodic $[\eta]$-minimal measure as well as of each ergodic $[\eta+\bar\mu]$-minimal measure has positive first component, then the set $\mathscr{C}_{\eta,\mu}$ is non-empty. For each curve γ: $\mathbb{R}\to M$ such that $\bar\gamma\in\mathscr{C}_{\eta,\mu}$, the ω-limit set $\omega(d\gamma)\subseteq\tilde{\mathcal{A}}([\eta+\bar\mu])$ and the α-limit set $\alpha(d\gamma)\subseteq\tilde{\mathcal{A}}([\eta])$.*

Proof. Denote by $\bar\gamma_{\eta,\mu}(t,\bar m,\bar m'):[-T,T]\to M$ the minimizer of the quantity $\inf_{T'>0}h^{T'}_{\eta,\mu}(\bar m,\bar m')$, i.e. $\bar\gamma_{\eta,\mu}(-T,\bar m,\bar m')=\bar m$, $\bar\gamma_{\eta,\mu}(T,\bar m,\bar m')=\bar m'$ and

$$[A(\bar\gamma_{\eta,\mu})]=\int_{-T}^{T}L_{\eta,\mu}(d\bar\gamma_{\eta,\mu}(t))dt=\inf_{T'>0}h^{T'}_{\eta,\mu}(\bar m,\bar m').$$

Because of the lemma 2.1, this infimum is attained for finite $T>0$ if $\bar m$ and $\bar m'$ are two different points in $\bar M$. The super-linear growth condition of L guarantees that $T\to\infty$ as $-\bar m_1,\bar m_1'\to\infty$, where $\bar m_1$ denotes the first coordinate of $\bar m$. Given an interval $[-T_0,T_0]$, for sufficiently large $-\bar m_1,\bar m_1$, the set $\{\bar\gamma_c(t,\bar m,\bar m')|_{[-T_0,T_0]}\}$ is pre-compact in $C^1([-T_0,T_0],M)$. Let $T_0\to\infty$, by diagonal extraction argument, we can find a subsequence of $\{\bar\gamma_c(t,\bar m,\bar m')\}$ which converges C^1-uniformly on each compact set to a C^1-curve $\bar\gamma$: $\mathbb{R}\to\bar M$. Obviously, this curve belongs to $\mathscr{C}_{\eta,\mu}$. On the other hand, each curve $\bar\gamma\in\mathscr{C}_{\eta,\mu}$ is an accumulation point of the set $\{\bar\gamma_c(t,\bar m,\bar m')|_{[-T_0,T_0]}\}$ ($T_0\to\infty$ as $-\bar m_1,\bar m_1'\to\infty$). In fact, given a curve $\bar\gamma\in\mathscr{C}_{\eta,\mu}$, the minimizer of $\inf_{T'>0}h^{T'}_{\eta,0}(\bar\gamma(-T),\bar\gamma(T))$ is $\bar\gamma|_{[-T,T]}$. Thus, $\bar\gamma$ is an accumulation point of the sequence $\{\bar\gamma|_{[-T,T]},T\to\infty\}$.

By the construction of $\mathscr{C}_{\eta,\mu}$, $\bar\gamma_1(t)\to-\infty$ as $t\to-\infty$ and $\bar\gamma_1(t)\to\infty$ as $t\to\infty$. Note $\varrho=0$ if $q\in\bar M^-\backslash(\Sigma^0_c+\delta)$ and $\varrho=1$ if $q\in\bar M^+\backslash(\Sigma^0_c+\delta)$, γ is $[\eta]$-semi-static for $q\in\bar M^-\backslash(\Sigma^0_c+\delta)$ and is $[\eta+\bar\mu]$-semi-static for $q\in\bar M^+\backslash(\Sigma^0_c+\delta)$, it is apparent that $\omega(d\gamma)\subseteq\tilde{\mathcal{A}}([\eta+\bar\mu])$ and $\alpha(d\gamma)\subseteq\tilde{\mathcal{A}}([\eta])$. □

Lemma 2.2. *If $\omega_1(\mu_c)>0$ for each ergodic c-minimal measure μ_c, then a curve $\bar\gamma\in\mathscr{C}_{\eta,0}$ if and only if its projection $\gamma=\pi\bar\gamma$: $\mathbb{R}\to M$ is c-semi-static.*

Proof. A curve in $\mathscr{C}_{\eta,0}$ implies that it is c-semi-static in $\bar M$. If γ is c-semi-static in M, its lift $\bar\gamma$ must be c-semi-static in its covering $\bar M$, i.e. $\bar\gamma\in\mathscr{C}_{\eta,0}$.

For the other direction of the proof, as the first step, we claim that there exists $a>0$ such that

$$\inf_{T'>0}h^{T'}_{\eta,0}(\bar\gamma(-T),\bar\gamma(T))<a$$

holds for each curve $\bar\gamma \in \mathscr{C}_{\eta,0}$ and each $T > 0$. In fact, we pick up the lift of one semi-static curve ζ, denoted by $\bar\zeta$, and join $\bar\gamma(-T)$ to an intersection point of $\bar\zeta$ with $\{q_1 = \bar\gamma_1(-T)\}$ by a curve ξ^-: $[0,1] \to \bar M$ and join the intersection point of $\bar\zeta$ with $\{q_1 = \bar\gamma_1(T)\}$ to the point $\bar\gamma(T)$ by a curve ξ^+: $[0,1] \to \bar M$. The action along the curve $\xi^- \cup \bar\zeta|_{\bar\gamma_1(-T)\le q_1\le\bar\gamma_1(T)} \cup \xi^+$ is uniformly bounded for all $T > 0$.

Secondly, for each small $\epsilon > 0$, some large $T = T(\epsilon)$ exists such that in each segment of $\bar\gamma|_{[-T,T]}$ there is a point $\bar q$ with $\pi\bar q \in \mathcal{N}(c) + \epsilon$. Otherwise, there would be an invariant measure approached by $d\pi\bar\gamma$ which is not minimal. Consequently, the action long $\bar\gamma$ would be unbounded, but this is absurd. Repeat this argument for infinite segments of this curve, we see that there are infinite many points $\bar q^i$ along $\bar\gamma$ such that $\pi\bar q^i \in \mathcal{N}(c) + \epsilon$. As $\mathcal{N}(c)$ is compact, there is a c-semi-static curve ζ such that along its lift $\bar\zeta$ there are infinitely many points $\bar q^{*,i}$ with $\bar q^i_1 = \bar q^{*,i}_1$ and $|\bar q^i - \bar q^{*,i}| < \epsilon$.

For the third step, we let $\bar\zeta|_{I_{i,j}}$ denote the segment of $\bar\zeta$ between $\bar q^{*,i}$ and $\bar q^{*,j}$, $\bar\gamma|_{I_{i,j}}$ the segment of $\bar\gamma$ between $\bar q^i$ and $\bar q^j$. It is easy to see

$$[A_c(\bar\gamma|_{I_{i,j}})] - [A_c(\bar\zeta|_{I_{i,j}})] < C\epsilon$$

where C is a constant independent of the length of $I_{i,j}$. Thus, if we assume that γ is not semi-static, there would be two points $m = \gamma(t_0)$ and $m' = \gamma(t_1)$ such that these two points could be joined by another curve ξ: $[t_0, t_1'] \to M$ and $[A_c(\gamma)|_{[t_0,t_1]}] - [A_c(\xi)] \ge E > 0$. There is a segment $\bar\gamma|_{I_{i,j}}$ which contains $\bar\gamma(t_0)$ and $\bar\gamma(t_1)$, this enable us to construct a curve γ^* which joins $q^{*,i} = \pi\bar q^{*,i}$ with $q^{*,j} = \pi\bar q^{*,j}$ and

$$[A_c(\bar\zeta|_{I_{i,j}})] - [A_c(\gamma^*)] > E - C\epsilon.$$

But it contradicts the minimality of ζ if ϵ is sufficiently small. Therefore, γ is semi-static. □

For the brevity of notation, we define two sets in the covering space. Let

$$\tilde{\mathcal{C}}_{\eta,\mu} = \bigcup_{\gamma\in\mathscr{C}_{\eta,\mu}, t\in\mathbb{R}} (\gamma(t), \dot\gamma(t)) \subset T\bar M, \qquad \mathcal{C}_{\eta,\mu} = \pi_q\tilde{\mathcal{C}}_{\eta,\mu} \subset \bar M,$$

where π_q denotes the operation $\pi_q(q,\dot q) = q$. According to the lemma 2.2, $\pi\mathcal{C}_{\eta,0} = \mathcal{N}(c)$. By standard argument (cf [Be1],[CY1]), the set $\tilde{\mathcal{C}}_{\eta,\mu}$ as well as $\mathcal{C}_{\eta,\mu}$ is upper semi-continuous with respect to $L_{\eta,\mu}$.

Theorem 2.1. *Assume the cohomology class c is c-equivalent to the class c'; for each $s \in [0,1]$, $\exists$ $\Sigma_{\Gamma(s)}$ admissible for all $\Gamma(s)$-minimal measure. Then there exist some classes $c = c_0, c_1, c_2, \cdots, c_k = c'$, closed 1-forms η_i, $\bar\mu_i$ on M with $[\eta_i] = c_i$, $[\bar\mu_i] = c_{i+1} - c_i$ and smooth functions ϱ_i on $\bar M$ for $i = 0, 1, \cdots, k-1$, such that the Lagrangian*

$$L_{\eta_i,\mu_i} = L - \eta_i - \mu_i$$

with $\mu_i=\varrho\bar\mu_i$ *determines the set* $\mathscr{C}_{\eta_i,\mu_i}$ *with the properties:*

(i), *each curve* $\bar\gamma\in\mathscr{C}_{\eta_i,\mu_i}$ *determines an orbit* $d\gamma=(\gamma,\dot\gamma)$ *of* ϕ^t_L*;*

(ii), *each of these orbits connects* $\tilde{\mathcal{A}}(c_i)$ *to* $\tilde{\mathcal{A}}(c_{i+1})$*, i.e., the* α*-limit set* $\alpha(d\gamma)\subseteq\tilde{\mathcal{A}}(c_i)$ *and* ω*-limit set* $\omega(d\gamma)\subseteq\tilde{\mathcal{A}}(c_{i+1})$*.*

Proof. By definition, there is a path Γ: $[0,1]\to H^1(M,\mathbb{R})$ with $\Gamma(0)=c$ and $\Gamma(1)=c'$ such that for each $s\in[0,1]$ there exists $\epsilon>0$ such that $\Gamma(s')-\Gamma(s)\in V^{\perp}_{\Gamma(s)}$ whenever $s'\in[0,1]$ and $|s-s'|<\epsilon$. Thus, there exist a closed form $\bar\mu_s$ and a neighborhood U of $\mathcal{N}_{\Gamma(s)}\cap\Sigma_{\Gamma(s)}$ such that $[\bar\mu_s]=\Gamma(s')-\Gamma(s)$ and $\mathrm{supp}\bar\mu_s\cap U=\varnothing$. We introduce a coordinate transformation $q=G^{-1}_{\Gamma(s)}x$ on torus as before and a smooth function ϱ: $\bar M\to[0,1]$ such that $\varrho=0$ if $q\in\bar M^-\backslash(\Sigma^0_{\Gamma(s)}+\delta)$, $\varrho=1$ if $q\in\bar M^+\backslash(\Sigma^0_{\Gamma(s)}+\delta)$. A sufficiently small $\delta>0$ can be chosen so that

$$(\Sigma^0_{\Gamma(s)}+\delta)\cap(\mathcal{C}_{\eta,0}+2\delta)\subset U,$$

here $[\eta]=\Gamma(s)$. Because of upper semi-continuity of $\mathcal{C}_{\eta,\mu}$ on $L_{\eta,\mu}$, we find

$$(\Sigma^0_{\Gamma(s)}+\delta)\cap(\mathcal{C}_{\eta,\mu}+\delta)\subset U, \tag{2.2}$$

if μ is C^0-sufficiently small. As μ is carefully chosen so that its support is disjoint from U, each curve $\bar\gamma\in\mathscr{C}_{\eta,\mu}$ is clearly a solution of the Euler-Lagrange equation determined by $L_{\eta,\mu}$, the term μ has no contribution to the Euler-Lagrange equation along $\bar\gamma$. In other words, each curve in $\mathscr{C}_{\eta,\mu}$ generates an orbit $d\gamma$ of ϕ^t_L: $\mathbb{R}\to TM$. By the proposition (2.1), we see that $d\gamma(t)\to\tilde{\mathcal{A}}(\Gamma(s))$ as $\bar\gamma_1\to-\infty$ and $d\gamma(t)\to\tilde{\mathcal{A}}(\Gamma(s'))$ as $\bar\gamma_1\to\infty$.

Because of the compactness of $[0,1]$, we can find finite $s_0,s_1,\cdots,s_k\in[0,1]$ such that above argument applies if s and s' are replaced by s_i and s_{i+1} respectively, let $c_i=\Gamma(s_i)$. ☐

Corollary 2.1. *Let* c_i*,* η_i*,* $\bar\mu_i$ *and* ϱ_i *be evaluated as in the theorem 2.1. Let* U_i *is an open neighborhood of* $\mathcal{N}_{c_i}\cap\Sigma^0_{c_i}$ *such that* $U_i\cap\mathrm{supp}\bar\mu_i=\varnothing$*. Then, there exist big* $K_i>0$*,* $T_i>0$ *and small* $\delta>0$ *such that for each* $\bar m,\bar m'\in\bar M$*, with* $-K_i\le\bar m_1\le-K_i+2\pi$*,* $K_i-2\pi\le\bar m'_1\le K_i$*, the quantity* $h^T_{\eta_i,\mu_i}(\bar m,\bar m')$ *reaches its minimum at some* $T<T_i$ *and the corresponding minimizer* $\bar\gamma_i(t,\bar m,\bar m')$ *satisfies the condition*

$$\mathrm{Image}(\bar\gamma_i)\cap(\Sigma^0_{c_i}+\delta)\subset U_i. \tag{2.3}$$

Proof. Recall that $\mathscr{C}_{\eta_i,\mu_i}$ is obtained as the set of accumulation points of the minimizers for of $\inf_{T'>0}h^{T'}_{\eta_i,\mu_i}(\bar m,\bar m')$ considered in $\bar M$. Assume by contradiction that given any $K>0$, if there are two points $\bar m,\bar m'\in\bar M$ with $-\bar m_1,\bar m'_1>K$ such that $\bar\gamma_i(t,\bar m,\bar m')$, the minimizer of $h^T_{\eta_i,\mu_i}(\bar m,\bar m')=\inf_{T'>0}h^{T'}_{\eta_i,\mu_i}(\bar m,\bar m')$, satisfies the condition that $(\mathrm{Image}(\bar\gamma_i)\cap(\Sigma^0_{c_i}+\delta))\backslash U_i$

is not empty. Thus, we obtain a sequence of curves, which has at least one accumulation point $\bar\gamma^\infty$. This curve is defined on the whole line $\mathbb{R} \to \bar M$, because $T \to \infty$ as $-m_1, m_1' \to \infty$. Obviously, $\bar\gamma^\infty \in \mathscr{C}_{\eta_i,\mu_i}$ and $(\mathrm{Image}(\bar\gamma^\infty) \cap (\Sigma^0_{c_i} + \delta))\backslash U_i$ is also not empty, but it contradicts to (2.2). The boundedness of T_i is the consequence of the lemma 2.1. □

We call the local connecting orbits obtained in the theorem 2.1 c-type local connecting orbits. In the construction of global connecting connecting orbits, we need to consider local connecting orbits between different Aubry sets for which the corresponding cohomology classes are not equivalent. In the following of this section, we introduce h-type of local connecting orbits, which look like heteroclinic orbits.

Let $N = \mathbb{T}^{n-1} \times \{-d < q_n < d\} \subset \mathbb{T}^n$ with suitably small $d > 0$. To each curve γ: $[-T, T] \to M$ with $\gamma(-T), \gamma(T) \in N$ we associate a homology $[\gamma] \in H_1(M, N, \mathbb{Z})$. If the Aubry set $\mathcal{A}(c)$ is contained in N, there exists at least one homoclinic orbit $d\gamma$ to $\tilde{\mathcal{A}}(c)$ with $[\gamma] \neq 0$ (cf. [Be0]). Remember $\alpha(c) = 0$, we define

$$h^\infty_{c,e_n}(m,m) = \lim_{T\to\infty} \inf_{\substack{\gamma(-T)=m\\ \gamma(T)=m\\ [\gamma]\neq 0}} \int_{-T}^{T} L_c(d\gamma(t))dt.$$

Here, we use the subscript e_n to show that the minimization is done among those curves whose homology class has non-zero n-th component. Obviously, $h^\infty_{c,e_n}(m,m) > 0$ for $m \in \mathcal{A}(c)$. If there is only one c-minimizing measure, some homoclinic orbit $d\gamma$ exists such that $[A_c(\gamma|_{[-T_i^-, T_i^+]})] \to h^\infty_{c,e_n}(m,m)$ where $T_i^\pm \to \infty$ is a sequence such that $\gamma(T_i^+) \to m$ and $\gamma(T_i^-) \to m$. We call $d\gamma$ minimal homoclinic orbit. γ is called (minimal) homoclinic curve if $d\gamma$ is a (minimal) homoclinic orbit. A homoclinic orbit is called isolated if restricted on each bounded interval of time I, there exists neighborhood of $d\gamma|_I$ in the phase space through which no other minimal homoclinic orbit passes.

We can think minimal homoclinic orbits in another way. If we choose a finite covering $\check\pi : \check M = \mathbb{T}^{n-1} \times (2\mathbb{T}) \to \mathbb{T}^n$, the lift of N has two connected components, denoted by N and N'. If we regard $\check M$ as the configuration manifold, the minimal measure has at least two components, one support is in N and another one is in N'. Consequently, the Mañé set with respect to $\check M$, $\check\pi\mathcal{N}(c, \check M)$ properly contains the lift of the original Mañé set, $\check\pi\mathcal{N}(c, \check M)\backslash\mathcal{N}(c)$ contains those minimal homoclinic curves ζ with $[\zeta] \neq 0$. Different from the case for Mañé set, the Aubry set keeps the same, $\check\pi\mathcal{A}(c, \check M) = \mathcal{A}(c)$.

Theorem 2.2. *We assume that the c-minimal measure is uniquely ergodic, $\mathcal{A}(c) \subset N$ and that there exists an isolated minimal homoclinic orbit $d\zeta$ to*

$\tilde{\mathcal{A}}(c)$ with $[\zeta] \neq 0$ and c' is sufficiently close to c with $\alpha(c) = \alpha(c')$, then there exists some orbit $d\gamma$ of ϕ_L^t which connects $\tilde{\mathcal{A}}(c)$ to $\tilde{\mathcal{A}}(c')$.

Proof. A cohomology class c with $\omega(\mu_c) = 0$ is the minimum point of the α-function. In autonomous case, $\tilde{\mathcal{A}}(c)$ can only be connected to those $\tilde{\mathcal{A}}(c')$ only if c' is also a minimum point, i.e. c and c' are in the horizontal flat of the α-function. Thus, $\tilde{\mathcal{A}}(c) \cap \tilde{\mathcal{A}}(c') \neq \emptyset$ (see [Mas]). It is trivial to connect an Aubry set to itself, thus, we only need to consider the case $\omega(\mu_c) \neq 0$.

By condition, $\omega(\mu_c) = (\omega_1, \cdots, \omega_{n-1}, 0)$. We can choose $\hat{g}_1^{-1} \in \mathbb{Z}^{n-1}$ such that $\langle \hat{g}_1^{-1}, (\omega_1, \cdots, \omega_{n-1})\rangle > 0$. Clearly, $\exists\ \hat{g}_2^{-1}, \cdots \hat{g}_{n-1}^{-1} \in \mathbb{Z}^{n-1}$ such that $\hat{G}^{-1} = [\hat{g}_1^{-1}, \cdots, \hat{g}_{n-1}^{-1}]^t$ is a uni-module matrix. Let $\hat{G}$ be the inverse of $\hat{G}^{-1}$, $G = \text{diag}(\hat{G}, 1)$ and $x = Gq$. In q-coordinates, we have $\omega_1 > 0$. It implies that $\bar{\zeta}_1(t) \to \pm\infty$ as $t \to \pm\infty$, if the orbit $d\zeta$ has $\tilde{\mathcal{A}}(c)$ as its ω- as well as α-limit set. Here $\bar{\zeta}$ denote the lift of ζ to the covering $\mathbb{R} \times \mathbb{T}^{n-1}$.

Because of the unique ergodicity, the set of all minimal homoclinic orbits with the homology $[\zeta]$ is a Lipschitz graph. By assumption, there exists an isolated minimal homoclinic orbit ζ with $[\zeta] \neq 0$ as well as an $(n-1)$-dimensional disk $D' \subset \{q_1 = q_1^0\}$ such that $D' \cap N = \varnothing$ and ζ passes through the center of this disk transversally with $\dot{\zeta}_1|_{q_1 = q_1^0} > 0$, and no other minimal homoclinic curve with the homology $[\zeta]$ passes through this disk. For simplification of notation, we assume that $q_1^0 = 0$.

Denote by $\hat{q} = (q_2, q_3, \cdots, q_n)$, we define a smooth function $w(\hat{q}) : \mathbb{T}^{n-1} \to \mathbb{R}^+$ such that supp$w = D'$, $w \equiv \lambda$ for $\hat{q} \in D$, where $\lambda > 0$, $D \subset D'$ is a smaller disk. Let $\Delta q_1 = \inf\{|\zeta_1| : \hat{\zeta} \notin D\}$ where ζ is the minimal curve of ϕ_L^t, homoclinic to $\mathcal{A}(c)$. Let $\varrho'(q_1)$ be a smooth function such that supp$\varrho' \subset \{|q_1| \leq \delta < \Delta q_1\}$, $\varrho'(0) > 0$, $0 \leq \varrho' \leq 1$ and define

$$L_{\eta,\psi} = L - \eta - \psi,$$

where $\psi = \varrho'(q_1)w(\hat{q})\dot{q}_1$. Recall $\alpha(c) = 0$, let

$$h_{\eta,\psi,e_n}^T(m, m') = \inf_{\substack{\xi(-T)=m \\ \xi(T)=m' \\ [\xi]=[\zeta]}} \int_{-T}^{T} L_{\eta,\psi}(d\xi(t))dt,$$

$$h_{\eta,\psi,e_n}^{\infty}(m, m') = \lim_{T\to\infty} h_{\eta,\psi,e_n}^T(m, m').$$

By the construction of ψ and note supp$\psi \cap N = \varnothing$, we see that, if $|w|$ is suitably small, the Aubry set for $L_{\eta,\psi}$ is the same as for L_η because $L_{\eta,\psi} = L_\eta$ when they are restricted in TN. It is not difficult to see that each minimal homoclinic orbit of ϕ_L^t to $\tilde{\mathcal{A}}(c)$ is still a homoclinic orbit of $\phi_{L_{\eta,\psi}}^t$. In fact, except for ζ, other minimal homoclinic curves do not pass through the support of ψ. Note $\varrho'(q_1)w(\hat{q})dq_1$ is closed on $\{|q_1| \leq \Delta q_1\} \times D$, this term has no contribution to the Euler-Lagrange equation along $d\zeta$. Thus $d\zeta$ is still a minimal homoclinic orbit of $\phi^t_{L_{\eta,\psi}}$.

Nevertheless, except for ζ, other homoclinic curves minimal for L_η are no longer minimal for $L_{\eta,\psi}$. To see it, let us recall that $\psi>0$ holds on a segment of ζ and $\psi=0$ holds along any other homoclinic curves minimal for L_η, the quantity $\min h^\infty_{\eta,\psi,e_n}(m,m)$ can only be achieved by the curve ζ which passes through the disk $\{q_1=0,\hat q\in D\}$. Thus, $d\zeta$ is the only minimal homoclinic orbit with nonzero homology in $H_1(M,N,\mathbb{Z})$ for $L_{\eta,\psi}$.

Let $U=\{|q_1|\le\Delta q_1\}\times D$. A closed 1-form $\bar\mu$ clearly exists such that $[\bar\mu]=c'-c$ and $\mathrm{supp}\bar\mu\cap\bar D=\varnothing$. We introduce a Lagrangian $L_{\eta,\mu,\psi}$: $T\bar M\to\mathbb{R}$

$$L_{\eta,\mu,\psi}(q,\dot q)=L-\eta-\mu-\psi,$$

where $\mu=\varrho(q_1)\bar\mu$, ϱ is a smooth function with $\varrho=0$ for $q_1\le-\Delta q_1$ and $\varrho=1$ for $q_1\ge\Delta q_1$, $\psi=\bar\varrho'(q_1)w(\hat q)\dot q_1$, $\bar\varrho'$ takes value of the lift of ϱ' for $|q_1|\le\pi$ and $\bar\varrho'=0$ for $|q_1|\ge\pi$.

Denote by $\bar N$ the lift of N in $\bar M$, let $\bar m,\bar m'\in\bar N$ such that $-\bar m_1$ and $\bar m'_1$ are rather large. To each curve $\bar\xi$ connecting $\bar m$ to $\bar m'$ we can associate a homology class $[\bar\xi]$ valued with the quantity $[\pi\bar\xi]\in H_1(M,N,\mathbb{Z})$. Let $\bar\gamma_{\eta,\mu,\psi}(t,m,m',e_n)$: $[-T,T]\to\bar M$ be the minimizer: $\bar\gamma_{\eta,\mu,\psi}(-T)=\bar m$, $\bar\gamma_{\eta,\mu,\psi}(T)=\bar m'$, $[\bar\gamma_{\eta,\mu,\psi}]=[\zeta]$ and

$$\int_{-T}^{T}L_{\eta,\mu,\psi}(d\bar\gamma_{\eta,\mu,\psi}(t))dt=\inf_{T'>0}h^{T'}_{\eta,\mu,\psi,e_n}(\bar m,\bar m'),$$

where

$$h^{T'}_{\eta,\mu,\psi,e_n}(\bar m,\bar m')=\inf_{\substack{\bar\xi(-T')=\bar m\\ \bar\xi(T')=\bar m'\\ [\bar\xi]=[\zeta]}}\int_{-T'}^{T'}L_{\eta,\mu,\psi}(d\bar\xi(t))dt.$$

From the proof of the lemma 2.1, we can see that T is finite for each fixed $(\bar m,\bar m')$ and $T\to\infty$ as $-\bar m_1,\bar m'_1\to\infty$. We denote by $\mathscr{C}_{\eta,\mu,\psi,e_n}$ the set of all accumulation points of $\{\bar\gamma_{\eta,\mu,\psi,}(t,\bar m,\bar m',e_n)|_{[-T_i,T_i]}\}$ ($T_i\to\infty$ as $i\to\infty$). For each curve $\bar\gamma\in\mathscr{C}_{\eta,\mu,\psi,e_n}$, let $\gamma=\pi\bar\gamma$, $d\gamma(t)$ approaches to $\tilde{\mathcal{A}}(c)$ as $t\to-\infty$ and approaches to $\tilde{\mathcal{A}}(c')$ as $t\to\infty$. Let

$$\mathcal{C}_{\eta,\mu,\psi,e_n}=\bigcup_{\substack{\bar\gamma\in\mathscr{C}_{\eta,\mu,\psi,e_n}\\ t\in\mathbb{R}}}\bar\gamma(t)\subset\bar M.$$

Clearly, this set is upper semi-continuous with respect to (η,μ,ψ).

By the choice of ψ, $\mathscr{C}_{\eta,0,\psi,e_n}$ contains only one curve which is a lift of the minimal homoclinic curve for L_η, passing through transversally the center of the disk. Due to the upper semi-continuity of $\mathcal{C}_{\eta,\mu,\psi,e_n}$ with respect to (η,μ,ψ), each curve in $\mathscr{C}_{\eta,\mu,\psi,e_n}$ passes through $U\backslash(\{|q_1|\le\Delta_q\}\times\partial D)$ if $|c'-c|$ is suitably small. Thus, each curve $\bar\gamma\in\mathscr{C}_{\eta,\mu,\psi,e_n}$ determines an orbit of

ϕ_L^t. Indeed, in the region $\{|q_1| \geq \Delta q_1\}$ $\psi = 0$ and $\varrho(q_1)\bar{\mu}$ is closed, while in U, ψ is closed and $\bar{\mu} = 0$. Therefore, along $\bar{\gamma}$ these terms have no contribution to the Lagrangian equation, thus $\alpha(d\gamma) \subset \tilde{\mathcal{A}}(c)$ and $\omega(d\gamma) \subset \tilde{\mathcal{A}}(c')$. $\square$

We call the local connecting orbit obtained in the theorem 2.2 h-type local connecting orbit. Such connecting orbit has some local minimal property which will be used for the construction of global connecting orbits. Let us introduce the definition first and give explanations afterwards.

Definition 2.2. *We assume that* $\mathcal{A}(c), \mathcal{A}(c') \subset N$*,* $[\eta] = c$ *and* $[\eta + \bar{\mu}] = c'$*.* $d\gamma$*:* $TM \to \mathbb{R}$ *is called local minimal orbit of* ϕ_L^t *of h-type that connects* $\tilde{\mathcal{A}}(c)$ *to* $\tilde{\mathcal{A}}(c')$ *if*

1, $0 \neq [\gamma] \in H_1(M, N, \mathbb{Z})$*.* $d\gamma(t)$ *is an orbit of the Euler-Lagrange flow* ϕ_L^t*, the* α*- and* ω*-limit set of* $d\gamma$ *are in* $\tilde{\mathcal{A}}(c)$ *and* $\tilde{\mathcal{A}}(c')$ *respectively.*

2, its lift $\bar{\gamma} \in \mathscr{C}_{\eta,\mu,\psi,e_n}$*, which is a local minimal curve of the Lagrangian* $L_{\eta,\mu,\psi} : T\bar{M} \to \mathbb{R}$ *in the following sense: there exist two* $(n-1)$*dimensional disks* V^-*,* $V^+ \subset \bar{M}$ *and positive numbers* $T, d > 0$ *such that* $\bar{\gamma}(-T) \in V^-$*,* $\bar{\gamma}(T) \in V^+$*,* $\pi V^- \subset N \backslash \mathcal{A}(c)$*,* $\pi V^+ \subset N \backslash \mathcal{A}(c')$ *and*

$$h_c^\infty(q^-, m_0) + h_{\eta,\mu,\psi,e_n}^{T'}(\bar{m}_0, \bar{m}_1) + h_{c'}^\infty(m_1, q^+) - \lim_{\substack{T_0^i \to \infty \\ T_1^i \to \infty}} \int_{-T_0^i}^{T_1^i} L_{\eta,\mu,\psi}(d\gamma(t))dt - (T_0^i + T_1^i)\alpha(c) > 0 \tag{2.4}$$

holds for each $(\bar{m}_0, \bar{m}_1, T') \in \partial(V^- \times V^+ \times [T-d, T+d])$*,* $q^- \in \mathcal{M}(c) \cap \pi_q(\alpha(d\gamma))$ *and* $q^+ \in \mathcal{M}(c') \cap \pi_q(\omega(d\gamma))$*. Where* $T_0^i \to \infty$ *and* $T_1^i \to \infty$ *are the sequences such that* $\gamma(-T_0^i) \to x^-$ *and* $\gamma(T_1^i) \to x^+$*.*

Here, the term $h_c^\infty(\xi, m_0) + h_{\eta,\mu,\psi,e_n}^{T'}(\bar{m}_0, \bar{m}_1) + h_{c'}^\infty(m_1, \zeta)$ measures the smallest action of $L_{\eta,\mu,\psi}$ along those curves $\{\chi\}$ which satisfy the following conditions:

1, each χ joining $\bar{m}_0$ with $\bar{m}_1$ with time $2T'$ and $[\chi] \neq 0$;

2, x^- is an accumulation point of χ as $t \to -\infty$ and x^+ is an accumulation point of χ as $t \to \infty$.

By requiring those curves passing through V^- and V^+ within time interval $2T'$ with $T' \in [T-d, T+d]$, we obtain a neighborhood of the curve γ. Once a curve touches the boundary of this neighborhood, the action of $L_{\eta,\mu,\psi}$ along this curve will be larger than the action along γ. As V^-, V^+ and therefore $d > 0$ can be chosen arbitrarily small, it is reasonable to call it locally minimal.

The orbit $d\gamma$ obtained in the theorem 2.2 is locally minimal in this sense. To see it, let us state a property of semi-static curves for autonomous systems:

Proposition 2.2. *Let $\gamma : \mathbb{R} \to M$ be a c-semi-static curve of some autonomous system. If γ is not periodic, then*

$$[A_c(\gamma|_{[a,b]})] < [A_c(\xi|_{[a',b']})]$$

holds for any absolutely continuous curve ξ if $\xi(a') = \gamma(a)$, $\xi(b') = \gamma(b)$ and $b - a \neq b' - a'$; If γ is periodic with least period $d \geq 0$, then the strict inequality also holds provided $b - a \neq b' - a'$ mod d.

Proof. Since the system is autonomous, we can assume that $a' = a$. The definition of semi-static curves guarantees that

$$[A_c(\gamma|_{[a,b]})] \leq [A_c(\xi|_{[a,b']})]$$

if $\xi(a) = \gamma(a)$, $\xi(b') = \gamma(b)$. To show that the equality can not hold for any $b' - a \neq b - a$ when γ is not periodic, let us assume the contrary, i.e. $[A_c(\gamma|_{[a,b]})] = [A_c(\xi|_{[a,b']})]$ holds for some $b' - a \neq b - a$ and ξ: $[a, b'] \to M$. The property $b - a \neq b' - a$ and the uniqueness of the solution guarantee that $\dot\gamma(a) \neq \dot\xi(a)$, $\dot\gamma(b) \neq \dot\xi(b')$. We join the point $\gamma(a - \delta)$ with the point $\xi(a + \delta)$ by the minimizer ζ: $[a - \delta, a + \delta] \to M$:

$$\int_{a-\delta}^{a+\delta} (L - \eta_c)(d\zeta(t))dt = \inf_{\substack{\chi(a-\delta)=\gamma(a-\delta) \\ \chi(a+\delta)=\xi(a+\delta)}} \int_{a-\delta}^{a+\delta} (L - \eta_c)(d\chi(t))dt.$$

By Tonelli lemma, the minimizer is a C^1-curve. Thus, the condition $\dot\gamma(a) \neq \dot\xi(a)$ implies $[A_c(\gamma|_{[a-\delta,a]})] + [A_c(\xi|_{[a,a+\delta]})] > [A_c(\zeta|_{[a-\delta,a+\delta]})]$. If we define a curve γ': $\mathbb{R} \to M$

$$\gamma'(t) = \begin{cases} \gamma(t) & \text{for } t \in (-\infty, a - \delta]; \\ \zeta(t) & \text{for } t \in [a - \delta, a + \delta]; \\ \xi(t) & \text{for } t \in [a + \delta, b']; \\ \gamma(t - b' + b) & \text{for } t \in [b', \infty), \end{cases}$$

we obtain that $\gamma(a_1) = \gamma'(a_1)$ for $a_1 \leq a$, $\gamma'(b_1) = \gamma(b_1 - b' + b)$ for $b_1 \geq b$ and

$$[A_c(\gamma|_{[a_1,b_1]})] > [A_c(\gamma'|_{[a_1,b_1-b'+b]})].$$

This is absurd as γ is assumed to be a c-semi-static curve.

To show the proposition in the case that γ is d-periodic, The condition $\dot\gamma(a) \neq \dot\xi(a)$, $\dot\gamma(b) \neq \dot\xi(b')$ holds only when $b - a \neq b' - a$ mod d. We have

$$[A_c(\gamma|_{[a,b]})] = [A_c(\gamma|_{[a,b']})], \qquad \forall\ b - a = b' - a \text{ mod } d.$$

□

The orbit $d\gamma$ obtained in the theorem 2.2 is locally minimal. Indeed, suitably small disk πV^- and πV^+ can be chosen such that the minimal homoclinic curve ζ intersects πV^- as well as πV^+ transversally. Once we fixed a lift $\bar\zeta$, $V^\pm$ is then automatically fixed. As ζ is an isolated minimal curve, by the upper semi-continuity of $\mathcal{C}_{\eta,\mu,\psi,e_n}$ in (η,μ,ψ), we can see that no curve in $\mathscr{C}_{\eta,\mu,\psi,e_n}$ touches the boundary of V^- as well of V^+ provided μ and ψ are suitably small. Consequently, (2.4) holds whenever $(\bar m_0,\bar m_1)\in\partial(V^-\times V^+)$, regardless what value T' takes.

To see (2.4) holds for all $(\bar m_0,\bar m_1)\in V^-\times V^+$ whenever T' touch the boundary of $[T-d,T+d]$, by applying the proposition 2.2 we find

$$h^{T'}_{\eta,\mu,\psi,e_n}(\bar\gamma(-T),\bar\gamma(+T))>h^{T}_{\eta,\mu,\psi,e_n}(\bar\gamma(-T),\bar\gamma(+T))$$

if $T'\neq T$ (γ is clearly not periodic in t). Considered as the function of T', $h^{T'}_{\eta,\mu,\psi,e_n}(\bar m_0,\bar m_1)$ is obviously a Lipschitz function. Although it reaches its minimum not necessarily at $T'=T$ unless $\bar m_0=\gamma(t-T)$ and $\bar m_1=\gamma(t+T)$ for some t, we have

$$h^{T\pm d}_{\eta,\mu,\psi,e_n}(\bar m_0,\bar m_1)>\min_{T'\in[T-d,T+d]}h^{T'}_{\eta,\mu,\psi,e_n}(\bar m_0,\bar m_1)$$

for each $(\bar m_0,\bar m_1)\in V^-\times V^+$, provided V^- and V^+ are chosen suitably small for given $d>0$.

3. CONSTRUCTION OF GLOBAL CONNECTING ORBITS

Roughly speaking, global connecting orbits are constructed along a series of local minimal connecting orbits. A sufficient condition to construct orbits connecting $\tilde{\mathcal{A}}(c)$ to $\tilde{\mathcal{A}}(c')$ is the existence of a generalized transition chain in $H^1(M,\mathbb{R})$ that connects c to c'.

Definition 3.1. *Let c and c' be two cohomolgy classes in $H^1(M,\mathbb{R})$. We say that c is joined with c' by a generalized transition chain if there is a continuous curve Γ: $[0,1]\to H^1(M,\mathbb{R})$ with $\Gamma(0)=c$, $\Gamma(1)=c'$, such that $\alpha(\Gamma(s))$ keeps constant for $s\in[0,1]$, and for each $s\in[0,1]$ at least one of the following cases takes place:*

(I), *$\Gamma(s)$-minimal measure is uniquely ergodic; there is a coordinate transformation $x\to q=G_s^{-1}x$ as well as $\delta_s>0$ such that in q-coordinates $\check M=\mathbb{T}^{n-1}\times(2\mathbb{T})$, $\mathcal{N}(\Gamma(s))+\delta_s\subset N$, and $\check\pi\mathcal{N}(\Gamma(s),\check M)\backslash(\mathcal{N}(\Gamma(s))+\delta_s)$ contains an isolated orbit;*

(II), *$\Gamma(s)$ is c-equivalent to $\Gamma(s')$ if s' is sufficiently close to s; $\exists$ $\Sigma_{\Gamma(s)}$ admissible for the $\Gamma(s)$-minimal measure.*

In the item (I), the unique ergodicity can be replaced by that the Aubry set has finitely many classes. This assumption is made for brevity of proof and it is enough for the application in *a priori* unstable systems.

Theorem 3.1. *Suppose that there is a generalized transition chain* $\Gamma : [0,1] \to H^1(M,\mathbb{R})$ *joining* c *to* c'. *Then there exists an orbit of the Lagrange flow* ϕ_L^t $d\gamma$: $\mathbb{R} \to TM$ *that connects the two Aubry sets:* $\alpha(d\gamma) \subseteq \tilde{\mathcal{A}}(c)$ *and* $\omega(d\gamma) \subseteq \tilde{\mathcal{A}}(c')$.

Proof. Roughly speaking, the global connecting orbits are constructed shadowing a sequence of local connecting orbits. According to the study in the last section, if s' is close to s, each Aubry set $\tilde{\mathcal{A}}(\Gamma(s))$ can be connected to some $\tilde{\mathcal{A}}(\Gamma(s'))$ by either h-type, or c-type of local minimal orbits. Thus, there is a sequence $0 = s_0 < s_1 < \cdots < s_k = 1$ such that for each $0 \le j < k$ $\tilde{\mathcal{A}}(\Gamma(s_j))$ is connected to $\tilde{\mathcal{A}}(\Gamma(s_{j+1}))$ by some local minimal orbits.

Let $c_j = \Gamma(s_j)$, $j = 0, 1, \cdots, k$. We divide the set $\{0, 1, \cdots, k\}$ into m partitions $\{0, 1, \cdots k\} = \{0, 1, \cdots, i_1, i_1+1, \cdots, i_2, \cdots, i_m-1, i_m = k\}$. The rule to make such a division is that for all $i = i_j, i_j+1, \cdots, i_{j+1}-1$, $\tilde{\mathcal{A}}(c_i)$ is connected to $\tilde{\mathcal{A}}(c_{i+1})$ by a local minimal orbit of the same type. Let Λ_c and Λ_h be the subset of $\{i_1, i_2, \cdots, i_m\}$, $\Lambda_c \cup \Lambda_h = \{i_1, i_2, \cdots i_m\}$, $\Lambda_c \cap \Lambda_h = \varnothing$. If $i_j \in \Lambda_\imath$, then for all $i = i_j, i_j+1, \cdots, i_{j+1}-1$, $\tilde{\mathcal{A}}(c_i)$ is connected to $\tilde{\mathcal{A}}(c_{i+1})$ by a local minimal orbit of the $\imath$- type ($\imath$ =c, or h).

Because of the upper semi-continuity $c \to \tilde{\mathcal{N}}(c, M)$, once the Mañé set $\tilde{\mathcal{N}}(\gamma(s))$ is in the case I (or II), then for those s' close to s, $\tilde{\mathcal{N}}(\Gamma(s'))$ is also in the case I (or II). Thus, for each $i_j \in \Lambda_h$, by choosing c_{i_j-1} and $c_{i_{j+1}}$ sufficiently close to c_{i_j} and $c_{i_{j+1}-1}$ respectively, we can assume that both c_{i_j-1} and $c_{i_{j+1}}$ satisfy the condition of the item (I) in the definition 3.1.

To each c_i we associate a coordinate system $x \to q = G_i^{-1}x$, write the inverse of G_i with $G_i^{-1} = [g_{i,1}^{-1}, g_{i,2}^{-1}, \cdots, g_{i,n}^{-1}]^t$. By the assumption, for each $i \in \Lambda_c$, we can choose G_i such that, in q-coordinates, the rotation vector of each c_i-minimal measure μ_{c_i} is positive. Note that each semi-static orbits approaches to the support of the minimal measure, and note the upper semi-continuity of Mañé set on c, we can assume that

$$\langle g_{j,1}^{-1}, \omega(\mu_{c_i})\rangle > 0, \qquad \forall\, j = i-1, i$$

holds for each ergodic component μ_{c_j} if c_{i-1} is chosen suitably close to c_i. Therefore, $\exists\, q_{i,1} > 0$ such that

$$\begin{aligned} &\langle g_{i,1}^{-1}, \Delta\tilde{\gamma}_i\rangle \ge 2\pi, \quad \text{whenever } \langle g_{i-1,1}^{-1}, \Delta\tilde{\gamma}_i\rangle \ge q_{i,1},\\ &\langle g_{i-1,1}^{-1}, \Delta\tilde{\gamma}_i\rangle \ge 2\pi, \quad \text{whenever } \langle g_{i,1}^{-1}, \Delta\tilde{\gamma}_i\rangle \ge q_{i,1} \end{aligned} \tag{3.1}$$

where $\tilde{\gamma}_i$ is any c_i-semi-static curve, $\Delta\tilde{\gamma}_i = \tilde{\gamma}_i(t') - \tilde{\gamma}_i(t)$ with $t' > t$.

For each integer $i\in\bigcup_{i_j\in\Lambda_h}\{i_j,i_j+1,\cdots,i_{j+1}-1\}$, we find a coordinate system $x\to q=G_i^{-1}x$ of $\mathbb{T}^n$ such that (3.1) holds and $N=\mathbb{T}^{n-1}\times\{|q_n|<d\}$. Let $U_i=\{|q_1|\le\Delta_i\}\times D_i$, there exist closed 1-forms η_i and $\bar\mu_i$ on $\mathbb{T}^n$ with $[\eta_i]=c_i$, $[\bar\mu_i]=c_{i+1}-c_i$ and $\mathrm{supp}\bar\mu_i\cap\bar D_i=\varnothing$. For the covering space $\bar M=\mathbb{R}\times\mathbb{T}^{n-1}$, η_i and $\bar\mu_i$ have their natural lift, denoted by the same symbol. We also use U_i to denote $\{|q_1|\le\Delta_i\}\times D_i\subset\bar M$. There is a non-negative function w_i of $\hat q$ with its support contained in a small neighborhood of D_i and $w_i|_{D_i}=$ constant, and two functions ϱ_i, $\bar\varrho_i'$ of q_1 such that $\varrho_i=\bar\varrho_i'=0$ when $q_1\in(-\infty,-\Delta_i]$, while for $q_1\in[\Delta_i,\infty)$ we have $\varrho_i=1$ and $\bar\varrho_i'=0$. Let $\mu_i=\varrho_i\bar\mu_i$, $\psi_i=\bar\varrho_i'(q_1)w_i(\hat q)\dot q_1$ and define a Lagrangian

$$L_{\eta_i,\mu_i,\psi_i}=L-\eta_i-\mu_i-\psi_i.$$

According to the theorem 2.2, we can see that each curve $\bar\gamma_i\in\mathscr{C}_{\eta_i,\mu_i,\psi_i,e_n}$ determines a local minimal orbit of ϕ_L^t of h-type, such that $d\gamma_i$ connects $\tilde{\mathcal{A}}(c_i)$ to $\tilde{\mathcal{A}}(c_{i+1})$. Note that $\bar\gamma_i$ is a curve $\mathbb{R}\to\bar M=\mathbb{R}\times\mathbb{T}^{n-1}$ passing through $U_i\backslash(\{|q_1|\le\Delta_i\}\times\partial D_i)$.

In this case, there exist two $(n-1)$-dimensional disks V_i^- and V_{i+1}^+ with $\pi V_i^-\subset(\mathcal{A}_0^-(c_i)+\delta)\backslash\mathcal{A}_0(c_i)$, $\pi V_{i+1}^+\subset(\mathcal{A}_0^+(c_{i+1})+\delta)\backslash\mathcal{A}_0(c_{i+1})$, big numbers $T_i^+>0$, suitably small $d_i>0$ and quite small $\epsilon_i^*>0$ such that $\bar\gamma_i(-T_i^+)\in V_i^-$, $\bar\gamma_i(T_i^+)\in V_{i+1}^+$ and

$$\begin{aligned}\min\Big\{&h_c^\infty(x^-,m_0)+h^T_{\eta_i,\mu_i,\psi_i,e_n}(\bar m_0,\bar m_1)+h_{c'}^\infty(m_1,x^+):\\&(\bar m_0,\bar m_1,T)\in\partial(V_i^-\times V_{i+1}^+\times[T_i^+-d_i,T_i^++d_i])\Big\}\\\ge\min\Big\{&h_c^\infty(x^-,m_0)+h^T_{\eta_i,\mu_i,\psi_i,e_n}(\bar m_0,\bar m_1)+h_{c'}^\infty(m_1,x^+):\\&(\bar m_0,\bar m_1,T)\in V_i^-\times V_{i+1}^+\times[T_i^+-d_i,T_i^++d_i]\Big\}+5\epsilon_i^*,\end{aligned}\tag{3.2}$$

where $x^-\in\mathcal{M}(c_i)\cap\pi_q\alpha(d\gamma)$, $x^+\in\mathcal{M}(c_{i+1})\cap\pi_q\omega(d\gamma)$. The disks V_i^- and V_{i+1}^+ are chosen so that $\bar\gamma$ intersects them transversally, for each $(\bar m_0,\bar m_1,T')\in V_i^-\times V_{i+1}^+\times[T_i^+-d_i,T_i^++d_i]$, the minimizer of $h^T_{\eta_i,\mu_i,\psi_i,e_n}(\bar m_0,\bar m_1)$, $\bar\gamma_i(t,\bar m_0,\bar m_1,T)$ has the property

$$\bar\gamma_i\in U_i\qquad\text{whenever}\ \ \bar\gamma_{i,1}\in[-\Delta_i,\Delta_i].\tag{3.3}$$

For each integer $i\in\bigcup_{i_j\in\Lambda_c}\{i_j,i_j+1,\cdots,i_{j+1}-1\}$, there exist an embedded $(n-1)$-torus Σ_{c_i}, a neighborhood U_i of $\mathcal{N}(c_i)\cap\Sigma_{c_i}$, two closed 1-forms η_i and $\bar\mu_i$ on M with $[\eta_i]=c_i$, $[\bar\mu_i]=c_{i+1}-c_i$ and $\mathrm{supp}\bar\mu_i\cap U_i=\varnothing$. Suitable coordinate systems $x\to q=G_i^{-1}x$ on M is chosen so that $\pi^{-1}\Sigma_{c_i}\subset\bar M=\mathbb{R}\times\mathbb{T}^{n-1}$ has infinitely many compact components $\Sigma_{c_i}^j$. $\bar M$ is separated by $\Sigma^0_{c_i}$ into upper part $\bar M_i^+$ and lower part $\bar M_i^-$. A smooth non-negative function ϱ_i of q is constructed such that $\varrho_i=0$ for $q\in\bar M_i^-\backslash(\Sigma^0_{c_i}+\delta_i)$ and $\varrho_i=1$

for $q \in \bar{M}_i^+ \backslash (\Sigma^0_{c_i} + \delta_i)$. Let $\mu_i = \varrho_i \bar{\mu}_i$ and

$$L_{\eta_i,\mu_i} = L - \eta_i - \mu_i.$$

$\delta_i > 0$ is so small such that $(\Sigma^0_{c_i} + \delta_i) \cap (\mathcal{N}_{\eta_i,\mu_i} + \delta_i) \subset U_i$, see formula (2.2).

Since c_i is equivalent to c_{i+1} and they are closed to each other, each curve $\bar{\gamma} \in \mathscr{C}_{\eta_i,\mu_i}$ determines a local minimal orbit $d\gamma$ of ϕ^t_L of c-type, which connects $\tilde{\mathcal{A}}(c_i)$ to $\tilde{\mathcal{A}}(c_{i+1})$.

Let $\bar{\gamma}_i(t, \bar{m}, \bar{m}') : [-T, T] \to \bar{M}$ be the minimizer of

$$h^T_{\eta_i,\mu_i}(\bar{m}, \bar{m}') = \inf_{T'>0} h^{T'}_{\eta_i,\mu_i}(m, m').$$

According to the lemma 2.1 and the corollary 2.1 $\exists\ K_i, T_i^+ > 0$, if $K_i \le -\bar{m}_1, \bar{m}_1' \le K_i + 2\pi$, then there exists $T < T_i^+$ such that $h^T_{\eta_i,\mu_i}(\bar{m}, \bar{m}') = \inf_{T'>0} h^{T'}_{\eta_i,\mu_i}(\bar{m}, \bar{m}')$ and its minimizer $\bar{\gamma}_i(t, m, m') : [-T, T] \to \bar{M}$ satisfies

$$\bar{\gamma}_i(t, \bar{m}, \bar{m}') \in U_i, \qquad \text{whenever } \ \bar{\gamma}_i(t, \bar{m}, \bar{m}') \in \Sigma^0_{c_i} + \delta_i. \tag{3.4}$$

To make notation simpler, we let $\psi_i = 0$, $V_i^- = \{q_1 = -K_i\} \times \mathbb{T}^{n-1}$ and $V_{i+1}^+ = \{q_1 = K_i\} \times \mathbb{T}^{n-1}$ for each integer $i \in \bigcup_{i_j \in \Lambda_c} \{i_j, i_j + 1, \cdots, i_{j+1} - 1\}$. (see the corollary 2.1 for the definition of K_i).

The condition (3.1) guarantees that V_i^- is not "perpendicular" to V_i^+ in the following sense:

$$\pi_1 V_i^+ = \pi_1 \bar{\pi} G_{i-1}^{-1} G_i \bar{\pi}^{-1} V_i^- = \mathbb{T}^{n-1}$$

where $\bar{\pi}$: $\mathbb{R}^n \to \bar{M} = \mathbb{R} \times \mathbb{T}^{n-1}$ and π_1: $\mathbb{R} \times \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$ are the standard projections. We define

$$V_{i,-}^+ = \bar{\pi} G_{i-1}^{-1} G_i \bar{V}_i^-, \qquad V_{i,+}^- = \bar{\pi} G_i^{-1} G_{i-1} \bar{V}_i^+ \tag{3.5}$$

in which $\bar{V}_i^- = \{q \in \mathbb{R}^n : q_1 = -K_i,\ q_i \in [0, 2\pi]\ \forall\ 2 \le i \le n\}$, $\bar{V}_i^+ = \{q \in \mathbb{R}^n : q_1 = K_{i-1},\ q_i \in [0, 2\pi]\ \forall\ 2 \le i \le n\}$. Clearly, there exists constant $q_{i,1} > 0$ such that

$$V_{i,\pm}^{\mp} \subset \{q_1^0 \le q_1 \le q_1^0 + q'_{i,1}\} \subset \bar{M}. \tag{3.6}$$

We consider an absolutely continuous curve γ: $[-K, K'] \to M$ joining m to m', i.e. $\gamma(-K) = m$ and $\gamma(K') = m'$. We divide the interval $[-K, K']$ into $2i_m + 1$ subintervals

$$[-K, K'] = [t_0^+, t_0^-] \cup [t_0^-, t_1^+] \cup \cdots \cup [t_{i_m}^+, t_{i_m}^-],$$

where $t_0^+ = -K$, $t_{i_m}^- = K'$. Correspondingly, we divide the curve into $2i_m + 1$ segments $\gamma_i^- = \gamma|_{[t_i^+, t_i^-]}$, $\gamma_i^+ = \gamma|_{[t_i^-, t_{i+1}^+]}$ for $i = 0, 1, 2, \cdots, i_m - 1$, and $\gamma_{i_m}^- = \gamma|_{[t_{i_m}^+, t_{i_m}^-]}$.

We take two coverings of $\mathbb{T}^n$, the universal covering $\mathbb{R}^n$ and $\bar{M} = \mathbb{R}\times\mathbb{T}^{n-1}$

$$\mathbb{R}^n \xrightarrow{\bar{\pi}} \mathbb{R}\times\mathbb{T}^{n-1} \xrightarrow{\pi} \mathbb{T}^n$$

and fix a lift $\tilde{\gamma}$ of γ to $\mathbb{R}^n$ by choosing $\bar{\pi}G_0^{-1}\tilde{\gamma}(t_0^-) \in V_0^-$ (G_i determines a coordinate transformation on $\mathbb{R}^n$). Correspondingly, each $\gamma_i^\pm$ has its lift $\tilde{\gamma}_i^\pm$ to $\mathbb{R}^n$.

The curve γ is required to satisfy the conditions:

1, for each $i = 0, 1, 2, \cdots i_m - 1$ there is some $k_i \in \mathbb{Z}$ such that

$$\begin{aligned}&\bar{\pi}G_i^{-1}\tilde{\gamma}_i^+(t_i^-) - (2k_i\pi, 0, \cdots, 0) \in V_i^-,\\ &\bar{\pi}G_i^{-1}\tilde{\gamma}_i^+(t_{i+1}^+) - (2k_i\pi, 0, \cdots, 0) \in V_{i+1}^+;\end{aligned}\tag{3.7}$$

2, for $i \in \bigcup_{i_j\in\Lambda_h}\{i_j, i_j+1, \cdots, i_{j+1}-1\}$

$$H_1(M, N, \mathbb{Z}) \ni [G_i^{-1}\gamma_i^+] \neq 0;\tag{3.8}$$

3, for $i \in \bigcup_{i_j\in\Lambda_c}\{i_j+1, \cdots, i_{j+1}-1\}$, because of (3.1), we can require

$$\begin{aligned}&2\pi \le \langle g_{i,1}^{-1}, \tilde{\gamma}_i^-(t_i^-) - \tilde{\gamma}_i^-(t_i^+)\rangle \le 2\pi + q_{i,1}^*\\ &2\pi \le \langle g_{i-1,1}^{-1}, \tilde{\gamma}_i^-(t_i^-) - \tilde{\gamma}_i^-(t_i^+)\rangle \le 2\pi + q_{i,1}^*,\end{aligned}\tag{3.9}$$

where $q_{i,1}^* = \max\{2q_{i,1}, q_{i,1}'\}$, (refer (3.6) for the definition of $q_{i,1}'$). This condition implies that $\gamma(t)$ passes through $G_{i-1}\pi V_i^+$ before it passes through $G_i\pi V_i^-$ as t increases. Let us consider the quantity

$$h_{c_i}^{\Delta t}(z_i^+, z_i^-) = \inf_{\substack{\xi(-\Delta t)=z_i^+\in G_{i-1}\pi V_i^+\\ \xi(\Delta t)=z_i^-\in G_i\pi V_i^-}} \int_{-\Delta t}^{\Delta t}(L-\eta_i)(d\xi(t))dt.$$

where the infimum is taken for those ξ whose lift satisfying the condition (3.9). In view of (3.1) and the proof of the lemma 2.1, we have $h_{c_i}^{\Delta t}(z_i^+, z_i^-) \to \infty$ as $\Delta t \to 0$ or $\to \infty$. Thus, if we take $T_i^- = T_i^-(z_i^+, z_i^-)$ as the quantity such that $h_{c_i}^{T_i^-}(z_i^+, z_i^-) = \min_{\Delta t} h_{c_i}^{\Delta t}(z_i^+, z_i^-)$, then we find $0 < T_i^-(x_i^+, x_i^-) < \infty$. Since both πV_i^+ and πV_i^- are compact, there exist $0 < \hat{T}_i^- < \breve{T}_i^- < \infty$ such that $\hat{T}_i^- < T_i^-(z_i^+, z_i^-) < \breve{T}_i^-$ for each $z_i^+ \in \pi V_i^+$ and $z_i^- \in \pi V_i^-$. Let

$$\Delta T_i^- = [\hat{T}_i^-, \breve{T}_i^-].\tag{3.10}$$

Recall μ_i and ψ_i are defined on $\mathbb{R}\times\mathbb{T}^{n-1}$, $G_i^*(\mu_i+\psi_i)(d\tilde{\gamma}) = (\mu_i+\psi_i)(\bar{\pi}G_i^{-1}d\tilde{\gamma})$ is well defined. We introduce a translation

$$L_{\eta_i,\mu_i,\psi_i} \to L - \eta_i - (k_iG_i)^*(\mu_i+\psi_i)$$

where k_i^* is a translation of q_1: $(k_i)^*\phi(q,\dot{q}) = \phi(q_1 - 2\pi k_i, \hat{q}, \dot{q})$ on $T\bar{M}$ and the integer k_i is chosen so that (3.7) holds.

Let $\vec{t}=(t_0^-,t_1^\pm,\cdots,t_{i_m-1}^\pm,t_{i_m}^+)$, $\vec{z}=(z_0^-,z_1^\pm,\cdots,z_{i_m-1}^\pm,z_{i_m}^+)$. We consider the minimal action

$$\begin{aligned} h_L^{K,K'}(m,m',\vec{z},\vec{t}) &= \inf\sum_{i=0}^{i_m}\int_{t_i^+}^{t_i^-}(L-\eta_i)(d\gamma_i^-(t))dt \\ &+\sum_{i=0}^{i_m-1}\int_{t_i^-}^{t_{i+1}^+}(L-\eta_i-(k_iG_i)^*(\mu_i+\psi_i))(d\tilde{\gamma}_i^+(t))dt \end{aligned} \tag{3.11}$$

where the infimum is taken over all C^1-curves γ: $[-K,K']\to TM$ with fixed boundary conditions $\bar{\pi}G_i^{-1}\tilde{\gamma}_i^+(t_i^-)-(2k_i\pi,0,\cdots,0)=z_i^-$, $\bar{\pi}G_i^{-1}\tilde{\gamma}_i^+(t_{i+1}^+)-(2k_i\pi,0,\cdots,0)=z_{i+1}^+$ for $i=0,1,\cdots,i_m-1$, $\gamma(-K)=m$, $\gamma(K')=m'$ and satisfying the conditions (3.7), (3.8) and (3.9).

As the system is autonomous, the quantity $h_L^{K,K'}(m,m',\vec{z},\vec{t})$ keeps constant if $(\vec{t},K,K')$ is subject to a translation. Thus, it is a function of $K'-t_{i_m}^+$, t_0^-+K and $\Delta\vec{t}=\{\Delta t_0^+,\Delta t_1^\pm,\cdots,\Delta t_{i_m-1}^\pm\}$ in which $\Delta t_i^+=\frac{1}{2}(t_{i+1}^+-t_i^-)$ and $\Delta t_i^-=\frac{1}{2}(t_i^--t_i^+)$.

Denote by $\gamma(t;K,K',m,m',\vec{z},\Delta\vec{t})$ the minimizer of (3.11). The purpose of the following steps is obviously to investigate when the minimizer is a solution of the Euler-Lagrange equation determined by L. Indeed, It is achieved by taking the minimum of $h_L^{K,K'}(m,m',\vec{z},\Delta\vec{t})$ in $\vec{z}$ and $\Delta\vec{t}$ over some set $\vec{V}$ and $\Delta\vec{T}$, in which the $\vec{V}=(V_0^-,V_1^\pm,\cdots,V_{i_m-1}^\pm,V_{i_m}^+)$ has been well defined in the previous demonstration. From now on, let us define the range where $\Delta\vec{t}$ takes its value.

Proposition 3.1. *Assume there is a unique c-minimizing measure. Then for each $m,m'\in M$ and each $x\in\mathcal{A}(c)$, we have*

$$h_c^\infty(m,x)+h_c^\infty(x,m')=h_c^\infty(m,m').$$

Refer Corollary 5.3.7 in [Fa2] for the proof.

Applying this proposition to c_i with $i\in\bigcup_{i_j\in\Lambda_h}\{i_j,i_j+1,\cdots,i_{j+1}-1\}$, noticing the compactness of M and $h_c^\infty=\lim_{T\to\infty}h_c^T$ in autonomous case [Fa1], we find that, for any $\epsilon_i^*>0$, there exists $T_i^-=T_i^-(\epsilon^*)>0$ such that

$$|h_{c_i}^T(m,m')-h_{c_i}^\infty(m,x)-h_{c_i}^\infty(x,m')|\le\epsilon_i^* \tag{3.12}$$

holds for each $T\ge T_i^-$, each $m,m'\in M$ and $x\in\mathcal{A}(c_i)$.

Denote by $\Delta\vec{T}$ the domain where $\Delta\vec{t}$ takes its value. Its definition is somehow implicit. Let

$$\Delta T_i^+=[T_i^+-d_i,T_i^++d_i],\qquad \forall i\in\bigcup_{i_j\in\Lambda_h}\{i_j,i_j+1,\cdots,i_{j+1}-1\} \tag{3.13}$$

$$\Delta T_i^+ = (0, T_i^+], \qquad \Delta T_i^- = [\hat{T}_i^-, \breve{T}_i^-], \qquad \text{for other } i \le i_m. \tag{3.14}$$

See (3.2), (3.4) for the definition of T_i^+ and (3.10) for the definition of ΔT_i^- respectively.

$\Delta\vec{t} \in \Delta\vec{T}$ implies that $\Delta t_i^+ \in \Delta T_i^+$ for all $0 \le i < i_m$, $\Delta t_i^- \in \Delta T_i^-$ for $i \in \bigcup_{i_j \in \Lambda_c}\{i_j+1, i_j+2, \cdots, i_{j+1}-1\}$, for other i Δt_i^- satisfies the following restrictions:

$$\begin{aligned} T_i^- + T_i^+ + T_{i-1}^+ + d_i + d_{i-1} &\le \Delta t_i^- + \Delta t_i^+ + \Delta t_{i-1}^+ \\ &\le T_i^- + T_i^+ + T_{i-1}^+ + 3(d_i + d_{i-1}) \end{aligned} \tag{3.15}$$

holds for $\Delta t_i^+ \in \Delta T_i^+$, $\Delta t_{i-1}^+ \in \Delta T_{i-1}^+$ (see (3.12) for the definition of T_i^-).

Let $h_L^{K,K'}(m,m')$ be the minimum of $h_L^{K,K'}(m,m',\vec{z},\vec{t})$ over $\vec{V}$ in $\vec{z}$ and over $\Delta\vec{T}$ in $\Delta\vec{t}$ respectively:

$$h_L^{K,K'}(m,m') = \min_{\Delta\vec{t} \in \Delta\vec{T}, \vec{z} \in \vec{V}} h_L^{K,K'}(m,m',\vec{z},\vec{t}),$$

denote the corresponding minimal curve by $\gamma(t;K,K',m,m')$, we claim that $d\gamma(t;K,K',m,m')$ is a solution of the Euler-Lagrange equation of L if K and K' are sufficiently large. To verify, we need to show

1, $d\gamma_i^+ = d\gamma|_{\Delta t_i^+}$ solves the Euler-Lagrange equation determined by L; Note $d\gamma_i^- = d\gamma|_{\Delta t_i^-}$ is clearly a solution of the equation;

2, $\gamma(t;K,K',m,m')$ has no corner at $G_i z_i^-$ and $G_{i-1} z_i^+$ for each $i = 0, 1, \cdots i_m - 1$, i.e. it is smooth for the whole $t \in [-K, K']$.

Indeed, if $i \in \bigcup_{i_j \in \Lambda_h}\{i_j, i_j+1, \cdots, i_{j+1}-1\}$, we obtained from (3.3) that

$$\bar{\gamma}_i^+(t) \in U_i \qquad \text{when } \ \bar{\gamma}_{i,1}^+(t) - 2k_i\pi \in [-\Delta_i, \Delta_i],$$

where $\bar{\gamma}_i^+ = \bar{\pi} G_i \tilde{\gamma}_i^+$. Since the support of $\bar{\mu}_i$ has no intersection with U_i and ψ_i is closed in U_i, while μ_i is closed and $\psi_i = 0$ in the region $\{\bar{\gamma}_{i,1}^+(t) - 2k_i\pi \notin [-\Delta_i, \Delta_i]\}$, the term μ_i and ψ_i have no contribution to the Euler-Lagrange equation along $\bar{\gamma}_i^+$. For other i, the conclusion is obtained from (3.4) by the same argument. This proves the first point.

We claim that γ does not touch the boundary of $G_i\pi V_i^- \times G_i\pi V_{i+1}^+ \times [T_i^+ - d_i, T_i^+ + d_i]$ for $i \in \bigcup_{i_j \in \Lambda_h}\{i_j, i_j+1, \cdots, i_{j+1}-1\}$. Let us assume the contrary, i.e. $(z_i^-, z_{i+1}^+, \Delta t_i^+) \in \partial(V_i^- \times V_{i+1}^+ \times [T_i^+ - d_i, T_i^+ + d_i])$ holds for some $i \in \bigcup_{i_j \in \Lambda_h}\{i_j, i_j+1, \cdots, i_{j+1}-1\}$ by noticing $z_i^- = \bar{\pi} G_i^{-1} \tilde{\gamma}_i^+(t_i^-) - (2k_i\pi, 0, \cdots, 0)$ and $z_i^+ = \bar{\pi} G_i^{-1} \tilde{\gamma}_i^+(t_i^+) - (2k_i\pi, 0, \cdots, 0)$. Then, in $q =$

$G_i^{-1}x$-coordinates, we obtain from (3.2) and (3.12) that

$$\begin{aligned}
&h_{c_i}^{\Delta t_i^-}(G_i^{-1}G_{i-1}\pi z_i^+,\pi z_i^-)+h_{\eta_i,\mu_i,\psi_i,e_n}^{\Delta t_i^+}(z_i^-,z_{i+1}^+)\\
&\quad+h_{c_{i+1}}^{\Delta t_{i+1}^-}(\pi z_{i+1}^+,G_i^{-1}G_{i+1}\pi z_{i+1}^-)\\
\geq{}&h_{c_i}^{\infty}(\xi,\pi z_i^-)+h_{\eta_i,\mu_i,\psi_i,e_n}^{\Delta t_i^+}(z_i^-,z_{i+1}^+)+h_{c_{i+1}}^{\infty}(\pi z_{i+1}^+,\zeta)\\
&\quad+h_{c_i}^{\infty}(G_i^{-1}G_{i-1}\pi z_i^+,\xi)+h_{c_{i+1}}^{\infty}(\zeta,G_i^{-1}G_{i+1}\pi z_{i+1}^-)-2\epsilon_i^*\\
\geq{}&h_{c_i}^{\infty}(\xi,\pi\bar{x}_i^-)+h_{\eta_i,\mu_i,\psi_i,e_n}^{T_i^+}(\bar{x}_i^-,\bar{x}_{i+1}^+)+h_{c_{i+1}}^{\infty}(\pi\bar{x}_{i+1}^+,\zeta)\\
&\quad+h_{c_i}^{\infty}(G_i^{-1}G_{i-1}\pi z_i^+,\xi)+h_{c_{i+1}}^{\infty}(\zeta,G_i^{-1}G_{i+1}\pi z_{i+1}^-)+3\epsilon_i^*\\
\geq{}&h_{c_i}^{\Delta t_i^-}(G_i^{-1}G_{i-1}\pi z_i^+,\pi\bar{x}_i^-)+h_{\eta_i,\mu_i,\psi_i,e_n}^{T_i^+}(\bar{x}_i^-,\bar{x}_{i+1}^+)\\
&\quad+h_{c_{i+1}}^{\Delta t_{i+1}^-}(\pi\bar{x}_{i+1}^+,G_i^{-1}G_{i+1}\pi z_{i+1}^-)+\epsilon_i^*
\end{aligned}$$

where $\bar{x}_i^-$ and $\bar{x}_{i+1}^+$ are the intersection point of a curve in $\mathscr{C}_{\eta_i,\mu_i,\psi_i,e_n}$ with V_i^- and with V_{i+1}^+ respectively, $\xi\in\mathcal{M}(c_{i-1})$ and $\zeta\in\mathcal{M}(c_i)$. It contradicts the minimality of γ, thus verifies our claim.

To see the curve $\bar{\pi}G_i^{-1}\tilde{\gamma}-(2k_i\pi,0,\cdots,0)$ is smooth at z_i^-, let us assume the contrary again. Let $z'=\bar{\pi}G_i^{-1}\tilde{\gamma}_i^-(t_i^--\delta)-(2k_i\pi,0,\cdots,0)$ and $z^*=\bar{\pi}G_i^{-1}\tilde{\gamma}_i^+(t_i^-+\delta)-(2k_i\pi,0,\cdots,0)$, here δ is chosen so small that $\Delta t_i^+\pm\delta\in[T_i^+-d_i,T_i^++d_i]$. This is possible since $(z_i^-,z_{i+1}^+,\Delta t_i^+)\notin\partial(V_i^-\times V_{i+1}^+\times[T_i^+-d_i,T_i^++d_i])$ implies that $T_i^+-d_i<\Delta t_i^+<T_i^++d_i$. We join these two points by a minimizer $\xi:[-\delta,\delta]\to M$ with $\xi(-\delta)=z'$ and $\xi(\delta)=z^*$

$$[A_{c_i}(\xi|_{[-\delta,\delta]})]=\inf_{\substack{\zeta(-\delta)=z'\\ \zeta(\delta)=z^*}}\int_{-\delta}^{\delta}(L-\eta_i)(d\xi(s))ds.$$

If ξ passes through V_i^-, we obtain a curve γ' by replacing the segment of the minimizer $\gamma|_{[t_i^--\delta,t_i^-+\delta]}$ with $G_i\xi:[-\delta,\delta]\to M$. Let $t_i'^-$ be the time for γ' passing through $G_i\pi V_i^-$, then $\frac{1}{2}(t_{i+1}^+-t_i'^-)\in\Delta T_i^+$ and $\frac{1}{2}(t_i'^--t_i^+)\in\Delta T_i^-$. Consequently, we would get an absolutely continuous curve which is admissible for each condition (see (3.13) and (3.15)). Along this curve we would get even smaller $h_L^{K,K'}(m,m')$, but this is absurd. So, we only need to show ξ passes through $G_iV_i^-$. Indeed, as V_i^- is chosen small and transversal to the local connecting curve in $\mathscr{C}_{\eta_i,\mu_i,\psi_i,e_n}$, $\gamma_i^-(t)$ and $\gamma_i^+(t)$ approaches $G_iV_i^-$ from different sides as $t\searrow t_i^-$ and $t\nearrow t_i^-$ respectively. Otherwise, the minimality of γ will be violated. One refers to [BCV] for the details of argument. The smoothness at $G_iz_{i+1}^+$ can be proved similarly.

To show the smoothness of γ at $t=t_i^\pm$ for $i\in\bigcup_{i_j\in\Lambda_c}\{i_j,i_j+1,\cdots,i_{j+1}-1\}$, let us consider first the case that c_{i-1} is connected to c_i also by an orbit of c-type. In the coordinate system $q=G_i^{-1}x$ we consider a covering space $\bar{M} = \mathbb{R} \times \mathbb{T}^{n-1}$. In $q = G_i^{-1}x$-coordinates, both $G_i^{-1}G_{i-1}V_i^+$ and $G_i^{-1}G_{i+1}V_{i+1}^-$ are codimension-1 sub-manifold which may extend to $-\infty$ as well as ∞ in q_1-direction. By construction however, $\bar{\pi}G_i^{-1}\tilde{\gamma}-(2k_i\pi, 0, \cdots, 0)$ passes through $V_{i,+}^-$, V_i^-, V_{i+1}^+ and $V_{i+1,-}^+$ successively (see (3.5) for the definition), they are compact in $\mathbb{R} \times \mathbb{T}^{n-1}$. Therefore, it satisfies the conditions

$$\begin{aligned}
&\bar{\pi}G_i^{-1}\tilde{\gamma}(t_i^+) - (2k_i\pi, 0, \cdots, 0) \in \{-K_i - 2\pi - q_{i,1}^* \le q_1 \le -K_i - 2\pi\},\\
&\bar{\pi}G_i^{-1}\tilde{\gamma}(t_i^-) - (2k_i\pi, 0, \cdots, 0) \in \{q_1 = -K_i\},\\
&\bar{\pi}G_i^{-1}\tilde{\gamma}(t_{i+1}^+) - (2k_i\pi, 0, \cdots, 0) \in \{q_1 = K_i\},\\
&\bar{\pi}G_i^{-1}\tilde{\gamma}(t_{i+1}^-) - (2k_i\pi, 0, \cdots, 0) \in \{K_i + 2\pi \le q_1 \le K_i + 2\pi + q_{i,1}^*\}.
\end{aligned}$$

If γ had corner at $\gamma(t_i^-)$, we would get a minimizer ξ: $[t_i^+, t_{i+1}^+] \to \bar{M}$ of $L - \eta_i - k_i^*\mu_i$ in $\bar{M}$ connecting from $\bar{\pi}G_i^{-1}\tilde{\gamma}(t_i^+)$ to $\bar{\pi}G_i^{-1}\tilde{\gamma}(t_{i+1}^+)$. As γ is assumed not smooth at $\gamma(t_i^-)$, the action of $L - \eta_i - k_i^*\mu_i$ along ξ is smaller than the action along $\bar{\pi}G_i^{-1}\tilde{\gamma}(t_i^+)|_{[t_i^+, t_{i+1}^+]}$. Replacing the segment of $\bar{\pi}G_i^{-1}\tilde{\gamma}|_{[t_i^+, t_{i+1}^+]}$ by ξ, we would obtain a curve γ' with smaller action. Indeed, let $t_i'^-$ be the time when ξ passes through $\{q_1 = -K_i\}$, we can see easily from the definition of $\Delta T_i^\pm$ that $t_i'^- - t_i^+ \in \Delta T_i^-$ and $t_{i+1}^+ - t_i'^- \in \Delta T_i^+$. This absurdity implies the smoothness of γ at $\gamma(t_i^-)$. In the same way we can show the smoothness of γ at $\gamma(t_{i+1}^+)$.

If c_{i-1} is connected to c_i by an orbit of h-type, our assumptions ensure the uniqueness of c_{i-1}-minimal measure. One obtains the smoothness of $\tilde{\gamma}$ at $G_{i-1}^{-1}V_i^+$ from the arguments for h-type and the smoothness at $G_i^-V_i^-$ from the arguments for c-type.

As the system is autonomous, the following limit exists

$$h_L^\infty(m, m') = \lim_{K, K' \to \infty} h_L^{K,K'}(m, m').$$

We pick out a sequence of $\gamma(t; K, K', m, m')$ for large K and K'. Obviously, the set $\{\gamma(t; K, K', m, m')\}$ has at least one accumulation point γ_∞: $\mathbb{R} \to M$ with the property $\alpha(d\gamma_\infty) \subseteq \tilde{\mathcal{A}}(c)$ and $\omega(d\gamma_\infty) \subseteq \tilde{\mathcal{A}}(c')$. As we have shown, it is an orbit of ϕ_L^t. ☐

4. APPLICATION TO A PRIORI UNSTABLE SYSTEMS

Arnold diffusion in time-dependent systems has been studied in many papers, for instance, see [Ar1], [CY1], [CY2], [DLS], [Tr] and [Be2]. It has been shown a generic phenomenon in [CY1], [CY2]. For autonomous systems, there are some examples of Arnold diffusion, for instance, see [Bs], [KL] and [Zh]. In this section, we apply the result in the last section to show that, in autonomous Hamiltonian systems of *a priori* unstable type, Arnold diffusion is generic phenomenon.

We consider the autonomous Hamiltonian of *a priori* unstable type

$$H_\epsilon(p,q,x,y) = R(p) + P(x,y) + h_\epsilon(p,q,x,y). \tag{4.1}$$

Here, $(q,p) \in \mathbb{T}^2 \times \mathbb{R}^2$, $x \in \mathbb{T}^k \times \mathbb{R}^k$ and H_ϵ is convex in (p,y), $h_\epsilon \to 0$ as $\epsilon \to 0$. Let $\Phi^t_{H_\epsilon}$ be the Hamiltonian flow determined by H_ϵ. When $\epsilon = 0$, we have $p(t) = p(t_0)$ along each orbit of $\Phi^t_{H_0}$. Let

$$\Gamma_e = \{p \in \mathbb{R}^2 : R(p) = e\}$$

which is a loop only if e is suitably large, we assume it for $e > 0$. A notable question is to ask whether there are some orbits of $\Phi^t_{H_\epsilon}$, along which p has substantial variation in a neighborhood of Γ_e.

Let L be the Lagrangian, corresponding to H_ϵ via Legendre transformation. It has the form

$$L(q,x,\dot q,\dot x) = \ell_q(\dot q) + \ell_x(x,\dot x) + \ell_\epsilon(q,x,\dot q,\dot x) \tag{4.2}$$

where $\ell_\epsilon \to 0$ as $\epsilon \to 0$. Without losing of generality, we assume $\ell_x(0,0) = 0$. The *a priori unstable* type assumes that

Hypothesis: there is an open disk $O_x \in H^1(\mathbb{T}^k,\mathbb{R})$ such that for each $c_x \in O_x$, the Aubry set for the Lagrangian ℓ_x is a single point: $\mathcal{A}(c_x) = \{x = 0\}$. $(x,\dot x) = (0,0)$ is a hyperbolic fixed point for the Hamiltonian flow Φ^t_P.

This condition implies that the subsystem determined by $P(x,y)$ can be interpreted as a pendulum with k degrees of freedom. The existence of such an open disk implies that

$$\int_{-\infty}^{\infty} \ell_x(d\gamma_x(t))dt > 0$$

holds for each orbit $d\gamma_x$ homoclinic to the origin, here we assume that $0 \in O_x$.

We use $\mathfrak{B}_{a,K}$ to denote a ball in the function space $C^r(\{(q,x,p,y) \in \mathbb{R}^{2+k} \times \mathbb{T}^{2+k} : \|(p,y)\| \le K\} \to \mathbb{R})$, centered at the origin with radius of a. Here $r = 3, \cdots, m, \cdots, \infty$.

Theorem 4.1. *We assume that the Hamiltonian (4.1) is positive definite and has super-linear growth in* (p,y). *It satisfies the hypothesis of a priori unstable type. Given small* $d > 0$ *and any two points* $p^*, p' \in \Gamma_e$ *for any* $e > 0$, *there is a suitably small* $\epsilon > 0$ *and a generic set* $\mathfrak{P} \subset \mathfrak{B}_{\epsilon,K}$ *such that for each* $h_\epsilon \in \mathfrak{P}$, $\Phi^t_{H_\epsilon}$ *has a trajectory* $\{p(t),q(t),x(t),y(t)\}$ *which connects* $\{\|p-p^*\| < d\}$ *to* $\{\|p-p'\| < d\}$.

Proof. We apply the results obtained in the last section to complete the proof of this theorem. The key step is to check whether there is a generalized transition chain. It is similar to our previous work in [CY1], [CY2].

When $\epsilon = 0$, The Hamiltonian flow $\Phi^t_{H_0}$ admits a normally hyperbolic manifold

$$\Pi_{0,e} = \{(p, q, x, y) \in \Gamma_e \times \mathbb{T}^2 \times (0, 0)\}.$$

According to the theorem of normally hyperbolic manifold, $\Phi^t_{H_\epsilon}$ also admits a normally hyperbolic manifold $\Pi_{\epsilon,e}$ provided ϵ is sufficiently small (cf. [HPS]). This manifold is a small deformation of $\Pi_{0,e}$. Locally, we can always choose a 2-D section in $\Pi_{\epsilon,e}$ such that Hamiltonian flow intersects it transversally. For example, we choose the section $\Pi_{\epsilon,e} \cap \{a_1q_1 + a_2q_2 = 0\}$ in the case that $a_1\partial_{p_1}R + a_2\partial_{p_2}R > 0$, where $(a_1, a_2) \in \mathbb{Z}^2\backslash\{0\}$. This section is diffeomorphic to a cylinder. In this way, we obtain a Poincaré return map on the cylinder. Obviously, restricted on the cylinder, this return map is twist and area-preserving (cf. [CY1]). The well-developed theory of twist map tells us that for each rotation number the Aubry-Mather set must be one of the following three cases:

1, there is an invariant curve which is a Lipschitz graph;

2, if the rotation number is irrational and there is no invariant curve, then there is a unique Cantorus which corresponds to the Aubry set;

3, it is generic that there is no invariant curve when the rotation number is rational, and the Aubry-Mather set contains only one minimal periodic orbit.

Let us consider it in the Lagrangian formalism. The α-function for ℓ_q is smooth, each 2-D torus $p \times \mathbb{T}^2$ for $p \in \Gamma_e$, under the Legendre transformation, is the support of some c_q-minimal measure $c_q = p \in \Gamma_e$. They have the same average action $\alpha_q(c_q) = e$. On the other hand, the α-function for ℓ_x has a flat containing the origin $c_x = 0$. If considering them in one system, we see that for $c = (c_q, c_x) \in \Gamma_e \times O_x \subset H^1(\mathbb{T}^{2+k}, \mathbb{R})$, $\mathcal{M} = \mathcal{A}(c) = \mathcal{N}(c) = \mathbb{T}^2 \times \{x = 0\}$.

Applying the upper semi-continuity of Mañé set on the Lagrangian, we see that, for a generic perturbation, there is loop $\Gamma_{\epsilon,e}$ and open disk $O_{\epsilon,x}$ such that

1, $d(\Gamma_{\epsilon,e}, \Gamma_e) \to 0$ and $O_{\epsilon,x} \to O_x$ as $\epsilon \to 0$,

2, for each $c \in \Gamma_{\epsilon,e} \times O_{\epsilon,x}$, the Aubry set for the perturbed system $\mathcal{A}_\epsilon(c) \subset \{x = 0\} + \delta$, the minimal measure has only one ergodic component, the support is either a 2-D torus, or a Cantori suspended by a Denjoy set, or a closed orbit.

3, for any loop γ with $0 \neq [\gamma] \in H_1(\mathbb{T}^{2+k}, \mathbb{T}^2, \mathbb{Z})$, we have

$$\int (L - \eta_c + \alpha(c))(d\gamma(t))dt > 0. \tag{4.3}$$

To verify the existence of a generalized transition chain, we consider first the case that there is no invariant curve in the cylinder for the corresponding $c \in \Gamma_{\epsilon,e} \times O_{\epsilon,x}$. For these cohomology classes, we can choose a vector (a_1, a_2) from the three candidates $\{(1,0),(1,1),(0,1)\}$ and choose $\Sigma_c = \{(q,x) \in \mathbb{T}^{k+2} : a_1q_1 + a_2q_2 = 0\}$ such that $\mathcal{N}(c) \cap \Sigma_c$ is totally disconnected. Thus, for each c under consideration, one of these three $k+1$ dimensional tori is admissible, each c is equivalent to any c' nearby.

If there is an invariant curve for c, we consider a finite covering of $\mathbb{T}^{2+k}$, $\tilde{M} = \mathbb{T}^{1+k} \times 2\mathbb{T}$. The condition (4.3) implies that $\pi_1\mathcal{N}(c, \tilde{M})\backslash(\mathcal{A}(c, M) + \delta)$ is not empty. Therefore, for the existence of a generalized transition chain, we need to verify that there is an isolated minimal homoclinic curve to each invariant curve.

Towards that goal, we fully invoke the strength of the technique used in [CY1]. Let us briefly describe it for the time-periodic case. Let $\mathfrak{Z}$ be the set of those functions whose minimal points are not totally disconnected, this set has "infinite codimensions" in C^0: there is a subspace with infinite dimensions $\mathfrak{N} \subset C^r$ such that $U_1 + U_2 \notin \mathfrak{Z}$ holds for each $U_1 \in \mathfrak{Z}$ and each $U_2 \in \mathfrak{N}\backslash\{0\}$. If the set of barrier functions $\mathfrak{B} = \cup B_c^*$ has finite box dimensions, one can expect that arbitrarily small translation $\mathfrak{B} \to \mathfrak{B} + u = \cup(B_c^* + u)$ exists such that $\mathfrak{Z} \cap (\mathfrak{B} + u) = \varnothing$, i.e. the set of minimal points of each $B_c^* + u$ is totally disconnected.

The finiteness of the box dimension is obtained by studying the regularity of the barrier functions in c. If B_c^* is α-Hölder continuous in c and if c takes its value on a smooth curve, the set of barrier function $\{B_c^*\}$ in C^0-function space has box-dimension not bigger than α^{-1}. We find $\frac{1}{2}$-Hölder regularity on those cohomology classes where the invariant curves exist. The proof is the same as in [CY1], we omit the details here.

For autonomous systems, the minimal point of $B_c^*|_{M\backslash\mathcal{A}(c,M)+\delta}$ corresponds to the minimal homoclinic curves in $\pi'\mathcal{N}(c, \tilde{M})\backslash(\mathcal{A}(c, M) + \delta)$. Since the system is autonomous, the barrier function reaches its minimum on the whole curve. What we need is that the set of the minimal points of each barrier function is totally disconnected when it is restricted on some co-dimension one section.

In virtue of the special form of the Hamiltonian (4.1), there exist m pairs of open disks $U_1 \subset U_1', U_2 \subset U_2', \cdots U_m \subset U_m' \subset \mathbb{R}^2$ such that

1, $U_1 \cup U_2 \cup \cdots \cup U_m \supset \Gamma_e$;

2, Restricted on each $U_i' \times \mathbb{R}^k \times \mathbb{T}^{k+2}$, there exist $j \in \{1,2\}$ and $d_i > 0$ such that $\partial_{p_j} H \geq d_i$ or $\partial_{p_j} H \leq -d_i$.

In the case that $j = 1$ in the region $U_i \times \mathbb{R}^k \times \mathbb{T}^{k+2}$, i.e. $\dot{q}_1 \geq d_i$. Let $K(p_2, x, y, T, q_2, e)$ be the solution of $H(p_1, p_2, -T, q_2, x, y) = e$, where $T = -q_1$, then the Hamiltonian flow Φ_K^T is equivalent to Φ_H^t in this region [Ar2]. By simple calculation, we see that K is positive definite in (p_2, y). Clearly, K is periodic in $T = -q_1$. Because of the special form of H, the variation of p is small along each orbit in the Mañé sets, along positive half of the forward semi-static orbits and along negative half of the backward semi-static orbits. By refining the division $U_1, U_2, \cdots, U_m$ we can assume that these orbits entirely stay in U_i' if they start from U_i. These orbits of K exactly correspond to those for H. Denote by $B_{K,c}$ the barrier functions corresponding to K. Clearly, each minimal point of $B_{K,c}$ exactly corresponds to a minimal point of the barrier function B_c for H in the section $\{q_1 = 0\}$. According to the study in [CY1] and [CY2]), outside a small neighborhood of Aubry set, any small perturbation $B_{K,c} \to B_{K,c} - u_\epsilon$ can be realized by a Hamiltonian perturbation $K \to K + K_\epsilon$. Note $\partial_{p_1} H \geq d_i > 0$, there exists h_ϵ such that $K + K_\epsilon$ is the solution of $H + h_\epsilon = e$. Thus, by the argument in [CY1] and [CY2], $\exists$ an open and dense set $\mathfrak{O}_{i,\epsilon,K} \subset \mathfrak{B}_{i,\epsilon,K}$ such that for each $h_\epsilon \in \mathfrak{O}_{i,\epsilon,K}$, the set of minimal homoclinic curves in $U_i \times \mathbb{R}^k \times \mathbb{T}^{k+2}$ are totally disconnected for each related cohomology class. Here, $\mathfrak{B}_{i,\epsilon,K}$ denote the set of Hamiltonians defined on $(U_1' \cup U_2' \cup \cdots \cup U_m') \times \mathbb{T}^{k+2} \times [-K, K]^k$ with the topology induced by the standard C^r-norm on $U_i' \times \mathbb{T}^{k+2} \times [-K, K]^k$. Obviously, the set $\cap_{i=1}^m \mathfrak{O}_{i,\epsilon,K} \subset C^r((U_1' \cup U_2' \cup \cdots \cup U_m') \times \mathbb{T}^{k+2} \times [-K, K]^k, \mathbb{R})$ is open and dense.

Therefore, given any two points in $\Gamma_{\epsilon,e} \times O_{\epsilon,x}$, generically, there exists a generalized transition chain joining these two points. This completes the proof. □

5. Appendix

Proposition 5.1. *Let a smooth injection φ: $\mathbb{T}^{n-1} \to \mathbb{T}^n$ such that the induced map φ_*: $H_1(\mathbb{T}^{n-1}, \mathbb{Z}) \to H_1(\mathbb{T}^n, \mathbb{Z})$ is an injection. Let $\{e_i\}_{1 \leq i \leq n-1}$ be the standard basis of $H_1(\mathbb{T}^{n-1}, \mathbb{Z})$, $g_{i+1} = \varphi_*(e_i) \in H_1(\mathbb{T}^n, \mathbb{Z})$. Then there exists $g_1 \in \mathbb{Z}^n$ such that the matrix $G = (g_1, g_2, \cdots, g_n)$ is uni-module, i.e.* $\det G = \pm 1$.

Proof. : For $\mathbb{R}^k$, we use e_j to denote the vector $(0, \cdots, 1, \cdots, 0)$ where the j-th entry is one and all other entries are zero, use I^k to denote the unit cube $[0,1]^k$. For a point $x = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k$, we use (x) to denote the point $((x_1), (x_2), \cdots, (x_k))$, where (a) denotes the decimal part of the number a. We define the projection map π: $\mathbb{R}^k \to I^k$ such that $\pi x = (x)$.

We assume that ϕ: $\mathbb{T}^{n-1} \to \mathbb{T}^n$ is a linear injection. Denote by $\bar{\phi}$: $\mathbb{R}^{n-1} \to \mathbb{R}^n$ the lift to the universal covering with $\bar{\phi}(0) = 0$. Let $g_i = \bar{\phi}(e_i)$ which is an integer vector. Thus, $\bar{\phi}(I^{n-1})$ is a $n-1$ dimensional parallel polyhedron, spanned by $(g_1, g_2, \cdots, g_{n-1})$. Restricted on $\bar{\phi}(I^{n-1})$, the projection π is one to one, because ϕ is an injection. Clearly, the image consists of some pieces of parallel hyperplane: $\pi\bar{\phi}(I^{n-1}) = \cup D_j, j = 1, 2, \cdots, J$, where the integer J depends on $\{g_i\}_{i \leq n-1}$. We use D_1 to denote the piece which contains the origin. Let Σ be the hyperplane containing $\bar{\phi}(I^{n-1})$, we consider some the parallel shift of Σ: $\Sigma_j, j = 2, \cdots, J$. This shift is made such that $\Sigma_j \supset D_j$ for each $j \leq J$. Clearly, each Σ_j passes some points in $\mathbb{Z}^n$. Being aware of codimension one, we see that there is some Σ_{j_0} such that the layer bounded by Σ and Σ_{j_0} does not contain any other Σ_j. Let $g_n \in \mathbb{Z}^n$ be a point in Σ_{j_0}, denote by V the n dimensional parallel polyhedron, spanned by $g_i = \bar{\phi}(e_i) \in \mathbb{Z}^n$ $(i = 1, 2, \cdots, n)$. We project this parallel polyhedron into the unit cube $\pi : V \to I^n$. Clearly, this map is onto, and one to one in $I^k \backslash \cup D_j$. Let $G = (g_1, g_2, \cdots, g_n)$ be an $n \times n$ matrix, the above argument proves that $\det G = \pm 1$. $\square$

Acknowledgement This work is partially supported by National Basic Research Program of China (973 Program, 2007CB814800), NNSF of China (Grant 10531050) and Basic Research Program of Jiangsu Province, China. The authors thank the referees for their comments and suggestions for the revise of the paper.

References

[Ar1] Arnol'd V. I., *Instability of dynamical systems with several degrees of freedom,* (Russian, English) Sov. Math., Dokl., **5**(1964), 581-585; translation from Dokl. Akad. Nauk SSSR, **156**(1964), 9-12.

[Ar2] Arnol'd V. I., *Mathematical Methods of Classical Mechanics* (1978) Springer.

[Be0] Bernard P., *Homoclinic orbits to invariant sets of quasi-integrable exact maps*, Ergod. Theory Dynam. Syst. **20** (2000), 1583-1601.

[Be1] Bernard P., *Connecting orbits of time dependent Lagrangian systems*, Ann. Inst. Fourier, **52** (2002), 1533-1568.

[Be2] Bernard P., *The dynamics of pseudographs in convex Hamiltonian systems*, J. Amer. Math. Soc. **21** (2008) 615-669.

[Bs] Bessi U. *Arnold's example with three rotators*, Nonlinearity **10** (1997) 763-781.

[BCV] Bessi U., Chierchia L. & Valdinoci E., *Upper bounds on Arnold diffusion times via Mather theory*, J. Math. Pure Appl. **80** (2001) 105-129.

[Ca] Carneiro M.J.D.,*On minimizing measures of the action of autonomous Lagrangians*, Nonlinearity **6** (1996) 1077-1085.

[CY1] Cheng C.-Q. & Yan J., *Existence of diffusion orbits in a priori unstable Hamiltonian systems,* J. Differential Geometry, **67** (2004), 457-517.

[CY2] Cheng C.-Q. & Yan J., *Arnold diffusion in Hamiltonian systems: a priori unstable case*, J. Differential Geometry, **82** (2009), 229-277.

[DLS] Delshams A., de la Llave R. and Seara T. M., *Geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristic and rigorous verification of a model*, Memoirs Amer. Math. Soc. **179**(844), (2006).

[Fa1] Fathi A., *Sur la convergence du semi-groupe de Lax-Oleinik*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1998) 267-270.

[Fa2] Fathi A., *Weak KAM Theorem in Lagrangian Dynamics* (Cambridge Studies in Advanced Mathematics), (2009) Cambridge University Press.

[FM] Fathi A. and Mather J., *Failure of convergence of the Lax-Oleinik semi-group in the time-periodic case,* Bull. Soc. Math. France, **128(3)**(2000), 473–483,

[HPS] Hirsch M.W. Pugh C.C. & Shub M., *Invariant Manifolds*, Lect. Notes Math. **583** (1977), Springer-Verlag.

[It] Iturriaga R., *Minimizing measures for time dependent Lagrangians*, Proc. London Math. Soc. **73** (1996) 216-240.

[KL] Kaloshin V. and Levi M., *Arnold diffusion for near-integrable Hamiltonians*, Bulletin AMS **45** (2008) 409-427.

[Mas] Massart D., *On Aubry sets and Mather's action functional*, Israel J. Math. **134** (2003), 157-171.

[Ma1] Mather J., *Action minimizing invariant measures for positive definite Lagrangian systems,* Math. Z., **207(2)**(1991), 169–207.

[Ma2] Mather J., *Variational construction of connecting orbits,* Ann. Inst. Fourier (Grenoble), **43(5)**(1993), 1349–1386.

[Me1] Mañé R., *Lagrangian flows: the dynamics of globally minimizing orbits*, Proceedings Int. Congress in Dynamical Systems (*Montevideo 1995*), Pitman Research Notes in Math. **362** (1996), 120-131.

[Me2] Mañé R., *Generic properties and problems of minimizing measures of Lagrangian systems*, Nonlinearity, **9** (1996),169-207.

[Tr] Treschev D.V., *Evolution of slow variables in a priori unstable Hamiltonian systems*, Nonlinearity, **17** (2004), 1803-1841.

[Zh] Zheng Y., Arnold diffusion in planar five-body problem, (2009) preprint.

Department of Mathematics, Nanjing University, Nanjing 210093, China,

E-mail address: `chengcq@nju.edu.cn`