CONVERGENCE OF RESONANCES ON THIN BRANCHED QUANTUM WAVE GUIDES

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Abstract. We prove an abstract criterion stating resolvent convergence in the case of operators acting in different Hilbert spaces. This result is then applied to the case of Laplacians on a family X_ε of branched quantum waveguides. Combining it with an exterior complex scaling we show, in particular, that the resonances on X_ε approximate those of the Laplacian with "free" boundary conditions on X_0, the skeleton graph of X_ε.

1. Introduction

In a few recent years there was a surge of interest to quantum mechanics on metric graphs. It is a subject with a long history reaching back to the paper of Ruedenberg and Scherr [RuS53] on spectra of aromatic carbohydrate molecules elaborating an idea of L. Pauling, but a systematic study motivated by the need to describe semiconductor graph-type structures began only at the end of the eighties, cf. [EŠ89]; a survey of the subsequent development with the appropriate bibliography can be found, e.g., in the papers [KoS99] or [Ku04].

Since quantum graphs are used in the first place to model various real graph-like structures whose transverse size is small but non-zero, one of the most important questions in the theory is how such system approximate an ideal graph in the limit of zero thickness. This problem is difficult and the answer is so far known in some cases only. In particular, compact "fat graphs" with Neumann boundary conditions has been analyzed, first in [FW93] and [F96], then in [KuZ01] and [RS01] where the eigenvalue convergence was demonstrated; an extension of this result to more general Neumann-type graph-like manifolds can be found in [EP05]. More recently, the resolvent convergence on non-compact graph-like manifolds of this type was dealt with in [P06]. Recall, however, that the analogous problem in the physically most important case of tube systems with Dirichlet boundary is more difficult and at the present moment far from being fully understood, although there are fresh results in this direction [P05], [MV06].

Apart of the spectral analysis, one of the most important questions we study on quantum graphs concerns the resonance scattering. It is usually easy to find resonances on a graph — see, e.g., [ETV01] and references therein — but *a priori* it is not clear how are these related to possible resonances on an approximating finite-thickness manifold. This is the main topic of the present paper.

An efficient way to study resonances understood as poles in the analytically continued resolvent is to rephrase the question as an eigenvalue problem. A time-honored trick to achieve this goal is based on the *complex scaling* — see, e.g., [C69, AC71, BC71, S72, CT73, S79, CDKS87, BCD89] or [RS80, Sec. XII.6 and XIII.10]

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— which transforms the Hamiltonian by a non-unitary operator with the aim to rotate the essential spectrum uncovering a part of the "second sheet" while leaving the poles at place[1]. Our aim here is to apply this method to the problem at hand. We will construct an *exterior* complex-scaling transformation for Hamiltonians on graph-like manifolds and show that some among its complex eigenvalues converge to the eigenvalues of the complex-scaled graph Hamiltonian[2]. In this way, resonances of the quantum graph are approximated by those of the corresponding family of "fat graphs" (cf. Theorems 2.1 and 6.2). Furthermore, graph Hamiltonians often have embedded eigenvalues, e.g. by rational relations between the edges, and these are again approximated, either by embedded eigenvalues or by resonances (one can conjecture that the latter case is generic).

As a by-product of our analysis we will prove, using the technique of [P06], that a magnetic Laplacian of a family of "fat *non-compact* graphs" converges to the one on the corresponding graph, this time without any complex scaling (cf. Theorem 6.4). This conclusion is rather important because it shows that nice results about fractal graph spectra such as the one discussed in [BGP07] can be observed in some form with more "realistic" systems. Needless to say, this is a goal which the experimental physicists vigorously pursue, see e.g. [ASvK$^+$01]. The convergence of the spectrum of the magnetic Laplacian on a *compact* graph was already established in [KuZ01].

Let us describe the contents of the paper. To explain our method in a simple setting first, we analyze in the next section an example of a "lasso" graph having one loop and one semi-infinite external link. After that we describe the two main objects of our approximation, quantum graphs in Section 3 and quantum wave guides in Section 4. The following section is devoted to explanation of the complex-dilation method in our setting, and in Section 6 we will state and prove our main results.

Since the arguments are rather technical and demand various auxiliary material, we collected it in a series of appendices. Appendix A contains facts about Hilbert scales associated to sectorial operators, Appendix B provides an abstract convergence theory for eigenvalues and eigenvectors of non-selfadjoint operators in different Hilbert spaces. Finally, we prove in Appendix C among other things the analyticity of the complex dilated operators.

2. A motivating example: a loop with a lead

Let us start with a slightly informal discussion of a simple example in order to show the main purpose and to motivate the general analysis presented in the forthcoming sections. Proofs and more precise definitions of the operators will also be given there.

2.1. **The graph and its neighbourhood.**

Denote by X_0 the metric graph consisting of a loop e_{int} with a finite length $\ell := \ell_{\mathrm{int}} \in (0, \infty)$ and one external line, i.e., an

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[1] While the complex-scaling method was formulated by mathematicians it became a practical and often used tool in atomic and molecular physics – see, e.g., the review [Moi98].

[2] Complex scaling was used to treat resonances of thin tubes also in [Ned97, DEM01], this time with Dirichlet boundary conditions. In that case the resonances of the limiting zero-thickness problem come from the tube curvature rather than the (trivial) graph structure. The complex scaling can be also used to demonstrate equivalence of the "resolvent" and scattering resonances for a wide class of quantum graphs including those discussed here [EL06].

edge e_{ext} of length $\ell_{\mathrm{ext}} = \infty$ attached to the loop e_{int} at the vertex v; sometimes also called a *lasso graph* [E97]. For simplicity we assume that the graph is planar, i.e., $e \subset \mathbb{R}^2$ and $v \in \mathbb{R}^2$ where e denotes either e_{int} or e_{ext} (cf. Figure 1), and furthermore, that the edges are straight in a neighbourhood of v; we will simply suppose that the exterior edge e_{int} is embedded as a straight half-line in $\mathbb{R}^2$. Denote by X_ε the open $\varepsilon/2$-neighbourhood of X_0. We decompose X_ε into three open, mutually disjoint sets $U_{\varepsilon,\mathrm{ext}}$, $U_{\varepsilon,v}$ and $U_{\varepsilon,\mathrm{int}}$ such that the union of their closures equals $\overline{X}_\varepsilon$. They are chosen in such a way that $v \in U_{\varepsilon,v}$ while $U_{\varepsilon,e}$ is the ε-tubular neighbourhood of the slightly shortened edge e. Since the edges are straight near v by assumption, $U_{\varepsilon,v}$ is ε-homothetic to a fixed set $U_v \subset \mathbb{R}^2$ and there exists an affine transformation

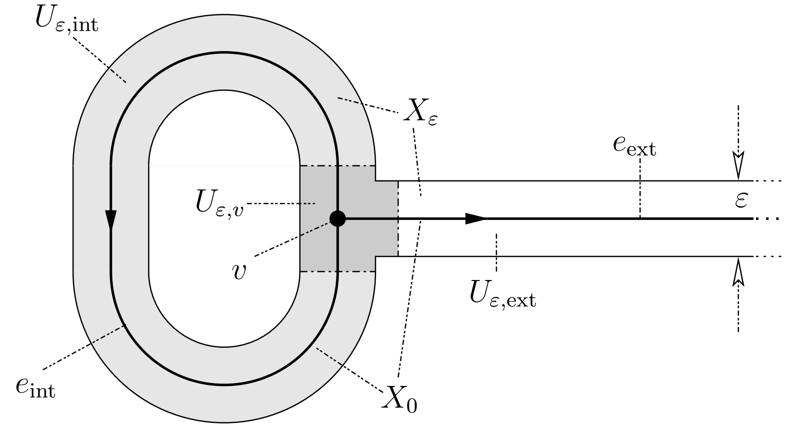

FIGURE 1. The metric graph X_0 consisting of one loop and one external line together with the $\varepsilon/2$-neighbourhood X_ε.

$$\tau_{\varepsilon,v} \colon U_v \longrightarrow U_{\varepsilon,v} \tag{2.1}$$

$$z \mapsto v + \varepsilon z. \tag{2.2}$$

The ε-tubular neighbourhood $U_{\varepsilon,e}$ is given by

$$\tau_{\varepsilon,e} \colon e \times F \longrightarrow U_{\varepsilon,e} \tag{2.3}$$

$$(x,y) \mapsto \gamma_e\bigl(\varphi_{\varepsilon,e}(x)\bigr) + \varepsilon y n_e\bigl(\varphi_{\varepsilon,e}(x)\bigr) \tag{2.4}$$

where $\gamma_e \colon (0,\ell_e) \longrightarrow X_\varepsilon \subset \mathbb{R}^2$ denotes the path parametrising the edge e by arclength (according to its orientation). Furthermore, $n_e \colon (0,\ell_e) \longrightarrow \mathbb{R}^2$ denotes one of the two possible unit vector fields along γ_e orthogonal to the tangent vector $\dot\gamma_e$. We can also identify e with the interval $(0,\ell_e)$ and set $F := (-1/2, 1/2)$. Since the graph is embedded into $\mathbb{R}^2$, we have to take a slightly smaller part of $e = e_{\mathrm{int}}$ instead of the full edge. This is needed when constructing the edge neighbourhood $U_{\varepsilon,e}$ in order to make room for the vertex neighbourhood $U_{\varepsilon,v}$. We therefore let

$$\varphi_{\varepsilon,e} \colon (0,\ell) \longrightarrow \bigl(\varepsilon\ell/2, (1-\varepsilon/2)\ell\bigr)$$

be the affine linear mapping from the full edge e onto the shortened edge where $\varepsilon\ell/2$ is the amount of e belonging to the vertex neighbourhoods; for the external edge a simple shift by $\varepsilon\ell/2$ will do the job.

Since we want to study the (non-relativistic) quantum dynamics on the graph in presence of external fields we have to introduce the latter. Denote by g_{eucl} the usual Euclidean metric in $\mathbb{R}^2$. The *vector potential* α in $\mathbb{R}^2$ is given by a real-valued 1-form

$\alpha = a_1 \mathrm{d}z_1 + a_2 \mathrm{d}z_2$ and we denote the corresponding vector field by $a = (a_1, a_2)$. Furthermore, let q be a real-valued function on $\mathbb{R}^2$, the *electric* potential. Their regularity properties will be specified below.

In the particular example of this section we could, of course, perform all the reasoning which follows in the coordinates given by the embedding. We will, however, employ the ε-independent sets U_v and $U_e := e \times F$, not only because the argument is simpler but also because it can be easily be generalized to the differential geometric setting which we will use in the general case below. Consequently, let us express the metric, the magnetic and electric potential in terms of the coordinates given on U_v and U_e. We set

$$\begin{aligned} g_{\varepsilon,v} &:= \tau^*_{\varepsilon,v} g_{\mathrm{eucl}}, & \alpha_{\varepsilon,v} &:= \tau^*_{\varepsilon,v}\alpha, & q_{\varepsilon,v} &:= \tau^*_{\varepsilon,v} q, \\ g_{\varepsilon,e} &:= \tau^*_{\varepsilon,e} g_{\mathrm{eucl}}, & \alpha_{\varepsilon,e} &:= \tau^*_{\varepsilon,e}\alpha, & q_{\varepsilon,e} &:= \tau^*_{\varepsilon,e} q, \end{aligned} \tag{2.5}$$

where $\tau^*_{\varepsilon,v}\omega$ denotes the usual pull-back of the tensor ω from (a subset of) $\mathbb{R}^2$ to $U_{\varepsilon,v}$, and the other map has the analogous meaning. A simple calculation now shows that quantities at left-hand sides are equal to

$$\begin{cases} g_{\varepsilon,v}(z) = \varepsilon^2 g_{\mathrm{eucl}}, \\ g_{\varepsilon,e}(x,y) = (1-\varepsilon)^2 \bigl(1 - \varepsilon y \kappa_e(\widetilde{x})\bigr)^2 \mathrm{d}x^2 + \varepsilon^2 \mathrm{d}y^2, \end{cases} \tag{2.6}$$

$$\begin{cases} \alpha_{\varepsilon,v}(z) = \varepsilon\alpha(v + \varepsilon z), \\ \alpha_{\varepsilon,e}(x,y) = (1-\varepsilon)\bigl(1 - \varepsilon y \kappa_e(\widetilde{x})\bigr) a^{\|}_e(x,y)\,\mathrm{d}x + \varepsilon a^{\perp}_e(x,y)\,\mathrm{d}y, \end{cases} \tag{2.7}$$

$$\begin{cases} q_{\varepsilon,v}(z) = q(v + \varepsilon z), \\ q_{\varepsilon,e}(x,y) = q\bigl(\gamma_e(\widetilde{x}) + \varepsilon y n_e(\widetilde{x})\bigr), \end{cases} \tag{2.8}$$

where $\widetilde{x} = \varphi_{\varepsilon,e}(x)$, $z \in U_v$, $(x,y) \in e \times F$ and

$$a^{\|}_e(x,y) := \dot\gamma_e(\widetilde{x}) \cdot a(\tau_{\varepsilon,e}(x,y)), \qquad a^{\perp}_e(x,y) := n_e(\widetilde{x}) \cdot a(\tau_{\varepsilon,e}(x,y)) \tag{2.9}$$

denote the *tangential* and *normal* component of the vector field a, respectively, taken at the (shortened) edge e parametrised by $\gamma \circ \varphi_{\varepsilon,e}$. Furthermore,

$$\kappa_e := \dot\gamma_{e,1}\ddot\gamma_{e,2} - \dot\gamma_{e,2}\ddot\gamma_{e,1} \tag{2.10}$$

denotes the (signed) curvature of the curve $\gamma_e = (\gamma_{e,1}, \gamma_{e,2})$ embedded in $\mathbb{R}^2$. As mentioned above we assume that $\kappa_e = 0$ on the external edge $e = e_{\mathrm{ext}}$, and therefore

$$g_{\varepsilon,\mathrm{ext}}(x,y) = \mathrm{d}x^2 + \varepsilon^2 \mathrm{d}y^2 \tag{2.11}$$

has a product structure. In addition, we suppose that the tangential component of the vector potential vanishes, $a^{\|}_e = 0$; notice that this can always be achieved by an appropriate gauge transformation (see Section 3.3 below). For simplicity, we assume also that there is no electric potential on the exterior edge e_{ext} as well as on its neighbourhood $U_{\varepsilon,\mathrm{ext}}$.

2.2. **Magnetic Hamiltonians.** After describing the graph and its neighbourhood we introduce now the corresponding magnetic Schrödinger operator for a vector potential $a = (a_1, a_2)$ (a vector field) and an electric potential q (a function). We shall assume that a_1, a_2, q and their first derivatives are bounded and, as we have said, that they vanish on the external edge neighbourhood.

Let us start with the "fat graph". The magnetic Hamiltonian H_ε in the Hilbert space $\mathsf{L}_2(X_\varepsilon)$ is given formally by the differential expression

$$H_\varepsilon := (\nabla - \mathrm{i}a)^*(\nabla - \mathrm{i}a) + q \tag{2.12}$$

acting on X_ε. To define it properly as a self-adjoint operator one has to specify its domain; namely, we assume *Neumann* boundary conditions. In terms of coordinates introduced on the edge and vertex neighbourhoods we have

$$H_{\varepsilon,e} = \bigl(-\partial_x + \mathrm{i}a_e^{\|} + \mathcal{O}(\varepsilon)\bigr)\bigl(\partial_x - \mathrm{i}a_e^{\|} + \mathcal{O}(\varepsilon)\bigr) + \frac{1}{\varepsilon^2}\bigl(-\partial_y + \mathrm{i}\varepsilon a_e^{\perp}\bigr)\bigl(\partial_y - \mathrm{i}\varepsilon a_e^{\perp}\bigr) + q_{\varepsilon,e}$$

for the internal edge $e = e_{\mathrm{int}}$ and

$$H_{\varepsilon,e} = -\partial_{xx} - \frac{1}{\varepsilon^2}\partial_{yy},$$

$$H_{\varepsilon,v} = \frac{1}{\varepsilon^2}\bigl(-\nabla + \mathrm{i}\varepsilon a_v\bigr)\bigl(\nabla - \mathrm{i}\varepsilon a_v\bigr) + q_{\varepsilon,v}$$

for the external edge $e = e_{\mathrm{ext}}$ and the vertex v, respectively, where a_v is the vector field corresponding to the 1-form α_v. The error term on the internal edge comes from the curvature and the shortened edge — cf. (2.6) and (2.7).

On the other hand, on the graph we consider the "limit" operator H_0 given by

$$H_{0,\mathrm{int}} = (-\partial_x + \mathrm{i}a_e)(\partial_x - \mathrm{i}a_e) + q_e, \quad e = e_{\mathrm{int}}$$
$$H_{0,\mathrm{ext}} = -\partial_{xx}.$$

To fix its domain we have to specify how the functions are related at the vertex v. We suppose that they satisfy the so-called *free* boundary conditions[3], namely

$$f_{\mathrm{int}}(0) = f_{\mathrm{int}}(\ell) = f_{\mathrm{ext}}(0),$$
$$(f' - \mathrm{i}af)_{\mathrm{int}}(0+) - (f' - \mathrm{i}af)_{\mathrm{int}}(\ell-) + f'_{\mathrm{ext}}(0+) = 0.$$

More general (self-adjoint) boundary conditions for a magnetic Hamiltonian on X_0 were discussed in [E97], in particular, from the point of view of resonances.

The magnetic and electric potential on the internal edge can be easily found, in particular, one can see from the "fat graph" Hamiltonian that

$$a_e(x) := \dot\gamma_e(x) \cdot a(\gamma_e(x)) = a_e^{\|}(\varphi_{\varepsilon,e}^{-1}(x), 0), \quad q_e(x) := q(\gamma_e(x)) \tag{2.13}$$

are the tangent component of a and the value of q, respectively, along the *full* edge $e = e_{\mathrm{int}}$. Indeed, on a heuristic level the choice of the potentials in the limiting operator is justified by the relations

$$\begin{aligned} |a_e^{\|}(x,y) + \mathcal{O}(\varepsilon) - a_e(x)| &\le \varepsilon c_1 \|a\|_{C^1}, \\ |\varepsilon a_{\varepsilon,v}(z)| &\le \varepsilon \|a\|_\infty, \end{aligned} \tag{2.14}$$

where $\|a\|_{C^1}$ denotes the supremum of $|a|$, $|\nabla a_1|$ and $|\nabla a_2|$ on X_ε, and

$$\begin{aligned} |q_{\varepsilon,e}(x,y) - q_e(x)| &\le \varepsilon c_2 \|q\|_{C^1}, \\ |q_{\varepsilon,v}(z) - q(v)| &\le \varepsilon c_3 \|q\|_\infty, \end{aligned} \tag{2.15}$$

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[3] They are often labelled as Kirchhoff boundary conditions, with an allusion to classical electrical circuits. The term is unfortunate, however, since *every* boundary condition giving rise to a self-adjoint graph Hamiltonian must preserve the (probability) current.

where the constants $c_i > 0$ depend only on ℓ and $\|\kappa_e\|_\infty$, $0 < \varepsilon \le 1$.

As in the previous work quoted in the introduction our aim is to give meaning to the intuitive notion that H_0 described above is in some sense a limit of the operators H_ε as $\varepsilon \to 0$ — now from the resonance point of view — despite the fact they act on different Hilbert spaces. There is no paradox here, of course, since only the lowest transverse eigenmode survives, in other words, all functions which are not constant in the transverse direction y will not contribute to the limit. We will make this vague observation precise in Section 6 and Appendix B below.

Note also that we have a somehow simpler, unitary equivalent magnetic Hamiltonian $\hat{H}$ on the graph obtained by the gauge transformation $\hat{f} = \Xi f$ where

$$\Xi_e(x) := \mathrm{e}^{-\mathrm{i}\Phi_e(x)} \qquad \text{and} \qquad \Phi_e(x) := \int_0^x a_e(s)\,\mathrm{d}s. \tag{2.16}$$

on the loop and $\Xi_e = 1$ on the external edge (cf. Section 3.3), where $\Phi = \Phi_e(\ell)$ is the *total flux* through the loop. The free boundary conditions under this unitary transformation become

$$\hat{f}_{\mathrm{int}}(0) = \mathrm{e}^{\mathrm{i}\Phi} \hat{f}_{\mathrm{int}}(\ell) = \hat{f}_{\mathrm{ext}}(0)\,, \tag{2.17a}$$

$$\hat{f}'_{\mathrm{int}}(0+) - \mathrm{e}^{\mathrm{i}\Phi} \hat{f}'_{\mathrm{int}}(\ell-) + \hat{f}'_{\mathrm{ext}}(0+) = 0\,; \tag{2.17b}$$

the price for the simpler expression of the Hamiltonian on an edge are more complicated boundary conditions, with *discontinuous* functions at the vertex.

2.3. **Complex dilations and resonances.** Let us recall briefly the essence of the complex exterior dilation argument — for more details we refer, e.g., to [RS80, Sec. XIII.10], [CDKS87] or [HS96]. We will do it in our setting, both on the graph and its neighbourhood, i.e., for a fixed $\varepsilon \ge 0$. Let us consider the one-parameter unitary group $U^\theta_{\varepsilon,e}$ on the *external* part $X_{\varepsilon,\mathrm{ext}} := U_{\varepsilon,e}$, in particular $X_{0,\mathrm{ext}} = e_{\mathrm{ext}}$ for the graph, whose element characterized by the parameter $\theta \in \mathbb{R}$ acts as

$$\begin{aligned} (U^\theta_0 f)_e(x) &:= \mathrm{e}^{\theta/2} f_e(\mathrm{e}^\theta x) \\ (U^\theta_\varepsilon u)_{\varepsilon,e}(x,y) &:= \mathrm{e}^{\theta/2} u(\mathrm{e}^\theta x, y) \end{aligned} \tag{2.18}$$

at the external edge $e = e_{\mathrm{ext}}$; note that U^θ_ε is unitary since the exterior edge $e = e_{\mathrm{ext}}$ is straight by assumption and therefore the metric on $U_{\varepsilon,e}$ has the product structure (2.11). The transformation can be extended to the whole Hilbert space acting as the identity operator on the *internal* parts $X_{\varepsilon,\mathrm{int}}$, in other words, a function on the graph or the fat graph is *longitudinally* dilated on the external edge and remains unchained on the remaining parts. A simple coordinate transformation shows that for a fixed $\varepsilon \ge 0$, the action of the dilated magnetic Hamiltonian $H^\theta_\varepsilon := U^\theta_\varepsilon H_\varepsilon (U^\theta_\varepsilon)^*$ is given by

$$\begin{aligned} H^\theta_\varepsilon u &= H_{\varepsilon,\mathrm{int}} u_{\mathrm{int}} + H^\theta_{\varepsilon,\mathrm{ext}} u_{\mathrm{ext}}, \\ H^\theta_0 f &= H_{0,\mathrm{int}} f_{\mathrm{int}} - \mathrm{e}^{-2\theta} \partial_{xx} f_{\mathrm{ext}}, \end{aligned} \tag{2.19}$$

where

$$H^\theta_{\varepsilon,\mathrm{ext}} := -\mathrm{e}^{-2\theta}\partial_{xx} - \frac{1}{\varepsilon^2}\partial_{yy}$$

The domain $\operatorname{dom} H_\varepsilon^\theta$ consists of all functions which are locally twice weakly L_2-differentiable and satisfy the conditions

$$u_{\mathrm{ext}} = \mathrm{e}^{\theta/2} u_{\mathrm{int}} \qquad \text{and} \qquad u'_{\mathrm{ext}} = \mathrm{e}^{3\theta/2} u'_{\mathrm{int}} \tag{2.20}$$

on Γ_ε, the common boundary of $X_{\varepsilon,\mathrm{int}}$ and $X_{\varepsilon,\mathrm{ext}}$, where $u'_{\mathrm{ext}} = \partial_x u$ and $u'_{\mathrm{int}} = \partial_x u$ denote the (normal) derivatives in the orientation of x, i.e., the outward normal derivative on $\partial X_{\varepsilon,\mathrm{int}}$ and the inward normal derivative on $\partial X_{\varepsilon,\mathrm{ext}}$.[4]

In the particular case of the graph, $\varepsilon = 0$, we have to specify the boundary conditions. Using the gauge described above we can write them as

$$\begin{aligned} \hat{f}_{\mathrm{int}}(0) = \mathrm{e}^{\mathrm{i}\Phi} \hat{f}_{\mathrm{int}}(\ell) &= \mathrm{e}^{-\theta/2} \hat{f}_{\mathrm{ext}}(0), \\ \hat{f}'_{\mathrm{int}}(0+) - \mathrm{e}^{\mathrm{i}\Phi} \hat{f}'_{\mathrm{int}}(\ell-) + \mathrm{e}^{-3\theta/2} \hat{f}'_{\mathrm{ext}}(0+) &= 0. \end{aligned}$$

In the next step we use (2.19) and (2.20) to perform the basic trick of the complex-scaling methods by extending the definition of H_ε^θ to complex θ with $2|\operatorname{Im}\theta| < \vartheta < \pi$. Note that such a perturbation is very singular with respect to θ, even for real θ, since not only the operator domain, but also the form domain depends on θ as we shall discuss in Appendix C below. In the spirit of [CDKS87] we are going to show there that $\{H_\varepsilon^\theta\}_\theta$ defines a self-adjoint family of operators (i.e., $(H_\varepsilon^\theta)^* = H_\varepsilon^{\overline{\theta}}$) with spectrum contained in the common sector Σ_ϑ for θ in the strip S_ϑ where

$$\Sigma_\vartheta := \bigl\{\, z \in \mathbb{C} \bigm| |\arg z| \le \vartheta \,\bigr\} \qquad \text{and} \qquad S_\vartheta := \bigl\{\, \theta \in \mathbb{C} \bigm| |\operatorname{Im}\theta| < \vartheta/2 \,\bigr\}. \tag{2.21}$$

Following the usual convention — see, e.g., [RS80, Sec. XII.6] — we define a *resonance* of H_ε with $\varepsilon \ge 0$ as the pole of the resolvent analytically continued over the cut given by the essential spectrum of the operator. The position of the cut changes once θ ceases to be real, in particular, for $\operatorname{Im}\theta > 0$ sufficiently large it may "expose" the pole which will just become a complex eigenvalue of H_ε^θ in the lower half-plane. Such eigenvalues will the main object of our interest.

In the example, the eigenvalues $\lambda = k^2$ of the quantum graph Hamiltonian H_0^θ with a magnetic field of total flux Φ through the loop (to make things simple we put $q = 0$), are obtained from the condition [E97]

$$2(\cos k\ell - \cos\Phi) = \mathrm{i} \sin k\ell\,.$$

If $\Phi \ne 0 \pmod{\pi}$ none of the solution is real, while for $\Phi = 0$ half of the solutions is on the real axis and the other half in the lower half-plane, explicitly

$$\lambda_j = \frac{1}{\ell^2}\bigl(2\pi j\bigr)^2, \quad \text{and} \quad \hat{\lambda}_j = \frac{1}{\ell^2}\bigl(2\pi j - \mathrm{i}\ln 3\bigr)^2. \tag{2.22}$$

for $j \in \mathbb{Z} \setminus \{0\}$ and $j \in \mathbb{Z}$, respectively; as expected the values of λ_j and $\hat{\lambda}_j$ are independent of the exterior scaling parameter θ. The real eigenvalues λ_j do not turn into resonances because they correspond to eigenfunctions on the loop which have a node at the vertex, and therefore do not "know" about the presence of the external lead, the half-line part of the eigenfunction being zero. The remain embedded into the essential spectrum of H_0 coming from the half-line, and naturally become isolated after the complex scaling whenever $\operatorname{Im}\theta > 0$ and $\sigma_{\mathrm{ess}}(H_0^\theta) = \mathrm{e}^{-2\theta}[0,\infty)$.

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[4] Here $\partial X_{\varepsilon,\bullet} = \partial_{\mathbb{R}^2} X_{\varepsilon,\bullet} \cap X_\varepsilon$ means the boundary w.r.t. the open set X_ε, *not* the boundary of X_ε as subset of $\mathbb{R}^2$

On the contrary, the solutions corresponding to $\hat{\lambda}_j$ are true resonances. Their half-line component is proportional to $\exp\bigl((\ln 3 + \mathrm{i}\cdot 2\pi j)x/\ell\bigr)$ and thus not square integrable, however, after a complex scaling with large enough $\operatorname{Im}\theta$ it will become (a part of) an L_2-eigenfunction of H_0^θ. Recall that $\sigma_{\mathrm{ess}}(H_0^\theta) = \mathrm{e}^{-2\theta}[0,\infty)$ is rotated into the lower half-plane by the angle $2\operatorname{Im}\theta$ and the resonances $\hat{\lambda}_j$ lie on the parabola

$$\operatorname{Im}\hat{\lambda} = -\frac{2\ln 3}{\ell}\sqrt{\operatorname{Re}\hat{\lambda} + \Bigl(\frac{\ln 3}{\ell}\Bigr)^2},$$

hence for complex scaling with $\operatorname{Im}\theta$ large enough *all* resonances are revealed.

On the other hand, for the "fat graph" X_ε one can check easily that $\sigma_{\mathrm{ess}}(H_\varepsilon^\theta) = \frac{1}{\varepsilon^2}\sigma(\Delta_F^{\mathrm{N}}) + \mathrm{e}^{2\theta}[0,\infty)$ consists of an infinite number of half-lines turned by $2\operatorname{Im}\theta$; each attached to the base point $(j\pi/2\varepsilon)^2 \in \sigma(\Delta_F^{\mathrm{N}})$. All these base points except the one with $j=0$ tend to ∞ as $\varepsilon \to 0$, so for any bounded set $B \subset \mathbb{C}$ we have

$$\sigma_{\mathrm{ess}}(H_0^\theta) \cap B = \sigma_{\mathrm{ess}}(H_\varepsilon^\theta) \cap B \tag{2.23}$$

provided $\varepsilon > 0$ is small enough, in other words, higher sheets of the Riemann surface associated the resolvent of H_ε^θ play no role. The question is whether the complex dilation reveals resonances of this system — manifested as complex eigenvalues of H_ε^θ — and what is their relation to the resonances of the graph. The answer which are going to demonstrate is the following.

Theorem 2.1. *Let $\lambda(0)$ be a resonance of the magnetic Hamiltonian H_0 with a multiplicity $m > 0$. Under the stated assumptions, for a sufficiently small $\varepsilon > 0$ there exist m resonances $\lambda_1(\varepsilon), \dots, \lambda_m(\varepsilon)$ of H_ε, satisfying* $\operatorname{Im}\lambda_j(\varepsilon) < 0$ *and not necessarily mutually different, which all converge to $\lambda(0)$ as $\varepsilon \to 0$. The same is true in the case when $\lambda(0)$ is an embedded eigenvalue of H_0, except that* $\operatorname{Im}\lambda_j(\varepsilon) \le 0$ *holds in general.*

In the following sections we will prove this claim in a considerably more general setting when the loop is replaced by a finite metric graph to which a finite number half-lines is attached — this will be the main result of this paper.

The indicated proof will be divided into several steps. First we will introduce generally in Section 3 and Section 4, respectively, the Hamiltonians of the quantum graph and on the corresponding graph-like waveguide. Next in Section 5 we present the exterior scaling argument. Finally, in Section 6 we conclude the proof by verifying conditions of abstract criteria given in Appendix B; the aim is to show the convergence of discrete eigenvalues — complex in general — for non-self-adjoint operators $\widetilde{H}^\theta = H_\varepsilon^\theta$ and $H^\theta = H_0^\theta$ having a "distance" which tends to zero. The difficult part of the argument, in comparison with [RS01, KuZ01, EP05], is that we cannot use the variational characterization of eigenvalues because our operators are not self-adjoint, nor even normal.

3. QUANTUM GRAPH MODEL

Passing to our main subject we define now the general model in which we are able to prove the convergence of resonances. We start with the quantum graph.

3.1. **Metric graphs.** Suppose X_0 is a connected metric graph given by (V, E, ∂, ℓ) where (V, E, ∂) is a usual graph, i.e., V denotes the set of vertices, E denotes the set of edges, $\partial\colon E \longrightarrow V \times V$ associates to each edge e the pair $(\partial_+ e, \partial_- e)$ of its terminal and initial point (and therefore an orientation). That X_0 is a *metric* graph (also called *quantum* graph) means that there is a *length function* $\ell\colon E \longrightarrow (0, \infty]$ associating to each edge e a length ℓ_e. We often identify the edge e with the interval $(0, \ell_e)$. Clearly, the length function makes X_0 into a metric space.

For each vertex $v \in V$ we set

$$E_v^\pm := \{\, e \in E \mid \partial_\pm e = v \,\} \qquad \text{and} \qquad E_v := E_v^+ \uplus E_v^-,$$

i.e., $E_v^\pm$ consists of all edges starting $(-)$ resp. ending $(+)$ at v and E_v their *disjoint* union. Note that the *disjoint* union is necessary in order to allow loops, i.e., edges having the same initial and terminal point as in the example in Section 2. We adopt the following uniform bounds on the degree $\deg v := |E_v|$ and the length function ℓ:

$$\deg v \le d_0, \qquad v \in V, \tag{H$_0$1}$$

$$\ell_e \ge \ell_0, \qquad e \in E, \tag{H$_0$2}$$

where $0 < d_0 < \infty$ and $0 < \ell_0 \le 1$. Of course, both assumptions are fulfilled if $|E|$ and $|V|$ are finite.

An edge e with $\ell_e = \infty$ will be called *external* and E_{ext} denotes the set of all external edges. Such edges are assumed to have only an initial point, i.e., ∂e consists only of the point $\partial_- e$ for $e \in E_{\mathrm{ext}}$. The remaining edges are called *internal* and their set will be denoted by $E_{\mathrm{int}} := E \setminus E_{\mathrm{ext}}$. We call the vertices connecting internal and external edges *boundary vertices*, denoted by

$$\Gamma_0 := \{\, \partial_- e \in V \mid e \in E_{\mathrm{ext}} \,\}. \tag{3.1}$$

Since we are not aware of reasonable models with an infinite number of external edges attached, we assume throughout this paper that Γ_0 (i.e., E_{ext}) is finite, namely

$$|E_{\mathrm{ext}}| = |\Gamma_0| < \infty. \tag{H$_0$3}$$

3.2. **Magnetic Hamiltonian on the graph.** Let $\mathcal{H} = \mathsf{L}_2(X_0) = \bigoplus_{e \in E} \mathsf{L}_2(e)$, and denote the corresponding norm by $\|f\|_0 = \|f\|$. Suppose that a and q are bounded, measurable functions on X_0, i.e,

$$\|a\|_\infty < \infty, \qquad \text{and} \qquad \|q\|_\infty < \infty. \tag{H$_0$4}$$

Without loss of generality, we assume that $q \ge 0$ and that a_e is a smooth function on each edge (cf. Remark 3.4). For simplicity, we also assume that q_e is smooth on each edge. We set

$$\mathfrak{h}(f) := \sum_{e \in E} \mathfrak{h}_e(f_e), \qquad \mathfrak{h}_e(f) := \int_e \bigl[|D_e f_e|^2 + q_e |f_e|^2 \bigr] \, \mathrm{d}x. \tag{3.2}$$

where $D_e f_e := f_e' - \mathrm{i} a_e f_e$. In particular, $\mathfrak{h}$ is non-negative, i.e, $\mathfrak{h}(f) \ge 0$ for all f. We specify its domain below.

Notation 3.1. Here and in the sequel, the subscript $(\cdot)_e$ refers to the restriction onto the edge e (sometimes also identified with the interval $(0, \ell_e)$, e.g., $f_e := f\restriction_e$, $\|\cdot\|_e$ denotes the norm on $\mathsf{L}_2(e)$, $\mathfrak{h}_e$ is the restriction of $\mathfrak{h}$ onto $\mathsf{L}_2(e)$ etc. We often omit the index if it is clear from the context (e.g., $\mathfrak{h}_e(f) = \mathfrak{h}_e(f_e)$).

Denote by $\mathsf{H}^k(e)$ the Sobolev space on the interval $e \cong (0, \ell_e)$ of k-times L_2-weakly differentiable functions.

Notation 3.2. Denote by $\|\cdot\|_{\mathfrak{q}}$ the norm associated to a closed, non-negative quadratic form $\mathfrak{q}$ in the Hilbert space $\mathcal{H}$, i.e.,

$$\|f\|_{\mathfrak{h}}^2 := \|f\|^2 + \mathfrak{q}(f). \tag{3.3}$$

This norm turns $\mathcal{H}^1 := \operatorname{dom} \mathfrak{h}$ into a complete Hilbert space.

Denote by $\mathfrak{d}$ the quadratic form $\mathfrak{h}$ where $a = 0$ and $q = 0$.

Lemma 3.3. *Assume that* $a, q \in \mathsf{L}_\infty(X_0)$. *Then* $\mathfrak{h}$ *and* $\mathfrak{d}$ *are closed forms on*

$$\mathcal{H}^1 := \mathsf{H}^1(X_0) := \mathsf{C}(X_0) \cap \bigoplus_{e \in E} \mathsf{H}^1(e). \tag{3.4}$$

Furthermore, the norms $\|\cdot\|_1 := \|\cdot\|_{\mathfrak{d}}$ *and* $\|\cdot\|_{\mathfrak{h}}$ *are equivalent.*

Proof. It can be quite easily seen that $\mathfrak{d}$ is a closed form on $\mathsf{H}^1(X_0)$ (this is clear by standard arguments for $\mathfrak{d}_e$, and the vertex condition remains true by the continuity of $f_e \mapsto f_e(v)$, $v \in \partial e$ (cf. (6.8)). In addition,

$$\mathfrak{h}_e(f) \le 2\mathfrak{d}_e(f) + (2\|a_e\|_\infty^2 + \|q_e\|_\infty^2)\|f_e\|_0^2$$

and a similar inequality holds with the roles of $\mathfrak{h}_e$ and $\mathfrak{d}_e$ interchanged, thus the norms $\|\cdot\|_{\mathfrak{h}}$ and $\|\cdot\|_{\mathfrak{d}}$ are equivalent. ☐

We denote the operators corresponding to $\mathfrak{h}$ and $\mathfrak{d}$ by H and Δ, respectively.[5]

Remark 3.4. We can always assume that a_e is a smooth function on each edge: We just have to replace a non-smooth magnetic potential a_e by a smooth function $\widetilde{a}_e$ having the same values at the endpoints and the same integral over e. Using the gauge transformation (2.16) we will see in Section 3.3 that the operators with magnetic potentials a and $\widetilde{a}$ are unitarily equivalent.

Nevertheless the domain $\mathcal{H}^2 := \operatorname{dom} H$ of H may depends on a in general, namely, a function f is in $\mathcal{H}^2$ iff (i) $f_e \in \mathsf{H}^2(e)$ (due to our smoothness assumption of a_e), (ii) f, $Hf \in \mathsf{L}_2(X_0)$ and (iii) the so-called *generalised free boundary conditions* (sometimes also labelled as Kirchhoff – see Footnote 3)

$$f_{e_1}(v) = f_{e_2}(v), \qquad e_1, e_2 \in E_v \tag{3.5a}$$

$$\sum_{e \in E_v} D_e f(v) = 0 \tag{3.5b}$$

are fulfilled for all $v \in V$ where $D_e f(v) := \vec{f}_e'(v) - \mathrm{i}\vec{a}_e(v) f_e(v)$ and

$$\vec{f}_e'(v) := \begin{cases} f_e'(0), & \text{if } v = \partial_- e, \\ -f_e'(\ell_e), & \text{if } v = \partial_+ e \end{cases} \tag{3.6}$$

defines the *outward* derivative of f_e at v, and similarly for $\vec{a}_e(v)$. The fact that we need different signs for incoming and outgoing edges is due to the fact that f' and a formally are 1-forms on the quantum graph. Note that 1-forms *do* see the orientation (in contrast to the second order operator H). Condition (3.5a) is the

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[5] We work in the "geometric" convention in which the Laplacian is a non-negative operator.

continuity at each vertex and (3.5b) is the conservation of the current generated by D_e.

If $\sum_{e\in E_v} \vec{a}_e(v) = 0$ for all $v \in V$ then $\mathcal{H}^2 = \operatorname{dom} H$, i.e. (3.5b) becomes the usual free boundary conditions, where $D_e f_e(v)$ is replaced by $\vec{f}'_e(v)$.

3.3. **Gauge transformations.** Without loss of generality, we may assume that $a_e = 0$ for external edges $e \in E_{\mathrm{ext}}$: Using the simple gauge transformation $\hat{f}_e = \Xi_e f_e$ on the external edge (cf. eq. (2.16)) one easily sees that $\mathfrak{h}_e(f_e) = \mathfrak{d}_e(\Xi_e f_e)$. In addition, $\Xi_e(0) = 1$ so that $(\Xi_e f_e)(v) = f(v)$, i.e., $\Xi_e f_e$ extends to a continuous function also onto interior edges (where $\Xi_e = 1$). In particular, $\hat{f} = \Xi f \in \mathsf{H}^1(X_0)$ and the assertion $\mathfrak{h}(f) = \hat{\mathfrak{h}}(\hat{f})$ holds where $\hat{\mathfrak{h}}$ is the quadratic form without magnetic potential on the external edges, which in turn implies that the corresponding operators are unitarily equivalent.

Similarly, we can always gauge away the magnetic potential on a *tree* graph (i.e., a graph without loops). On a general graph, we can use a gauge transformation to eliminate the vector potential on each edge; the price for that is a less convenient quadratic form domain, now consisting of functions generally *discontinuous* at the vertices. Specifically, the values $\mathrm{e}^{\mathrm{i}\Phi_e(v)} f_e(v)$ have to be equal for all $e \in E_v$. Note that $\Phi_e(\partial_- e) = \Phi_e(0) = 0$, but $\Phi_e(\partial_+ e) = \Phi_e(\ell_e) \notin 2\pi\mathbb{Z}$ in general. Furthermore, the condition for the operator domain is the free condition with $D_e f(v)$ replaced by $\mathrm{e}^{\mathrm{i}\Phi_e(v)} f'_e(v)$ (cf. eq. (2.17)).

In addition, a magnetic Hamiltonian on a quantum graph is completely determined (up to a unitary equivalence) by the values of the magnetic flux $\Phi_L := \Phi_{e_1}(\ell_{e_1}) \cdot \ldots \cdot \Phi_{e_n}(\ell_{e_n}) \pmod{2\pi}$ through all its primary loops $L = (e_1, \ldots, e_n)$ by Stokes theorem. For a general treatment of magnetic perturbations on quantum graphs we refer to [KoS03].

4. Quantum wave guide model

4.1. **Branched quantum wave guides.** Let X_ε be a d-dimensional manifold. If X_ε has boundary, we denote it by ∂X_ε. We assume that X_ε and ∂X_ε are disjoint, i.e., X_ε is the interior of $\overline{X}_\varepsilon = X_\varepsilon \cup \partial X_\varepsilon$. In addition, we assume that X_ε can be decomposed into open sets $U_{\varepsilon,e}$ and $U_{\varepsilon,v}$, i.e,

$$X_\varepsilon = \overline{\biguplus_{e\in E}} U_{\varepsilon,e} \,\overline{\uplus}\, \overline{\biguplus_{v\in V}} U_{\varepsilon,v}.$$

Notation 4.1. Here and in the sequel, $A = \overline{\biguplus}_i A_i$ means that A_i are open (in A), mutually disjoint and the interior of $\bigcup_i \overline{A}_i$ equals A.

We have introduced this notion to avoid mentioning boundaries of dimension $d-1$ which are unimportant in an L_2-decomposition. Note that it suffices to consider a chart cover of X_ε *up to a set of measure* 0 when dealing with L_2-theory.

Denote the metric on X_ε by g_ε. We assume that $U_{\varepsilon,e}$ and $U_{\varepsilon,v}$ are isometric to $(U_e, g_{\varepsilon,e})$ and $(U_v, g_{\varepsilon,v})$, respectively, where the underlying manifolds are independent of $\varepsilon > 0$. In addition, we assume that $U_e = e \times F$ where F is a compact m-dimensional manifold with $m := (d-1)$. The cross section manifold F has boundary depending on whether X_ε has a boundary or not.

Notation 4.2. Here and in the sequel, the subscripts $(\cdot)_{\varepsilon,e}$ and $(\cdot)_{\varepsilon,v}$ (or sometimes only $(\cdot)_e$ and $(\cdot)_v$) denotes the restriction of objects living on X_ε to $U_{\varepsilon,e}$ and $U_{\varepsilon,v}$, respectively. For example, $g_{\varepsilon,e} := g\restriction_{U_{\varepsilon,e}}$ or $u_v := u\restriction_{U_v}$. We will switch between different charts (e.g., $U_{\varepsilon,v}$ and $U_e = e \times F \cong (0,\ell_e) \times F$) without mentioning. If no confusion can occur, we also omit the subscripts.

Notation 4.3. As a Riemannian manifold, U_e carries the metric $g_{\varepsilon,e}$ with $\varepsilon = 1$. Similarly, $\hat{U}_{\varepsilon,e} = (U_e, \hat{g}_{\varepsilon,e})$ and $U_v = (U_v, g_v)$.

Motivated by our example in Section 2 we assume that the metric components satisfy

$$\begin{aligned} g_{\varepsilon,e} &= (1+\mathcal{O}(\varepsilon))^2 \mathrm{d}x^2 + \varepsilon^2 h, & g_{\varepsilon,v} &\approx \varepsilon^2 g_v \\ \hat{g}_{\varepsilon,e} &= \mathrm{d}x^2 + \varepsilon^2 h, & \hat{g}_{\varepsilon,v} &= \varepsilon^2 g_v \end{aligned} \tag{$\mathrm{H}_\varepsilon 1$}$$

where g_v and h are fixed metrics on U_v and F, respectively. For simplicity, we suppose that $\operatorname{vol}_m F = 1$. Clearly, we have

$$\mathrm{d}U_{\varepsilon,e} = (1+\mathcal{O}(\varepsilon))\,\mathrm{d}\hat{U}_{\varepsilon,e} \tag{4.1}$$

for the Riemannian densities w.r.t. $g_{\varepsilon,e}$ and $\hat{g}_{\varepsilon,e}$. To keep the model simple, we also assume that an *exterior* edge neighbourhood $U_{\varepsilon,e}$ has exact product structure, i.e., that $g_{\varepsilon,e} = \hat{g}_{\varepsilon,e}$ for $e \in E_{\mathrm{ext}}$.

Notation 4.4. Here and in the following, $b_\varepsilon = \mathcal{O}(\varepsilon^\alpha)$ means that $|b_\varepsilon \varepsilon^{-\alpha}|$ is bounded by some constant $c > 0$ for $0 < \varepsilon < \varepsilon_0$. Similarly, $b_\varepsilon \approx \hat{b}_\varepsilon$ means that there exist constants $c_\pm > 0$ such that $c_- b_\varepsilon \le \hat{b}_\varepsilon \le c_+ b_\varepsilon$ for all sufficiently small $\varepsilon > 0$. The constants c and $c_\pm$ are supposed to be independent of $\varepsilon > 0$, $z \in X_\varepsilon$, $e \in E$ and $v \in V$; e.g., $g_{\varepsilon,v} \approx \hat{g}_{\varepsilon,v}$ means that $b_\varepsilon = g_{\varepsilon,v}(z)(w,w)$ and $\hat{b}_\varepsilon = \hat{g}_{\varepsilon,v}(z)(w,w)$ satisfy $b_\varepsilon \approx \hat{b}_\varepsilon$ *uniformly* in $\varepsilon > 0$, $v \in V$, $z \in U_v$ and $w \in T^*_z U_v$.

Condition ($\mathrm{H}_\varepsilon 1$) means that on the edge neighbourhood, the metric $g_{\varepsilon,e}$ differs from $\hat{g}_{\varepsilon,e}$ only by a small longitudinal error. On the vertex neighbourhood, we are closed to the ε-homothetic metric $\hat{g}_{\varepsilon,v}$. Note that the embedded case of Section 2 is included in this setting. The estimate $g_{\varepsilon,v} \approx \varepsilon^2 g_v$ allows us to consider also *non*-homothetic vertex neighbourhoods $U_{\varepsilon,v}$ occurring e.g. if the edges are curved up to the vertex, cf. [P06, Sec. 3.1]. We can indeed treat a slightly more general model with off-diagonal terms in the metric (coming e.g. from non-constant radii along the edge neighbourhood) and a slightly slower scaling at the vertex neighbourhood. We refer to [EP05, P06] and keep the simpler model here, since it already covers the main example, the embedded quantum graph.

The metric $\hat{g}_\varepsilon$ on X_ε, close to the original one, is more adapted to the reduction onto the quantum graph. Note that $(X_\varepsilon, \hat{g}_\varepsilon)$ consists of straight cylinders $(U_e, \hat{g}_{\varepsilon,e})$ of radius ε and *fixed* length ℓ_e joined by ε-homothetic vertex neighbourhoods $(U_v, \hat{g}_{\varepsilon,v})$. The manifold $(X_\varepsilon, \hat{g}_\varepsilon)$ does *not* form an ε-neighbourhood of an quantum graph embedded in some ambient space, since the vertex neighbourhoods cannot be fixed in the ambient space unless one allows slightly shortened edge neighbourhoods as we described in the example in Section 2. Nevertheless, introducing ε-independent coordinates simplifies the comparison of the Laplacian on the quantum graph and the manifold.

In addition, we assume the following uniformity conditions:

$$c_{\mathrm{vol}} := \sup_{v\in V} \operatorname{vol}_d U_v < \infty, \qquad \lambda_2 := \inf_{v \in V} \lambda_2^{\mathrm{N}}(U_v) > 0, \tag{$\mathrm{H}_\varepsilon 2$}$$

where $\lambda_2^{\mathrm{N}}(U_v)$ denotes the second (first non-zero) Neumann eigenvalue of (U_v, g_v). In addition, we assume that X_ε is of bounded geometry, i.e., we have a global lower bound on the injectivity radius and the Ricci curvature, namely

$$r_0(\varepsilon) := \operatorname{inj\,rad} X_\varepsilon > 0, \qquad \kappa_0(\varepsilon) := \inf_{\substack{x\in X_\varepsilon\\ v \in T_x X_\varepsilon \setminus\{0\}}} \frac{g_\varepsilon\bigl(\mathrm{Ric}(x)v, v\bigr)}{g_\varepsilon(v,v)} > -\infty. \tag{$\mathrm{H}_\varepsilon 3$}$$

Both constants will in general depend on ε. Roughly speaking, Condition ($\mathrm{H}_\varepsilon 2$) means that U_v remains small (cf. the discussion in [P06, Rem. 2.7]). The assumption ($\mathrm{H}_\varepsilon 2$)–($\mathrm{H}_\varepsilon 3$) are trivially satisfied once the vertex set V is finite. Assumption ($\mathrm{H}_\varepsilon 3$) still remains true for example if the set of "building blocks", i.e., the sets of isometry classes of $\{U_v\}_{v\in V}$ and $\{U_e\}_{e\in E}$ are finite. This assumption is only needed in (C.23) in order to assure elliptic regularity.

For further purposes, we need a finer decomposition of U_v into

$$U_v = \overline{\biguplus_{e\in E_v}} A_{v,e} \mathbin{\overline{\uplus}} U_v^- \tag{4.2}$$

where $A_{v,e} \cong (0, \ell_0/2) \times F$ with coordinates $(\check x, y)$. Note that we have $x \approx \varepsilon \check x$ (if we extend the coordinate x to $A_{v,e}$ and $\check x$ to U_e), and therefore $\mathrm{d}x = \varepsilon \mathrm{d}\check x$. In particular, $g_{\varepsilon,v,e} \approx \varepsilon^2(\mathrm{d}\check x^2 + h)$ where $g_{\varepsilon,v,e}$ is the restriction of g_ε to $A_{\varepsilon,v}$. Note that this decomposition always exists. If necessary, we have to remove a small part (of length $\mathcal{O}(\varepsilon)$) of the adjacent edge neighbourhood and rescale the coordinates on the shortened edge neighbourhood in order to obtain again ε-independent coordinates on the edge neighbourhood.

Notation 4.5. We denote $\partial_e U_v$ the boundary part of U_v meeting $\overline{U}_e$ and similarly, $\partial_v U_e$ the boundary part meeting $\overline{U}_v$ (if $v \in \partial e$). Similarly, $\partial_e U_v^-$ denotes the common part of $\overline{U_v^-}$ and $\overline{A}_{v,e}$.

4.2. **Magnetic Hamiltonian on the quantum wave guide.** We now determine the assumptions on the magnetic and electric potentials. Here, the magnetic potential is a 1-form on X_ε and q_ε is a function on X_ε such that

$$\alpha_\varepsilon \in \mathsf{L}_\infty(T^*X_\varepsilon) \qquad \text{and} \qquad q_\varepsilon \in \mathsf{L}_\infty(X_\varepsilon), \tag{4.3}$$

i.e., $|\alpha_\varepsilon|_{g_\varepsilon}$ and $|q_\varepsilon|$ are essentially bounded functions on X_ε. As on the quantum graph, we assume for simplicity that $q_\varepsilon \geq 0$ and that α_ε, q_ε vanish on the exterior edge neighbourhoods. To avoid any difficulties with the operator domain and elliptic regularity in (C.23) we assume that α_ε is smooth.

In order to compare the magnetic and electric potential with the one on the quantum graph, we introduce another magnetic and electric potential $\hat\alpha_\varepsilon$ and $\hat q_\varepsilon$, respectively. The fact that $\hat\alpha_\varepsilon$ is no longer smooth does not matter since we use $\hat\alpha_\varepsilon$ only as intermediate step in the verification of the closeness assumptions in Section 6.

Again, motivated by the loop example in Section 2 we assume that

$$\begin{aligned} \alpha_{\varepsilon,e} &= (a_e + \mathcal{O}(\varepsilon))\mathrm{d}x + \varepsilon \omega_{\varepsilon,e}, & \alpha_{\varepsilon,v} &\approx \varepsilon \alpha_v \\ \hat\alpha_{\varepsilon,e} &= a_e \mathrm{d}x, & \hat\alpha_{\varepsilon,v} &= 0 \end{aligned} \tag{$\mathrm{H}_\varepsilon 4$}$$

where a_e is the magnetic potential on the quantum graph and where

$$\omega_{\varepsilon,e} = \mathrm{d}_F \vartheta_{\varepsilon,e} = \mathcal{O}(1), \qquad \vartheta_{\varepsilon,e} = \mathcal{O}(1), \qquad \partial_x \vartheta_{\varepsilon,e} = \mathcal{O}(1). \tag{H$_\varepsilon$5}$$

In particular, $\omega_{\varepsilon,e}$ is an exact 1-form on F and α_v is a fixed 1-form on T^*U_v.

For the electric potential, we assume that

$$\begin{aligned} q_{\varepsilon,e} &= q_e + \mathcal{O}(\varepsilon), \qquad & q_{\varepsilon,v} &= \mathcal{O}(1) \\ \hat{q}_{\varepsilon,e} &= q_e, & \hat{q}_{\varepsilon,v} &= 0 \end{aligned} \tag{H$_\varepsilon$6}$$

where q_e is the electric potential on the quantum graph. From these assumptions, it is clear, that global bounds on the quantum graph potentials a and q are enough to ensure that α_ε and q_ε are bounded.

We define the magnetic Hamiltonian H_ε acting in the the Hilbert space $\mathcal{H}_\varepsilon := \mathsf{L}_2(X_\varepsilon, g_\varepsilon)$ (with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$) via the quadratic form

$$\mathfrak{h}_\varepsilon(u) := \|D_{\varepsilon,e} u\|^2 + \langle u, q_\varepsilon u\rangle, \tag{4.4}$$

where $D_{\varepsilon,e} := (\mathrm{d} - \mathrm{i}\alpha_\varepsilon)$. Denote by $\mathfrak{d}_\varepsilon$ the quadratic form given by $\mathfrak{h}_\varepsilon$ without field, i.e, $\alpha_\varepsilon = 0$ and $q_\varepsilon = 0$. The proof of the following lemma is straightforward (cf. Lemma 3.3):

Lemma 4.6. *Assume that $\alpha_\varepsilon \in \mathsf{L}_\infty(T^*X_\varepsilon)$ and $q_\varepsilon \in \mathsf{L}_\infty(X_\varepsilon)$, i.e., $|\alpha_\varepsilon|_{g_\varepsilon}$ and q_ε are essentially bounded functions on X_ε. Then $\mathfrak{h}_\varepsilon$ and $\mathfrak{d}_\varepsilon$ are closed forms on*

$$\mathcal{H}^1_\varepsilon := \mathsf{H}^1(X_\varepsilon) := \bigl\{\, u \in \mathsf{L}_2(X_\varepsilon) \bigm| |\mathrm{d}u|_{g_\varepsilon} \in \mathsf{L}_2(X_\varepsilon) \,\bigr\} \tag{4.5}$$

where the derivative is understood in the weak sense. Furthermore, the norms $\|\cdot\|_1 := \|\cdot\|_{\mathfrak{d}_\varepsilon}$ and $\|\cdot\|_{\mathfrak{h}_\varepsilon}$ satisfy $\|u\|_{\mathfrak{d}_\varepsilon} \approx \|u\|_{\mathfrak{h}_\varepsilon}$ (independently of ε). In particular, the norms are equivalent.

We denote by H_ε and Δ_ε the corresponding operators associated to $\mathfrak{h}_\varepsilon$ and $\mathfrak{d}_\varepsilon$. Note that $\Delta_\varepsilon = \Delta_{X_\varepsilon} \ge 0$ is the usual (Neumann) Laplacian on X_ε. Since we assumed that α_ε is smooth also the operator domains of H_ε and Δ_ε agree, namely they equal

$$\mathcal{H}^2_\varepsilon := \mathsf{H}^2(X_\varepsilon) := \bigl\{\, u \in \mathsf{L}_2(X_\varepsilon) \bigm| |\mathrm{d}u|_{g_\varepsilon}, \Delta_\varepsilon u \in \mathsf{L}_2(X_\varepsilon),\, \partial_{\mathrm{n}} u = 0 \text{ on } \partial X_\varepsilon \,\bigr\}. \tag{4.6}$$

Note that we include the Neumann boundary condition in the definition of the second order Sobolev space if $\partial X_\varepsilon \neq \emptyset$.

4.3. **Intermediate product model.** When comparing the magnetic Laplacian H_ε on the branched quantum wave guide with the magnetic Laplacian H on the graph, it will be convenient to use also the magnetic Laplacian $\hat{H}_\varepsilon$ defined via the hat-quantities:

Notation 4.7. Here and in the sequel, the label $\hat{\cdot}$ refers to the product metric $\hat{g}_\varepsilon$ and the simplified potentials $\hat{\alpha}_\varepsilon$ and $\hat{q}_\varepsilon$ defined as above. Similarly, a Hilbert space defined via $\hat{g}_\varepsilon$ will carry the label $\hat{\cdot}$, e.g., $\hat{\mathcal{H}}_\varepsilon := \mathsf{L}_2(X_\varepsilon, \hat{g}_\varepsilon)$ with norm and inner product $\widehat{\|\cdot\|}$, $\widehat{\langle\cdot,\cdot\rangle}$, resp. The quadratic form $\hat{\mathfrak{h}}_\varepsilon$ is defined as in (4.4) but with $\hat{g}_\varepsilon$, $\hat{\alpha}_\varepsilon$ and $\hat{q}_\varepsilon$, instead.

The main reason why we introduced the intermediate model operator $\hat{H}_\varepsilon$ on $\hat{\mathcal{H}}_\varepsilon$ is to split the reduction onto the quantum graph into two steps: In the first step, we discard the error terms coming from the failure of the metric to be an exact product as well as from the transverse magnetic and electric potential terms. Once having established some closeness estimates on H_ε and $\hat{H}_\varepsilon$ in Lemma 4.8, we will show in Section 6.2 that H_ε approaches the quantum graph Hamiltonian H using the intermediate operator $\hat{H}_\varepsilon$; this will simplify the estimates used there.

Using our assumptions on the metric and the fields, we have (taking Notation 4.2 into account):

$$\begin{aligned} \|u\|^2_{\varepsilon,e} &= \int_{U_{\varepsilon,e}} |u|^2 \,\mathrm{d}U_{\varepsilon,e}, & \|u\|^2_{\varepsilon,v} &= \int_{U_{\varepsilon,v}} |u|^2 \,\mathrm{d}U_{\varepsilon,v}, \\ \|u\widehat{\|}^2_{\varepsilon,e} &= \varepsilon^m \int_{U_e} |u|^2 \,\mathrm{d}F \,\mathrm{d}x, & \|u\widehat{\|}^2_{\varepsilon,v} &= \varepsilon^d \int_{U_v} |u|^2 \,\mathrm{d}U_v, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \mathfrak{h}_{\varepsilon,e}(u) = {} & \int_{U_{\varepsilon,e}} \Big[g^{xx}_{\varepsilon,e} \big|(D_e + \mathcal{O}(\varepsilon))u\big|^2 \\ & \qquad + \frac{1}{\varepsilon^2} \big|(\mathrm{d}_F - \mathrm{i}\varepsilon\omega_{\varepsilon,e})u\big|^2_h + q_{\varepsilon,e}|u|^2 \Big] \,\mathrm{d}U_{\varepsilon,e}, \\ \hat{\mathfrak{h}}_{\varepsilon,e}(u) = {} & \varepsilon^m \int_{U_e} \Big[|D_e u|^2 + \frac{1}{\varepsilon^2} |\mathrm{d}_F u|^2_h + q_e |u|^2 \Big] \,\mathrm{d}F \,\mathrm{d}x, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \mathfrak{h}_{\varepsilon,v}(u) &= \int_{U_{\varepsilon,v}} \big[|(\mathrm{d} - \mathrm{i}\alpha_{\varepsilon,v})u|^2_{g_{\varepsilon,v}} + q_{\varepsilon,v}|u|^2 \big] \,\mathrm{d}U_{\varepsilon,v}, \\ \hat{\mathfrak{h}}_{\varepsilon,v}(u) &= \varepsilon^{d-2} \int_{U_v} |\mathrm{d}u|^2_{g_v} \,\mathrm{d}U_v \end{aligned} \tag{4.9}$$

where $D_e := \partial_x - \mathrm{i}a_e$ and $g^{xx}_{\varepsilon,e} := g_{\varepsilon,e}(\mathrm{d}x, \mathrm{d}x) = 1 + \mathcal{O}(\varepsilon)$ due to (H_ε1). To discard the transversal magnetic potential $\omega_{\varepsilon,e}$, we need to introduce an approximate gauge function, namely

$$\Theta_{\varepsilon,e}(x,y) := \mathrm{e}^{\mathrm{i}\vartheta_{\varepsilon,e}(x,y)} \tag{4.10}$$

$$\Theta_{\varepsilon,v}(z) := \begin{cases} \chi_{v,e}(\check{x}) + (1 - \chi_{v,e}(\check{x}))\Theta_{\varepsilon,e}(v,y), & z = (\check{x}, y) \in A_{v,e} \\ 1, & z \in U_v^- \end{cases}$$

where $\chi_{v,e}$ equals 0 on $\partial_e U_v$ and 1 on $\partial_e U_v^-$. The function $\vartheta_{\varepsilon,e}$ was introduced in (H_ε5); we also recall (4.2) for a definition of $A_{v,e}$, U_v^-, $\check{x}$, and Notation 4.5 for the definition of the boundary $\partial_e U_v$ etc. In particular, we can choose $\chi_{v,e}$ in such a way that $|\chi'_{v,e}| \le 4/\ell_0$ (since the length of $A_{v,e}$ is $\ell_0/2$). Note that the "gauge" function Θ_ε is unitary only on $U_{\varepsilon,e}$ since $|\Theta_{\varepsilon,e}| = 1$, while on the vertex neighbourhood we have just $|\Theta_{\varepsilon,v}| \le 1$. Note, in addition, that the components $\Theta_{\varepsilon,e}$ give together a global *Lipschitz-continuous* function Θ_ε. We will need this fact in Section 6.2. A simple estimate shows that

$$\begin{aligned} \|\Theta_{\varepsilon,e} - 1\|_\infty &= \mathcal{O}(\varepsilon), & \mathrm{d}\Theta_{\varepsilon,e} &= \mathrm{i}\varepsilon(\partial_x \vartheta_{\varepsilon,e} + \omega_{\varepsilon,e})\Theta_{\varepsilon,e}, \\ \|\Theta_{\varepsilon,v} - 1\|_\infty &= \mathcal{O}(\varepsilon), & |\mathrm{d}\Theta_{\varepsilon,v}|_{g_{\varepsilon,v}} &= \mathcal{O}(1) \end{aligned} \tag{4.11}$$

where e.g. $\mathcal{O}(\varepsilon) = \varepsilon\|\vartheta_{\varepsilon,e}\|_\infty$ and $\mathcal{O}(1) = 4\|\vartheta_{\varepsilon,e}\|_\infty/\ell_0 + \||\omega_{\varepsilon,e}|_h\|_\infty$ (cf. (H_ε5)). Now we are going to provide some estimates which will be used when comparing the Hamiltonian on the quantum wave guide with the one on the quantum graph:

Lemma 4.8. *We have*

$$\big|\langle u, \hat{u}\widehat{\rangle}_{\varepsilon,e} - \langle u, \hat{u}\rangle_{\varepsilon,e}\big| = \mathcal{O}(\varepsilon)\|u\|_{\varepsilon,e}\|\hat{u}\widehat{\|}_{\varepsilon,e} \tag{4.12}$$

$$\big|\hat{\mathfrak{h}}_{\varepsilon,e}(u,\hat{u}) - \mathfrak{h}_{\varepsilon,e}(u, \Theta_{\varepsilon,e}\hat{u})\big| = \mathcal{O}(\varepsilon)\|u\|_{\mathfrak{d}_{\varepsilon,e}}\|\hat{u}\widehat{\|}_{\hat{\mathfrak{d}}_{\varepsilon,e}} \quad (\textit{if } \mathrm{d}_F\hat{u} = 0) \tag{4.13}$$

$$\big|\mathfrak{h}_{\varepsilon,v}(u, \Theta_{\varepsilon,v}\hat{u})\big| = \mathcal{O}(1)\|u\|_{\mathfrak{d}_{\varepsilon,v}}\|\hat{u}\widehat{\|}_{\hat{\mathfrak{d}}_{\varepsilon,v}} \tag{4.14}$$

for all functions u, $\hat u$ in the appropriate spaces. Here, $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(1)$ depend only on the error terms $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(1)$ in (H_0) and (H_ε).

Proof. The inner product estimate follows immediately from (4.1). For the second assertion note that

$$D_{\varepsilon,e}(\Theta_{\varepsilon,e}\hat u) = \bigl(\partial_x \hat u - \mathrm{i}(a_e + \mathcal{O}(\varepsilon) - \varepsilon \partial_x \vartheta_{\varepsilon,e})\hat u\bigr)\Theta_{\varepsilon,e}\mathrm{d}x$$

where the y-component vanishes due to the fact that $\mathrm{d}_F \hat u = 0$ and that $\mathrm{d}_F \Theta_{\varepsilon,e} = \mathrm{i}\varepsilon\Theta_{\varepsilon,e}\omega_{\varepsilon,e}$ cancels the transversal magnetic potential. Furthermore, the difference of the $\mathrm{d}x$-components is

$$\begin{aligned}\overline{D_e u}\, D_e \hat u - (1+\mathcal{O}(\varepsilon))\overline{(D_e u + \mathrm{i}\mathcal{O}(\varepsilon))u}\,(D_e(\Theta_{\varepsilon,e}\hat u) + \mathrm{i}\mathcal{O}(\varepsilon)\Theta_{\varepsilon,e}\hat u)&\\ = \mathcal{O}(\varepsilon)(\partial_x \overline{u} + \overline{u})(\partial_x \hat u + \hat u)&\end{aligned}$$

where $1+\mathcal{O}(\varepsilon)$ is the error factor in the metric $g_{\varepsilon,e}$ and $\mathcal{O}(\varepsilon)$ in the last line depends only on the errors given in assumptions (H_0) and (H_ε). In addition, the y-component does not occur. The last estimate follows in a similar way using $|\mathrm{d}\Theta_{\varepsilon,v}|_{g_{\varepsilon,v}} = \mathcal{O}(1)$ (cf. (4.11)). ☐

The requirement $\mathrm{d}_F \hat u = 0$ in the second estimate is due to the fact that we used u instead of $\overline{\Theta}_{\varepsilon,e} u$ in $\hat h_{\varepsilon,e}$. This is exactly the situation we will need in Section 6.2. We will also see that our rough estimate $\mathcal{O}(1)$ in (4.14) is already sufficient to ensure that H_ε approaches the quantum graph Hamiltonian H.

5. Complex dilation

Next we are going to explain the complex dilation argument. We use an exterior scaling on the external edges only.

5.1. Space decomposition.

We start with the space decomposition into an interior and exterior part. Recall that we assumed that each edge neighbourhood of an external edge $e \in E_{\mathrm{ext}}$ has exact product structure (i.e., $g_{\varepsilon,e} = \hat g_{\varepsilon,e}$) and no field (i.e., $\alpha_{\varepsilon,e} = 0$ and $q_{\varepsilon,e} = 0$).

Notation 5.1. Here and in the sequel, the subscript $(\cdot)_{\mathrm{int}}$ stands for the *internal* component and $(\cdot)_{\mathrm{ext}}$ for the external component of an element in the Hilbert space, respectively, for the restriction to the subspace $\mathcal{H}_{\mathrm{int}}$ of a quadratic form or an operator. We often omit the label $(\cdot)_{\mathrm{int}}$ or $(\cdot)_{\mathrm{ext}}$ on a function, if it is clear (e.g., we write $\mathfrak{h}_{\mathrm{int}}(f)$ instead of $\mathfrak{h}_{\mathrm{int}}(f_{\mathrm{int}})$ etc.).

To avoid difficulties with a cut into an internal and external part at a vertex, we can introduce artificial vertices of degree 2 on the external edges. Note that such vertices do not change the domain of the graph Hamiltonian since a vertex of degree 2 with free boundary conditions means nothing else then continuity of a function and its derivative at the vertex (cf. (3.5)). Remember that there is no potential on the external edges.

Without loss of generality we can therefore assume that each boundary vertex $\partial_- e$ of an external edge $e \in E_{\mathrm{ext}}$ has degree 2 and distance ℓ_0 from any other vertex in V. If this were not the case for an external edge e, just introduce a new boundary vertex at distance ℓ_0 from $\partial_- e$ on e.

We can also assume that the manifold X_ε has product structure near the boundary vertices since we assumed that the edge neighbourhood $U_{\varepsilon,e}$ has *exact* metric product structure for external edges e. This means in particular, that we do not associate a vertex neighbourhood to a *boundary* vertex.

We remind the user that we used a different decomposition in Section 2.3. For computational reasons, it is easier to keep the number of vertices minimal on a quantum graph, but for our purposes, it is easier to be away from the inner vertices. From an abstract point of view, of course, both models lead to the same definition of resonances, cf. Lemma 5.10.

We denote by $X_{0,\mathrm{int}} := (V, E_{\mathrm{int}}, \ell)$ the *internal* and by $X_{0,\mathrm{ext}} := (\Gamma_0, E_{\mathrm{ext}}, \ell)$ the *external* metric graph. Note that $X_{0,\mathrm{ext}}$ corresponds to the disjoint union of $|\Gamma_0| = |E_{\mathrm{ext}}|$ many half-lines. The boundary vertices Γ_0 form the common boundary of $X_{0,\mathrm{int}}$ and $X_{0,\mathrm{ext}}$.

Similarly, we decompose the manifold X_ε into

$$X_{\varepsilon,\mathrm{int}} := \overline{\biguplus_{e\in E_{\mathrm{int}}}} U_{\varepsilon,e} \uplus \overline{\biguplus_{v\in V}} U_{\varepsilon,v} \quad \text{and} \quad X_{\varepsilon,\mathrm{ext}} := \overline{\biguplus_{e\in E_{\mathrm{ext}}}} U_{\varepsilon,e}$$

(remind Notation 4.1) and denote the common boundary of $X_{\varepsilon,\mathrm{int}}$ and $X_{\varepsilon,\mathrm{ext}}$ by Γ_ε. Again, $X_{\varepsilon,\mathrm{ext}}$ consists of $|E_{\mathrm{ext}}|$ many disjoint half-infinite cylinders $(0,\infty)\times F_\varepsilon$.

Notation 5.2. For a boundary vertex $v = \partial_- e \in \Gamma_0$ with external edge $e \in E_{\mathrm{ext}}$ we set

$$\begin{aligned} f_{\mathrm{int}}(v) &:= f_e(-0), & u_{\mathrm{int}}(v,\cdot) &:= u_e(-0,\cdot)\\ f_{\mathrm{ext}}(v) &:= f_e(+0), & u_{\mathrm{ext}}(v,\cdot) &:= u_e(+0,\cdot)\\ f'_{\mathrm{int}}(v) &:= f'_e(-0), & u'_{\mathrm{int}}(v,\cdot) &:= \partial_x u_e(-0,\cdot)\\ f'_{\mathrm{ext}}(v) &:= f'_e(+0), & u'_{\mathrm{ext}}(v,\cdot) &:= \partial_x u_e(+0,\cdot) \end{aligned}$$

where we identify a neighbourhood of v with a neighbourhood of $0 \in \mathbb{R}$ (positive numbers corresponding to the external part) and where $g(\pm 0)$ denotes the left/right limit. Note that the sign convention for $f'_{\mathrm{int}}(v)$ differs from the one for internal vertices in (3.6).

We split the Hilbert space $\mathcal{H}$ and $\mathcal{H}_\varepsilon$ into two components, namely we take

$$\mathcal{H} = \mathcal{H}_{\mathrm{int}} \oplus \mathcal{H}_{\mathrm{ext}} \tag{5.1}$$

and the analogous decomposition for $\mathcal{H}_\varepsilon$ where

$$\begin{aligned} \mathcal{H}_{\mathrm{int}} &= \mathsf{L}_2(X_{0,\mathrm{int}}), & \mathcal{H}_{\mathrm{ext}} &= \mathsf{L}_2(X_{0,\mathrm{ext}})\\ \mathcal{H}_{\varepsilon,\mathrm{int}} &= \mathsf{L}_2(X_{\varepsilon,\mathrm{int}}), & \mathcal{H}_{\varepsilon,\mathrm{ext}} &= \mathsf{L}_2(X_{\varepsilon,\mathrm{ext}}) \end{aligned} \tag{5.2}$$

on the quantum graph and the branched quantum wave guide, respectively.

5.2. **Dilated operators.**

Now we introduce the exterior dilation operator. For $\theta \in \mathbb{R}$ we define by

$$\Phi_e^\theta(x) := \mathrm{e}^\theta x, \qquad x > 0$$

a non-smooth flow on an external edge $e \in E_{\mathrm{ext}}$. Clearly, Φ_e^θ extends (by identity) to a (non-smooth) flow on the graph X_0. Similarly, $\Phi_{\varepsilon,e}^\theta(x,y) := (\Phi_e^\theta(x), y)$ defines a non-smooth flow on the external edge neighbourhood, again extended to a flow Φ_ε^θ on X_ε. For a *smooth* version of exterior dilation we refer to [HS89, HS96].

Remark 5.3. The smooth dilation argument seems to be less technical, at least, one does not have to deal with θ-dependent domains (see the appendix). The price to pay is a more complicated expression of the dilated operator between the interior and exterior part. Since most of the technical details are hidden in the abstract criterion, the verification of the convergence assumptions for the non-smooth dilation is simpler. Moreover, on a graph it is in a sense natural to have a "constant" scaling at each edge. In addition, both dilation arguments leads to the same definition of resonances (cf. Lemma 5.10).

On an edge $e \in E$ we have then the following group action

$$U^\theta f := (\det D\Phi^\theta)^{1/2}(f \circ \Phi^\theta) \tag{5.3}$$

where

$$(\det D\Phi^\theta)^{1/2} = \begin{cases} 1 & \text{on } X_{0,\mathrm{int}}, \\ \mathrm{e}^{\theta/2} & \text{on } X_{0,\mathrm{ext}} \end{cases}$$

and similarly for U^θ_ε. Clearly, U^θ and U^θ_ε are 1-parameter unitary groups with respect to $\theta \in \mathbb{R}$, acting non-trivially on the *external* part only.

Notation 5.4. For a quadratic form $\mathfrak{h}$ and an operator H in $\mathcal{H}$ we set

$$\mathfrak{h}^\theta(f) := \mathfrak{h}(U^{-\theta} f) \qquad \text{and} \qquad H^\theta := U^\theta H U^{-\theta}$$

with domains $\operatorname{dom} \mathfrak{h}^\theta := U^\theta(\operatorname{dom} \mathfrak{h})$ and $\operatorname{dom} H^\theta := U^\theta(\operatorname{dom} H)$ for *real* θ.

Clearly, $\mathfrak{h}^0 = \mathfrak{h}$ and $H^0 = H$. A simple calculation shows that for an external edge $e \in E_{\mathrm{ext}}$ we have

$$\mathfrak{h}^\theta_e(f) = \mathrm{e}^{-2\theta}\mathfrak{h}_e(f), \qquad (H^\theta f)_e = -\mathrm{e}^{-2\theta} f''_e \tag{5.4a}$$

on the quantum graph and

$$\mathfrak{h}^\theta_{\varepsilon,e}(u) = \mathrm{e}^{-2\theta}\|\partial_x u\|^2_{\varepsilon,e} + \frac{1}{\varepsilon^2}\big\||\mathrm{d}_F u|_h\big\|^2_{\varepsilon,e},$$

$$(H^\theta_\varepsilon u)_e = -\mathrm{e}^{-2\theta}\partial_{xx} u_e + \frac{1}{\varepsilon^2}\Delta_F u_e \tag{5.4b}$$

on the manifold. Of course, the action on *internal* edges remains unchanged. On the quantum graph, the domains are given for a real θ by

$$\mathcal{H}^{1,\theta} := \operatorname{dom} \mathfrak{h}^\theta = \big\{\, f \in \mathsf{H}^1(X_{0,\mathrm{int}}) \oplus \mathsf{H}^1(X_{0,\mathrm{ext}}) \,\big|\, f_{\mathrm{ext}} = \mathrm{e}^{\theta/2} f_{\mathrm{int}} \text{ on } \Gamma_0 \,\big\} \tag{5.5a}$$

and

$$\mathcal{H}^{2,\theta} := \operatorname{dom} H^\theta = \Big\{\, f \in \mathsf{H}^2(X_{0,\mathrm{int}}) \oplus \mathsf{H}^2(X_{0,\mathrm{ext}}) \,\Big| \; f_{\mathrm{ext}} = \mathrm{e}^{\theta/2} f_{\mathrm{int}},\ f'_{\mathrm{ext}} = \mathrm{e}^{3\theta/2} f'_{\mathrm{int}} \text{ on } \Gamma_0. \Big\} \tag{5.5b}$$

Here,

$$\mathsf{H}^1(X_{0,\mathrm{int}}) := \mathsf{C}(X_{0,\mathrm{int}}) \cap \bigoplus_{e \in E_{\mathrm{int}}} \mathsf{H}^1(e) \tag{5.6a}$$

and

$$\mathsf{H}^2(X_{0,\mathrm{int}}) := \big\{\, f \in \mathsf{C}(X_{0,\mathrm{int}}) \cap \bigoplus_{e \in E_{\mathrm{int}}} \mathsf{H}^2(e) \,\big|\, \sum_{v \in E_v} D_e f = 0,\ v \in V \,\big\} \tag{5.6b}$$

on the internal part and

$$\mathsf{H}^1(X_{0,\mathrm{ext}}) := \bigoplus_{e \in E_{\mathrm{ext}}} \mathsf{H}^1(e) \qquad \text{and} \qquad \mathsf{H}^2(X_{0,\mathrm{ext}}) := \bigoplus_{e \in E_{\mathrm{ext}}} \mathsf{H}^2(e) \tag{5.6c}$$

on the external part. Note that due to Assumption ($\mathrm{H}_0$3) we have $\mathsf{H}^k(X_{0,\mathrm{ext}}) \cong \mathsf{H}^k(0,\infty)^{|E_{\mathrm{ext}}|}$.

On the manifold, we have a very similar definition for $\mathcal{H}^{1,\theta}_\varepsilon$ and $\mathcal{H}^{2,\theta}_\varepsilon$, with first order Sobolev spaces $\mathsf{H}^1(X_{\varepsilon,\bullet})$ defined as in (4.5) and second order Sobolev spaces and second order Sobolev spaces

$$\mathsf{H}^2(X_{\varepsilon,\bullet}) := \{\, u\restriction_{X_{\varepsilon,\bullet}} \,|\, u \in \mathsf{H}^2(X_\varepsilon)\,\} \tag{5.7}$$

where $\mathsf{H}^2(X_\varepsilon)$ already includes the Neumann boundary conditions on ∂X_ε (if non-empty), i.e., we impose these boundary conditions only on $\partial X_\varepsilon \cap \overline{X}_{\varepsilon,\bullet}$, *not* on Γ_ε.

Roughly speaking, the domain of the quadratic form consists of functions having a jump of magnitude $\mathrm{e}^{\theta/2}$ from the internal to the external part. The operator domain in addition requires that the derivative along the common boundary of the internal and external part has a jump of magnitude $\mathrm{e}^{3\theta/2}$. In particular, even the quadratic form domain depends on θ.

The expression of H^θ now serves as a generalization for θ in the strip $S_\vartheta = \{\,\theta \in \mathbb{C} \,|\, |\operatorname{Im}\theta| < \vartheta/2\,\}$ where $0 \le \vartheta < \pi$. We call H^θ the *complex dilated* Hamiltonian, and similarly for H^θ_ε. We will show in Appendix C that $\{H^\theta\}_\theta$ is a self-adjoint family with spectrum contained in the common sector Σ_ϑ. In addition, we show that $R^\theta(z) := (H^\theta - z)^{-1}$ is an analytic family in θ (for z not in the ϑ-sector $\Sigma_\vartheta = \{\, z \in \mathbb{C} \,|\, |\arg z| \le \vartheta\,\}$, cf. Lemmas C.12 and C.13). This is a highly non-trivial fact since H^θ is neither of type A nor of type B, i.e., both sesquilinear form and operator domain depend on θ even for real θ. In other words, the non-smooth exterior scaling as defined here is a very singular perturbation of the operator $H = H^0$. The same statements hold for the complex dilated Hamiltonian H^θ_ε on $\mathcal{H}_\varepsilon$.

The sesquilinear form $\mathfrak{h}^\theta$ associated with the operator H^θ is defined via

$$\mathfrak{h}^\theta(f,g) := \langle f, H^\theta g\rangle = \mathfrak{h}_{\mathrm{int}}(f_{\mathrm{int}}, g_{\mathrm{int}}) + \mathrm{e}^{-2\theta}\mathfrak{h}_{\mathrm{ext}}(f_{\mathrm{ext}}, g_{\mathrm{ext}}) \tag{5.8}$$

for $f \in \mathcal{H}^{1,\overline{\theta}}$ and $g \in \mathcal{H}^{2,\theta}$ with domains as in (5.5), where

$$\mathfrak{h}_{\mathrm{int}} := \bigoplus_{e \in E_{\mathrm{int}}} \mathfrak{h}_e, \qquad \text{and} \qquad \mathfrak{h}_{\mathrm{ext}} := \bigoplus_{e \in E_{\mathrm{ext}}} \mathfrak{h}_e.$$

Similarly, the sesquilinear form $\mathfrak{h}^\theta_\varepsilon$ associated to H^θ_ε is

$$\mathfrak{h}^\theta_\varepsilon(u,w) := \mathfrak{h}_{\varepsilon,\mathrm{int}}(u_{\mathrm{int}}, w_{\mathrm{int}}) + \mathrm{e}^{-2\theta}\mathfrak{h}_{\varepsilon,\mathrm{ext}}(u_{\mathrm{ext}}, w_{\mathrm{ext}}) \tag{5.9}$$

for $u \in \mathcal{H}^{1,\overline{\theta}}$ and $w \in \mathcal{H}^{2,\theta}$. We show in Lemma C.14 how these sesquilinear forms can be extended to bounded sesquilinear forms on $\mathcal{H}^{1,\overline{\theta}} \times \mathcal{H}^{1,\theta}$ and this is actually all we need in order to show the convergence in the appendices.

Remark 5.5. We naturally have to introduce the sesquilinear forms on *mixed* pairs $\mathcal{H}^{1,\overline{\theta}} \times \mathcal{H}^{1,\theta}$ in order to formally preserve the analyticity in θ. This is exactly the setting we need in order to apply our abstract convergence result provided in Appendix B.

The difficulty here is to find a good norm on the natural quadratic form domain $\mathcal{H}^{1,\theta}$. The corresponding expression defined via $\mathfrak{q}^\theta(f) := \langle f, H^\theta f\rangle$ contains boundary terms of the form $\overline{f}(v)f'(v)$ (on the quantum graph) which are not obviously defined on $\mathcal{H}^{1,\theta}$.

In addition, it seems to be very difficult to estimate errors in terms of the corresponding norm $\|\cdot\|_{\mathfrak{q}^\theta}$. There has been some confusion on the *quadratic* form domain on $\mathcal{H}^{1,\theta}$ due to the anti-linearity of a sesquilinear form in its first argument (cf. the Mathematical Reviews entry for [GY83]).

To avoid these difficulty, we use a simpler norm on $\mathcal{H}^{1,\theta}$ related with the unperturbed form $\mathfrak{h}$ by a simple multiplication operator. In this case, we have to assure that the corresponding spaces behave like a "natural" scale of Hilbert spaces associated to H^θ (cf. Appendix A).

5.3. **Essential and discrete spectrum.** We collect some facts about the family of dilated operators $\{H^\theta\}_\theta$. Note that we cannot directly apply the perturbation theory of such operators developped in [RS80, XIII.10] since the form domain of H_θ (cf. (5.5a)) contains *discontinuous* functions and is therefore not included in the form domain $\mathcal{H}^1 = \mathsf{H}^1(X_\varepsilon)$ of the free operator, even not for real $\theta \neq 0$. In particular, we cannot directly use the $\mathcal{H}^{\pm 1}$-scale of Hilbert spaces associated to the free operator. Nevertheless, most of the conclusions of [RS80, XIII.10] remain true since $(H^\theta - z)^{-1}$ depends analytically on $\theta \in S_\vartheta$ for $z \notin \Sigma_\vartheta$ as we will see in Appendix C.

We first determine the essential spectrum of H^θ and H^θ_ε. Note that the essential spectrum is determined by the behaviour of X_ε at infinity. Namely, it does not matter if we change the operator on a compact set due the invariance of the essential spectrum under compact perturbations (decomposition principle). Recall that we assumed in ($\mathrm{H}_0$3) that we only have finitely many external edges.

Proposition 5.6. *The essential spectrum is given by*

$$\sigma_{\mathrm{ess}}(H^\theta) \setminus (0,\infty) = \mathrm{e}^{2\theta}[0,\infty).$$

If, in addition, E_{int} *is also finite (i.e., the internal graph* $X_{0,\mathrm{int}}$ *is compact), then*

$$\sigma_{\mathrm{ess}}(H^\theta) = \mathrm{e}^{-2\theta}[0,\infty).$$

Similarly, we can prove on the manifold:

Proposition 5.7. *The essential spectrum is given by*

$$\sigma_{\mathrm{ess}}(H^\theta_\varepsilon) \setminus (0,\infty) = \frac{1}{\varepsilon^2} \bigcup_{k\in\mathbb{N}} \lambda^{\mathrm{N}}_k(F) + \mathrm{e}^{-2\theta}[0,\infty).$$

If, in addition, E_{int} *is also finite, then*

$$\sigma_{\mathrm{ess}}(H^\theta_\varepsilon) = \frac{1}{\varepsilon^2} \bigcup_{k\in\mathbb{N}} \lambda^{\mathrm{N}}_k(F) + \mathrm{e}^{-2\theta}[0,\infty).$$

In particular, since $\lambda^{\mathrm{N}}_1(F) = 0$*, for any bounded set* $B \subset \mathbb{C} \setminus (0,\infty)$*,*

$$\sigma_{\mathrm{ess}}(H^\theta_\varepsilon) \cap B = \mathrm{e}^{-2\theta}[0,\infty) \cap B$$

provided ε *is small enough.*

Next, we make some general observations on the spectrum of H^θ_ε ($\varepsilon \geq 0$) which are true for both models, the quantum graph and manifold model:

Proposition 5.8. *Assume that* E_{int} *is finite.*

(i) *The spectrum* $\sigma(H_\varepsilon^\theta)$ *depends only on* $\operatorname{Im}\theta$ *and* $\sigma(H_\varepsilon^{\overline{\theta}}) = \overline{\sigma(H_\varepsilon^\theta)}$.

(ii) *The discrete spectrum* $\sigma_{\mathrm{d}}(H_\varepsilon^\theta)$ *is locally constant in* θ, *i.e., if* $0 < \operatorname{Im}\theta_1 \leq \operatorname{Im}\theta_2 < \vartheta/2$ *then* $\sigma_{\mathrm{d}}(H_\varepsilon^{\theta_1}) \subset \sigma_{\mathrm{d}}(H_\varepsilon^{\theta_2})$.

(iii) *We have* $\sigma(H_\varepsilon^\theta) \cap [0,\infty) = \sigma_{\mathrm{p}}(H_\varepsilon)$ *where* $\sigma_{\mathrm{p}}(H_\varepsilon)$ *denotes the set of eigenvalues of* H_ε *(which are embedded in the continuous spectrum).*

(iv) *The singular continuous spectrum of* H_ε *is empty.*

(v) *There is a subspace* $\mathcal{A}$ *satisfying* (5.10) *such that* $\Psi_f(z) := \langle f, (H_\varepsilon - z)^{-1} f\rangle$ *has a meromorphic continuation onto the Riemann surface defined by* $w \to \sqrt{w}$ *if* $\varepsilon = 0$ *resp.* $w \to \sqrt{w - \lambda_k^{\mathrm{N}}(F)/\varepsilon^2}$, $k \in \mathbb{N}$ *if* $\varepsilon > 0$.

(vi) $\lambda \in \Sigma_\vartheta$ *is a discrete eigenvalue of* H_ε^θ *iff there exists* $f \in \mathcal{H}_\varepsilon$ *such that the meromorphic continuation of* Ψ_f *has a pole in* λ.

Proof. The proof follows closely the proof of [RS80, Thm. XIII.36], so we only comment on the differences. We omit the dependency on ε here. The basic ingredient in the proof is first, the analyticity of the family $\{H^\theta\}_\theta$ in the sense that the resolvents are analytic in θ, and second, the unitary equivalence

$$H^{\theta_1+\theta_2} = U^{\theta_1} H^{\theta_2} U^{-\theta_1} \tag{5.10}$$

for real θ_1 and complex $\theta_2 \in S_\vartheta$. This unitary equivalence holds a priori only for real θ_1 and θ_2. But since both sides are analytic in θ_2, equality (5.10) extends therefore to complex $\theta_2 \in S_\vartheta$. From this and the fact that $\{H^\theta\}_\theta$ is a self-adjoint family, (i) follows immediately. In a similar way, (ii) follows noting the fact that an eigenvalue of H^θ depends analytically on θ since $(H^\theta + 1)^{-1}$ is analytic (cf. [Ka66, Thm. VII.1.8]). In order to prove (iii) and (iv) as in [RS80], we need the notion of *analytic vectors* with respect to the unitary group U^θ (namely w.r.t. its self-adjoint generator given by $A := (x\partial_x + \partial_x x)\mathrm{i}/2$ on each external edge). The subspace of analytic vectors is defined as

$$\mathcal{A} := \Big\{ f \in \bigcap_{k\in\mathbb{N}} \operatorname{dom} A^k \,\Big|\, \sum_n \frac{t^n}{n!}\|A^n f\| < \infty \Big\}$$

for some $t \geq \vartheta/2$. It then follows that

$$\begin{cases} U^\theta(\mathcal{A}) & \text{is dense in } \mathcal{H} \text{ and} \\ \theta \mapsto U^\theta f & \text{extends analytically as map } S_\vartheta \to \mathsf{L}_2(X_\varepsilon) \end{cases} \tag{5.11}$$

for all $f \in \mathcal{A}$ using (5.10) (cf. [RS80, Ch. X.6]). The analytic extension is then

$$f^\theta := U^\theta f = \sum_n \frac{\theta^n}{n!}(iA)^n f.$$

To prove (v) we just note that a meromorphic continuation of Ψ_f is given by

$$\Psi_f^\theta(z) = \langle f^{\overline{\theta}}, (H^\theta - z)^{-1} f^\theta\rangle$$

since we have $\Psi_f^0(z) = \Psi^\theta(f)$ a priori only for real θ but by analyticity also for $\theta \in S_\vartheta$. For (vi) we argue as follows: If g is an eigenvector of H^θ with eigenvalue λ then let $f := U^{-\theta} g$. Again, by analyticity, we have $U^\theta f = g$ not only for real, but also for complex θ. In particular, Ψ_f has a pole at λ. On the other side, if $\Psi_f = \Psi_f^\theta$ has a pole at λ, then $\langle f^\theta, \mathbb{1}_{\{\lambda\}} f^\theta\rangle \neq 0$ and in particular, f^θ is an eigenvector of H^θ. $\square$

Motivated by (iii) and (vi) of the last lemma, we make the following definition.

Definition 5.9. A *resonance* of H_ε is a non-real eigenvalue of the dilated operator H_ε^θ for some $\theta \in S_\vartheta$ and $0 < \vartheta < \pi$.

Finally, we assure that our definition of resonances does not depend on where we cut the spaces into an internal and external part (cf. also [HeM87]):

Lemma 5.10. *The definition of resonances Definition 5.9 does not depend on where we cut the graph and the manifold into an external and internal part. Furthermore, the definition of resonances is the same if we use a smooth flow as in* [HS89].

Proof. Denote by U^θ and $\widetilde{U}^\theta$ the exterior dilation operators associated to the flow Φ^θ and $\widetilde{\Phi}^\theta$, respectively (cf. (5.3)), where the flow is either a (non-smooth) flow with cut at some point $x_0 \ge 0$ on the external edge or smooth. The main point is to show that there exists a subspace $\mathcal{A}$ which satisfies (5.11) for *both* U^θ *and* $\widetilde{U}^\theta$. But since we have $A_{x_0} = ((x - x_0)\partial_x + \partial_x(x - x_0))\mathrm{i}/2 = A_0$ for the generator, the set of analytic vectors of A_0 forms such a subspace. Then an eigenvalue of the dilated operator with respect to U^θ or $\widetilde{U}^\theta$ is a pole of the meromorphic continuation of $\Psi_f(z) = \langle f, (H - z)^{-1} f\rangle$ for some $f \in \mathcal{A}$, and the latter definition is clearly independent of the dilation operators. ☐

6. Closeness of graph and wave-guide model

6.1. **Quasi-unitary operators.**

We now define quasi-unitary operators mapping from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$ and vice versa, as well as their analogues on the compatible scales of order 1, namely $\mathcal{H}^{1,\theta}$ and $\widetilde{\mathcal{H}}^{1,\theta}$ (cf. Definition B.2 and Definition B.3). Here,

$$\mathcal{H} := \mathsf{L}_2(X_0) \qquad\qquad \widetilde{\mathcal{H}} := \mathcal{H}_\varepsilon = \mathsf{L}_2(X_\varepsilon) \tag{6.1}$$

and we define $\mathcal{H}^{1,\theta}$ and $\widetilde{\mathcal{H}}^{1,\overline{\theta}}$ as in (5.5a), but now for *complex* $\theta \in S_\vartheta$. Using the map

$$\begin{aligned} T^\theta &\colon \mathcal{H} \longrightarrow \mathcal{H}, & T^\theta f &:= f_{\mathrm{int}} \oplus \mathrm{e}^{-\theta/2} f_{\mathrm{ext}} \\ \widetilde{T}^\theta &\colon \mathcal{H} \longrightarrow \mathcal{H}, & \widetilde{T}^\theta u &:= u_{\mathrm{int}} \oplus \mathrm{e}^{-\theta/2} u_{\mathrm{ext}}, \end{aligned} \tag{6.2}$$

we have $\mathcal{H}^{1,\theta} = T^{-\theta}(\mathcal{H}^1)$ where $\mathcal{H}^1 = \operatorname{dom} \mathfrak{h} = \mathsf{H}^1(X_0)$ is the quadratic form domain of the undilated Hamiltonian. On $\mathcal{H}^{1,\theta}$, we use the (complete) norm

$$\|f\|_{1,\theta} := \|T^\theta f\|_1 \tag{6.3}$$

where $\|\cdot\|_1$ is the norm associated to $\mathfrak{d}$. Similarly, we define $\widetilde{T}^\theta$ and a norm on $\widetilde{\mathcal{H}}^{1,\theta}$ via $\|u\|_{1,\theta} := \|\widetilde{T}^\theta u\|_1$ where $\|\cdot\|_1$ is the norm associated to the free quadratic form $\mathfrak{d}_\varepsilon$. We show in Appendix C, namely in Lemmas C.14–C.16 that we obtain a scale of order 1 in the sense of Definition A.4. In particular, we also show in Appendix C that the various constants $\widetilde{C}_i^\theta$ in Appendices A and B associated to the manifold case are ε-independent.

Let $J \colon \mathcal{H} \longrightarrow \widetilde{\mathcal{H}}$ be given on the components of X_ε by

$$(J_e f)(x, y) := \varepsilon^{-m/2} f_e(x) \qquad \text{and} \qquad (J_v f) := 0, \tag{6.4}$$

i.e., as an extension independent of the transverse variable. Recall that $d = m+1 \geq 2$ is the dimension of the manifold X_ε. Next we define $J^1 \colon \mathcal{H}^{1,\theta} \longrightarrow \widetilde{\mathcal{H}}^{1,\theta}$ by

$$\begin{aligned} (J^1_e f)(x,y) &:= \varepsilon^{-m/2} \Theta_{\varepsilon,e}(x,y) f_e(x), \\ (J^1_v f)(z) &:= \varepsilon^{-m/2} \Theta_{\varepsilon,v}(z) f(v) \end{aligned} \tag{6.5}$$

for internal vertices $v \in V \setminus \Gamma_0$ where Θ_ε is given in (4.10). Note that we did not associate a vertex neighbourhood to the boundary vertices since they have degree 2. Note in addition that the latter operator is well defined: the function $J^1 f$ matches along the different internal components (recall that Θ_ε is a Lipschitz function on X_ε) and has a jump of relative magnitude $e^{\theta/2}$ from the internal to the external part. Finally, $f(v)$ is defined for H^1-functions (cf. (6.8)).

Concerning the mappings in the opposite direction, we first introduce the following averaging operators

$$(N_e u)(x) := \int_F u_e(x,y) \,\mathrm{d}F(y) \qquad \text{and} \qquad C_v u := \frac{1}{\operatorname{vol}_d U_v} \int_{U_v} u \,\mathrm{d}U_v$$

for $u \in \widetilde{\mathcal{H}} = \mathsf{L}_2(X_\varepsilon)$. Recall that $\operatorname{vol}_m F = 1$. The map in the opposite direction $J' \colon \widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$ is given by

$$(J'u)_e(x) := \varepsilon^{m/2} (N_e u)(x), \qquad x \in e. \tag{6.6}$$

Furthermore, we define $J'_1 \colon \widetilde{\mathcal{H}}^{1,\overline{\theta}} \longrightarrow \mathcal{H}^{1,\overline{\theta}}$ by

$$(J'^1_e u)(x) := \varepsilon^{m/2} \Big[N_e u(x) + \sum_{v \in \partial e, v \notin \Gamma_0} \rho_{v,e}(x) \big[C_v u - N_e u(v) \big] \Big] \tag{6.7}$$

for $x \in e$ on an internal edge e and $J'^1_e u := J'_e u$ for an external edge e. Here $\rho_{v,e}$ is a smooth cut-off function such that $\rho_{v,e}(v) = 1$ and $\rho_{v,e}(x) = 0$ if $d(v,x) \geq \ell_e/2$. Moreover, $J'^1_e u(v) = \varepsilon^{m/2} C_v(u)$ so that $J'^1_e u$ fits to a continuous function at each internal vertex $v \in V \setminus \Gamma_0$ of the the quantum graph. In addition $J'^1_e u$ has a jump of magnitude $\mathrm{e}^{\overline{\theta}/2}$ at the boundary Γ_0 since we assumed that at a boundary vertex $v \in \Gamma_0$ there is no additional vertex neighbourhood. The role of the conjugation will become clear in Appendix B.

Note that the definition of J, J' and J'_1 are the same as in the absence of the fields, $\alpha = 0$ and $q = 0$ (cf. e.g. [KuZ01, EP05, P06]), while the definition of J_1 has an additional phase factor due to the magnetic potential.

6.2. **Closeness assumptions.**

Now we are in position to demonstrate that the two Hamiltonians are close to each other:

Theorem 6.1. *Assume* (H$_0$1)–(H$_0$4) *on the quantum graph* X_0 *and* (H$_\varepsilon$1)–(H$_\varepsilon$6) *on the manifold* X_ε*. Then the dilated magnetic Hamiltonians* H^θ_ε *and* H^θ *are* $\mathcal{O}(\varepsilon^{1/2})$*-close in the sense of Definition B.3 where the error depends only on the constants and errors in the hypotheses (except on* $r_0(\varepsilon)$ *and* $\kappa_0(\varepsilon)$*), and on* $\operatorname{Re}\theta$.

Proof. We start with Condition (B.5) of the closeness assumptions: we estimate

$$\begin{aligned}
&\|Jf - J^1 f\|^2 \\
&= (1+\mathcal{O}(\varepsilon)) \sum_{e\in E} \int_{U_e} |\Theta_{\varepsilon,e} - 1|^2 |f_e|^2 \, \mathrm{d}F \, \mathrm{d}x + \mathcal{O}(\varepsilon) \sum_{v\in V} \int_{U_v} |\Theta_{\varepsilon,v}|^2 \, \mathrm{d}U_v |f(v)|^2 \\
&\qquad = \mathcal{O}(\varepsilon) \sum_{e\in E} \|f_e\|_e^2 + \mathcal{O}(\varepsilon) c_{\mathrm{vol}} \sum_{v\in V} |f(v)|^2 = \mathcal{O}(\varepsilon) \sum_{e\in E} \bigl(\|f_e\|_e^2 + \|f_e'\|_e^2\bigr)
\end{aligned}$$

where we have used (4.11) and our assumptions ($\mathrm{H}_0 2$), ($\mathrm{H}_\varepsilon 1$), ($\mathrm{H}_\varepsilon 2$) and ($\mathrm{H}_\varepsilon 5$). In addition, we have used the standard Sobolev estimate

$$|f(0)|^2 \le \frac{4}{\ell_0} \bigl(\|f\|^2_{(0,\ell_0/2)} + \|f'\|^2_{(0,\ell_0/2)}\bigr). \tag{6.8}$$

which implies

$$\sum_{v\in V} |f(v)|^2 \le \frac{4}{\ell_0} \sum_{e\in E} \bigl(\|f_e\|_e^2 + \|f_e'\|_e^2\bigr) \tag{6.9}$$

by choosing an edge $e \in E_v$ for each vertex v. Clearly, we can estimate the right-hand side by $\|f\|_{1,\theta} = \|T^\theta f\|_1$ and we obtain $\|Jf - J^1 f\| = \mathcal{O}(\varepsilon^{1/2}) \|f\|_{1,\theta}$ where $\mathcal{O}(\varepsilon^{1/2})$ depends now also on $\operatorname{Re}\theta$.

Next we have

$$\|J'u - J'^1 u\|^2 = \sum_{e\in E} \sum_{v\in\partial e} \varepsilon^m \int_e \rho_{v,e}^2 \, \mathrm{d}x \, |C_v u - N_e u(v)|^2$$

since the supports of $\rho_{v,e}$, $v \in \partial e$, are disjoint by construction. As in [EP05, Lemma 5.5] (cf. also [P06, Lem. 2.10] for the non-compact case) we can show

$$\varepsilon^m |C_v u - N_e u(v)|^2 = \mathcal{O}(\varepsilon) \|\mathrm{d}u\|^2_{U_{\varepsilon,v}}$$

for $u \in \mathsf{H}^1(U_{\varepsilon,v})$, $v \in \partial e$. Here $\mathcal{O}(\varepsilon)$ depends on the errors in ($\mathrm{H}_\varepsilon 1$), on λ_2 in ($\mathrm{H}_\varepsilon 2$) and on ℓ_0. Reordering the sum $\sum_{e\in E} \sum_{v\in\partial e} \|u\|^2_{U_{\varepsilon,v}}$, we gain a factor d_0 (the maximal degree of a vertex, cf. ($\mathrm{H}_0 1$)). In particular,

$$\|J'u - J'^1 u\|^2 = \mathcal{O}(\varepsilon) d_0 \sum_{v\in V} \|\mathrm{d}u\|^2_{\varepsilon,v} = \mathcal{O}(\varepsilon) \|u\|^2_{1,\overline{\theta}}$$

for $u \in \widetilde{\mathcal{H}}^{1,\overline{\theta}}$ so that $\|J'u - J'^1 u\| = \mathcal{O}(\varepsilon^{1/2}) \|u\|_{1,\overline{\theta}}$ where again the error term $\mathcal{O}(\varepsilon^{1/2})$ depends also on $\operatorname{Re}\theta$.

Assumption (B.2) follows easily from (4.12), i.e.,

$$|\langle Jf, u\rangle - \langle f, J'u\rangle| \le \mathcal{O}(\varepsilon) \|f\| \|u\|$$

for $f \in \mathsf{L}_2(X_0)$ and $u \in \mathsf{L}_2(X_\varepsilon)$. In the same way, Assumption (B.4) follows from

$$\|Jf\|^2 \le (1+\mathcal{O}(\varepsilon)) \|f\|^2 \qquad \text{and} \qquad \|J'u\|^2 \le (1+\mathcal{O}(\varepsilon)) \|u\|^2.$$

Assumption (B.3) follows from $J'Jf = f$ and

$$\|JJ'u - u\|^2 = \sum_{e\in E} \|N_e u - u\|^2_{U_{\varepsilon,e}} + \sum_{v\in V} \|u\|^2_{U_{\varepsilon,v}}.$$

Now, as in [EP05, Lemmas 3.1, 4.4] (cf. [P06, Lem. 2.11] for the non-compact case), we can show

$$\|N_e u - u\|^2_{U_{\varepsilon,e}} \le \mathcal{O}(\varepsilon^2) \|\mathrm{d}u\|^2_{U_{\varepsilon,e}}$$

where $\mathcal{O}(\varepsilon^2)$ depends on the error in ($\mathrm{H}_\varepsilon1$) and on $\lambda_2^{\mathrm{N}}(F)$. Next we have

$$\|u\|^2_{U_{\varepsilon,v}} = \mathcal{O}(\varepsilon)(\|u\|^2_{U^+_{\varepsilon,e}} + \|du\|^2_{U^+_{\varepsilon,e}})$$

(cf. [EP05, Cor. 5.8] or [P06, Lem. 2.12] for the non-compact case)) with an error depending on ℓ_0 and the errors in ($\mathrm{H}_\varepsilon1$) and ($\mathrm{H}_\varepsilon2$). Here,

$$U^+_{\varepsilon,v} = U_{\varepsilon,v} \uplus \biguplus_{e \in E_v} U_{\varepsilon,e}$$

is the vertex neighbourhood together with its adjacent edge neighbourhoods. The last two estimates mean that a function orthogonal to the constant transversal function or being concentrated at a vertex neighbourhood cannot be spectrally bounded. Summing all these error terms, we obtain $\|JJ'u - u\| = \mathcal{O}(\varepsilon^{1/2})\|u\|_1$ for $u \in \mathcal{H}^1$.

We finally prove (B.6') in our model. On each internal edge, we have the contribution

$$\mathfrak{h}^\theta_e({J'_e}^1 u, f) - \mathfrak{h}^\theta_{\varepsilon,e}(u, J^1_e f) = \mathfrak{h}^\theta_e({J'_e}^1 u, f) - \hat{\mathfrak{h}}^\theta_{\varepsilon,e}(u, J_{\mathrm{e}} f) + \mathcal{O}(\varepsilon)\|u\|_{\mathfrak{d}_{\varepsilon,e}}\|f\|_{\mathfrak{d}_e},$$

where we used (4.13). Note that $J^1_e f = \Theta_{\varepsilon,e} J_e f$, $\mathrm{d}_F J_e f = 0$ and $\|J_e f\|_{\hat{\mathfrak{d}}_{\varepsilon,e}} = \|f\|_{\mathfrak{d}_e}$ and that $\mathcal{O}(\varepsilon) = 0$ if $e \in E_{\mathrm{ext}}$. Recall that $\|u\|^2_{\mathfrak{d}_{\varepsilon,e}} = \|u\|^2_{\varepsilon,e} + \|\mathrm{d}u\|^2_{\varepsilon,e}$ and similarly for the other norms. Now

$$\mathfrak{h}^\theta_e({J'_e}^1 u, f) - \hat{\mathfrak{h}}^\theta_{\varepsilon,e}(u, J_{\mathrm{e}} f) = \sum_{v \in \partial e, v \notin \Gamma_0} \langle D_e \rho_{v,e}, f\rangle_{\varepsilon,e} \varepsilon^{m/2}\bigl(C_v \overline{u} - N_e \overline{u}(v)\bigr)$$

since the longitudinal terms cancel due to the simple form of $\mathfrak{h}^{\hat{\theta}}_{\varepsilon,e}$ and $\mathrm{d}_F J_e f = 0$. On external edges we even have $\mathfrak{h}^\theta_e({J'_e}^1 u, f) = \mathfrak{h}^\theta_{\varepsilon,e}(u, J_{\mathrm{e}} f)$ since $U_{\varepsilon,e}$ has exact product structure there.

The vertex contribution is

$$\mathfrak{h}^\theta_{\varepsilon,v}(u, J^1_v f) = \varepsilon^{-m/2}\mathfrak{h}^\theta_{\varepsilon,v}(u, \Theta_{\varepsilon,v} f(v)) = \mathcal{O}(\varepsilon^{1/2})\|u\|_{\mathfrak{d}_{\varepsilon,v}}|f(v)|,$$

where we have used (4.14). Note that $J^1_v f = \varepsilon^{-m/2}\Theta_{\varepsilon,v} f(v)$, $\mathrm{d}f(v) = 0$, and therefore $\|f(v)\|_{\hat{\mathfrak{d}}_{\varepsilon,v}} = \varepsilon^{d/2}|f(v)|$.

Finally, summing up all the error terms, we obtain (B.6') with $\delta = \mathcal{O}(\varepsilon^{1/2})$, again depending also on d_0 and $\operatorname{Re}\theta$. ☐

Using the additional information of Theorem C.17 we can conclude from Appendix B our main result:

Theorem 6.2. *Let* $0 \le \vartheta < \pi$ *and* $\theta \in S_\vartheta$, *i.e.,* $|\operatorname{Im}\theta| < \vartheta/2$. *Assume in addition* ($\mathrm{H}_0 1$)–($\mathrm{H}_0 4$) *on the quantum graph* X_0 *and* ($\mathrm{H}_\varepsilon1$)–($\mathrm{H}_\varepsilon6$) *on the manifold* X_ε. *If* $\lambda(0)$ *denotes a resonance of the magnetic Hamiltonian* H_0 *with a multiplicity* $m > 0$ *then for a sufficiently small* $\varepsilon > 0$ *there exist* m *resonances* $\lambda_1(\varepsilon), \dots, \lambda_m(\varepsilon)$ *of* H_ε, *satisfying* $\operatorname{Im}\lambda_j(\varepsilon) < 0$ *and not necessarily mutually different, which all converge to* $\lambda(0)$ *as* $\varepsilon \to 0$. *The same is true in the case when* $\lambda(0)$ *is an embedded eigenvalue of* H_0, *except that only* $\operatorname{Im}\lambda_j(\varepsilon) \le 0$ *holds in general.*

Note that if the internal part is compact (i.e., if there are only finitely many vertices), then the assumptions ($\mathrm{H}_0 1$)–($\mathrm{H}_0 2$) and ($\mathrm{H}_\varepsilon2$) are automatically fulfilled.

We can even conclude stronger results from Appendix B using the identification maps J and J' defined in (6.4) and (6.6), namely the resolvent convergence and the convergence of the eigenprojections.

Theorem 6.3. *Under the same assumptions as in the previous theorem, we have*

$$\|J(H_0^\theta - z)^{-1} - (H_\varepsilon^\theta - z)^{-1}J\| = \mathcal{O}(\varepsilon^{1/2}), \tag{6.10a}$$

$$\|J(H_0^\theta - z)^{-1}J' - (H_\varepsilon^\theta - z)^{-1}\| = \mathcal{O}(\varepsilon^{1/2}) \tag{6.10b}$$

for $z \notin \Sigma_\vartheta$. The error depends on the same quantities as the error in Theorem 6.1, and also on ϑ and z.

In addition, suppose that $\lambda^\theta(0)$ is a discrete eigenvalue of H_0^θ. Let D be an open disc such that D contains λ but no other spectral point of H_0^θ. Then (6.10) *holds when the resolvent is replaced by the spectral projection $\mathbb{1}_D(H_0^\theta)$ resp. $\mathbb{1}_D(H_\varepsilon^\theta)$. If the multiplicity of $\lambda^\theta(0)$ is* 1 *with normalised eigenfunction ψ_0^θ (a* resonance *or* eigenstate *for H_0) then there exists a normalised eigenfunction ψ_ε^θ (a* resonance *or* eigenstate *for H_ε) on the manifold such that*

$$\|J\psi_0^\theta - \psi_\varepsilon^\theta\| = \mathcal{O}(\varepsilon^{1/2}) \qquad \text{and} \qquad \|J'\psi_\varepsilon^\theta - \psi_0^\theta\| = \mathcal{O}(\varepsilon^{1/2}).$$

As a by-product, we also have shown that the spectrum of a magnetic Hamiltonian on a *non-compact* manifold converges to the associated non-compact quantum graph Hamiltonian provided our uniformity assumptions are fulfilled. In particular, we could approximate fractal spectra such as studied, e.g., in [BGP07] as we have mentioned in the introduction.

Theorem 6.4. *Assume* ($\mathrm{H}_0$1)–($\mathrm{H}_0$4) *on the quantum graph X_0 and* (H_ε1)–(H_ε6) *on the manifold X_ε. Then the spectrum of H_ε converges to H_0 on any finite energy interval. The same is true for the essential and discrete spectrum.*

Proof. The spectral convergence is a direct consequence of the closeness, as it follows from the general theory developed in [P06, Appendix]. ☐

Appendices

A. Scale of Hilbert spaces

A.1. **Scale of Hilbert spaces associated with a self-adjoint operator.** Denote by Δ a non-negative, self-adjoint operator in the Hilbert space $\mathcal{H}$. We sometimes refer to Δ as the *free* operator. Throughout the paper we use the convention that $\langle\cdot,\cdot\rangle$ and other sesquilinear forms are *anti-linear* in the first and linear in the second argument.

A scale of Hilbert spaces can be associated with Δ as follows: For fixed $k \ge 0$ we set $\mathcal{H}^k := \operatorname{dom}\Delta^{k/2}$ equipped with the norm $\|u\|_k := \|(\Delta+1)^{k/2}u\|$. For negative powers, we set $\mathcal{H}^{-k} := (\mathcal{H}^k)^*$ where

$$(\mathcal{H}^k)^* := \{\, \varphi\colon \mathcal{H}^k \longrightarrow \mathbb{C} \mid \varphi \text{ anti-linear and bounded}\,\}, \tag{A.1a}$$

with the norm

$$\|\varphi\|_{-k} := \sup_{f\in\mathcal{H}^k} \frac{|\varphi(f)|}{\|f\|_k} \tag{A.1b}$$

and $\mathcal{H}$ is embedded in $\mathcal{H}^{-k}$ via $f \mapsto \langle\cdot, f\rangle$. For more details we refer e.g. to [KPS82].

A.2. **Scale of Hilbert spaces associated with a self-adjoint family of operators.** Since our dilated operators are no longer self-adjoint, we also need a scale of Hilbert spaces associated with a particular class of non-self-adjoint operators, namely *sectorial* operators. Most of the material on such operators is standard and can be found e.g. in [Ka66]. We also introduce a scale of order 1 which is not associated to the natural quadratic form, but easier to handle in the present application.

Let $\{H^\theta\}_\theta$ with $\theta \in S = \{\, w \in \mathbb{C} \,|\, |\operatorname{Im} w| < b \,\}$ be a family of closed operators acting in the Hilbert space $\mathcal{H}$.

Definition A.1. We say that the family $\{H^\theta\}_\theta$ is *self-adjoint*[6] if $(H^\theta)^* = H^{\overline{\theta}}$.

The family $\{H^\theta\}$ is called *(spectrally) uniformly ϑ-sectorial*[7] if $\sigma(H^\theta)$ is contained in the common sector $\Sigma_\vartheta = \{\, z \in \mathbb{C} \,|\, |\arg z| \le \vartheta \,\}$.

We allow values $0 \le \vartheta < \pi$, although operators with spectrum *not* contained in the right half-plane are no longer semi-bounded. The only point we need here is, that -1 belongs to the resolvent set and that we can control the norm of the corresponding resolvent (denoted by the constant C_0^θ).

From now on we assume that $\{H^\theta\}$ is a self-adjoint, uniformly ϑ-sectorial family of operators. We start defining the scales of order 2, 0 and -2:

Let $\mathcal{H}^0 := \mathcal{H}$, $\|\cdot\|_0 := \|\cdot\|$ and

$$\mathcal{H}^{2,\theta} := \operatorname{dom} H^\theta, \qquad \|f\|_{2,\theta} := \|(H^\theta + 1)f\| \tag{A.2}$$

be the spaces of order 0 and 2. Since H^θ is closed and $-1 \notin \sigma(H^\theta)$, $\mathcal{H}^{2,\theta}$ with norm $\|\cdot\|_{2,\theta}$ is also a Hilbert space. The dual space is defined by

$$\mathcal{H}^{-2,\theta} := (\mathcal{H}^{2,\overline{\theta}})^* \tag{A.3}$$

similarly as in (A.1). Note the complex conjugation of θ in order to compensate the anti-linearity in the definition of the dual. In the next two lemmas, we want to assure that H^θ and its resolvent extend to maps on the scale of order $-2, 0, 2$:

Lemma A.2. *The embedding $\iota\colon \mathcal{H} \longrightarrow \mathcal{H}^{-2,\theta}$, $g \mapsto \langle \cdot, g\rangle$ is continuous. Furthermore, $\|(H^\theta+1)^{-1}g\| = \|\iota g\|_{-2,\theta}$ for $g \in \mathcal{H}$, i.e., $\mathcal{H}^{-2,\theta}$ can be considered as the completion of $\mathcal{H}$ in the norm $\|g\|_{-2,\theta} := \|(H^\theta+1)^{-1}g\|$.*

Proof. We have

$$\|\iota g\|_{2,-\theta} = \sup_{f \in \mathcal{H}^{2,\overline{\theta}}} \frac{|\langle f, g\rangle|}{\|f\|_{2,\overline{\theta}}} = \sup_{h \in \mathcal{H}} \frac{|\langle h, (H^\theta+1)^{-1} g\rangle|}{\|h\|} = \|(H^\theta+1)^{-1}g\| \tag{A.4}$$

where $h = (H^{\overline{\theta}} + 1)f$ and the claims follow. ☐

Lemma A.3. *The maps*

$$(H^\theta + 1)\colon \mathcal{H}^{2,\theta} \longrightarrow \mathcal{H} \qquad \text{and} \qquad (H^\theta+1)^{-1}\colon \mathcal{H} \longrightarrow \mathcal{H}^{2,\theta} \tag{A.5}$$

are isometries and inverse to each other. Similarly,

$$(H^\theta + 1)\colon \mathcal{H} \longrightarrow \mathcal{H}^{-2,\theta} \qquad \text{and} \qquad (H^\theta+1)^{-1}\colon \mathcal{H}^{-2,\theta} \longrightarrow \mathcal{H} \tag{A.6}$$

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[6] In general, H^θ is self-adjoint only for *real* θ.

[7] Usually, an operator is called *sectorial*, if $\vartheta < \pi/2$, and if one requires in addition that for all $\vartheta_1 \in (\vartheta, \pi/2)$ there is a constant $C_0 = C_0(\vartheta, \vartheta_1)$ such that $\|(H^\theta - z)^{-1}\| \le C_0/|z|$ for all $z \notin S_{\vartheta_1}$. We do not need this fact here.

are isometries and inverse to each other. Here $(H^\theta+1)g := \langle (H^{\overline{\theta}}+1)(\cdot), g\rangle$ *and* $(H^\theta+1)^{-1}\iota g := (H^\theta+1)^{-1} g$ *extend to an isometry on* $\mathcal{H}^{-2,\theta}$*. Finally,*

$$H^\theta \colon \mathcal{H}^{2,\theta} \longrightarrow \mathcal{H} \quad \textit{and} \quad H^\theta \colon \mathcal{H} \longrightarrow \mathcal{H}^{-2,\theta} \tag{A.7}$$

are bounded maps with norm bounded by $1+C_0^\theta$*, where* $C_0^\theta = \|(H^\theta+1)^{-1}\|$ *in general depends on* θ*.*

Proof. The first two assertions are almost obvious. The last one follows from the fact that since $-1 \notin \sigma(A)$, we have $\|f\| \le C_0^\theta \|(H^\theta+1)f\|$, and therefore

$$\|H^\theta f\| \le \|(H^\theta+1)f\| + \|f\| \le (1+C_0^\theta)\|(H^\theta+1)f\| = (1+C_0^\theta)\|f\|_{2,\theta}.$$

Similarly

$$\begin{aligned} |\langle H^{\overline{\theta}} f, g\rangle| &\le |\langle (H^{\overline{\theta}}+1)f, g\rangle| + |\langle f, g\rangle| \\ &\le (\|f\|_{2,\overline{\theta}} + \|f\|)\|g\| \le (1+C_0^\theta)\|f\|_{2,\overline{\theta}}\|g\| \end{aligned}$$

and therefore $\|H^\theta g\|_{-2,\theta} \le (1+C_0^\theta)\|g\|$. ☐

So far, we have defined a scale of Hilbert spaces $\{\mathcal{H}^{k,\theta}\}_{k,\theta}$, $k=-2,0,2$, associated to the self-adjoint, uniformly ϑ-sectorial family $\{H^\theta\}_\theta$, i.e., for $k=0$ and $k=2$, the inclusion map

$$\iota \colon \mathcal{H}^{k,\theta} \longrightarrow \mathcal{H}^{k-2,\theta} \tag{A.8}$$

is continuous, $\mathcal{H}^{k,\theta}$ is dense in $\mathcal{H}^{k-2,\theta}$ and the maps

$$H^\theta \colon \mathcal{H}^{k,\theta} \longrightarrow \mathcal{H}^{k-2,\theta}, \tag{A.9a}$$

$$(H^\theta+1)^{-1} \colon \mathcal{H}^{k-2,\theta} \longrightarrow \mathcal{H}^{k,\theta} \tag{A.9b}$$

are continuous.

Since in our context, the domain $\operatorname{dom} H^\theta$ will depend on the *complex* parameter θ, the natural quadratic form associated to H^θ is not well-adopted to our application (especially its natural norm). We therefore define the norm on the Hilbert space of order 1 in a different way:

Definition A.4. Let $\mathcal{H}^{1,\theta}$ be a linear subspace of $\mathcal{H}$, and let $\Delta \ge 0$ be a self-adjoint, non-negative operator on $\mathcal{H}$. We say that $\mathcal{H}^{1,\theta}$ defines a *compatible scale of order* 1 *w.r.t.* Δ if the following conditions are fulfilled:

(i) There is a family of bounded, invertible operators $T^\theta \colon \mathcal{H} \longrightarrow \mathcal{H}$, called *compatibility operators* such that

$$(T^\theta)^* = T^{\overline{\theta}} \qquad \text{and} \qquad (T^\theta)^{-1} = T^{-\theta} \tag{A.10}$$

for $\theta \in S$. We assume that $\mathcal{H}^{1,\theta} = T^{-\theta}(\mathcal{H}^1)$ and define a norm

$$\|u\|_{1,\theta} := \|T^\theta u\|_1 = \|(\Delta+1)^{1/2} T^\theta u\| \tag{A.11}$$

where $\mathcal{H}^1$ is the element of the Hilbert space scale of order 1 associated with Δ.

(ii) We assume that $\mathcal{H}^{2,\theta}$ is a dense subspace of $\mathcal{H}^{1,\theta}$.

(iii) We assume that the embedding $\mathcal{H}^{2,\theta} \hookrightarrow \mathcal{H}^{1,\theta}$ is continuous with norm bounded by C_2^θ.

(iv) Finally, we assume that the *sesquilinear form associated to* H^θ defined by $\mathfrak{h}^\theta(f,g) := \langle f, H^\theta g\rangle$ is continuous on $\mathcal{H}^{1,\overline{\theta}} \times \mathcal{H}^{1,\theta}$, i.e., there exists a constant C_1^θ such that

$$|\mathfrak{h}^\theta(f,g)| \le C_1^\theta \|f\|_{1,\overline{\theta}} \|g\|_{1,\theta} \tag{A.12}$$

for all $f \in \mathcal{H}^{1,\overline{\theta}}$ and $g \in \mathcal{H}^{2,\theta} = \operatorname{dom} H^\theta$.

Clearly, by the construction, $\mathcal{H}^{1,\theta}$ is complete, since $\mathcal{H}^1$ is. In addition, since $\mathcal{H}^{2,\theta}$ is dense in $\mathcal{H}^{1,\theta}$, the sesquilinear form $\mathfrak{h}^\theta$ extends uniquely to a bounded one on $\mathcal{H}^{1,\overline{\theta}} \times \mathcal{H}^{1,\theta} \to \mathbb{C}$ which we denote by the same symbol.

We define the dual space as before by

$$\mathcal{H}^{-1,\theta} := (\mathcal{H}^{1,\overline{\theta}})^* \tag{A.13a}$$

with the canonical norm $\|\cdot\|_{-1,\theta}$ as in (A.1). Note that we can consider $\mathcal{H}^{-1,\theta}$ as the completion of $\mathcal{H}$ in the norm

$$\|u\|_{-1,\theta} = \|(\Delta+1)^{-1/2} T^{-\theta} u\|. \tag{A.13b}$$

There are simple equivalent characterisations of the last two conditions (iii) and (iv) following from the definitions:

Lemma A.5. *Condition* (iii) *is equivalent to the fact that*

$$(H^\theta + 1)^{-1} \colon \mathcal{H} \longrightarrow \mathcal{H}^{1,\theta} \tag{A.14a}$$

is norm-bounded by C_2^θ *or equivalently,*

$$(\Delta+1)^{1/2} T^\theta (H^\theta+1)^{-1} \tag{A.14b}$$

is a bounded operator in $\mathcal{H}$ *with bound* C_2^θ.

We also have a sufficient condition:

Lemma A.6. *Condition* (iii) *follows from the fact that*

$$(H^\theta + 1)^{-1} \colon \mathcal{H}^{-1,\theta} \longrightarrow \mathcal{H}^{1,\theta} \tag{A.15a}$$

is norm-bounded or equivalently,

$$(\Delta+1)^{1/2} T^\theta (H^\theta+1)^{-1} T^\theta (\Delta+1)^{1/2} \tag{A.15b}$$

is a bounded operator in $\mathcal{H}$.

Lemma A.7. *The continuity of the sesquilinear form*

$$\mathfrak{h}^\theta \colon \mathcal{H}^{1,\overline{\theta}} \times \mathcal{H}^{1,\theta} \longrightarrow \mathbb{C} \tag{A.16}$$

is equivalent to the fact that

$$H^\theta \colon \mathcal{H}^{1,\theta} \longrightarrow \mathcal{H}^{-1,\theta}, \qquad g \mapsto \mathfrak{h}^\theta(\cdot, g) \tag{A.17a}$$

is norm-bounded by C_1^θ *or equivalently,*

$$(\Delta+1)^{-1/2} T^{-\theta} H^\theta T^{-\theta} (\Delta+1)^{-1/2} \tag{A.17b}$$

is a bounded operator in $\mathcal{H}$ *with bound* C_1^θ.

These observations show that $\{\mathcal{H}^{k,\theta}\}_{k,\theta}$ behaves almost like a natural scale of Hilbert spaces; in particular, $\mathcal{H}^{k,\theta}$ is dense in $\mathcal{H}^{k-1,\theta}$: This follows for $k = 1$ by the construction of $\mathcal{H}^{1,\theta}$ and for $k = 2$ by Definition A.4 (ii). Furthermore, the inclusions

$$\iota \colon \mathcal{H}^{k,\theta} \longrightarrow \mathcal{H}^{k-1,\theta}$$

are continuous for $k = 1$ by the construction of $\mathcal{H}^{1,\theta}$ and for $k = 2$ by Definition A.4 (iii). By duality, the same statements hold for $k = 0$ and $k = -1$. In addition, (A.9) is valid for $k = 0, 1, 2$ (by Lemma A.3 and Lemma A.7) except that the resolvent is only a continuous map from $\mathcal{H}$ to $\mathcal{H}^{1,\theta}$ (Lemma A.5). Therefore, the following definition is natural:

Definition A.8. We call $\{\mathcal{H}^{k,\theta}\}_{k,\theta}$, $k = -2, -1, 0, 1, 2$, a *compatible* scale if $\mathcal{H}^{1,\theta}$ is a compatible scale of order 1 (Definition A.4) and $\{\mathcal{H}^{k,\theta}\}_{k,\theta}$, $k = -2, 0, 2$, is a scale in the sense of (A.8)–(A.9).

B. AN ABSTRACT CONVERGENCE CRITERIA FOR NON-SELFADJOINT OPERATORS

In this section we are going to prove the resolvent convergence for self-adjoint, uniformly ϑ-sectorial families of (closed) operators $\{H^\theta\}_\theta$ and $\{\widetilde{H}^\theta\}_\theta$ acting in $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, respectively, for all θ in the strip S_ϑ (i.e. $|\operatorname{Im}\theta| < \vartheta/2$).

Notation B.1. We will use the obvious notation $\|A\|_{k\to m}$ for the norm of the operator $A \colon \mathcal{H}^k \longrightarrow \mathcal{H}^m$ where $\mathcal{H}^k$ is an element of the scale w.r.t. the self-adjoint operator $\Delta \geq 0$. Similarly, we write $\|A\|_{k,\theta\to m,\theta}$ for the norm of the operator $A \colon \mathcal{H}^{k,\theta} \longrightarrow \mathcal{H}^{m,\theta}$ where $\{\mathcal{H}^{k,\theta}\}_{k,\theta}$, $k = -2, \dots, 2$, is a compatible scale associated to the operator H^θ (cf. Definitions A.4 and A.8).

Furthermore, we employ the analogous *tilded* notation for the respective objects acting in the Hilbert space $\widetilde{\mathcal{H}}$, namely the self-adjoint operator $\widetilde{\Delta} \geq 0$ with the scale $\{\widetilde{\mathcal{H}}^k\}_k$ and the operator $\widetilde{H}^\theta$ giving rise to the scale $\{\widetilde{\mathcal{H}}^{k,\theta}\}_{k,\theta}$.

Next we introduce the notion of *quasi-unitarity* up to an error $\delta > 0$. In our application, $\delta = \delta(\varepsilon)$ where ε is the parameter appearing in the operators *and* domains and Hilbert spaces. We prefer to formulate the results below without mentioning explicitly the parameter ε.

Definition B.2. Suppose that we have linear operators

$$J \colon \mathcal{H} \longrightarrow \widetilde{\mathcal{H}} \qquad \text{and} \qquad J' \colon \widetilde{\mathcal{H}} \longrightarrow \mathcal{H}. \tag{B.1}$$

We say that J and J are *δ-quasi-unitary* w.r.t. the operators Δ and $\widetilde{\Delta}$ iff the following conditions hold for $\delta > 0$:

$$\|J - J'^*\| \leq \delta, \tag{B.2}$$

$$\|\mathbb{1} - J'J\|_{1\to 0} \leq \delta, \qquad \|\mathbb{1} - JJ'\|_{1\to 0} \leq \delta, \tag{B.3}$$

$$\|J\| \leq 2, \qquad \|J'\| \leq 2, \tag{B.4}$$

where $\|A\|_{1\to 0} = \|A(\Delta + 1)^{-1/2}\|$ is the norm of $A \colon \mathcal{H}^1 \longrightarrow \mathcal{H}$ and the analogous norm is used on $\widetilde{\mathcal{H}}$.

This allows us to specify what we mean by closeness of operators H^θ and $\widetilde{H}^\theta$:

Definition B.3. We say that the operators H^θ and $\widetilde{H}^\theta$ in $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, respectively, are *δ-close* w.r.t. the δ-quasi-unitary operators J and J' or briefly, *δ-close* if there exist compatible scales $\mathcal{H}^{1,\theta}$ and $\widetilde{\mathcal{H}}^{1,\theta}$ of order 1 associated to H^θ and $\widetilde{H}^\theta$ with compatibility operators T^θ and $\widetilde{T}^\theta$ in the sense of Definition A.4, and if there exist operators[8]

$$J^1 = J^{1,\theta} \colon \mathcal{H}^{1,\theta} \longrightarrow \widetilde{\mathcal{H}}^{1,\theta} \quad \text{and} \quad J'^1 = J'^{1,\overline{\theta}} \colon \widetilde{\mathcal{H}}^{1,\overline{\theta}} \longrightarrow \mathcal{H}^{1,\overline{\theta}}$$

such that

$$\|J^1 - J\|_{1,\theta\to 0} \le \delta \qquad \text{and} \qquad \|J'^1 - J'\|_{1,\overline{\theta}\to 0} \le \delta. \tag{B.5}$$

and

$$\|(J'^1)^* H^\theta - \widetilde{H}^\theta J^1\|_{1,\theta\to -1,\theta} \le \delta, \tag{B.6}$$

$$\widetilde{T}^\theta J = J T^\theta, \qquad T^\theta J' = J' \widetilde{T}^\theta, \tag{B.7}$$

where $\|A\|_{1,\theta\to 0} = \|A T^{-\theta}(\Delta+1)^{-1/2}\|$ and similarly on $\widetilde{\mathcal{H}}$ and where

$$\|V\|_{1,\theta\to -1,\theta} = \|(\widetilde{\Delta}+1)^{-1/2}\widetilde{T}^{-\theta} V T^{-\theta}(\Delta+1)^{-1/2}\|.$$

Remark B.4.

(i) We do not exclude that δ depends on θ.

(ii) Note that H^θ in (B.6) is a bounded operator as map $\mathcal{H}^{1,\theta}$ to $\mathcal{H}^{-1,\theta}$ (cf. (A.17)) and similarly for $\widetilde{H}^\theta$.

(iii) Denote the associated sesquilinear forms to H^θ and $\widetilde{H}^\theta$ by $\mathfrak{h}^\theta$ and $\widetilde{\mathfrak{h}}^\theta$, respectively (cf. Definition A.4 (iv)). Then (B.6) is equivalent to

$$\left|\mathfrak{h}^\theta(J'^1 u, f) - \widetilde{\mathfrak{h}}^\theta(u, J^1 f)\right| \le \delta \|u\|_{1,\overline{\theta}} \|f\|_{1,\theta} \tag{B.6'}$$

for $u \in \widetilde{\mathcal{H}}^{1,\overline{\theta}}$ and $f \in \mathcal{H}^{1,\theta}$. In fact, we will see in the proof of Theorem B.6 (the only point, where J^1 and J'^1 enter), that it is enough to have (B.6') only for f and u in the *operator* domains, i.e., $f \in \mathcal{H}^{2,\theta}$ and $u \in \widetilde{\mathcal{H}}^{2,\overline{\theta}}$. Since $\mathcal{H}^{2,\theta}$ is dense in $\mathcal{H}^{1,\theta}$ and similarly on $\widetilde{\mathcal{H}}$ by Definition A.4 (ii), this implies of course (B.6).

An immediate consequence is the following:

Lemma B.5. *With the previous notation we have*

$$\|(J'J - \mathbb{1})f\| \le C_3^\theta \delta \|f\|_{2,\theta} \tag{B.8}$$

$$\left|\|Jf\|^2 - \|f\|^2\right| \le C_4^\theta \delta \|f\|_{2,\theta}^2 \tag{B.9}$$

for $f \in \mathcal{H}^{2,\theta}$ and similarly on $\widetilde{\mathcal{H}}$.

Proof. We estimate

$$\|(J'J - \mathbb{1})f\| \le \|(J'J - \mathbb{1})T^{-\theta}\|_{1\to 0} \|T^\theta f\|_1 \le \|T^{-\theta}\| \delta C_2^\theta \|f\|_{2,\theta} =: C_3^\theta \delta \|f\|_{2,\theta}$$

using (B.3), (B.7) and Definition A.4 (iii). Similarly,

$$\begin{aligned}
\left|\|Jf\|^2 - \|f\|^2\right| &= \left|\langle (J^*J - \mathbb{1})f, f\rangle\right| \\
&\le \left|\langle T^{-\theta}(J^* - J')JT^\theta f, T^{-\theta}T^\theta f\rangle\right| + \left|\langle T^{-\theta}(J'J - \mathbb{1})T^\theta f, T^{-\theta}T^\theta f\rangle\right| \\
&\le 3\|T^{-\theta}\|^2 \delta \|T^\theta f\|_1^2 \le 3\|T^{-\theta}\|^2 (C_2^\theta)^2 \delta \|f\|_{2,\theta}^2 =: C_4^\theta \delta \|f\|_{2,\theta}^2
\end{aligned}$$

---

[8] The operators J^1 and J'^1 need not to be bounded.

using again assumptions in Definition A.4, Definition B.2 and Definition B.3. The estimates on $\widetilde{\mathcal{H}}$ follow similarly. □

We can now state the convergence of the resolvents:

Theorem B.6. *Assume that the families* (H_θ) *and* $(\widetilde{H}_\theta)$ *are* δ*-close w.r.t. the quasi-unitary operators* J *and* J'*, then*

$$\|\widetilde{R}^\theta J - JR^\theta\| \le C_5^\theta \delta, \tag{B.10}$$

where $R^\theta := (H^\theta + 1)^{-1}$, $\widetilde{R}^\theta := (\widetilde{H}^\theta + 1)^{-1}$ *and* $C_5^\theta := (1 + C_0^\theta + C_2^\theta)(1 + \widetilde{C}_0^{\overline{\theta}} + \widetilde{C}_2^{\overline{\theta}})$.

Proof. We write

$$\widetilde{R}^\theta J - JR^\theta = \widetilde{R}^\theta \bigl[JH^\theta - \widetilde{H}^\theta J\bigr] R^\theta$$

where the operator in the bracket maps from $\mathcal{H}^{2,\theta}$ to $\mathcal{H}^{-2,\theta} = (\mathcal{H}^{2,\overline{\theta}})^*$. This operator can be decomposed into

$$\begin{aligned} &JH^\theta - \widetilde{H}^\theta J \\ &\qquad = (J - J'^*)H^\theta + (J' - J'^1)^* H^\theta + \bigl((J'^1)^* H^\theta - \widetilde{H}^\theta J^1\bigr) + \widetilde{H}^\theta (J^1 - J) \end{aligned}$$

where

$$(J'^1)^* \colon \mathcal{H}^{-1,\theta} = (\mathcal{H}^{1,\overline{\theta}})^* \longrightarrow \widetilde{\mathcal{H}}^{-1,\theta} = (\widetilde{\mathcal{H}}^{1,\overline{\theta}})^*. \tag{B.11}$$

Now H^θ is bounded as a map from $\mathcal{H}^{2,\theta}$ to $\mathcal{H}$, as well as $\widetilde{H}^\theta$ is bounded as map from $\mathcal{H}$ to $\mathcal{H}^{-2,\theta}$ with the bounds $C_0^\theta + 1$ and $\widetilde{C}_0^\theta + 1$, respectively, cf. Lemma A.3. Next, the inclusion $\mathcal{H}^{2,\theta} \hookrightarrow \mathcal{H}^{1,\theta}$ is bounded with bound C_2^θ, and similarly in the space $\widetilde{\mathcal{H}}$ (cf. Definition A.4 (iii)). Finally, we can sum up all the error terms to arrive at the given bound. □

Denote by $\rho(H)$ the *resolvent set* of H. A simple argument allows us to deal with all z in $\rho(H^\theta)$ *and* $\rho(\widetilde{H}^\theta)$:

Theorem B.7. *Suppose that* $z_0, z \in \rho(H^\theta) \cap \rho(\widetilde{H}^\theta)$*, then*

$$\|V(z)\| \le C_5^\theta(z) \|V(z_0)\|$$

where $V(z) := \widetilde{R}^\theta(z) J - JR^\theta(z)$ *and* $R^\theta(z) := (H^\theta - z)^{-1}$ *for* $z \in \rho(H^\theta)$*, and similarly for* $\widetilde{H}^\theta$*. In particular,*

$$\|V(z)\| \le C_6^\theta(z)\delta \tag{B.12}$$

under the assumptions of Theorem B.6. The constants $C_5^\theta(z)$ *and* $C_6^\theta(z)$ *depend continuously on* z*.*

Proof. Setting for brevity $R := R^\theta$ and $\widetilde{R} := \widetilde{R}^\theta$, we have

$$\begin{aligned} V(z) &= V(z_0) + (z - z_0)\bigl(\widetilde{R}(z)\widetilde{R}(z_0)J - JR(z)R(z_0)\bigr) \\ &= V(z_0) + (z - z_0)\bigl(\widetilde{R}(z)V(z_0) + V(z)R(z_0)\bigr) \end{aligned}$$

where we have used the second resolvent identity. Reordering the terms we get

$$V(z)\bigl[\mathbb{1} - (z - z_0)R(z_0)\bigr] = \bigl[\mathbb{1} + (z - z_0)\widetilde{R}(z)\bigr] V(z_0)$$

Since $\mathbb{1} + (z - z_0)R(z)$ is the inverse of $\mathbb{1} - (z - z_0)R(z_0)$, we obtain

$$V(z) = \bigl[\mathbb{1} + (z - z_0)\widetilde{R}(z)\bigr] V(z_0) \bigl[\mathbb{1} + (z - z_0)R(z)\bigr] \tag{B.13}$$

and the estimate follows with

$$C_5^\theta(z) := \Big(1 + \frac{|z - z_0|}{\widetilde{d}(z)}\Big)\Big(1 + \frac{|z - z_0|}{d(z)}\Big) \tag{B.14}$$

where $d(z) := \|R(z)\|^{-1}$ and similarly for $\widetilde{d}(z)$. Estimate (B.12) follow immediately from (B.10) with $C_6^\theta(z) := C_5^\theta C_5^\theta(z)$. ☐

We can now easily extend the convergence results to a suitable class of holomorphic functions of the operators:

Theorem B.8. *Suppose that φ is a holomorphic functions in a neighbourhood of a simply connected domain $D \subset \mathbb{C}$ such that D is disjoint from $\sigma(H^\theta)$ and $\sigma(\widetilde{H}^\theta)$ for H^θ and $\widetilde{H}^\theta$ being δ-close and δ small enough. Suppose in addition that $\varphi \in \mathsf{L}_1(\partial D, C_5^\theta(z)\,\mathrm{d}|z|)$ (cf.* (B.14)*). Then*

$$\|\varphi(\widetilde{H}^\theta)J - J\varphi(H^\theta)\| \le C_7^\theta \delta \tag{B.15}$$

where the constant depends only on θ and φ. The integrability condition on φ is in particular satisfied if the curve is compact.

Proof. Since D is contained in the resolvent set of both operators and due to our integrability assumption on φ, the holomorphic spectral calculus applies,

$$\varphi(H^\theta) = \frac{1}{2\pi\mathrm{i}} \oint_{\partial D} \frac{\varphi(z)}{z - H^\theta}\,\mathrm{d}z,$$

and a similar claim is valid for $\widetilde{H}^\theta$. This implies

$$J\varphi(H^\theta) - \varphi(\widetilde{H}^\theta)J = -\frac{1}{2\pi\mathrm{i}} \oint_{\partial D} \big(JR^\theta(z) - \widetilde{R}^\theta(z)J\big)\varphi(z)\,\mathrm{d}z.$$

and therefore,

$$\|J\varphi(H^\theta) - \varphi(\widetilde{H}^\theta)J\| \le \frac{\delta}{2\pi} \int_{\widetilde{\gamma}} C_6^\theta(z)|\varphi(z)|\,\mathrm{d}|z| =: C_7^\theta \delta$$

Since $C_6^\theta(z) = C_5^\theta C_5^\theta(z)$ depends continuously on z, the right-hand side is in particular finite if ∂D is compact. ☐

Now we are able to demonstrate the main result of this section namely the convergence of eigenprojections and eigenvalues. For the discrete spectrum of H^θ it is not necessary to consider the *whole* spectrum of $\widetilde{H}^\theta$, we only need to make sure that we are away from its *essential* spectrum.

Theorem B.9. *Suppose that λ is a discrete eigenvalue of H^θ with multiplicity $m > 0$. Let $D \subset \rho(H^\theta)$ be an open disc such that D contains λ but no other spectral point of H^θ. If $\overline{D} \cap \sigma_{\mathrm{ess}}(\widetilde{H}^\theta) = \emptyset$ for $\widetilde{H}^\theta$ being δ-close to H^θ, then*

$$\|J\mathbb{1}_{\{\lambda\}}(H^\theta) - \mathbb{1}_D(\widetilde{H}^\theta)J\| \le C_7^\theta \delta$$

where C_7^θ depends only on θ and D.

In particular, if m denotes the multiplicity of λ, then there exist m discrete eigenvalues $\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_m$ (not necessarily mutually distinct) in the discrete spectrum of $\widetilde{H}^\theta$ such that

$$|\widetilde{\lambda}_j - \lambda| \le \eta(\delta), \qquad j = 1, \dots, m,$$

where $\eta(\delta) \to 0$ as $\delta \to 0$.

Proof. We choose a sequence $\widetilde{H}_n := \widetilde{H}^\theta_n$ which is δ_n-close to $H = H^\theta$ where $\delta_n \to 0$. Since $\overline{D} \cap \bigcup_n \sigma_{\mathrm{ess}}(\widetilde{H}_n) = \emptyset$ and $\bigcup_n \sigma_{\mathrm{d}}(\widetilde{H}_n)$ is countable, there exists a closed curve $\widetilde{\gamma}$ in D, disjoint from $\bigcup_n \sigma(\widetilde{H}_n)$ enclosing λ but no other spectral point of H. Denote by $\widetilde{D}$ the enclosed region in $\mathbb{C}$, i,e., $\partial \widetilde{D} \subset \rho(H) \cap \rho(\widetilde{H}_n)$ is parametrised by $\widetilde{\gamma}$ and $\widetilde{D} \cap \sigma(H) = \{\lambda\}$. Then we can apply the previous theorem and obtain

$$\|J\mathbb{1}_{\{\lambda\}}(H) - \mathbb{1}_{\widetilde{D}}(\widetilde{H}_n)J\| = \|J\mathbb{1}_{\widetilde{D}}(H) - \mathbb{1}_{\widetilde{D}}(\widetilde{H}_n)J\| \le C^\theta_7 \delta_n \tag{B.16}$$

where C^θ_7 is finite and depend only on θ and D.

For the eigenvalue convergence we first denote by $P = \mathbb{1}_D(H) = \mathbb{1}_{\{\lambda\}}(H)$ and $\widetilde{P} = \mathbb{1}_D(\widetilde{H})$ the corresponding spectral projections. We start proving that $\dim P(\mathcal{H}) = \dim \widetilde{P}(\widetilde{\mathcal{H}})$. Note first that $P(\mathcal{H}) \subset \mathcal{H}^{2,\theta}$ for $f = Pf \in P(\mathcal{H})$ since $\|f\|_{2,\theta} = |\lambda + 1| \|f\|$. Then we estimate

$$\|\widetilde{P}Jf\| \ge \|Jf\| - \|\widetilde{P}J - JP\| \|f\| \ge \bigl(1 - \sqrt{C^\theta_4 \delta}\, |\lambda + 1| - C^\theta_7 \delta\bigr) \|f\| \tag{B.17}$$

using (B.9) and (B.16). If δ is small enough, the right-hand side is still positive. In particular, $\widetilde{P}J$ is injective on $P(\mathcal{H})$ so that $(\widetilde{P}J)(P(\mathcal{H}))$ has at least the dimension of $P(\mathcal{H})$, i.e., $\dim P(\mathcal{H}) \le \dim \widetilde{P}(\widetilde{\mathcal{H}})$. The opposite inequality follows similarly.

Now it is almost obvious that in every neighbourhood $\widetilde{D}$ of λ satisfying the above assumption there are m (not necessarily mutually distinct) eigenvalues $\widetilde{\lambda}_j$ of $\widetilde{H}^\theta$ provided δ is small enough (cf. [Ka66, Ch. II.5.1]). ☐

In the case of one-dimensional projections we can even show the convergence of the corresponding eigenvectors. Note that generically, the eigenvalues are simple (cf. [U76]):

Theorem B.10. *Suppose that ψ is a normalized eigenvector of H^θ with eigenvalue λ of multiplicity 1 and that $\lambda \notin \sigma_{\mathrm{ess}}(\widetilde{H}^\theta)$. Then there exist an eigenvalue $\widetilde{\lambda}$ of $\widetilde{H}^\theta$ of multiplicity 1 arbitrary close to λ and a unique eigenvector $\widetilde{\psi}$ (up to a unitary scalar factor) and constants C^θ_8, $\widetilde{C}^\theta_8 > 0$ depending only on θ and λ such that*

$$\|J\psi - \widetilde{\psi}\| \le C^\theta_8 \delta, \qquad \|J'\widetilde{\psi} - \psi\| \le \widetilde{C}^\theta_8 \delta$$

provided H^θ and $\widetilde{H}^\theta$ are δ-close and $\delta > 0$ is small enough.

Proof. The first assertion follows from the previous theorem. Denote the corresponding eigenprojections by P and $\widetilde{P}$, respectively. For the eigenvector convergence, note that

$$\widetilde{\psi} = \frac{1}{\langle \widetilde{P}J\psi, J\psi\rangle} \widetilde{P}J\psi$$

since $\widetilde{P}$ is a one-dimensional projection. Note in addition that

$$\langle \widetilde{P}J\psi, J\psi\rangle = \|\widetilde{P}J\psi\|^2 \ge \frac{1}{4}\|\psi\|^2 = \frac{1}{4}, \qquad 0 < \delta < \delta_0 \tag{B.18}$$

for some $\delta_0 > 0$ due to (B.17). Now,

$$\begin{aligned}\|J\psi - \widetilde{\psi}\| &= \Big\| JP\psi - \frac{1}{\langle \widetilde{P}J\psi, J\psi\rangle}\,\widetilde{P}J\psi \Big\| \\ &\le \|(JP - \widetilde{P}J)\psi\| + \Big|1 - \frac{1}{\langle \widetilde{P}J\psi, J\psi\rangle}\Big| \|\widetilde{P}J\psi\| \\ &\le C_7^\theta \delta + 8\big|\langle(\widetilde{P}J - JP)\psi, J\psi\rangle + \|J\psi\|^2 - \|\psi\|^2\big| \\ &\qquad\qquad \le (17C_7^\theta + 8C_4^\theta|1+\lambda|^2)\delta =: C_8^\theta\delta\end{aligned}$$

since $\psi = P\psi$ and $\|\psi\| = 1$ using the previous theorem, (B.4), (B.9) and (B.18). The second estimate follows immediately from

$$\|J'\widetilde{\psi} - \psi\| \le \|J'(\widetilde{\psi} - J\psi)\| + \|(J'J - \mathbb{1})\psi\| \le \big(2C_8^\theta + C_3^\theta|1+\lambda|\big)\delta =: \widetilde{C}_8^\theta\delta$$

using (B.8) All the estimates are valid for $0 < \delta < \delta_0$. ☐

C. Analyticity and a resolvent estimate

Here we sketch the proof of analyticity of the complex dilated Hamiltonian H^θ as given in Section 5. We follow closely the proof given in [CDKS87]. We repeat the arguments here, since we are not aware of a proof in the quantum graph case. In addition, we need a stronger assertion, namely an explicit control of the norm of the resolvent $R^\theta := (H^\theta + 1)^{-1}$ as a map from $\mathcal{H}$ to $\mathcal{H}^{1,\theta}$ (cf. Lemma A.5). On the quantum graph, it is enough to show that the operator is bounded, but on the manifold, we need a uniform control of the constant with respect to the shrinking parameter ε. Since the proof of the analyticity and the resolvent estimate is basically the same, we state it in an abstract way for both models at the same time. The main idea in showing the analyticity is to compare H^θ with the decoupled operator $H^{\theta,\mathrm{D}}$ where the decoupling is achieved via an additional Dirichlet condition at the boundary between the interior and exterior part.

We first need some notation. Assume that the Hilbert space splits into an interior and exterior part, namely $\mathcal{H} = \mathcal{H}_{\mathrm{int}} \oplus \mathcal{H}_{\mathrm{ext}}$ (cf. Section 5.1 and (5.2)).

Notation C.1. We constantly use the subscripts $(\cdot)_{\mathrm{int}}$ and $(\cdot)_{\mathrm{ext}}$ for the *interior* and *exterior part*, respectively. Similarly, quadratic forms, operators and functions with these subscripts are understood in the obvious way. In addition, $\bullet$ stands either for "int" or "ext".

Decomposition and quadratic forms. We will make common use of minimal and maximal quadratic form domains which corresponds to Neumann and Dirichlet boundary conditions for the associated operators. Note that the classical Neumann boundary conditions appear only in the domain of the associated operator (for details, see e.g. [RS80]).

Suppose that $\mathfrak{h}$ is a quadratic form of the magnetic Hamiltonian on $\mathcal{H}$ (either on the quantum graph or the manifold). Denote by $\mathcal{H}_\bullet = \mathsf{L}_2(X_\bullet)$ the corresponding subspace of $\mathcal{H}$ for $X_\bullet = X_{\mathrm{int}}$ or X_{ext}.

The quadratic form $\mathfrak{h}_\bullet^{\mathrm{N}} = \mathfrak{h}_\bullet$ with domain $\mathcal{H}_\bullet^{1,\mathrm{N}} = \mathcal{H}_\bullet^1$ associated to the Neumann operator on $X_\bullet$ consists of those functions $u \in \mathcal{H}_\bullet$ such that $\mathfrak{h}_\bullet(u)$ is defined and finite. In particular, we have

$$\mathcal{H}_\bullet^1 = \mathsf{H}^1(X_{0,\bullet}) \qquad \text{and} \qquad \mathcal{H}_\bullet^1 = \mathsf{H}^1(X_{\varepsilon,\bullet}) \tag{C.1}$$

where the Sobolev spaces $\mathsf{H}^1(X_{\varepsilon,\bullet})$ are defined in (3.4) and (4.5) on the quantum graph and the manifold, respectively.

We often omit the superscript $(\cdot)^{\mathrm{N}}$ on the Neumann quadratic form and its domain (when the boundary does not separate the domain into separate parts), since Neumann boundary conditions mean no restriction on the quadratic form domain.

We assume that the quadratic forms can be written as

$$\mathfrak{h}_{\mathrm{int}}(u) = \|\partial_{\mathrm{int}}(\chi u)\|^2 + \mathfrak{h}^{\perp}_{\mathrm{int}}(\chi u) + \mathfrak{h}^{\mathrm{rest}}_{\mathrm{int}}(u) \tag{C.2a}$$

$$\mathfrak{h}_{\mathrm{ext}}(u) = \|\partial_{\mathrm{ext}} u\|^2 + \mathfrak{h}^{\perp}_{\mathrm{ext}}(u) \tag{C.2b}$$

for $u = u_{\mathrm{int}} \oplus u_{\mathrm{ext}} \in \mathcal{H}^{1,\mathrm{N}}$ where $\partial_\bullet u = \partial_x u$ is the derivative w.r.t. the coordinate x (oriented towards infinity on the external edge) and. In addition, χ is assumed to be a smooth cut-off function such that $\chi = 1$ near the common boundary and equals 0 away from it. Furthermore, we assume that $\mathfrak{h}^{\mathrm{rest}}_{\mathrm{int}}(u) = 0$ for functions with support near the boundary. To be more concrete, we give the expressions in our examples: On the manifold we have

$$\mathfrak{h}^{\perp}_{\varepsilon,\bullet}(u) = \frac{1}{\varepsilon^2}\int_{X_{\varepsilon,\bullet}} |\mathrm{d}_F u|^2_h \,\mathrm{d}X_\varepsilon \qquad \text{and} \qquad \mathfrak{h}^{\mathrm{rest}}_{\varepsilon,\mathrm{int}}(u) = \|\mathrm{d}((1-\chi)u)\|^2_{X_{\varepsilon,\mathrm{int}}}, \tag{C.3a}$$

where we can choose χ independently of ε in the manifold case due to our decomposition *away* from the internal vertices: Namely, Γ_ε has distance ℓ_0 from any internal vertex due to our assumptions in Section 5.1. On the quantum graph, we simply have

$$\mathfrak{h}^{\perp}_{0,\bullet}(f) = 0 \qquad \text{and} \qquad \mathfrak{h}^{\mathrm{rest}}_{0,\mathrm{int}}(f) = \|(1-\chi)f)'\|^2_{X_{0,\mathrm{int}}} \tag{C.3b}$$

The quadratic form $\mathfrak{h}^{\mathrm{D}}_\bullet$ with domain $\mathcal{H}^{1,\mathrm{D}}$ associated to the Dirichlet operator on $X_\bullet$ is defined as restriction of $\mathfrak{h}_\bullet$ on the subset of functions in $\mathcal{H}^{1,\mathrm{N}}_\bullet$ vanishing on the common boundary Γ of $X_\bullet$ and $X \setminus X_\bullet$.

Notation C.2. The superscripts $(\cdot)^{\mathrm{N}}$ and $(\cdot)^{\mathrm{D}}$ will always refer to *Neumann* and *Dirichlet boundary conditions* on the common boundary Γ of the interior and exterior part, respectively.

We also need the corresponding forms on the whole space, namely

$$\mathcal{H}^{1,\mathrm{N}} := \mathcal{H}^{1,\mathrm{N}}_{\mathrm{int}} \oplus \mathcal{H}^{1,\mathrm{N}}_{\mathrm{ext}} \qquad \text{and} \qquad \mathcal{H}^{1,\mathrm{D}} := \mathcal{H}^{1,\mathrm{D}}_{\mathrm{int}} \oplus \mathcal{H}^{1,\mathrm{D}}_{\mathrm{ext}} \tag{C.4a}$$

together with their natural quadratic forms

$$\mathfrak{h}^{\mathrm{N}} := \mathfrak{h}^{\mathrm{N}}_{\mathrm{int}} \oplus \mathfrak{h}^{\mathrm{N}}_{\mathrm{ext}} \qquad \text{and} \qquad \mathfrak{h}^{\mathrm{D}} := \mathfrak{h}^{\mathrm{D}}_{\mathrm{int}} \oplus \mathfrak{h}^{\mathrm{D}}_{\mathrm{ext}}. \tag{C.4b}$$

Notation C.3. For a non-negative quadratic form (i.e., $\mathfrak{h}(u) \ge 0$ for $u \in \operatorname{dom}\mathfrak{h}$) we define the associated natural norm as

$$\|u\|_1 := \bigl(\|u\|^2 + \mathfrak{h}(u)\bigr)^{1/2}. \tag{C.5}$$

We refer to $\mathcal{H}^1 = \operatorname{dom}\mathfrak{h}$ with norm $\|\cdot\|_1$ as *space of order* 1. We use similar notation for the various quadratic forms defined in this section.

In our application, all the quadratic forms defined here, are closed, so that the corresponding scales of order 1 are indeed Hilbert spaces.

Boundary maps. We also need the boundary maps in order to express the various boundary conditions. It will be convenient to use an ε-independent space in the manifold case:

Notation C.4. Let $S_\bullet \colon \mathcal{H}^1_\bullet \longrightarrow \mathcal{G}$ be the *the restriction* or *boundary map* onto the boundary $\Gamma = \partial X_\bullet$ given by

$$u \longmapsto \{u(v)\}_{v \in \Gamma_0} \qquad \text{and} \qquad u \longmapsto \varepsilon^{m/2} u{\restriction}_{\Gamma_1} \tag{C.6}$$

where

$$\mathcal{G} = \ell_2(\Gamma_0) \cong \mathbb{C}^{|\Gamma_0|} \qquad \text{and} \qquad \mathcal{G} = \mathsf{L}_2(\Gamma_1), \tag{C.7}$$

in the quantum graph and manifold cases, respectively. Here, Γ_1 is the rescaled boundary Γ_ε with $\varepsilon = 1$. Note that the number of boundary vertices equals the number of external edges which we assumed to be finite in (H$_0$3).

In the manifold case, we also have a scale on the boundary Hilbert space $\mathcal{G} = \mathsf{L}_2(\Gamma_1)$, namely $\mathcal{G}^k = \mathsf{H}^k(\Gamma_1)$. In particular, we can define the dual $\mathcal{G}^{-1/2}$ of $\mathcal{G}^{1/2}$ with respect to the pairing $(\cdot,\cdot)_\mathcal{G} \colon \mathcal{G}^{-k} \times \mathcal{G}^k \longrightarrow \mathbb{C}$. In addition to the boundary map $S_\bullet = S^1_\bullet$ we need a similar map of order 0, namely

$$S^0_\bullet \colon \mathcal{H}_\bullet \longrightarrow \mathcal{G}^{-1/2}, \quad u \longmapsto \varepsilon^{m/2} u{\restriction}_{\Gamma_1}. \tag{C.8}$$

Note that on the quantum graph case, there is no such scale since $\mathcal{G} = \ell_2(\Gamma_0) \cong \mathbb{C}^{|\Gamma_0|}$. This means in particular, that $\mathcal{G}^* = \mathcal{G}$ and $S^*_\bullet$ is a map from $\mathcal{G}$ into $\mathcal{H}^{-1}_\bullet$.

We have to make sure that $\|S_\bullet\|$ and $\|S^0_\bullet\|$ do not depend on ε in the manifold case:

Lemma C.5. *The norm of the restriction maps*

$$S_\bullet = S^1_\bullet \colon \mathcal{H}^{1,\mathrm{N}}_\bullet = \mathsf{H}^1(X_{\varepsilon,\bullet}) \longrightarrow \mathcal{G}^{1/2} = \mathsf{H}^{1/2}(\Gamma_1), \qquad u \longmapsto \varepsilon^{m/2} u{\restriction}_{\Gamma_1} \tag{C.9}$$

$$S^0_\bullet \colon \mathcal{H} = \mathsf{L}_2(X_{\varepsilon,\bullet}) \longrightarrow \mathcal{G}^{-1/2} = \mathsf{H}^{-1/2}(\Gamma_1), \qquad u \longmapsto \varepsilon^{m/2} u{\restriction}_{\Gamma_1} \tag{C.10}$$

are bounded independently of ε.

Proof. We have $\|S^1_\bullet u\| = \varepsilon^{m/2} \|\tau^1_\bullet \iota^1_\varepsilon u\|$ where $\tau^1_\bullet \colon \mathsf{H}^1(X_{1,\bullet}) \longrightarrow \mathsf{H}^{1/2}(\Gamma_1)$ is the trace map, $\mathsf{H}^1(X_{1,\bullet})$ is the Sobolev space $\mathsf{H}^1(X_{\varepsilon,\bullet})$ with $\varepsilon = 1$ fixed and $\iota^1_\varepsilon \colon \mathsf{H}^1(X_{\varepsilon,\bullet}) \longrightarrow \mathsf{H}^1(X_{1,\bullet})$. Now, $\|\iota^1_\varepsilon\| = \varepsilon^{-m/2}$, so that $\|S^1_\bullet\| \le \|\tau^1_\bullet\|$. Clearly, the latter norm is independent of ε. Similarly, $S^0_\bullet$ is the composition of $\tau^0_\bullet$ and ι^0_ε with $\tau^0_\bullet \colon \mathsf{L}_2(X_{1,\bullet}) \longrightarrow \mathsf{H}^{-1/2}(\Gamma_1)$ and $\iota^0_\varepsilon \colon \mathsf{L}_2(X_{\varepsilon,\bullet}) \longrightarrow \mathsf{L}_2(X_{1,\bullet})$. Again, $\|\iota^0_\varepsilon\| = \varepsilon^{-m/2}$ and the result follows. □

Coupled quadratic forms. With the help of the boundary maps, we can express the Dirichlet quadratic form domain as

$$\mathcal{H}^{1,\mathrm{D}}_\bullet = \ker S_\bullet \subset \mathcal{H}^{1,\mathrm{N}}_\bullet. \tag{C.11}$$

We define the *undilated* Hamiltonian via its quadratic form $\mathfrak{h}$ on

$$\mathcal{H}^1 := \{\, u \in \mathcal{H}^{1,\mathrm{N}} \mid S_{\mathrm{ext}} u = S_{\mathrm{ext}} u \,\} = \ker(-S_{\mathrm{int}} + S_{\mathrm{ext}}) \tag{C.12a}$$

with form given by

$$\mathfrak{h}(u) := \mathfrak{h}_{\mathrm{int}}(u) + \mathfrak{h}_{\mathrm{ext}}(u), \tag{C.12b}$$

where

$$(-S_{\mathrm{int}} + S_{\mathrm{ext}})u := -S_{\mathrm{int}} u_{\mathrm{int}} + S_{\mathrm{ext}} u_{\mathrm{ext}}$$

for $u = u_{\mathrm{int}} \oplus u_{\mathrm{ext}}$. We will often omit the subscripts $u = u_{\mathrm{int}}$ etc. if they are clear from the context.

Similarly, we define the *dilated* quadratic form $\mathfrak{h}^\theta$, for the moment for *real* θ only, on the space

$$\mathcal{H}^{1,\theta} := \{\, u \in \mathcal{H}^{1,\mathrm{N}} \mid S_{\mathrm{ext}} u = \mathrm{e}^{-\theta/2} S_{\mathrm{ext}} u \,\} = \ker(-S_{\mathrm{int}} + \mathrm{e}^{-\theta/2} S_{\mathrm{ext}}) \tag{C.13}$$

and we set

$$\mathfrak{h}^\theta(u) = \mathfrak{h}_{\mathrm{int}}(u) + \mathfrak{h}^\theta_{\mathrm{ext}}(u), \tag{C.14a}$$

$$\mathfrak{h}^\theta_{\mathrm{ext}}(u) = \mathrm{e}^{-2\theta} \|\partial_{\mathrm{ext}} u\|^2 + \mathfrak{h}^\perp_{\mathrm{ext}}(u) \tag{C.14b}$$

(cf. (C.2)). Note that the dilated form $\mathfrak{h}^\theta$ agrees with the free form $\mathfrak{h}$ if $\theta = 0$.

The various quadratic form domains satisfy the following inclusions, also called *Dirichlet-Neumann bracketing*, namely,

$$\mathcal{H}^{1,\mathrm{D}} := \mathcal{H}^{1,\mathrm{D}}_{\mathrm{int}} \oplus \mathcal{H}^{1,\mathrm{D}}_{\mathrm{ext}} \subset \mathcal{H}^{1,\theta} \subset \mathcal{H}^{1,\mathrm{N}}_{\mathrm{int}} \oplus \mathcal{H}^{1,\mathrm{N}}_{\mathrm{ext}} =: \mathcal{H}^{1,\mathrm{N}} \tag{C.15}$$

If we equip the spaces with their canonical quadratic form norm as in Notation C.3, these inclusions are also bounded and induce bounded maps on the corresponding dual spaces (cf. (A.1)), e.g.

$$\mathcal{H}^{1,\mathrm{D}} \xrightarrow{\iota_{1,\theta}} \mathcal{H}^{1,\theta} \qquad \text{and} \qquad \mathcal{H}^{-1,\theta} \xrightarrow{\iota_{-1,\theta}} \mathcal{H}^{-1,\mathrm{D}} \tag{C.16}$$

where $\iota_{-1,\theta} = (\iota_{1,\overline{\theta}})^*$.

The following estimate follows immediately from (C.2):

Lemma C.6. *We have*

$$\|\partial_\bullet u\|^2 \le \mathfrak{h}_\bullet(u) \tag{C.17}$$

for functions $u \in \mathcal{H}^1_\bullet$ (with support close to the boundary if $\bullet = \mathrm{int}$). In particular, the operator[9] *$\partial_\bullet \colon \mathcal{H}^1_\bullet \longrightarrow \mathcal{H}_\bullet$ has a norm bounded by* 1*.*

Associated operators and Sobolev spaces of second order. We denote the Dirichlet operator on $X_\bullet$ corresponding to the Dirichlet quadratic form $\mathfrak{h}^{\mathrm{D}}_\bullet$ by $H^{\mathrm{D}}_\bullet$ with domain $\mathcal{H}^{2,\mathrm{D}}_\bullet$.

If we are on the exterior part, we also need the dilated version, namely we denote by $H^{\theta,\mathrm{D}}_{\mathrm{ext}}$ the operator associated to the form $\mathfrak{h}^{\theta,\mathrm{D}}_{\mathrm{ext}}$ which is the restriction of $\mathfrak{h}^\theta_{\mathrm{ext}}$ (cf. (C.14b)) onto $\mathcal{H}^{1,\mathrm{D}}$. Note that the domains of $H^{\mathrm{D}}_{\mathrm{ext}}$ and $H^{\theta,\mathrm{D}}_{\mathrm{ext}}$ agree and that the operators agree for $\theta = 0$. The *decoupled* operator $H^{\theta,\mathrm{D}}$ is then the direct sum, namely $H^{\theta,\mathrm{D}} = H^{\mathrm{D}}_{\mathrm{int}} \oplus H^{\theta,\mathrm{D}}_{\mathrm{ext}}$.

Before defining the *coupled* dilated operators we introduce minimal and maximal (non-selfadjoint) operators with respect to the common boundary Γ of the internal and external part.

Let $H^{\theta,\min}_\bullet$ be the restriction of $H^{\theta;\mathrm{D}}_\bullet$ to

$$\mathcal{D}^2_\bullet := \{\, u \in \mathcal{H}^{2,\mathrm{D}}_\bullet \mid S_\bullet \partial u = 0 \,\} \tag{C.18}$$

(i.e., the intersection of the Dirichlet and Neumann operator domain) and set

$$\mathcal{D}^2 := \mathcal{D}^2_{\mathrm{int}} \oplus \mathcal{D}^2_{\mathrm{ext}} \qquad \text{and} \qquad H^{\theta,\min} := H^{\min}_{\mathrm{int}} \oplus H^{\theta,\min}_{\mathrm{ext}}. \tag{C.19}$$

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[9] Strictly speaking, ∂_{int} is defined only on the subset of functions with support close to the boundary.

The corresponding maximal operators are defined by

$$H_\bullet^{\theta,\max} := (H_\bullet^{\theta,\min})^*, \qquad H^{\theta,\max} := H_{\mathrm{int}}^{\max} \oplus H_{\mathrm{ext}}^{\theta,\max} \tag{C.20}$$

with domains

$$\mathcal{W}_\bullet^2 := \operatorname{dom} H_\bullet^{\max}, \qquad \mathcal{W}^2 := \mathcal{W}_{\mathrm{int}}^2 \oplus \mathcal{W}_{\mathrm{ext}}^2. \tag{C.21}$$

independent of θ also for $\bullet = \mathrm{ext}$.

Since we assumed that the magnetic potential on the manifold α_ε is smooth and that a on the graph is smooth inside each edge, we can characterise $\mathcal{W}_\bullet^2$ via the Sobolev spaces already introduced earlier, namely $\mathcal{W}_\bullet^2 = \mathsf{H}^2(X_{\varepsilon,\bullet})$ for $\varepsilon \geq 0$ (see (5.6) for $\varepsilon = 0$ and (5.7) for $\varepsilon > 0$).

We now define the *dilated* operator H^θ as the operator associated to the quadratic form $\mathfrak{h}^\theta$ for *real* θ. Its domain is given by

$$\begin{aligned}\mathcal{H}^{2,\theta} := \{\, u \in \mathcal{W}^2 \mid S_{\mathrm{ext}} u = \mathrm{e}^{-\theta/2} S_{\mathrm{ext}} u,\ S_{\mathrm{ext}} \partial_{\mathrm{int}} u = \mathrm{e}^{-3\theta/2} S_{\mathrm{ext}} \partial_{\mathrm{ext}} u \,\} \\ = \ker(-S_{\mathrm{int}} + \mathrm{e}^{-\theta/2} S_{\mathrm{ext}}) \cap \ker[(-S_{\mathrm{int}} + \mathrm{e}^{-3\theta/2} S_{\mathrm{ext}})\partial]\end{aligned} \tag{C.22}$$

where $\partial := \partial_{\mathrm{int}} \oplus \partial_{\mathrm{ext}}$. In our application, H^θ acts formally on exterior edges as in (5.4). As before, we denote by $H = H^0$ and $\mathcal{H}^2 = \mathcal{H}^{2,0}$ the undilated operator and domain, respectively.

We will need in Lemma C.13 the following facts from elliptic regularity. In the manifold case, we have the *continuous* embeddings

$$\iota_{\mathrm{ell}} \colon \mathcal{H}^2 \longrightarrow \mathcal{W}^2 \quad \text{and} \quad \iota_{\mathrm{ell}} \colon \mathcal{H}^{2,\mathrm{D}} \longrightarrow \mathcal{W}^2 \tag{C.23}$$

where the space $\mathcal{W}^2$ is endowed with a suitable Sobolev norm. On the quantum graph, such estimates are almost trivial (under Assumption ($\mathrm{H}_0 2$)) and on the manifold, we refer e.g. to [Aub76, Prop. 3]. Note that on the manifold, we need to consider this embedding only for *fixed* ε; we do not need a global constant for all $\varepsilon > 0$. In general, ι_{ell} has a finite norm *depending on* ε, since X_ε is of bounded geometry by Assumption ($\mathrm{H}_\varepsilon 3$) with constants depending on ε and since we imposed no bounds on (general) derivatives of the magnetic vector potential α_ε.

Analyticity. The first aim in this section is to show that the family $\{H^\theta\}_\theta$ with domain $\mathcal{H}^{2,\theta}$ can be extended analytically into the complex strip S_ϑ (i.e., $|\operatorname{Im}\theta| < \vartheta/2$) and has spectrum contained in Σ_ϑ (i.e., each z in the spectrum satisfies $|\arg z| \leq \vartheta$). Analyticity here means, that the resolvent

$$R^\theta(z) := (H^\theta - z)^{-1} \tag{C.24}$$

for $z \notin \Sigma_\theta$ depends analytically on θ as operator in $\mathcal{H}$.

To this aim we introduce the decoupled dilated operator

$$H^{\theta,\mathrm{D}} := H_{\mathrm{int}}^{\mathrm{D}} \oplus H_{\mathrm{ext}}^{\theta,\mathrm{D}}, \tag{C.25}$$

where $H_{\mathrm{int}}^{\mathrm{D}}$ and $H_{\mathrm{ext}}^{\theta,\mathrm{D}}$ denotes the operator with *Dirichlet* boundary conditions at the boundary Γ of X_{int} and X_{ext}. We are now able to state the first lemma on analytic dependence.

Lemma C.7. *The decoupled dilated Hamiltonian* $\{H^{\theta,\mathrm{D}}\}_\theta$ *extends to an analytic family of type A into the strip* $\theta \in S_\vartheta = \{\, \theta \in \mathbb{C} \mid 2|\operatorname{Im}\theta| < \vartheta \,\}$. *In addition,* $\sigma(H^{\theta,\mathrm{D}})$ *is contained in the sector* $\Sigma_\vartheta = \{\, z \in \mathbb{C} \mid |\arg z| \leq \vartheta \,\}$ *and therefore a self-adjoint, uniformly* ϑ*-sectorial family in the sense of Definition A.1*

Proof. The proof is almost obvious due to the explicit expression of the operators and the fact that

$$\sigma(H^{\theta,\mathrm{D}}) = \sigma(H^{\mathrm{D}}_{\mathrm{int}}) \cup \sigma(H^{\theta,\mathrm{D}}_{\mathrm{ext}}) \subset \Sigma_\vartheta$$

since $H^{\theta,\mathrm{D}}$ has numerical range in the sector Σ_ϑ by (C.14). Note that the domain of $H^{\theta,\mathrm{D}}$ is independent of θ due to the decoupling. ☐

Now we are going to extend the definition of the coupled operators H^θ for real θ to the complex strip S_ϑ. We follow closely [CDKS87]. We want to compare the resolvent $R^\theta(z) := (H^\theta - z)^{-1}$ with the decoupled resolvent $R^{\theta,\mathrm{D}}(z) := (H^{\theta,\mathrm{D}}(z))^{-1}$. To do so, want to express the difference $R^\theta(z) - R^{\theta,\mathrm{D}}(z)$ in terms of an explicit sequence of bounded and analytic operators, for the moment for *real* θ only. Since this expression will serve as generalisation for *complex* θ, we formulate it already for the complex case in order to formally respect analyticity.

Denote by

$$\hat{R}(z) \colon \mathcal{H}^{-1} \longrightarrow \mathcal{H}^1 \tag{C.26}$$

the undilated resolvent $(H - z)^{-1}$ of H as an operator in the natural scale of Hilbert spaces $\mathcal{H}^k$ associated to the self-adjoint operator H (cf. Section A.1). Since on $\mathcal{H}^1$, the boundary values on the internal and external part agree by (C.12), we can define a bounded map

$$S \colon \mathcal{H}^1 \longrightarrow \mathcal{G}, \quad f \mapsto S_{\mathrm{int}} f = S_{\mathrm{ext}} f \tag{C.27}$$

with dual $S^* \colon \mathcal{G} \longrightarrow \mathcal{H}^{-1}$.

The following arguments for the quantum graph and the manifold differ slightly due to the fact that the boundary space $\mathcal{G}$ allows a natural scale of Sobolev spaces only on the manifold.

We start on the quantum graph and define a bounded operator

$$B^\theta(z) := S^{1,\theta} \partial R^{\theta,\mathrm{D}}(z) \colon \mathcal{H} \xrightarrow{R^{\theta,\mathrm{D}}(z)} \mathcal{H}^{2,\mathrm{D}} \overset{\iota_{\mathrm{ell}}}{\hookrightarrow} \mathcal{W}^2 \xrightarrow{\partial} \mathcal{H}^{1,\mathrm{N}} \xrightarrow{S^{1,\theta}} \mathcal{G} \tag{C.28}$$

for $\theta \in S_\vartheta$ and $z \notin \Sigma_\vartheta$ where

$$S^{1,\theta} \colon \mathcal{H}^{1,\mathrm{N}} \longrightarrow \mathcal{G} \cong \mathbb{C}^{|\Gamma_0|}, \qquad f \longmapsto -S_{\mathrm{int}} f + \mathrm{e}^{-3\theta/2} S_{\mathrm{ext}} f,$$
$$\partial := \partial_{\mathrm{int}} \oplus \partial_{\mathrm{ext}} \colon \mathcal{W}^2 \longrightarrow \mathcal{H}^{1,N}.$$

For further purposes, we need to express the adjoint operator as a solution operator.

Lemma C.8. *On the quantum graph, the adjoint* $(B^{\overline{\theta}}(\overline{z}))^* \colon \mathcal{G} \longrightarrow \mathcal{H}$ *of* $B^\theta(z)$ *is given as follows: If* $f = (B^{\overline{\theta}}(\overline{z}))^* F$ *with* $F \in \mathcal{G}$*, then* $f \in \mathcal{W}^2$ *and* f *is the unique solution of the Dirichlet problem*

$$(H^{\theta,\max} - z) f = 0, \quad f_{\mathrm{int}}(v) = F(v), \quad f_{\mathrm{ext}}(v) = \mathrm{e}^{\theta/2} F(v) \tag{C.29}$$

for all boundary vertices $v \in \Gamma_0$*. In particular,* f *satisfies all inner boundary conditions and the jump condition along* Γ_0*.*

Proof. Let $\widetilde{g} \in \mathcal{H}$, $F \in \mathcal{G}$ and denote $g := (H^{\overline{\theta},\mathrm{D}} - z)^{-1} \widetilde{g}$. Then

$$\langle \widetilde{g}, f \rangle_{\mathcal{H}} = \big((B^{\overline{\theta}}(z))^* \widetilde{g}, F \big)_{\mathcal{G}} = \sum_{v \in \Gamma_0} \big(-\overline{g}'_{\mathrm{int}}(v) + \mathrm{e}^{-3\theta/2} \overline{g}'_{\mathrm{ext}}(v) \big) F(v).$$

In particular, $0 = \langle \widetilde{g}, f \rangle = \langle (H^{\overline{\theta},\mathrm{D}} - \overline{z}) g, f \rangle$ for functions $\widetilde{g}$ with support away from the boundary vertices. Choosing $\widetilde{g} \in \mathsf{C}^\infty_{\mathrm{c}}(e)$, and since we assumed that the potentials a_e and q_e are smooth inside an internal edge e we conclude that f_e is smooth as solution of the ODE $(-\partial_x + \mathrm{i}a_e)(\partial - \mathrm{i}a_e)f_e + q_e f_e = z f_e$ and $-f_e'' = z f_e$ on external edges. To conclude that f also satisfies all inner boundary conditions we use the arguments of [KoS99, Lem. 2.2]. In particular, we conclude that $f \in \mathcal{W}^2$ and $(H^{\theta,\max} - z)f = 0$. Now, using general functions $\widetilde{g}$, we have

$$\langle \widetilde{g}, f\rangle = \langle (H^{\overline{\theta},\mathrm{D}} - \overline{z})g, f\rangle = \sum_{v\in\Gamma_0} \bigl(-\overline{g}'_{\mathrm{int}}(v) f_{\mathrm{int}}(v) + \mathrm{e}^{-2\theta}\overline{g}'_{\mathrm{ext}}(v) f_{\mathrm{ext}}(v)\bigr)$$

since g vanishes on Γ_0. It follows that $F(v) = f_{\mathrm{int}}(v)$ and $f_{\mathrm{ext}}(v) = \mathrm{e}^{\theta/2} f_{\mathrm{int}}(v)$ for boundary vertices $v \in \Gamma_0$. □

Lemma C.9. *We can factorize the adjoint map on the quantum graph by the bounded maps*

$$(B^{\overline{\theta}}(\overline{z}))^* = \bigl(\mathrm{id}_{\mathcal{W}^2} - (H^{\theta,\mathrm{D}} - z)^{-1}(H^{\theta,\max} - z)\bigr)E^\theta \colon \mathcal{G} \longrightarrow \mathcal{W}^2, \tag{C.30}$$

where

$$E^\theta \colon \mathcal{G} \longrightarrow \mathcal{W}^2$$

is a bounded extension map such that $\widetilde{f} := E^\theta F$ *is constant near* Γ_0 *and* $F(v) = \widetilde{f}_{\mathrm{ext}}(v) = \mathrm{e}^{\theta/2}\widetilde{f}_{\mathrm{int}}(v)$ *for boundary vertices* $v \in \Gamma_0$. *In particular,*

$$(B^{\overline{\theta}}(\overline{z}))^* \colon \mathcal{G} \longrightarrow \mathcal{W}^2 \tag{C.31}$$

is a bounded map and depends analytically on θ *(and* z*). Finally, the dual of* (C.31), *namely*

$$(B^{\overline{\theta}}(\overline{z}))^{**} \colon \mathcal{W}^{-2} \longrightarrow \mathcal{G}, \tag{C.32}$$

is bounded and an extension of $B^{\overline{\theta}}(\overline{z}) \colon \mathcal{H} \longrightarrow \mathcal{G}$.

The extension map can for example be defined as

$$E^\theta F(x) := \chi^\theta(x) F(v)$$

near the boundary vertex v where χ^θ is a smooth map with compact support and derivatives bounded in terms of $1/\ell_0$ (cf. ($\mathrm{H}_0$2)) such that $\chi^\theta_{\mathrm{int}} = 1$ and $\chi^\theta_{\mathrm{ext}} = \mathrm{e}^{\theta/2}$ near v.

Proof. A direct calculation shows that (C.30) defines the (unique) solution of the Dirichlet problem (see e.g. [HP06, Lem. D.1]). The boundedness follows since the maps in the factorization (C.30) are bounded. The analyticity is a consequence of the explicit form of (C.30) and (5.4a). Note that no space in the factorization depend on θ. □

Lemma C.10. *For* $\theta \in S_\vartheta$ *and* $z \notin \Sigma_\vartheta$, *the map*

$$W^\theta(z) := \bigl(B^{\overline{\theta}}(\overline{z})\bigr)^* S\widehat{R}(z)S^* B^\theta(z) \colon \mathcal{H} \longrightarrow \mathcal{H} \tag{C.33}$$

on the quantum graph is bounded and analytic. Furthermore, it is also bounded and analytic considered as map $\widetilde{W^\theta}(z)$ *from* $\mathcal{H}$ *into* $\mathcal{W}^2$.

For real θ, *we have*

$$W^\theta(z) = R^\theta(z) - R^{\theta,\mathrm{D}}(z). \tag{C.34}$$

Proof. The boundedness and analyticity follows from the preceding two lemmas. The proof of (C.34) is essentially the same as in [CDKS87, Lem. A.2] and basically an application of Greens formula for *real* θ. □

On the manifold, we have a similar assertion:

Lemma C.11. *For $\theta \in S_\vartheta$ and $z \notin \Sigma_\vartheta$, the map $W^\theta(z)$ from $\mathcal{H}$ into $\mathcal{H}$ defined as in* (C.33) *but now with*

$$B^\theta(z) := S^{0,\theta}\partial R^{\theta,\mathrm{D}}(z) \colon \mathcal{H} \xrightarrow{R^{\theta,\mathrm{D}}(z)} \mathcal{H}^{1,\mathrm{D}} \xrightarrow{\partial} \mathcal{H} \xrightarrow{S^{0,\theta}} \mathcal{G}^{-1/2}, \tag{C.35}$$

$$(B^{\overline{\theta}}(\overline{z}))^* \colon \mathcal{G}^{1/2} \longrightarrow \mathcal{H}$$

$$S^{0,\theta} \colon \mathcal{H} \longrightarrow \mathcal{G}^{-1/2}, \quad f \longmapsto -S^0_{\mathrm{int}} f + \mathrm{e}^{-3\theta/2} S^0_{\mathrm{ext}} f,$$

$$\partial := \partial_{\mathrm{int}} \oplus \partial_{\mathrm{ext}} \colon \mathcal{H}^{1,N} \longrightarrow \mathcal{H},$$

$$S \colon \mathcal{H}^1 \longrightarrow \mathcal{G}^{1/2}, \quad f \mapsto S_{\mathrm{int}} f = S_{\mathrm{ext}} f, \qquad S^* \colon \mathcal{G}^{-1/2} \longrightarrow \mathcal{H}^{-1}.$$

on the quantum graph is bounded and analytic. In addition, $\|W^\theta(z)\|$ is bounded w.r.t. ε.

For real θ *and $z \notin \Sigma_\vartheta$, we have again* (C.34).

Proof. Again, the boundedness and analyticity follows from the explicit representation. The ε-independence of the norm follows from Lemmas C.5 and C.6. The last assertion is again similar to the proof of [CDKS87, Lem. A.2]. ☐

As in [CDKS87], we now define the operator $R^\theta(z)$ also for *complex* $\theta \in S_\vartheta$ via the formula (C.34), i.e.,

$$R^\theta(z) := W^\theta(z) + R^{\theta,\mathrm{D}}(z) \colon \mathcal{H} \longrightarrow \mathcal{H}. \tag{C.36}$$

In particular, we have:

Lemma C.12.

(i) *For $z \notin \Sigma_\vartheta$, the family of $R^\theta(z)$ is analytic in $\theta \in S_\vartheta$.*
(ii) *The operators $R^\theta(z)$ satisfy the resolvent equation for $z \notin \Sigma_\vartheta$.*
(iii) *The kernel of $R^\theta(z)$ is trivial.*

In particular, $R^\theta(z)$ is the resolvent of an operator

$$H^\theta u := (R^\theta)^{-1} u - u, \qquad u \in \operatorname{dom} H^\theta := \mathcal{H}^{2,\theta} := R^\theta(\mathcal{H}) \tag{C.37}$$

where $R^\theta := R^\theta(-1)$ and the family $\{H^\theta\}_\theta$ is self-adjoint with spectrum contained in the sector Σ_ϑ. Finally, the norm of R^θ as operator on $\mathcal{H} = \mathsf{L}_2(X_\varepsilon)$ is independent of ε in the manifold case.

Proof. (i) The first assertion follows immediately from Lemma C.7 and the explicit formula for $W^\theta(z)$ given in (C.33) (now for complex θ). (ii) The resolvent equation is fulfilled for real θ, since then, the operator is the resolvent of a (self-adjoint) operator. Due to analyticity, the resolvent equation remains true for all $\theta \in S_\vartheta$. (iii) To prove the third assertion, we claim that

$$\big\langle (H^{\overline{\theta},\mathrm{D}} - \overline{z})\varphi, W^\theta(z) v \big\rangle = 0 \tag{C.38}$$

for all $\varphi \in \mathcal{D}^2$, for all $v \in \mathcal{H}$, all $z \in (\Sigma_\vartheta)^{\mathrm{c}}$ and all $\theta \in S_\vartheta$ where $\mathcal{D}^2$ was defined in (C.19). The claim can easily be seen from

$$\big\langle (H^{\overline{\theta},\mathrm{D}} - \overline{z})\varphi, W^\theta(z) v \big\rangle_{\mathcal{H}} = \big(B^{\overline{\theta}}(\overline{z})(H^{\overline{\theta},\mathrm{D}} - \overline{z})\varphi, S^1 R(z)(S^1)^* B^\theta(z) v \big)_{\mathcal{G}}$$

using (C.33) for complex θ, where $(\cdot,\cdot)_{\mathcal{G}}$ is the pairing $\mathcal{G} \times \mathcal{G}$ in the quantum graph case and $\mathcal{G}^{-1/2} \times \mathcal{G}^{1/2}$ in the manifold case. Now the left-hand side of the last inner product vanishes since $B^{\overline{\theta}}(\overline{z})(H^{\overline{\theta},\mathrm{D}} - \overline{z})\varphi = (-S^0_{\mathrm{int}} + \mathrm{e}^{-3\theta/2} S^0_{\mathrm{ext}})\varphi = 0$ for $\varphi \in \mathcal{D}^2$ and therefore, (C.38) is fulfilled.

Suppose finally, that $R^\theta(z)v = 0$. Then we have

$$0 = \big\langle (H^{\overline{\theta},\mathrm{D}} - \overline{z})\varphi, R^\theta(z)v \big\rangle = \big\langle (H^{\overline{\theta},\mathrm{D}} - \overline{z})\varphi, R^{\theta,\mathrm{D}}(z)v \big\rangle$$

for $\varphi \in \mathcal{D}^2$ due to (C.38) . Since $(H^{\overline{\theta},\mathrm{D}} - \overline{z})(\mathcal{D}^2)$ is dense it follows that $R^{\theta,\mathrm{D}}(z)v = 0$ and therefore $v = 0$ since the latter operator is injective as resolvent.

To conclude we observe that (ii) and (iii) imply that $R^\theta(z)$ is a resolvent (cf. [Ka66, p. 428]) for all $z \notin \Sigma_\vartheta$, i.e., $\sigma(H^\theta) \subset \Sigma_\vartheta$. In addition, the family $\{H^\theta\}_\theta$ is self-adjoint since $(W^\theta)^* = W^{\overline{\theta}}$ and $(R^{\theta,\mathrm{D}})^* = R^{\overline{\theta},\mathrm{D}}$. Finally, $\|R^\theta\| \le \|R^{\theta,\mathrm{D}}\| + \|W^\theta\|$ is bounded independently of ε by the spectral calculus and the preceding lemma. ☐

We finally characterize the domain of H^θ.

Lemma C.13. *For* complex $\theta \in S_\vartheta$, *the domain of* H^θ *is given by* $\mathcal{H}^{2,\theta}$ *as in* (C.22) *and* $H^\theta u = H^{\max} u$ *where* $H^{\max}$ *is defined in* (C.20).

Proof. Let $u = R^\theta v \in \operatorname{dom} H^\theta$ then

$$\big\langle (H^{\overline{\theta},\min} + 1)\varphi, u \big\rangle = \big\langle (H^{\overline{\theta},\mathrm{D}} + 1)\varphi, R^{\theta,\mathrm{D}} v \big\rangle = \langle \varphi, v \rangle$$

for all $\varphi \in \mathcal{D}^2$. In particular, due to the definition of the adjoint operator, we have $u \in \operatorname{dom} H^{\theta,\max}$ and $H^{\theta,\max} u + u = v$. In particular, $H^{\theta,\max} u = (R^\theta)^{-1} u - u$ so that finally, $H^\theta u = H^{\theta,\max} u$ using the definition (C.37).

To show that u belongs to the set defined on the right-hand side of (C.22) we will first show that $R^\theta = R^{\theta,\mathrm{D}} + W^\theta$ defines a bounded and analytic map from $\mathcal{H}$ into $\mathcal{W}^2$ denoted by $\widetilde{R}^\theta$. For $R^{\theta,\mathrm{D}}$ this follows from the sequence of maps

$$\widetilde{R}^{\theta,\mathrm{D}} \colon \mathcal{H} \xrightarrow{R^{\theta,\mathrm{D}}} \mathcal{H}^{2,\mathrm{D}} \xrightarrow{\iota_{\mathrm{ell}}} \mathcal{W}^2 \tag{C.39}$$

on the quantum graph and the manifold (cf. (C.23)) and Lemma C.7).

The fact that $\widetilde{W}^\theta$ defines a bounded and analytic map from $\mathcal{H} = \mathsf{L}_2(X_0)$ into $\mathcal{W}^2$ in the quantum graph case was already shown in Lemma C.10 (setting $z = -1$). On the manifold, we decompose W^θ into the sequence of bounded maps (denoted again by $\widetilde{W}^\theta$ when considered as map $\mathcal{H}$ into $\mathcal{W}^2$)

$$\mathcal{H} \xrightarrow{B^{0,\theta}} \mathcal{G}^{1/2} \xrightarrow{(S^0)^*} \mathcal{H} \xrightarrow{R} \mathcal{H}^2 \xrightarrow{\iota_{\mathrm{ell}}} \mathcal{W}^2 \xrightarrow{S^2} \mathcal{G}^{3/2} \xrightarrow{(B^{-2,\overline{\theta}})^*} \mathcal{W}^2$$

where $S^k \colon \mathcal{H}^k \longrightarrow \mathcal{G}^{k-1/2}$ are the usual trace maps (note that S^{-1} is *not* an inverse of S) and $B^{0,\theta}$ resp. $B^{-2,\overline{\theta}}$ are defined as

$$B^{0,\theta} \colon \mathcal{H} \xrightarrow{R^{\theta,\mathrm{D}}} \mathcal{H}^{2,\mathrm{D}} \xrightarrow{\partial} \mathcal{H}^{1,\mathrm{N}} \xrightarrow{S^1} \mathcal{G}^{1/2}$$
$$B^{-2,\overline{\theta}} \colon \mathcal{W}^{-2} \xrightarrow{(\iota_{\mathrm{ell}})^*} \mathcal{H}^{-2,\mathrm{D}} \xrightarrow{R^{\theta,\mathrm{D}}} \mathcal{H} \xrightarrow{\partial} \mathcal{H}^{-1,\mathrm{N}} \xrightarrow{S^{-1}} \mathcal{G}^{-3/2}.$$

Now all these maps are bounded and also analytic.

Since now in both cases, $\widetilde{R}^\theta$ and $\widetilde{W}^\theta + \widetilde{R}^{\theta,\mathrm{D}}$ are analytic, and since they agree for *real* θ by Lemma C.10 and Lemma C.11, they agree for *all* $\theta \in S_\vartheta$. Finally, since $\widetilde{W}^\theta + \widetilde{R}^{\theta,\mathrm{D}}$ is bounded, the same is true for $\widetilde{R}^\theta$.

To finish the proof that u belongs to the set defined on the right-hand side of (C.22), we note that

$$\big[(-S_{\mathrm{int}}) \oplus \mathrm{e}^{-\theta/2} S_{\mathrm{ext}}\big] \widetilde{R}^\theta \quad \text{and} \quad \big[(-S_{\mathrm{int}}\partial_{\mathrm{int}}) \oplus \mathrm{e}^{-3\theta/2} S_{\mathrm{ext}}\partial_{\mathrm{ext}}\big] \widetilde{R}^\theta$$

are bounded and analytic as operators in $\mathcal{H}$ since the operators in the brackets are bounded and analytic from $\mathcal{W}^2$ to $\mathcal{H}$. These operators vanish for real θ due to (C.22) and vanish therefore for all $\theta \in S_\vartheta$.

For the opposite inclusion, we have to check that a function u belonging to the set on the right-hand side of (C.22) is of the form $u = R^\theta v$. A straightforward calculation using similar arguments as in Lemmas C.10 and C.11 shows that $v := H^{\theta,\max} u + u$ is the right candidate. ☐

Finally, we have shown that H^θ with the above domain $\mathcal{H}^{2,\theta}$ is a self-adjoint, analytic family of operators with spectrum in the sector Σ_ϑ either on the quantum graph as well as on the manifold.

Compatible scales. To conclude this section, we have to check that there is a compatible scale of order 1 w.r.t. the operator $\Delta \geq 0$ in the sense of Definition A.4.

We define the compatibility operators $\{T^\theta\}_\theta$ as

$$T^\theta \colon \mathcal{H} \longrightarrow \mathcal{H}, \qquad T^\theta u := u_{\mathrm{int}} \oplus \mathrm{e}^{-\theta/2} u_{\mathrm{ext}}. \tag{C.40}$$

Clearly, these operators are bounded, invertible and satisfy (A.10). As in the abstract setting of Appendix A we define

$$\mathcal{H}^{1,\theta} := T^{-\theta}(\mathcal{H}^1), \qquad \|u\|_{1,\theta} := \|T^\theta u\|_1 = \|(\Delta+1)^{1/2} T^\theta u\| \tag{C.41}$$

for *complex* $\theta \in S_\vartheta$ and $\mathcal{H}^{-1,\theta} := (\mathcal{H}^{1,\overline{\theta}})^*$. Note that we could also use the undilated *magnetic* Hamiltonian H instead of the Laplacian Δ in (C.41), since their quadratic forms are equivalent (cf. Lemmas 3.3 and 4.6). In addition, the definition (C.41) agrees with the one given in (C.13) (a priori only for *real* θ).

The density assumption Definition A.4 (ii), namely the density of $\mathcal{H}^{2,\theta}$ in $\mathcal{H}^{1,\theta}$ (or equivalently, that $T^\theta(\mathcal{H}^{2,\theta}) \subset T^\theta(\mathcal{H}^{1,\theta}) = \mathcal{H}^1$ is dense in $\mathcal{H}^1$) follows by standard arguments . We omit the proof since we do not need the density in order to apply the results of Appendix B: The only point where the density enters is the unique continuation of $\mathfrak{h}^\theta$ to $\mathcal{H}^{1,\overline{\theta}} \times \mathcal{H}^{1,\theta}$ (cf. Definition A.4 (iv)), but we only need the sesquilinear form on $\mathcal{H}^{1,\overline{\theta}} \times \mathcal{H}^{2,\theta}$ (resp. in the dual form $\widetilde{\mathcal{H}}^{2,\overline{\theta}} \times \widetilde{\mathcal{H}}^{1,\theta}$) (cf. Remark B.4 (iii)).

Next, we have to show that the sesquilinear form $\mathfrak{h}^\theta$ associated to the operator H^θ satisfies Definition A.4 (iv):

Lemma C.14. *The sesquilinear form* $\mathfrak{h}^\theta(f,g) := \langle f, H^\theta g\rangle$, $f \in \mathcal{H}^{1,\overline{\theta}}$, $g \in \mathcal{H}^{2,\theta}$ *extends to a bounded, sesquilinear form*

$$\mathfrak{h}^\theta \colon \mathcal{H}^{1,\overline{\theta}} \times \mathcal{H}^{1,\theta} \longrightarrow \mathbb{C} \tag{C.42}$$

with bound C_1^θ *depending only on* $\operatorname{Re}\theta$ *and the bounds on the potentials* $\|a\|_\infty$ *and* $\|q\|_\infty$.

Proof. Partial integration shows that

$$\langle f, H^\theta g\rangle = \mathfrak{h}_{\mathrm{int}}(f,g) + \mathfrak{h}^\theta_{\mathrm{ext}}(f,g)$$

for $f \in \mathcal{H}^{1,\overline{\theta}}$ and $g \in \mathcal{H}^{2,\theta}$; in particular, there are no boundary terms. On the internal part, we have $\mathfrak{h}_{\mathrm{int}}(f,g) \leq (\mathfrak{h}_{\mathrm{int}}(f)\, \mathfrak{h}_{\mathrm{int}}(g))^{1/2}$ and $\mathfrak{h}_{\mathrm{int}}(f) \leq \|f\|^2_{\mathfrak{h}_{\mathrm{int}}} \approx$

$\|f\|^2_{\mathfrak{d}_{\mathrm{int}}} = \|f_{\mathrm{int}}\|^2_{1,\theta}$ using Lemmas 3.3 and 4.6. On the external part, we estimate $|\mathfrak{h}^\theta_{\mathrm{ext}}(f,g)| \le 2\cosh(\operatorname{Re}\theta)\|f_{\mathrm{ext}}\|_{1,\overline{\theta}}\|g_{\mathrm{ext}}\|_{1,\theta}$ and the claim follows. □

Finally, we have to check the embedding assumption Definition A.4 (iii) or the equivalent resolvent estimate (A.14). We can prove even more, namely the stronger resolvent estimate (A.15). Again, the proofs differ slightly in the quantum graph and manifold case.

Lemma C.15. *There exists a constant* C_2^θ *depending only on* $\operatorname{Re}\theta$, $\|a\|_\infty$ *and* $\|q\|_\infty$ *such that the resolvent* R^θ *extends to a bounded map* $\widehat{R}^\theta \colon \mathcal{H}^{-1,\theta} \longrightarrow \mathcal{H}^{1,\theta}$ *with norm bounded by* C_2^θ. *In particular, we have*

$$\|f\|^2_{1,\theta} \le C_2^\theta \|f\|^2_{2,\theta} \tag{C.43}$$

for all $f \in \mathcal{H}^{2,\theta}$ *on the quantum graph.*

Proof. Similar as in the proof of Lemma C.13 we show that each summand on the RHS of $R^\theta = W^\theta + R^{\theta,\mathrm{D}}$ extends individually to bounded maps $\mathcal{H}^{-1,\theta} \longrightarrow \mathcal{H}^{1,\theta}$. We start with W^θ and note that

$$\widehat{W}^\theta := (B^{\overline{\theta}})^* S\widehat{R}S^* B^\theta \colon \mathcal{H}^{-1,\theta} \longrightarrow \mathcal{H}^{1,\theta} \tag{C.44}$$

is bounded where we consider B^θ as bounded map $\mathcal{G} \longrightarrow \mathcal{W}^{-2}$ together with its dual (cf. Lemma C.9). In that lemma we have also seen that the adjoint $(B^{\overline{\theta}})^*$ maps into $\mathcal{H}^{1,\theta}$. Since the inclusion $\mathcal{W}^2 \cap \mathcal{H}^{1,\theta}$ (with $\mathcal{W}^2$-norm) into $\mathcal{H}^{1,\theta}$ is continuous, $\widehat{W}^\theta$ is bounded. Furthermore, $\widehat{W}^\theta$ and W^θ coincide on the dense set $\mathcal{H}$ and $\widehat{W}^\theta$ is the unique extension of W^θ onto $\mathcal{H}^{-1,\theta}$.

Similarly, $\iota_{1,\theta}\widehat{R}^{\theta,\mathrm{D}}\iota_{-1,\theta}$ is bounded and agrees with $R^{\theta,\mathrm{D}}$ on $\mathcal{H}$: Here, the inclusion map $\iota_{1,\theta}$ defined in (C.16) is bounded (see Lemma 3.3) and $\widehat{R}^{\theta,\mathrm{D}} \colon \mathcal{H}^{-1,\mathrm{D}} \longrightarrow \mathcal{H}^{1,\mathrm{D}}$. Note that the norm of $\iota_{1,\theta}$ depends on $\operatorname{Re}\theta$ and on a resp. q since $H^{\theta,\mathrm{D}}$ is the decoupled *magnetic* Hamiltonian and the norm on $\mathcal{H}^{1,\theta}$ (cf. (C.41)) is defined with the *free* Hamiltonian. Finally we have seen that R^θ extends to a bounded map $\widehat{R}^\theta \colon \mathcal{H}^{-1,\theta} \longrightarrow \mathcal{H}^{1,\theta}$ as desired. □

Recall that due to (H$_\varepsilon$4) and (H$_\varepsilon$6), the magnetic potential α_ε and the electric potential q_ε are bounded independently of the squeezing parameter ε.

Lemma C.16. *There exists a constant* $\widetilde{C}_2^\theta$ *depending only on* $\operatorname{Re}\theta$, $\|\alpha_\varepsilon\|_\infty$ *and* $\|q_\varepsilon\|_\infty$*, and* not *on* ε*, such that the resolvent* R^θ *extends to a bounded map* $\widehat{R}^\theta \colon \mathcal{H}^{-1,\theta} \longrightarrow \mathcal{H}^{1,\theta}$ *with norm bounded by* $\widetilde{C}_2^\theta$. *In particular, we have*

$$\|u\|^2_{1,\theta} \le \widetilde{C}_2^\theta \|u\|^2_{2,\theta} \tag{C.45}$$

for all $u \in \mathcal{H}^{2,\theta}$ *on the manifold.*

Proof. Similar as in the previous proof, we show first that $\widehat{W}^\theta \colon \mathcal{H}^{-1,\mathrm{D}} \longrightarrow \mathcal{H}^{1,\mathrm{D}}$ defined as in (C.44), where now $B^\theta = B^{-1,\theta}$ with

$$B^{-1,\theta} := S^{0,\theta}\partial R^{\theta,\mathrm{D}} \colon \mathcal{H}^{-1,\mathrm{D}} \xrightarrow{R^{\theta,\mathrm{D}}} \mathcal{H}^{1,\mathrm{D}} \xrightarrow{\partial} \mathcal{H} \xrightarrow{S^{0,\theta}} \mathcal{G}^{-1/2}, \tag{C.46}$$

is bounded: From Lemmas C.5 and C.6 we see that the norm of $B^{-1,\theta}$ is bounded independently of ε, and therefore, the same is true for the norm of $\widehat{W}^\theta$. It can easily be seen by the very definition that $\|\widehat{R}^{\theta,\mathrm{D}}\| = 1$. As before, $\widehat{W}^\theta$ and $\widehat{R}^{\theta,\mathrm{D}}$ are extensions of the corresponding operators on $\mathcal{H}$. Finally, the norm of the inclusion map $\iota_{1,\theta}$ depends on $\|\alpha_\varepsilon\|_\infty$ and $\|q_\varepsilon\|_\infty$, but can be bounded *independently* on ε by our assumptions. ☐

Summarizing the results of Lemmas C.12 and C.13 and Lemmas C.15 and C.16 we have shown the following theorem:

Theorem C.17. *The family* $\{H^\theta\}_{\theta\in S_\vartheta}$ *is a self-adjoint, analytic family of operators with domain given by* (C.22)*. In addition,* $\{\mathcal{H}^{1,\theta}\}_\theta$ *is a compatible scale of order* 1 *with respect to the free operator* $\Delta = \Delta_{X_\varepsilon}$ *in both the quantum graph and manifold case. Finally, the constant* $C_0^\theta = \|(H^\theta+1)^{-1}\|$ *in Lemma A.3 and the constant* C_2^θ *in Definition A.4 do not depend on* ε *in the manifold case. In particular, the results of Appendix A and Appendix B apply.*

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