Semiclassical Analysis for Magnetic Scattering by Two Solenoidal Fields

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1. Introduction

In this paper we make a semiclassical study of magnetic scattering by two solenoidal magnetic fields (point–like fields) in two dimensions. Let $\Lambda(x) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the vector potential defined by

$$\Lambda(x) = \left(-x_2/|x|^2, x_1/|x|^2\right) = \left(-\partial_2 \log |x|, \partial_1 \log |x|\right), \quad \partial_j = \partial/\partial x_j,$$

which has the point–like field

$$\nabla \times \Lambda = \left(\partial_1^2 + \partial_2^2\right) \log |x| = \Delta \log |x| = 2\pi\delta(x)$$

at the origin. The quantum particle moving in two solenoidal magnetic fields with centers at e_1 and e_2 is governed by the Hamiltonian

$$H_h = (-ih\nabla - A)^2 = \sum_{j=1,2} \left(-ih\partial_j - a_j\right)^2, \quad 0 < h \ll 1, \tag{1.1}$$

where $A(x) = (a_1(x), a_2(x)) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ takes the form

$$A(x) = \alpha_1 \Lambda(x - e_1) + \alpha_2 \Lambda(x - e_2), \quad e_1 \ne e_2$$

and the real number α_j is called the flux of the field $2\pi\alpha_j\delta(x - e_j)$. The aim is to study the asymptotic behavior of scattering amplitude for H_h in the semiclassical limit $h \to 0$. The potential $A(x)$ does not necessarily fall off rapidly at infinity, but it has the long–range property. In quantum mechanics, vector potentials are known to have a direct significance to the motion of particles, although the motion in classical mechanics is only described by magnetic fields. This is called the Aharonov–Bohm effect ([3]). The classical mechanical system of particles moving in two solenoidal fields has the trajectory oscillating between two centers of fields and not scattering far away. We derive the asymptotic formula with the first three terms and make clear how such a trapping trajectory is reflected in the obtained formula through the Aharonov–Bohm effect.

We make a quick review on the scattering by a single solenoidal field before stating the obtained result. We study it in some detail in section 3. In particular, we there make use of microlocal analysis in the phase space to represent the scattering amplitude in terms of resolvent. This representation plays a basic role in the discussion in the sequel.

We now consider the Hamiltonian

$$H_{\alpha h} = (-ih\nabla - A_\alpha)^2\,, \quad A_\alpha(x) = \alpha\Lambda(x),$$

which governs the particle moving in the field $2\pi\alpha\delta(x)$. The potential $A_\alpha(x)$ has a strong singularity at the origin. For this reason, $H_{\alpha h}$ is not essentially self–adjoint over $C_0^\infty(\boldsymbol{R}^2\backslash\{0\})$ ([1, 5]) and hence a boundary condition at the origin is required to define a self–adjoint realization in the space $L^2 = L^2(\boldsymbol{R}^2)$. In fact, it is a symmetric operator with type $(2, 2)$ of deficiency indices. We know that $H_{\alpha h}$ is self–adjoint with domain

$$\mathcal{D} = \{u \in L^2 : (-ih\nabla - A_\alpha)^2 u \in L^2,\ \lim_{|x|\to 0} |u(x)| < \infty\}, \tag{1.2}$$

where $(-ih\nabla - A_\alpha)^2 u$ is understood in $\mathcal{D}'(\boldsymbol{R}^2\backslash\{0\})$ (distribution sense). If α/h is not an integer, then this condition is equivalent to saying that $\lim_{|x|\to 0} |u(x)| = 0$, and it means that the particle never touches the solenoid at the origin. The scattering amplitude for $H_{\alpha h}$ has been explicitly calculated by [3] (see [2, 14] also). We use the Gauss notation $[\beta]$ to denote the greatest integer not exceeding β, and we often identify the coordinates over the unit circle S^1 with the azimuth angles from the positive x_1 axis. Then the amplitude $f_h = f_h(\omega_- \to \omega_+; \alpha)$ for the scattering from incident direction $\omega_- \in S^1$ to final one $\omega_+ (\neq \omega_-)$ at energy $E > 0$ is known to take the form

$$f_h = (ic(E)/\pi)\, h^{1/2} \sin(\alpha/h)\pi\, \exp\left(i[\alpha/h](\omega_+ - \omega_-)\right) F_0(\omega_+ - \omega_-), \tag{1.3}$$

where

$$c(E) = \left(2\pi/i\sqrt{E}\right)^{1/2} = (2\pi)^{1/2} e^{-i\pi/4} E^{-1/4}$$

and $F_0(\theta) = e^{i\theta}/\left(1 - e^{i\theta}\right)$ for $\theta \neq 0$. We drop the dependence on energy E of amplitude f_h, E being fixed throughout. The amplitude $f_h = f_h(\omega_- \to \omega_+; \alpha)$ appears in the asymptotic form

$$\varphi_+ \sim e^{i(\alpha/h)(\gamma(x;\omega_-)-\pi)} \exp(ih^{-1}\sqrt{E} x \cdot \omega_-) + f_h \exp(ih^{-1}\sqrt{E}|x|)|x|^{-1/2} \tag{1.4}$$

as $x = |x|\omega_+ \to \infty$ of eigenfunction $\varphi_+ = \varphi_+(x)$, $H_{\alpha h}\varphi_+ = E\varphi_+$, where $\cdot$ is the scalar product in $\boldsymbol{R}^2$ and

$$\gamma(x; \omega) = \gamma(\hat{x}; \omega), \quad 0 \le \gamma < 2\pi, \quad \hat{x} = x/|x|,$$

denotes the azimuth angle from ω to $\hat{x}$. The incident wave described by the first term on the right side of (1.4) takes a form different from the usual plane wave $\exp(ih^{-1}\sqrt{E}x\cdot\omega_-)$. The modified factor $e^{i(\alpha/h)(\gamma(x;\omega_-)-\pi)}$ is required because of the long–range property of $A_\alpha(x)$. The amplitude $f_h(\omega_-\to\omega_+;\alpha,p)$ for the scattering by the field $2\pi\alpha\delta(x-p)$ with center at $p\in\boldsymbol{R}^2$ is given by

$$f_h(\omega_-\to\omega_+;\alpha,p)=e^{-ih^{-1}\sqrt{E}p\cdot(\omega_+-\omega_-)}f_h(\omega_-\to\omega_+;\alpha), \tag{1.5}$$

because $|x-p|=|x|-p\cdot\omega_++O(|x|^{-1})$ as $|x|\to\infty$ along direction ω_+.

We return to the scattering by two solenoidal fields. The operator H_h formally defined by (1.1) is self–adjoint under the boundary conditions $\lim_{|x-e_j|\to 0}|u(x)|<\infty$ at two centers e_1 and e_2 of fields. We denote by $f_h(\omega_-\to\omega_+)$ the scattering amplitude at energy E for H_h. The precise representation for $f_h(\omega_-\to\omega_+)$ is given in section 2. It is defined through the integral kernel of the scattering matrix. The asymptotic behavior as $h\to 0$ of $f_h(\omega_-\to\omega_+)$ is formulated as the theorem below. The statement uses the notation

$$\tau(\omega;\omega_1,\omega_2)=\gamma(\omega;\omega_1)-\gamma(\omega;-\omega_2) \tag{1.6}$$

for $\omega_1\neq\omega$ and $\omega_2\neq-\omega$. If $\omega_1=\omega$ or $\omega_2=-\omega$, then we interpret $\gamma(\omega;\omega)$ as $\gamma(\omega;\omega)=(0+2\pi)/2=\pi$ and define $\tau(\omega;\omega_1,\omega_2)$ by (1.6).

Theorem 1.1 *Let the notation be as above and let $f_h(\omega\to\theta;\alpha_j,e_j)$ be defined by (1.5) for $j=1,2$. Set $\hat{e}=e/|e|$ for $e=e_2-e_1\neq 0$. Assume that $\omega_-\neq\pm\hat{e}$ and $\omega_+\neq\pm\hat{e}$. If $\omega_-\neq\omega_+$, then $f_h=f_h(\omega_-\to\omega_+)$ behaves like*

$$\begin{aligned}
f_h &= \tau_1(h)f_h(\omega_-\to\omega_+;\alpha_1,e_1)+\tau_2(h)f_h(\omega_-\to\omega_+;\alpha_2,e_2)\\
&+ \tau_{12}(h)|e|^{-1/2}f_h(\omega_-\to\hat{e};\alpha_1,e_1)f_h(\hat{e}\to\omega_+;\alpha_2,e_2)\\
&+ \tau_{21}(h)|e|^{-1/2}f_h(\omega_-\to-\hat{e};\alpha_2,e_2)f_h(-\hat{e}\to\omega_+;\alpha_1,e_1)\\
&+ \tau_{121}(h)|e|^{-1}f_h(\omega_-\to\hat{e};\alpha_1,e_1)f_h(\hat{e}\to-\hat{e};\alpha_2,e_2)f_h(-\hat{e}\to\omega_+;\alpha_1,e_1)\\
&+ \tau_{212}(h)|e|^{-1}f_h(\omega_-\to-\hat{e};\alpha_2,e_2)f_h(-\hat{e}\to\hat{e};\alpha_1,e_1)f_h(\hat{e}\to\omega_+;\alpha_2,e_2)\\
&+ o(h^{3/2}),\qquad h\to 0,
\end{aligned}$$

where $\tau_1(h)=\exp(i(\alpha_2/h)\tau(-\hat{e};\omega_-,\omega_+))$, $\tau_2(h)=\exp(i(\alpha_1/h)\tau(\hat{e};\omega_-,\omega_+))$ and

$$\begin{aligned}
\tau_{12}(h) &= \exp(i(\alpha_2/h)\tau(-\hat{e};\omega_-,\hat{e}))\exp(i(\alpha_1/h)\tau(\hat{e};\hat{e},\omega_+))\\
\tau_{21}(h) &= \exp(i(\alpha_1/h)\tau(\hat{e};\omega_-,-\hat{e}))\exp(i(\alpha_2/h)\tau(-\hat{e};-\hat{e},\omega_+))\\
\tau_{121}(h) &= \exp(i(\alpha_2/h)\tau(-\hat{e};\omega_-,\hat{e}))\exp(i(\alpha_2/h)\tau(-\hat{e};-\hat{e},\omega_+))\\
\tau_{212}(h) &= \exp(i(\alpha_1/h)\tau(\hat{e};\omega_-,-\hat{e}))\exp(i(\alpha_1/h)\tau(\hat{e};\hat{e},\omega_+)).
\end{aligned}$$

We make a comment on the factors $\tau_{121}(h)$ and $\tau_{212}(h)$ in front of the third terms. For example, we had better understand $\tau_{121}(h)$ as

$$\tau_{121}(h)=e^{i(\alpha_2/h)\tau(-\hat{e};\omega_-,\hat{e})}e^{i(\alpha_1/h)\tau(\hat{e};\hat{e},-\hat{e})}e^{i(\alpha_2/h)\tau(-\hat{e};-\hat{e},\omega_+)}$$

by inserting $e^{i(\alpha_1/h)\tau(\hat{e};\hat{e},-\hat{e})} = 1$ between the first and last factors.

The proof of the theorem is reduced to the study on the scattering by two solenoidal fields with centers at large separation. We write $\gamma(x) = \gamma(\hat{x})$ for the azimuth angle from the positive x_1 axis. Then we have the relation

$$\nabla\gamma(x) = \Lambda(x) = \left(-x_2/|x|^2, x_1/|x|^2\right). \tag{1.7}$$

Let $U_1(h)$ and $U_2(h) : L^2 \to L^2$ be the unitary operators defined by

$$\left(U_1(h)f\right)(x) = h^{-1}f(h^{-1}x), \quad \left(U_2(h)f\right)(x) = \exp(i\zeta(x))f(x), \tag{1.8}$$

where

$$\zeta(x) = \rho_1\gamma(x-d_1) + \rho_2\gamma(x-d_2), \quad d_j = e_j/h, \tag{1.9}$$

with $\rho_1 = [\alpha_1/h]$ and $\rho_2 = [\alpha_2/h]$. The function $\exp(i\zeta(x))$ is single valued over $\boldsymbol{R}^2 \setminus \{d_1, d_2\}$, and $\zeta(x)$ satisfies $\nabla\zeta = \rho_1\Lambda(x-d_1) + \rho_2\Lambda(x-d_2)$ by (1.7). By definition, it follows that

$$U_1(h)^*\left(h\nabla\right)U_1(h)f = \nabla f, \qquad U_1(h)^*a(x)U_1(h)f = a(hx)f.$$

If we further make use of the relation $U_2(h)^*\nabla U_2(h)f = \left(\nabla + i\nabla\zeta\right)f$, then we obtain

$$K_d := U(h)^*H_hU(h) = \left(-i\nabla - B_d\right)^2 \tag{1.10}$$

with $B_d = \beta_1\Lambda(x-d_1) + \beta_2\Lambda(x-d_2)$, where $U(h) = U_1(h)U_2(h)$ and

$$\beta_j = \beta_j(h) = \alpha_j/h - \rho_j = \alpha_j/h - [\alpha_j/h], \quad 0 \le \beta_1, \ \beta_2 < 1. \tag{1.11}$$

We denote by $g_d(\omega_- \to \omega_+; E)$ the scattering amplitude at energy E for pair (K_d, H_0), where $H_0 = -\Delta$ is the free Hamiltonian. Then we can show (Proposition 2.1) that

$$f_h(\omega_- \to \omega_+) = (-1)^\rho \exp\left(i\rho(\omega_+ - \omega_-)\right)h^{1/2}g_d(\omega_- \to \omega_+; E), \tag{1.12}$$

where

$$\rho = \rho_1 + \rho_2 = [\alpha_1/h] + [\alpha_2/h]. \tag{1.13}$$

Thus the proof of Theorem 1.1 is reduced to analyzing the behavior as $|d| \to \infty$ of $g_d(\omega_- \to \omega_+; E)$, where $d = d_2 - d_1 = \left(e_2 - e_1\right)/h = e/h$.

We take a look at the asymptotic formula in the theorem. Even if the problem is reduced to the scattering by two solenoidal fields with centers at large separation, the leading term is not necessarily split into the sum of two scattering amplitudes by single solenoidal fields. We can find the Aharonov–Bohm effect in the factors $\tau_1(h)$ and $\tau_2(h)$ in front of the two leading terms. For example, $\tau_1(h)$ depends on the flux α_2 of the field with center at e_2. This implies that the vector potential $\alpha_2\Lambda(x-e_2)$ has an influence on the scattering by the field with center at e_1. The two second terms with magnitude of order $O(h)$ are involved with the trajectory joining the two centers e_1 and e_2. For example, the first term is associated with the trajectory of particle which is scattered from $\hat{e}$ to ω_+ by the field with center at e_2 after it has been scattered from ω_- to $\hat{e}$ by the field with center at e_1. A similar interpretation applies to the other second term and also the two third terms with magnitude of order $O(h^{3/2})$ are interpreted in a similar way. The factors $\tau_{12}(h)$, $\tau_{21}(h)$ and $\tau_{121}(h)$, $\tau_{212}(h)$ can be seen to arise in the same manner as $\tau_1(h)$ and $\tau_2(h)$.

The proof of Theorem 1.1 is completed in section 5. In section 2 we prove the relation (1.12), and in section 3 we represent $g_d(\omega_- \to \omega_+; E)$ in terms of the boundary value

$$R(E + i0; K_d) = \lim_{\varepsilon \downarrow 0} R(E + i\varepsilon; K_d) : L^2_s(\boldsymbol{R}^2) \to L^2_{-s}(\boldsymbol{R}^2), \quad s > 1/2, \tag{1.14}$$

of resolvent $R(z; K_d) = (K_d - z)^{-1}$ with $\operatorname{Im} z \neq 0$, where $L^2_s(\boldsymbol{R}^2) = L^2(\boldsymbol{R}^2; \langle x \rangle^{2s} dx)$ with $\langle x \rangle = (1 + |x|^2)^{1/2}$. If we denote by (,) the L^2 scalar product and if we write $\varphi_0(\omega)$ for $\exp(i\sqrt{E}\, x \cdot \omega)$ with $\omega \in S^1$, then $g_d(\omega_- \to \omega_+; E)$ takes the form

$$g_d(\omega_- \to \omega_+; E) \sim \sum_{1 \le j,k \le 2} \left(R(E + i0; K_d) Q_{-j} \varphi_0(\omega_-), Q_{+k} \varphi_0(\omega_+) \right),$$

where $Q_{\pm j}$ is a differential operator of first order with coefficients supported in the ball $S_j = \{|x - d_j| < K|d|^\sigma\}$, $\sigma > 0$ being fixed small enough. In section 4 we analyze the behavior as $|d| \to \infty$ of operator $X_{jk} = Q^*_{+k} R(E + i0; K_d) Q_{-j}$. We can show that the leading and third terms of the asymptotic formula come from X_{jk} with $j = k$ and the second term comes from X_{jk} with $j \neq k$. The analysis is based on the resolvent estimate

$$\| s_j R(E + i0; K_d) s_k \| \sim O(|d|^{-1/2}), \quad j \neq k, \quad |d| \to \infty,$$

established in the previous works [8, 9], where $\| \ \|$ denotes the norm of bounded operators acting on L^2 and $s_j(x)$ is the characteristic function of S_j.

We further add three remarks to the theorem in the last section (section 6). In Remark 6.1 we discuss the case when $\omega_- = \pm\hat{e}$ or $\omega_+ = \pm\hat{e}$. Even in this case, we can derive the asymptotic formula with only the leading terms, provided that $\omega_- \neq \omega_+$. The obtained formula takes a different form. The second term seems to be difficult to calculate. In fact, the second term contains the forward amplitude such as $f_h(\hat{e} \to \hat{e}, E; \alpha_1, e_1)$ or $f_h(-\hat{e} \to -\hat{e}, E; \alpha_2, e_2)$ which is divergent in general. The other two remarks are devoted to the generalization to the case of scattering by many solenoidal fields. The results obtained heavily depend on the location of centers. In particular, the Aharonov–Bohm effect is strongly reflected in the case that centers of solenoidal fields are on an even line. The Aharonov–Bohm effect problem in many solenoidal fields has also been discussed in the physical literatures (see for example [2, 12, 15, 16] and references there). Among these works, Stovicek

[15, 16] has studied a problem similar to that in the present work and developed a quite different idea to propose the higher order expansion in a rather formal way.

There are a lot of works devoted to the semiclassical analysis on the asymptotic behavior of amplitudes in the case of potential scattering. We refer to the recent work [4] for this subject. An extensive survey on earlier results can be found there. As a closely related reference, we should also refer to [10] which has studied the asymptotic behavior as $|d_1 - d_2| \to \infty$ of amplitude for the scattering by potential $V_1(\cdot - d_1) + V_2(\cdot - d_2)$ with V_1 and V_2 falling off rapidly at infinity.

2. Scattering by two solenoidal fields at large separation

We begin by introducing the new notation. For two self–adjoint operators H and K acting on $L^2 = L^2(\boldsymbol{R}^2)$, we denote by $W_\pm(H, K)$ the wave operator

$$W_\pm(H, K) = s - \lim_{t \to \pm\infty} \exp(itH) \exp(-itK)$$

and by $S(H, K)$ the scattering operator

$$S(H, K) = W_+(H, K)^* W_-(H, K).$$

We also define $W_\pm(H, K; J)$ by

$$W_\pm(H, K; J) = s - \lim_{t \to \pm\infty} \exp(itH) J \exp(-itK) \tag{2.1}$$

for a bounded operator J on L^2.

Let $K_d = (-i\nabla - B_d)^2$ be defined by (1.10). The operator is self–adjoint under the boundary conditions $\lim_{|x-d_j| \to 0} |u(x)| < \infty$. We have established the basic spectral properties of K_d in section 7 of [8]. According to the results there, K_d has no bound states, and there exists the boundary value $R(E + i0; K_d)$ in (1.14). This has been verified by use of the commutator method due to Mourre [11]. Moreover we can prove that $W_\pm(K_d, H_0)$ exists and is asymptotically complete ($\operatorname{Ran} W_\pm(K_d, H_0) = L^2$). Hence $S(K_d, H_0)$ can be defined as a unitary operator.

Let $U_1(h)$ and $U_2(h)$ be defined by (1.8). Then $H_0 = U_1(h)^* H_{0h} U_1(h)$ for $H_{0h} = -h^2\Delta$, and hence

$$\exp(-itH_0) = U_1(h)^* \exp(-itH_{0h}) U_1(h).$$

We use the notation (2.1) to set

$$W_+(h) = W_+(H_0, H_0; U_2(h)), \quad W_-(h) = W_-(H_0, H_0; U_2(h)^*). \tag{2.2}$$

Then $S(H_h, H_{0h})$ admits the decomposition

$$S(H_h, H_{0h}) = U_1(h) W_+(h) S(K_d, H_0) W_-(h) U_1(h)^*. \tag{2.3}$$

The existence of $W_\pm(h)$ follows as a consequence of Lemma 2.1 below.

We use the abbreviation that integrations with no domain attached are taken over the whole space. Let

$$\varphi_0(x;\lambda,\omega) = \exp(i\sqrt{\lambda}\, x\cdot\omega), \quad \lambda > 0, \quad \omega \in S^1,$$

be the generalized eigenfunction of the free Hamiltonian H_0. We define the unitary mapping $F : L^2 \to L^2((0,\infty); d\lambda) \otimes L^2(S^1)$ by

$$(Fu)(\lambda,\omega) = 2^{-1/2}(2\pi)^{-1}\int \bar{\varphi}_0(x;\lambda,\omega)u(x)\,dx = 2^{-1/2}\widehat{u}(\sqrt{\lambda}\omega) \tag{2.4}$$

and $F(h)$ by

$$(F(h)u)(\lambda,\omega) = 2^{-1/2}(2\pi h)^{-1}\int \bar{\varphi}_0(x/h;\lambda,\omega)u(x)\,dx,$$

where $\widehat{u}(\xi)$ is the Fourier transform of u. Then a simple computation shows that

$$F = F(h)U_1(h). \tag{2.5}$$

The mapping $F(h)$ decomposes $S(H_h, H_{0h})$ into the direct integral

$$S(H_h, H_{0h}) \simeq F(h)S(H_h, H_{0h})F(h)^* \simeq \int_0^\infty \oplus\, S(\lambda; H_h, H_{0h})\,d\lambda,$$

where the fibre $S(\lambda; H_h, H_{0h}) : L^2(S^1) \to L^2(S^1)$ is called the scattering matrix at energy $\lambda > 0$ and it acts as

$$\left(S(\lambda; H_h, H_{0h})(F(h)u)(\lambda,\,\cdot\,)\right)(\omega) = \left(F(h)S(H_h, H_{0h})u\right)(\lambda,\omega)$$

for $u \in L^2$. Similarly we have

$$S(K_d, H_0) \simeq FS(K_d, H_0)F^* \simeq \int_0^\infty \oplus\, S(\lambda; K_d, H_0)\,d\lambda.$$

The next lemma is well known (see [13, Theorem IX. 31] for example).

Lemma 2.1 *The free solution* $\exp(-itH_0)f$ *with initial state* $f \in L^2$ *behaves like*

$$(\exp(-itH_0)f)(x) = (2it)^{-1}\exp(i|x|^2/4t)\widehat{f}(x/2t) + o(1), \quad |t| \to \infty,$$

on L^2.

We recall that $\gamma(x) = \gamma(\hat{x})$ denotes the azimuth angle from the positive x_1 axis. Let $\zeta(x)$ be defined by (1.9). It behaves like $\zeta(x) \sim \rho\gamma(\hat{x})$ as $|x| \to \infty$, where ρ is the integer defined by (1.13). Hence the lemma above implies that $W_+(h)$ defined by (2.2) acts as the multiplication

$$FW_+(h)F^* = \exp(i\rho\gamma(\omega))\times = \exp(i\rho\omega)\times$$

on $L^2((0,\infty); d\lambda) \otimes L^2(S^1)$. If we note that $\gamma(-\omega) = \omega + \pi$ or $\omega - \pi$ according as $0 \le \omega < \pi$ or $\pi \le \omega < 2\pi$, then

$$FW_-(h)F^* = \exp(-i\rho(\gamma(-\omega)))\times = (-1)^\rho \exp(-i\rho\omega)\times .$$

We denote by $S(\theta,\omega;\lambda,H_h,H_{0,h})$ and $S(\theta,\omega;\lambda,K_d,H_0)$ the kernels of scattering matrices $S(\lambda;H_h,H_{0h})$ and $S(\lambda;K_d,H_0)$ respectively. Then it follows from (2.3) and (2.5) that

$$S(\theta,\omega;E,H_h,H_{0h}) = (-1)^\rho \exp(i\rho(\theta-\omega))S(\theta,\omega,E;K_d,H_0).$$

The scattering amplitude $f_h(\omega_- \to \omega_+)$ is defined by

$$f_h(\omega_- \to \omega_+) = c(E/h^2)S(\omega_+,\omega_-;E,H_h,H_{0h})$$

through the scattering kernel and also we have

$$g_d(\omega_- \to \omega_+;E) = c(E)S(\omega_+,\omega_-;E,K_d,H_0)$$

as the special case with $h = 1$, where $c(E)$ is defined in (1.3). Since $c(E/h^2) = h^{1/2}c(E)$, we obtain the following proposition as the first step toward the proof of Theorem 1.1.

Proposition 2.1 *Let $g_d(\omega_- \to \omega_+;E)$ be the scattering amplitude at energy E for pair (K_d,H_0) and let ρ be as in (1.13). Then*

$$f_h(\omega_- \to \omega_+) = (-1)^\rho \exp(i\rho(\omega_+ - \omega_-))h^{1/2}g_d(\omega_- \to \omega_+;E).$$

We end the section by making a comment on the amplitude $f_h(\omega_- \to \omega_+;\alpha,p)$ in (1.5) for later reference. Let $H_\beta = (-i\nabla - A_\beta)^2$ with $A_\beta = \beta\Lambda(x)$. This is self–adjoint under the same boundary condition at the origin as in (1.2). We denote by $g(\omega_- \to \omega_+;E,\beta)$ the scattering amplitude at energy E for pair (H_β,H_0) and by $g(\omega_- \to \omega_+;E,\beta,p)$ the amplitude for the scattering by the field $2\pi\beta\delta(x-p)$. Then we can show

$$f_h(\omega_- \to \omega_+;\alpha,p) = (-1)^{[\alpha/h]}e^{i[\alpha/h](\omega_+ - w_-)}h^{1/2}g(\omega_- \to \omega_+;E,\beta,p/h) \tag{2.6}$$

in exactly the same way as above, where $\beta = \alpha/h - [\alpha/h]$.

3. Representation for scattering amplitudes

As already stated in section 1, the kernel of scattering matrix $S(\lambda; H_\beta, H_0)$ at energy $\lambda > 0$ is explicitly calculated (see [2, 3, 14] for example). It takes the form

$$S(\theta, \omega; \lambda, H_\beta, H_0) = (i/\pi) \sin \beta\pi \, e^{i[\beta](\omega-\theta)} F_0(\omega - \theta) \tag{3.1}$$

for $\theta \neq \omega$, where $F_0(\theta)$ is as in (1.3). By definition, the amplitude $g(\omega_- \to \omega_+; E, \beta)$ is related to $S(\omega_+, \omega_-; E, H_\beta, H_0)$ through the relation

$$g = c(E)S(\omega_+, \omega_-; E, H_\beta, H_0) = (ic(E)/\pi) \sin \beta\pi \, e^{i[\beta](\omega_+-\omega_-)} F_0(\omega_+ - \omega_-),$$

so that (1.3) follows from (2.6) with $p = 0$. In this section we represent the scattering amplitude $g(\omega_- \to \omega_+; E, \beta)$ in terms of resolvent $R(E+i0; H_\beta)$. A similar argument applies to the amplitude $g_d(\omega_- \to \omega_+; E)$ for the scattering by two solenoidal fields. We formulate the result as Proposition 3.1 at the end of the section.

We introduce a cut–off function $\chi \in C_0^\infty[0, \infty)$ such that $\chi(s) \geq 0$ and

$$\chi(s) = 1 \quad \text{for } 0 \leq s \leq 1, \qquad \chi(s) = 0 \quad \text{for } s > 2. \tag{3.2}$$

We define $p_-(\xi) = \chi(|\xi - \sqrt{E}\omega_-|/\delta^2)$ for δ, $0 < \delta \ll 1$, small enough and take $q_- \in C^\infty(\boldsymbol{R}^2)$ in such a way that

$$q_-(x) \geq 0, \quad \operatorname{supp} q_- \subset \Sigma(R, \omega_-, \delta/2), \quad q_- = 1 \text{ on } \Sigma(2R, \omega_-, \delta) \tag{3.3}$$

and $\partial_x^\mu q_-(x) = O(|x|^{-|\mu|})$ at infinity, where

$$\Sigma(R, \omega, \delta) = \{x : |x| > R, \ |\hat{x} - \omega| > \delta\}, \quad R > 0.$$

Recall the notation (2.1). We set $J_- = q_-^2 p_-^2$, where $p_- = p_-(D_x)$. Then we have

$$W_-(H_\beta, H_0)p_-^2 = W_-(H_\beta, H_0; J_-)$$

by Lemma 2.1. Since $\gamma(x; \omega)$ satisfies (1.7), we get

$$\exp(-i\beta\gamma(x; \omega_-))H_\beta \exp(i\beta\gamma(x; \omega_-)) = H(A_\beta - \beta\nabla\gamma) = H_0 \tag{3.4}$$

on $\Sigma(R, \omega_-, \delta/2)$. We further set

$$J_{-0} = q_- \exp(i\beta\gamma(x; \omega_-))p_-, \quad J_{-1} = q_- \exp(-i\beta\gamma(x; \omega_-))p_-.$$

Then $W_-(H_\beta, H_0; J_{-0})$ and $W_-(H_0, H_0; J_{-1})$ exist. In fact, the existence of the first and second limits follows from relation (3.4) and Lemma 2.1 respectively. Thus $W_-(H_\beta, H_0)p_-^2$ admits the decomposition

$$W_-(H_\beta, H_0)p_-^2 = W_-(H_\beta, H_0; J_{-0})W_-(H_0, H_0; J_{-1}). \tag{3.5}$$

The same argument applies to ω_+. We define $p_+(\xi) = \chi(|\xi - \sqrt{E}\omega_+|/\delta^2)$ and take $q_+ \in C^\infty(\boldsymbol{R}^2)$ in such a way that

$$q_+(x) \geq 0, \quad \operatorname{supp} q_+ \subset \Sigma(R, -\omega_+, \delta/2), \quad q_+ = 1 \text{ on } \Sigma(2R, -\omega_+, \delta). \tag{3.6}$$

We set

$$J_{+0} = q_+ \exp(i\beta\gamma(x; -\omega_+))p_+, \quad J_{+1} = q_+ \exp(-i\beta\gamma(x; -\omega_+))p_+,$$

where $p_+ = p_+(D_x)$. Then we get

$$W_+(H_\beta, H_0)p_+^2 = W_+(H_\beta, H_0; J_{+0})W_+(H_0, H_0; J_{+1}). \tag{3.7}$$

We combine (3.5) and (3.7) to obtain that

$$p_+^2 S(H_\beta, H_0)p_-^2 = W_+(H_0, H_0; J_{+1})^* S_0(H_\beta, H_0)W_-(H_0, H_0; J_{-1}), \tag{3.8}$$

where

$$S_0(H_\beta, H_0) = W_+(H_\beta, H_0; J_{+0})^* W_-(H_\beta, H_0; J_{-0}).$$

The operator $S_0(H_\beta, H_0)$ also has the direct integral decomposition, because it commutes with H_0. We denote the fibre by $S_0(\lambda; H_\beta, H_0) : L^2(S^1) \to L^2(S^1)$ and its kernel by $S_0(\theta, \omega; \lambda, H_\beta, H_0)$. We see by Lemma 2.1 that $W_-(H_0, H_0; J_{-1})$ acts as the multiplication

$$FW_-(H_0, H_0; J_{-1})F^* = \exp(-i\beta\gamma(-\theta; \omega_-))p_-(\sqrt{\lambda}\theta)\times$$

on $L^2((0, \infty); d\lambda) \otimes L^2(S^1)$, where F is defined by (2.4). Similarly

$$FW_+(H_0, H_0; J_{+1})F^* = \exp(-i\beta\gamma(\theta; -\omega_+))p_+(\sqrt{\lambda}\theta) \times .$$

Hence it follows from (3.8) that

$$S(\omega_+, \omega_-; E, H_\beta, H_0) = S_0(\omega_+, \omega_-; E, H_\beta, H_0), \tag{3.9}$$

because $e^{-i\beta\gamma(-\omega_-;\omega_-)}p_-(\sqrt{E}\omega_-) = e^{-i\beta\pi}$ and $e^{-i\beta\gamma(\omega_+;-\omega_+)}p_+(\sqrt{E}\omega_+) = e^{-i\beta\pi}$.

We derive the representation for kernel $S_0(\omega_+, \omega_-; E, H_\beta, H_0)$ on the right side of (3.9). The derivation is based on the idea due to [6, 7]. We calculate $T_- = H_\beta J_{-0} - J_{-0}H_0$ as

$$T_- = \exp(i\beta\gamma(x; \omega_-))\left(H_0 q_- - q_- H_0\right)p_- = \exp(i\beta\gamma(x; \omega_-))[H_0, q_-]p_-$$

by use of (3.4). Similarly we have

$$T_+ = H_\beta J_{+0} - J_{+0}H_0 = \exp(i\beta\gamma(x; -\omega_+))[H_0, q_+]p_+.$$

Since $W_+(H_\beta, H_0; J_{-0}) = 0$ by Lemma 2.1, it follows that

$$W_-(H_\beta, H_0; J_{-0}) = -i \int \exp(itH_\beta)T_- \exp(-itH_0)\, dt.$$

If we use this relation, then we get

$$S_0(\lambda; H_\beta, H_0) = 2\pi i\Gamma(\lambda)\left(-J_{+0}^* T_- + T_+^* R(\lambda + i0; H_\beta)T_-\right)\Gamma(\lambda)^*$$

in exactly the same way as [7, Theorem 3.3] (see [17, subsection 7.3] also), where $\Gamma(\lambda) : L^2_s(\boldsymbol{R}^2) \to L^2(S^1)$ is the trace operator defined by

$$(\Gamma(\lambda)u)(\theta) = (Fu)(\mu,\theta)|_{\mu=\lambda}$$

for $s > 1/2$. This relation yields the representation for $S_0(\omega_+,\omega_-; E, H_\beta, H_0)$, and the lemma below is obtained from (3.9).

Lemma 3.1 *Assume that* $\omega_- \neq \omega_+$. *Then* $g = g(\omega_- \to \omega_+; E, \beta)$ *is represented as*

$$\begin{aligned} g &= -(ic(E)/4\pi)\,(T_-\varphi_0(\omega_-), J_{+0}\varphi_0(\omega_+)) \\ &+ (ic(E)/4\pi)\,(R(E+i0;H_\beta)T_-\varphi_0(\omega_-), T_+\varphi_0(\omega_+))\,, \end{aligned}$$

where $\varphi_0(\omega) = \exp(i\sqrt{E}x\cdot\omega)$ *and* $(\ ,\ )$ *is the* L^2 *scalar product.*

We analyze the two terms on the right side of the relation in the lemma. We fix σ, $0 < \sigma \ll 1$, small enough and we take R as $R = |d|^\sigma$ in (3.3) and (3.6), where $d = (e_2 - e_1)/h$ again. Then we may assume that $q_\pm$ obeys

$$|\partial_x^\mu q_\pm(x)| \le C_\mu\,(|x| + |d|^\sigma)^{-|\mu|}$$

uniformly in d. Since $\omega_- \neq \omega_+$, we can choose δ so small that the support of symbols $T_-(x,\xi)$ and $J_{+0}(x,\xi)$ of pseudodifferential operators T_- and J_{+0} does not intersect with each other. Hence the first term obeys

$$(T_-\varphi_0(\omega_-), J_{+0}\varphi_0(\omega_+)) = O(|d|^{-N}), \quad |d| \to \infty,$$

for any $N \gg 1$. We move to the second term. We decompose $T_- = T_-(x, D_x)$ into the sum

$$T_- = \chi_0 T_- + (1-\chi_0)T_- = T_{-0} + T_{-\infty},$$

where $\chi_0(x) = \chi(|d|^{-\sigma}|x|/M)$, $M \gg 1$, for the cut–off function $\chi \in C_0^\infty[0,\infty)$ with property (3.2). Since ∇q_- vanishes on $\Sigma(2|d|^\sigma, \omega_-, \delta)$, the symbol $T_{-\infty}(x,\xi)$ has the support in the outgoing region

$$\operatorname{supp} T_{-\infty} \subset \{(x,\xi) : |x| > M|d|^\sigma,\ |\xi - \sqrt{E}\omega_-| < 2\delta^2,\ \hat{x}\cdot\hat{\xi} > 1 - 2\delta\}.$$

The particle with initial state $(x,\xi) \in \operatorname{supp} T_{-\infty}$ at $t = 0$ moves like the free particle and it passes away from the origin for $t \ge 0$. In fact, we have

$$|x + t\xi| \ge c\left(|x| + |d|^\sigma + t\sqrt{E}\right), \quad (x,\xi) \in \operatorname{supp} T_{-\infty}, \quad t \ge 0,$$

for some $c > 0$. Thus the outgoing particle does not take momentum around $\sqrt{E}\omega_+$. If we decompose T_+ into $T_+ = T_{+0} + T_{+\infty}$ in a similar way, then $T_{+\infty}(x,\xi)$ has the support in the incoming region

$$\operatorname{supp} T_{+\infty} \subset \{(x,\xi) : |x| > M|d|^\sigma,\ |\xi - \sqrt{E}\omega_+| < 2\delta^2,\ \hat{x}\cdot\hat{\xi} < -1 + 2\delta\}.$$

The particle with initial state $(x, \xi) \in \operatorname{supp} T_{+\infty}$ at $t = 0$ moves like the free particle and does not take momentum around $\sqrt{E}\omega_-$ for $t \le 0$. Hence we can show

$$\begin{aligned}(R(E+i0; H_\beta)T_{-\infty}\varphi_0(\omega_-), T_+\varphi_0(\omega_+)) = O(|d|^{-N})\\ (T_{-0}\varphi_0(\omega_-), R(E-i0; H_\beta)T_{+\infty}\varphi_0(\omega_+)) = O(|d|^{-N})\end{aligned}$$

by the microlocal resolvent estimate, and it follows that

$$g = (ic(E)/4\pi)\,(R(E+i0; H_\beta)T_{-0}\varphi_0(\omega_-), T_{+0}\varphi_0(\omega_+)) + O(|d|^{-N}).$$

The rather rough argument above can be justified by the microlocal analysis developed in section 4. The proof is done by constructing an approximation to $R(E + i0; H_\beta)$ as in the proof of Lemma 4.3.

We continue to analyze the term on the right side. The operator T_{-0} acts as

$$T_{-0}\varphi_0(\omega_-) = \exp(i\beta\gamma(x; \omega_-))\chi_0[H_0, q_-]\varphi_0(\omega_-)$$

on $\varphi_0(\omega_-) = \varphi_0(x; \omega_-, E)$, and T_{+0} acts as

$$T_{+0}\varphi_0(\omega_+) = \exp(i\beta\gamma(x; -\omega_+))\chi_0[H_0, q_+]\varphi_0(\omega_+).$$

We set

$$\psi_0(x) = 1 - \chi(|d|^{-\sigma}|x|) \tag{3.10}$$

and define

$$\zeta_{-0}(x) = \eta(\gamma(x; \omega_-)), \qquad \zeta_{+0}(x) = \eta(\gamma(x; -\omega_+)),$$

where $\eta \in C^\infty(\boldsymbol{R})$ is a real periodic function with period 2π such that $\eta(s) = \beta s$ on the interval $[\delta, 2\pi - \delta]$. We may assume that $q_-(x) = \psi_0(x)$ on the conical set $\{|\hat{x} - \omega_-| > \delta\}$ and $q_+(x) = \psi_0(x)$ on $\{|\hat{x} + \omega_+| > \delta\}$. Then

$$\exp(i\beta\gamma(x; \omega_-))\chi_0(x)[H_0, q_-] = -\exp(i\zeta_{-0}(x))[\psi_0, H_0]$$

on $\Sigma(|d|^\sigma, \omega_-, \delta)$, and

$$\exp(i\beta\gamma(x; -\omega_+))\chi_0(x)[H_0, q_+] = -\exp(i\zeta_{+0}(x))[\psi_0, H_0]$$

on $\Sigma(|d|^\sigma, -\omega_+, \delta)$. If $|\hat{x} - \omega_-| < \delta$ and $|\xi - \sqrt{E}\omega_-| < 2\delta^2$, then $\hat{x} \cdot \hat{\xi} > 1 - 2\delta$. The outgoing particle with initial state in

$$\{(x, \xi) : |x| > |d|^\sigma,\ \ |\xi - \sqrt{E}\omega_-| < 2\delta^2,\ \ \hat{x} \cdot \hat{\xi} > 1 - 2\delta\}$$

moves like the free particle and it does not take momentum around $\sqrt{E}\omega_+$ for $t \ge 0$. On the other hand, the incoming particle with initial state in

$$\{(x, \xi) : |x| > |d|^\sigma,\ \ |\xi - \sqrt{E}\omega_+| < 2\delta^2,\ \ \hat{x} \cdot \hat{\xi} < -1 + 2\delta\}$$

does not take momentum around $\sqrt{E}\omega_-$ for $t \le 0$. Thus we obtain the following lemma by the microlocal resolvent estimate again.

Lemma 3.2 *Let the notation be as above. If $\omega_- \neq \omega_+$, then $g = g(\omega_- \to \omega_+; E, \beta)$ behaves like*

$$g = (ic(E)/4\pi)\,(R(E+i0; H_\beta)Q_{-0}\varphi_0(\omega_-), Q_{+0}\varphi_0(\omega_+)) + O(|d|^{-N})$$

as $|d| \to \infty$ for any $N \gg 1$, where $Q_{\pm 0} = \exp\left(i\zeta_{\pm 0}(x)\right)[\psi_0, H_0]$.

Remark 3.1. Let $\tilde{\psi}_0(x) = 1 - \chi(|d|^{-\sigma}|x|/K)$ for $K > 1$. It is easy to see that Lemma 3.2 remains true for $\tilde{Q}_{+0} = e^{i\zeta_{+0}}[\tilde{\psi}_0, H_0]$.

A similar argument applies to the amplitude $g_d(\omega_- \to \omega_+; E)$. We fix the new notation. Let d_j and β_j be as in (1.9) and (1.11) respectively, and let $\eta_j \in C^\infty(\boldsymbol{R})$ be a real periodic function with period 2π such that $\eta_j(s) = \beta_j s$ on the interval $[\delta, 2\pi - \delta]$. Then we define

$$\begin{aligned}\zeta_-(x) &= \eta_1(\gamma(x-d_1;\omega_-)) + \eta_2(\gamma(x-d_2;\omega_-))\\ \zeta_+(x) &= \eta_1(\gamma(x-d_1;-\omega_+)) + \eta_2(\gamma(x-d_2;-\omega_+))\end{aligned}$$

and set

$$\psi_j(x) = \psi_0(x-d_j) = 1 - \chi(|d|^{-\sigma}|x-d_j|) \tag{3.11}$$

for $\psi_0(x)$ defined by (3.10). Since $\omega_- \neq \pm\hat{e}$ and $\omega_+ \neq \pm\hat{e}$ by assumption, we can take $\delta > 0$ so small that neither of the conical sets $\{|(x-d_1)/|x-d_1| \pm \omega_\pm| < \delta\}$ with d_1 as a vertex contains d_2, and similarly for the conical sets with d_2 as a vertex. If we take account of these facts, then the argument used for the proof of Lemma 3.2 enables us to prove the following proposition as the second step toward the proof of Theorem 1.1.

Proposition 3.1 *Let the notation be as above. Assume that $\omega_- \neq \pm\hat{e}$ and $\omega_+ \neq \pm\hat{e}$. If $\omega_- \neq \omega_+$, then $g_d = g_d(\omega_- \to \omega_+; E)$ obeys*

$$g_d = (ic(E)/4\pi) \sum_{1\le j,k\le 2} (R(E+i0;K_d)Q_{-j}\varphi_0(\omega_-), Q_{+k}\varphi_0(\omega_+)) + O(|d|^{-N})$$

as $|d| \to \infty$ for any $N \gg 1$, where $Q_{\pm j} = \exp\left(i\zeta_\pm(x)\right)[\psi_j, H_0]$.

We skip the proof. It uses the resolvent estimate uniform in d on $R(E+i0;K_d)$ (Lemma 4.1) and a similar argument can be found in [9, section 4].

4. Asymptotic analysis on resolvents

The third step is to analyze the behavior as $|d| \to \infty$ of

$$Z_{jk} = s_k R(E+i0;K_d)s_j, \quad 1 \le j,\ k \le 2,$$

where $s_j(x)$ is the characteristic function of the ball $S_j = \{|x-d_j| < K|d|^\sigma\}$, $K > 0$ being fixed arbitrarily. The result obtained is applied to $Q_{+k}^* R(E+i0; K_d)Q_{-j}$ in Proposition 3.1.

We introduce some auxiliary operators. Let $\eta_j \in C^\infty(\boldsymbol{R})$ be as in the previous section, and we define two real smooth functions

$$\zeta_1 = \eta_1(\gamma(x-d_1; -\hat{e})) \text{ over } |x-d_1| > |d|^\sigma, \quad \zeta_1 = 0 \text{ over } |x-d_1| < |d|^\sigma/2$$
$$\zeta_2 = \eta_2(\gamma(x-d_2; \hat{e})) \text{ over } |x-d_2| > |d|^\sigma, \quad \zeta_2 = 0 \text{ over } |x-d_2| < |d|^\sigma/2.$$

Then we introduce the following operators :

$$\begin{aligned} K_{jd} &= (-i\nabla - B_{jd})^2, \quad B_{jd} = \beta_j \Lambda(x-d_j) \\ L_{0d} &= (-i\nabla - A_{0d})^2, \quad A_{0d} = \nabla\zeta_1(x) + \nabla\zeta_2(x) \\ L_{1d} &= (-i\nabla - A_{1d})^2, \quad A_{1d} = B_{1d}(x) + \nabla\zeta_2(x) \\ L_{2d} &= (-i\nabla - A_{2d})^2, \quad A_{2d} = B_{2d}(x) + \nabla\zeta_1(x). \end{aligned}$$

These operators satisfy the relations

$$H_0 = e^{-i(\zeta_1+\zeta_2)} L_{0d} e^{i(\zeta_1+\zeta_2)}, \quad K_{1d} = e^{-i\zeta_2} L_{1d} e^{i\zeta_2}, \quad K_{2d} = e^{-i\zeta_1} L_{2d} e^{i\zeta_1}. \tag{4.1}$$

The present section is devoted to proving the following two propositions.

Proposition 4.1 *Let the notation be as above. Then*

$$\begin{aligned} Z_{11} &= s_1 R(E+i0; L_{1d}) s_1 + O_p(|d|^{-1+c\sigma}) \\ Z_{22} &= s_2 R(E+i0; L_{2d}) s_2 + O_p(|d|^{-1+c\sigma}) \end{aligned}$$

for $c > 0$ independent of σ, $0 < \sigma \ll 1$, where $O_p(|d|^{-\nu})$ denotes bounded operators the norm of which obeys the bound $O(|d|^{-\nu})$ as $|d| \to \infty$.

Proposition 4.2 *Let $\tilde{\psi}_j$ be defined by $\tilde{\psi}_j(x) = 1 - \chi(|d|^{-\sigma}|x-d_j|/M)$ for $M \gg 1$ large enough. Then*

$$\begin{aligned} Z_{12} &= s_2 R(E+i0; L_{2d}) \Gamma_{12} R(E+i0; L_{1d}) s_1 + O_p(|d|^{-3/2+c\sigma}) \\ Z_{21} &= s_1 R(E+i0; L_{1d}) \Gamma_{21} R(E+i0; L_{2d}) s_2 + O_p(|d|^{-3/2+c\sigma}) \end{aligned}$$

for $c > 0$ independent of σ, where

$$\begin{aligned} \Gamma_{12} &= e^{i(\zeta_1+\zeta_2)}[\tilde{\psi}_2, H_0] R(E+i0; H_0) [H_0, \tilde{\psi}_1] e^{-i(\zeta_1+\zeta_2)} \\ \Gamma_{21} &= e^{i(\zeta_1+\zeta_2)}[\tilde{\psi}_1, H_0] R(E+i0; H_0) [H_0, \tilde{\psi}_2] e^{-i(\zeta_1+\zeta_2)}. \end{aligned}$$

The proof of two propositions is done through a series of lemmas. We use the notation $\|\ \ \|$ to denote the norm of bounded operators acting on L^2.

Lemma 4.1 *The operator* $R(E \pm i0; K_d)$ *obeys the following estimates.*

(1) *There exists* $L \gg 1$ *large enough such that*

$$\|r_L R(E \pm i0; K_d) r_L\| = O(1), \quad \|r_L R(E \pm i0; K_d) s_j\| = O(1), \quad |d| \to \infty,$$

where r_L *is the multiplication operator by* $r_L(x) = (|x| + |d|)^{-L}$.

(2) *If* $j \neq k$, *then*

$$\|s_j R(E \pm i0; K_d) s_k\| = O(d^{-1/2+c\sigma})$$

for $0 < \sigma \ll 1$ *small enough, where* $c > 0$ *is independent of* σ.

This lemma has been proved as [9, Theorem 4.1] and plays the most important role in proving the propositions. We make a brief comment on the above lemma. The flux $\beta_j = \beta_j(h) = \alpha_j/h - [\alpha_j/h]$, $0 \le \beta_j < 1$, depends on h. However, the order estimates in the lemma are uniform in h, which is seen from the argument used in proving Theorem 4.1 in [9]. The next lemma is easy to prove. We skip the proof.

Lemma 4.2 *Let* $\tilde{\psi}_j \in C^\infty(\boldsymbol{R}^2)$ *be as in Proposition 4.2. Then*

$$\begin{aligned} R(E \pm i\varepsilon; K_d)\tilde{\psi}_1 &= \tilde{\psi}_1 R(E \pm i\varepsilon; L_{2d}) + R(E \pm i\varepsilon; K_d) V_1 R(E \pm i\varepsilon; L_{2d}) \\ R(E \pm i\varepsilon; K_d)\tilde{\psi}_2 &= \tilde{\psi}_2 R(E \pm i0; L_{1d}) + R(E \pm i\varepsilon; K_d) V_2 R(E \pm i\varepsilon; L_{1d}), \end{aligned}$$

where $V_1 = [\tilde{\psi}_1, L_{2d}] - (K_d - L_{2d})\,\tilde{\psi}_1$ *and* $V_2 = [\tilde{\psi}_2, L_{1d}] - (K_d - L_{1d})\,\tilde{\psi}_2$.

Remark 4.1. The relations in the above lemma remain true for the limit case $\varepsilon \to 0$ as operators between appropriate weighted L^2 spaces. This is seen from Lemma 4.3 below. For example, we use the relations

$$\begin{aligned} R(E + i0; K_d)\tilde{\psi}_2 &= \tilde{\psi}_2 R(E + i0; L_{1d}) + R(E + i0; K_d) V_2 R(E + i0; L_{1d}) \\ \tilde{\psi}_2 R(E + i0; K_d) &= R(E + i0; L_{1d})\tilde{\psi}_2 + R(E + i0; L_{1d}) V_2^* R(E + i0; K_d) \end{aligned}$$

for the proof of Propositions 4.1, 4.2 and Theorem 1.1.

Lemma 4.3 *Let the notation be as above. Then*

$$\begin{aligned} \|s_1 R(E + i0; K_d) W_2 R(E + i0; L_{1d}) s_1\| &= O(|d|^{-N}) \\ \|s_2 R(E + i0; K_d) W_1 R(E + i0; L_{2d}) s_2\| &= O(|d|^{-N}) \end{aligned}$$

for any $N \gg 1$, *where* $W_1 = (K_d - L_{2d})\tilde{\psi}_1$ *and* $W_2 = (K_d - L_{1d})\tilde{\psi}_2$.

Proof. We prove only the first relation. For brevity, we assume that $d_2 = 0$ (origin), and we write W for W_2. We denote by $W(x, \xi)$ the symbol of operator $W(x, D_x)$, which is a differential operator of first order with coefficients having the support in

$$\Sigma = \{x : |x| > M|d|^\sigma,\ |\hat{x} - \hat{e}| < \delta\}, \quad M \gg 1. \tag{4.2}$$

We introduce a smooth nonnegative partition $\{\rho_\pm,\ \rho_\infty\}$ of unity normalized by

$$\rho_+(\xi) + \rho_-(\xi) + \rho_\infty(\xi) = 1. \tag{4.3}$$

The partition has the following properties : $\operatorname{supp} \rho_\infty \subset \{|\xi|^2 < E/2 \text{ or } |\xi|^2 > 2E\}$ and

$$\operatorname{supp} \rho_\pm \subset \{E/3 < |\xi|^2 < 3E,\ \pm\hat{\xi} \cdot \hat{e} > -\delta\}.$$

According to (4.3), W is decomposed into the sum of three operators

$$W = w_+(x, D_x) + w_-(x, D_x) + w_\infty(x, D_x),$$

where

$$w_\pm(x, D_x) = W(x, D_x)\rho_\pm(D_x), \quad w_\infty(x, D_x) = W(x, D_x)\rho_\infty(D_x).$$

The symbol $w_+(x, \xi)$ of pseudodifferential operator w_+ has support in the outgoing region

$$G_+ = \{(x, \xi) : x \in \Sigma,\ E/3 < |\xi|^2 < 3E,\ \hat{x} \cdot \hat{\xi} > -2\delta\}$$

of the phase space $\boldsymbol{R}^2_x \times \boldsymbol{R}^2_\xi$, where Σ is as in (4.2). On the other hand, $w_-(x, \xi)$ has support in the incoming region

$$G_- = \{(x, \xi) : x \in \Sigma,\ E/3 < |\xi|^2 < 3E,\ \hat{x} \cdot \hat{\xi} < 2\delta\}.$$

These two symbols satisfy

$$|\partial_x^l \partial_\xi^m w_\pm(x, \xi)| \le C_{lm} \left(|\xi| + 1\right)^{-|m|} \left(|x| + d^\sigma\right)^{-|l|}.$$

We now assert that

$$\begin{aligned} \|r_{-L} w_\infty R(E + i0; L_{1d}) s_1\| &= O(|d|^{-N}) && (4.4)\\ \|s_1 R(E + i0; K_d) w_+ r_{-L}\| &= O(|d|^{-N}) && (4.5)\\ \|r_{-L} w_- R(E + i0; L_{1d}) s_1\| &= O(|d|^{-N}), && (4.6) \end{aligned}$$

where r_{-L} is the multiplication operator by $r_{-L}(x) = (|x| + |d|)^L$ with $L \gg 1$ as in Lemma 4.1. If these assertions are verified, then the lemma follows from Lemma 4.1 at once.

We begin by proving (4.4). If $|x| \ge M|d|^\sigma$ and $|\hat{x} - \hat{e}| < \delta$ and if $|\xi|^2 < E/2$ or $|\xi|^2 > 2E$, then the symbol $L_{1d}(x, \xi) - E$ of operator

$$L_{1d} - E = e^{i\zeta_2} \left(K_{1d} - E\right) e^{-i\zeta_2} = e^{i\zeta_2} \left((-i\nabla - B_{1d})^2 - E\right) e^{-i\zeta_2} \sim H_0 - E$$

has a bounded inverse. This enables us to construct an approximation to $w_\infty R(E+i0;L_{1d})$ in the standard way used in constructing parametrices for elliptic operators. If we take account of the fact that $\Sigma \cap S_1 = \emptyset$ ($S_1 = \operatorname{supp} s_1$) and if we make use of Lemma 4.1 to control an error operator rising from the approximate operator, then we can show (4.4).

We proceed to proving (4.5). This is intuitively obvious. In fact, a classical particle with initial state $(x,\xi) \in G_+$ at $t=0$ does not touch the two centers d_1 and d_2 for $t>0$, and it behaves as if it were a free particle. Hence the particle never passes over S_1 for $t>0$. Thus we can give a rough proof for (4.5). We prove it rigorously by constructing an approximation to $R(E+i0;K_d)w_+$. Recall that d_2 is fixed as $d_2 = 0$. We define the real function $\tilde{\zeta}_2 \in C^\infty(\boldsymbol{R}^2)$ by

$$\tilde{\zeta}_2(x) = \eta_2(\gamma(x;-\hat{e})) \quad \text{over } \{|x| > |d|^\sigma\}, \quad \tilde{\zeta}_2 = 0 \quad \text{over } \{|x| < |d|^\sigma/2\}.$$

Then it follows that

$$K_d = e^{i\tilde{\zeta}} H_0 e^{-i\tilde{\zeta}}, \qquad \tilde{\zeta}(x) = \zeta_1(x) + \tilde{\zeta}_2(x), \tag{4.7}$$

on Σ, Σ being as in (4.2). Let $\tilde{\rho}_+(\xi)$ be a smooth real symbol with the property that $\tilde{\rho}_+$ has a slightly larger support than ρ_+ and $\tilde{\rho}_+ = 1$ on the support of ρ_+, and let $q(x)$ be a smooth function such that $q = 1$ on

$$\Sigma_1 = \{|x| > M|d|^\sigma/2,\ |\hat{x}+\hat{e}| > 2\delta\} \supset \Sigma$$

and q has support in $\Sigma_2 = \{|x| > M|d|^\sigma/3,\ |\hat{x}+\hat{e}| > \delta\}$. The relation (4.7) remains true over Σ_2. Hence we have

$$(K_d - E)e^{i\tilde{\zeta}}q = e^{i\tilde{\zeta}}(H_0 - E)q = e^{i\tilde{\zeta}}\left(q(H_0-E) + [H_0,q]\right).$$

We denote by $\tilde{r}_N$ bounded operators such that $(|x|+|d|)^N\,\tilde{r}_N$ and $\tilde{r}_N\,(|x|+|d|)^N$ are both bounded uniformly in d. Since $qw_+(x,D_x) = w_+(x,D_x)$ and since

$$(1-\tilde{\rho}_+)e^{-i\tilde{\zeta}}w_+ = \tilde{r}_N, \quad \tilde{\rho}_+ = \tilde{\rho}_+(D_x),$$

for any $N \gg 1$, we have

$$\begin{aligned}(K_d - E)&e^{i\tilde{\zeta}}qR(E+i0;H_0)\tilde{\rho}_+e^{-i\tilde{\zeta}}w_+\\ &= w_+ + \tilde{r}_N + e^{i\tilde{\zeta}}[H_0,q]R(E+i0;H_0)\tilde{\rho}_+e^{-i\tilde{\zeta}}w_+.\end{aligned}$$

We can take $0 < \delta \ll 1$ so small and $M \gg 1$ so large that the free particle with initial state in G_+ at $t=0$ never passes over $\operatorname{supp}\nabla q$ for $t>0$. This enables us to put

$$[H_0,q]R(E+i0;H_0)\tilde{\rho}_+e^{-i\tilde{\zeta}}w_+ = \tilde{r}_N. \tag{4.8}$$

To see it, we represent the resolvent

$$R(E+i0;H_0) = i\int_0^\infty e^{itE}\exp(-itH_0)\,dt$$

in the integral form of propagator $\exp(-itH_0)$. If x is in the support of ∇q and if $(y,\xi) \in G_+$, then

$$|x-y-2t\xi| \geq c\,(t+|x|+|y|+|d|^\sigma)$$

for $t \geq 0$, and hence

$$\|(|x|+|d|)^N [H_0,q]\exp(-itH_0)\tilde{\rho}_+ e^{-i\tilde{\zeta}} w_+ (|x|+|d|)^N\| \leq c(1+t)^{-2}$$

by partial integration. This implies (4.8). Thus $R(E+i0;K_d)w_+$ takes the form

$$R(E+i0;K_d)w_+ = e^{i\tilde{\zeta}} q R(E+i0;H_0)\tilde{\rho}_+ e^{-i\tilde{\zeta}} w_+ + R(E+i0;K_d)\tilde{r}_N.$$

Since s_1 vanishes on Σ_2, (4.5) follows from Lemma 4.1.

We conclude the proof by showing (4.6). This is verified in almost the same way as (4.5). But we require a slight modification to construct an approximation to

$$w_- R(E+i0;L_{1d}) = \left(R(E-i0;L_{1d})w_-^*\right)^*.$$

We expand w_-^* as $w_-^* = w_{-N} + \tilde{r}_N$ by pseudodifferential calculus, where $w_{-N}(x,\xi)$ has support in G_-. We apply the same argument as above and construct an approximation to $R(E-i0;L_{1d})w_{-N}$. If we have only to take account of the fact that the free particle with initial state in G_- never passes $\operatorname{supp}\nabla q$ for $t<0$, then we get

$$w_- R(E+i0;L_{1d}) = w_{-N}^* e^{i\tilde{\zeta}} \tilde{\rho}_- R(E+i0;H_0) e^{-i\tilde{\zeta}} q + \tilde{r}_N R(E+i0;L_{1d}),$$

where $\tilde{\rho}_-$ has a property similar to that of $\tilde{\rho}_+$. This proves (4.6) and the proof is complete. $\square$

The next lemma is verified in almost the same way as Lemma 4.3. We skip the detail.

Lemma 4.4 *Let the notation be as above. Then*

$$\begin{aligned} \|s_2 R(E+i0;K_d) W_2 R(E+i0;L_{1d}) s_1\| &= O(|d|^{-N}) \\ \|s_1 R(E+i0;K_d) W_1 R(E+i0;L_{2d}) s_2\| &= O(|d|^{-N}) \end{aligned}$$

for any $N \gg 1$.

Lemma 4.5 *Let the notation be as above. Then*

$$\begin{aligned} \|s_1 \left(R(E+i0;K_d) - R(E+i0;L_{1d})\right) s_1\| &= O(|d|^{-1+c\sigma}) \\ \|s_2 \left(R(E+i0;K_d) - R(E+i0;L_{2d})\right) s_2\| &= O(|d|^{-1+c\sigma}) \end{aligned}$$

for some $c>0$ independent of σ.

Proof. Since $s_1\tilde{\psi}_2 = s_1$, we have by Lemma 4.2 (see Remark 4.1 also) that

$$s_1\left(R(E+i0;K_d) - R(E+i0;L_{1d})\right)s_1 = s_1R(E+i0;K_d)V_2R(E+i0;L_{1d})s_1.$$

The commutator $[\tilde{\psi}_2, L_{1d}]$ is a differential operator of first order. The coefficients are bounded uniformly in d and have support in $\{M|d|^\sigma < |x-d_2| < 2M|d|^\sigma\}$. Hence, by elliptic estimate, it follows from Lemma 4.1 (2) that

$$\|s_1R(E+i0;K_d)[\tilde{\psi}_2, L_{1d}]R(E+i0;L_{1d})s_1\| = O(|d|^{-1+c\sigma}).$$

This, together with Lemma 4.3, implies the first estimate in the lemma. The second one can be shown in a similar way, and the proof is complete. □

We are now in a position to prove the two propositions mentioned at the beginning of the section.

Proof of Proposition 4.1. The proposition is nothing but Lemma 4.5. □

Proof of Proposition 4.2. We prove only the first relation. We may assume $s_1\tilde{\psi}_1 = s_2\tilde{\psi}_2 = 0$ and $s_1\tilde{\psi}_2 = s_1$. Then Lemmas 4.1, 4.2, 4.4 and 4.5 imply that

$$\begin{aligned} Z_{12} &= s_2R(E+i0;K_d)[\tilde{\psi}_2, L_{1d}]R(E+i0;L_{1d})s_1 + O_p(|d|^{-N}) \\ &= s_2R(E+i0;L_{2d})[\tilde{\psi}_2, L_{1d}]R(E+i0;L_{1d})s_1 + O_p(|d|^{-3/2+c\sigma}). \end{aligned}$$

We can easily obtain the relation

$$R(E-i0;L_{1d})\tilde{\psi}_1 = \tilde{\psi}_1R(E-i0;L_{0d}) + R(E-i0;L_{1d})V_3R(E-i0;L_{0d}),$$

as in Lemma 4.2, where $V_3 = [\tilde{\psi}_1, L_{0d}] - W_3$ with $W_3 = (L_{1d} - L_{0d})\tilde{\psi}_1$. Since $s_1\tilde{\psi}_1 = 0$, it follows that

$$\tilde{\psi}_1R(E+i0;L_{1d})s_1 = R(E+i0;L_{0d})V_3^*R(E+i0;L_{1d})s_1.$$

We can also show in the same way as in the proof of Lemma 4.3 that

$$\|[\tilde{\psi}_2, L_{1d}]R(E+i0;L_{0d})W_3^*R(E+i0;L_{1d})s_1\| = O(|d|^{-N}).$$

Thus Z_{12} takes the form

$$Z_{12} \sim s_2R(E+i0;L_{2d})[\tilde{\psi}_2, L_{1d}]R(E+i0;L_{0d})[L_{0d}, \tilde{\psi}_1]R(E+i0;L_{1d})s_1$$

with error operator $O_p(|d|^{-3/2+c\sigma})$. By (4.1), we have

$$[L_{0d}, \tilde{\psi}_1] = e^{i(\zeta_1+\zeta_2)}[H_0, \tilde{\psi}_1]e^{-i(\zeta_1+\zeta_2)}$$

and $R(E+i0;L_{0d}) = e^{i(\zeta_1+\zeta_2)}R(E+i0;H_0)e^{-i(\zeta_1+\zeta_2)}$. If we further write $K_{1d} = e^{i\zeta_1}H_0e^{-i\zeta_1}$ on the support of $\nabla\tilde{\psi}_2$, then

$$[\tilde{\psi}_2, L_{1d}] = e^{i\zeta_2}[\tilde{\psi}_2, K_{1d}]e^{-i\zeta_2} = e^{i(\zeta_1+\zeta_2)}[\tilde{\psi}_2, H_0]e^{-i(\zeta_1+\zeta_2)}$$

there. We combine these relations to obtain that Γ_{12} takes the desired form in the proposition. □

5. Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We write f_h for the amplitude $f_h(\omega_- \to \omega_+)$ in question. It follows from Propositions 2.1 and 3.1 that

$$f_h = (-1)^\rho \exp\left(i\rho(\omega_+ - \omega_-)\right) \sum_{1 \le j,k \le 2} g_{jk} + O(h^N)$$

for any $N \gg 1$, where ρ is defined by (1.13) and

$$g_{jk} = (ic(E)/4\pi)h^{1/2}\left(R(E+i0;K_d)Q_{-j}\varphi_0(\omega_-), Q_{+k}\varphi_0(\omega_+)\right).$$

We analyze the behavior as $h \to 0$ of g_{jk}. We can show that the leading and third terms of the asymptotic formula come from g_{jk} with $j = k$ and that the second term comes from g_{jk} with $j \ne k$.

We first deal with the case $j = k$. We have

$$R(E+i0;L_{1d}) = \exp(i\zeta_2)R(E+i0;K_{1d})\exp(-i\zeta_2) \tag{5.1}$$

by (4.1) and also it follows that

$$\begin{aligned} \zeta_+(x) - \zeta_2(x) &= \eta_1(\gamma(x-d_1;-\omega_+)) + \eta_2(\gamma(x-d_2;-\omega_+)) - \eta_2(\gamma(x-d_2;\hat{e})) \\ &= \eta_1(\gamma(x-d_1;-\omega_+)) + \beta_2(\gamma(-\hat{e};-\omega_+) - \pi) \\ \zeta_-(x) - \zeta_2(x) &= \eta_1(\gamma(x-d_1;\omega_-)) + \beta_2(\gamma(-\hat{e};\omega_-) - \pi) \end{aligned}$$

over the support of $\nabla\psi_1$. We make use of Proposition 4.1 to obtain that

$$g_{11} = (ic(E)/4\pi)\kappa_1(h)h^{1/2}\left(R(E+i0;K_{1d})T_{-1}\varphi_0(\omega_-), T_{+1}\varphi_0(\omega_+)\right) + o(h),$$

where $\kappa_1(h) = \exp(i\beta_2\tau(-\hat{e};\omega_-,\omega_+))$ and

$$T_{-1} = e^{i\eta_1(\gamma(x-d_1;\omega_-))}[\psi_1, H_0], \qquad T_{+1} = e^{i\eta_1(\gamma(x-d_1;-\omega_+))}[\psi_1, H_0].$$

Hence we have

$$g_{11} = \kappa_1(h)h^{1/2}g(\omega_- \to \omega_+; E, \beta_1, d_1) + o(h)$$

by Lemma 3.2, where $g(\omega \to \theta; E, \beta, p)$ is the amplitude for the scattering by $2\pi\beta\delta(x-p)$. Since

$$\kappa_1(h)(-1)^\rho \exp\left(i\rho(\omega_+ - \omega_-)\right) = \tau_1(h)(-1)^{\rho_1}\exp(i\rho_1(\omega_+ - \omega_-))$$

with $\rho_1 = [\alpha_1/h]$, this, together with (2.6), implies that

$$(-1)^\rho \exp(i\rho(\omega_+ - \omega_-))g_{11} = \tau_1(h)f_h(\omega_- \to \omega_+; \alpha_1, e_1) + o(h).$$

A similar argument applies to g_{22} and we get

$$(-1)^\rho \exp(i\rho(\omega_+ - \omega_-))g_{22} = \tau_2(h) f_h(\omega_- \to \omega_+; \alpha_2, e_2) + o(h).$$

Thus we can derive the leading term of the asymptotic formula.

The second term comes from g_{jk} with $j \ne k$. We consider g_{12} only. The derivation is based on the lemma below, which is proved after completing the proof of the theorem.

Lemma 5.1

$$\begin{array}{rcl} s_2 R(E+i0; H_0) s_1 & = & (ic(E)/4\pi)\,|d|^{-1/2} s_2 \Phi_+ s_1 + O_p(|d|^{-3/2+c\sigma}) \\ s_1 R(E+i0; H_0) s_2 & = & (ic(E)/4\pi)\,|d|^{-1/2} s_1 \Phi_- s_2 + O_p(|d|^{-3/2+c\sigma}), \end{array}$$

where $\Phi_\pm = \varphi_0(\pm\hat{e}) \otimes \varphi_0(\pm\hat{e})$ *acts as*

$$\Phi_\pm u = (u, \varphi_0(\pm\hat{e}))\,\varphi_0(\pm\hat{e}) = \left(\int u(y)\varphi_0(y; E, \mp\hat{e})\,dy\right)\varphi_0(x; E, \pm\hat{e}).$$

We write $R(E+i0; L_{jd})$ as in (5.1). Since

$$\zeta_+(x) - \zeta_1(x) = \eta_2(\gamma(x - d_2; -\omega_+)) + \beta_1(\gamma(\hat{e}; -\omega_+) - \pi)$$

over the support of $\nabla\psi_2$ and since

$$\zeta_-(x) - \zeta_2(x) = \eta_1(\gamma(x - d_1; \omega_-)) + \beta_2(\gamma(-\hat{e}; \omega_-) - \pi)$$

over the support of $\nabla\psi_1$, it follows from Lemmas 5.1, 3.2 (see Remark 3.1 also) and Proposition 4.2 that

$$g_{12} = \kappa_{12}(h) h g(\omega_- \to \hat{e}; E, \beta_1, d_1) g(\hat{e} \to \omega_+; E, \beta_2, d_2)|e|^{-1/2} + o(h^{2-c\sigma}),$$

where

$$\kappa_{12}(h) = \exp(-i(\beta_2 - \beta_1)\pi)\exp(i\beta_2\gamma(-\hat{e}; \omega_-))\exp(-i\beta_1\gamma(\hat{e}; -\omega_+)).$$

If we interpret $\gamma(\omega; \omega)$ as $\gamma(\omega; \omega) = (0 + 2\pi)/2 = \pi$, then

$$\tau_{12}(h) = \kappa_{12}(h)\exp(i\rho_2(\hat{e} - \omega_-))\exp(-i\rho_1(\hat{e} - \omega_+)),$$

so that $\kappa_{12}(h) = \exp(-i\rho_2(\hat{e} - \omega_-))\exp(i\rho_1(\hat{e} - \omega_+))\tau_{12}(h)$. By (2.6), we have

$$\begin{array}{rcl} h^{1/2} g(\omega_- \to \hat{e}; E, \beta_1, d_1) & = & (-1)^{\rho_1}\exp(-i\rho_1(\hat{e} - \omega_-)) f_h(\omega_- \to \hat{e}; \alpha_1, e_1) \\ h^{1/2} g(\hat{e} \to \omega_+; E, \beta_2, d_2) & = & (-1)^{\rho_2}\exp(-i\rho_2(\omega_+ - \hat{e})) f_h(\hat{e} \to \omega_+; \alpha_2, e_2). \end{array}$$

The product

$$\exp(-i\rho_1(\hat{e}-\omega_-))\exp(-i\rho_2(\omega_+-\hat{e}))\exp(-i\rho_2(\hat{e}-\omega_-))\exp(i\rho_1(\hat{e}-\omega_+))$$

equals $\exp(-i\rho(\omega_+-\omega_-))$ and hence the second term is obtained.

We give only a sketch for the derivation of the third term. As stated above, this is obtained from g_{jk} with $j=k$ by repeated use of the resolvent identities in Lemma 4.2. We consider only the case $j=k=1$. We begin with the relation

$$Q_{+1}^*R(E+i0;K_d)Q_{-1}\sim Q_{+1}^*R(E+i0;L_{1d})Q_{-1}+X$$

with negligible error operator, where

$$X=Q_{+1}^*R(E+i0;K_d)[\tilde{\psi}_2,L_{1d}]R(E+i0;L_{1d})Q_{-1}.$$

The third term comes from X. By Lemma 4.2, we have

$$\tilde{\psi}_1R(E+i0;L_{1d})\sim R(E+i0;L_{0d})\tilde{\psi}_1+R(E+i0;L_{0d})[L_{0d},\tilde{\psi}_1]R(E+i0;L_{1d}).$$

Since $[\tilde{\psi}_2,L_{1d}]\tilde{\psi}_1=[\tilde{\psi}_2,L_{1d}]$ and since $\tilde{\psi}_1Q_{-1}=0$ for $M\gg 1$, it follows that

$$[\tilde{\psi}_2,L_{1d}]R(E+i0;L_{1d})Q_{-1}\sim[\tilde{\psi}_2,L_{1d}]R(E+i0;L_{0d})[L_{0d},\tilde{\psi}_1]R(E+i0;L_{1d})Q_{-1}.$$

This shows that

$$X\sim Q_{+1}^*R(E+i0;K_d)[\tilde{\psi}_2,L_{1d}]R(E+i0;L_{0d})[L_{0d},\tilde{\psi}_1]R(E+i0;L_{1d})Q_{-1}.$$

We repeat a similar argument for $Q_{+1}^*R(E+i0;K_d)[\tilde{\psi}_2,L_{1d}]$. We set

$$\tilde{\psi}_3(x)=1-\chi(|d|^{-\sigma}|x-d_1|/M^2),\quad \tilde{\psi}_4(x)=1-\chi(|d|^{-\sigma}|x-d_2|/M^2).$$

We have

$$\tilde{\psi}_4R(E+i0;K_d)\sim R(E+i0;L_{1d})\tilde{\psi}_4+R(E+i0;L_{1d})[L_{1d},\tilde{\psi}_4]R(E+i0;K_d)$$

by Lemma 4.2. Since $\tilde{\psi}_3[L_{1d},\tilde{\psi}_4]=[L_{1d},\tilde{\psi}_4]$ and since $\tilde{\psi}_4[\tilde{\psi}_2,L_{1d}]=Q_{+1}^*\tilde{\psi}_3=0$ for $M\gg 1$, we get

$$\begin{aligned}X\ \sim\ & Q_{+1}^*R(E+i0;L_{1d})[\tilde{\psi}_3,L_{0d}]R(E+i0;L_{0d})[L_{1d},\tilde{\psi}_4]R(E+i0;K_d)\times\\ &\times\ [\tilde{\psi}_2,L_{1d}]R(E+i0;L_{0d})[L_{0d},\tilde{\psi}_1]R(E+i0;L_{1d})Q_{-1}.\end{aligned}$$

By Proposition 4.1, we see that

$$[L_{1d},\tilde{\psi}_4]\left(R(E+i0;K_d)-R(E+i0;L_{2d})\right)[\tilde{\psi}_2,L_{1d}]=O_p(|d|^{-1+c\sigma})$$

for some $c>0$, and also it follows from Lemma 5.1 that

$$\begin{aligned}[\tilde{\psi}_3,L_{0d}]R(E+i0;L_{0d})[L_{1d},\tilde{\psi}_4]&=O_p(|d|^{-1/2+c\sigma})\\ [\tilde{\psi}_2,L_{1d}]R(E+i0;L_{0d})[L_{0d},\tilde{\psi}_1]&=O_p(|d|^{-1/2+c\sigma})\end{aligned}$$

with another $c > 0$. Hence we have

$$\begin{aligned} X &= Q_{+1}^* R(E+i0; L_{1d})[\tilde{\psi}_3, L_{0d}]R(E+i0; L_{0d})[L_{1d}, \tilde{\psi}_4]R(E+i0; L_{2d}) \times \\ &\qquad \times\ [\tilde{\psi}_2, L_{1d}]R(E+i0; L_{0d})[L_{0d}, \tilde{\psi}_1]R(E+i0; L_{1d})Q_{-1} + O_p(|d|^{-2+c\sigma}). \end{aligned}$$

The operator on the right side yields one of the third terms with factor $\tau_{121}(h)$ as a coefficient, while the other term is obtained from g_{22} in a similar way. The proof of the theorem is now complete. $\square$

It remains to prove Lemma 5.1.

Proof of Lemma 5.1. The resolvent $R(E+i0; H_0)$ has the integral kernel

$$G_0(x, y; E) = (i/4)H_0^{(1)}(\sqrt{E}|x-y|)$$

and the Hankel function $H_0^{(1)}(z)$ of first kind and order zero is known to behave like

$$H_0^{(1)}(z) = (2/\pi)^{1/2}\exp(i(z-\pi/4))z^{-1/2}\left(1 + O(|z|^{-1})\right)$$

at infinity. Hence

$$G_0(x, y; E) = c_0(E)\exp(i\sqrt{E}|x-y|)|x-y|^{-1/2}\left(1 + O(|x-y|^{-1})\right)$$

as $|x-y| \to \infty$, where

$$c_0(E) = \left(8\pi\sqrt{E}\right)^{-1/2} e^{i\pi/4} = ic(E)/(4\pi).$$

If $x \in S_2$ and $y \in S_1$, then it follows that

$$|x-y| = |d| + \hat{e}\cdot((x-d_2)-(y-d_1)) + O(|d|^{-1+2\sigma})$$

and hence we have

$$\exp(i\sqrt{E}|x-y|) = \exp(i\sqrt{E}\hat{e}\cdot x)\exp(-i\sqrt{E}\hat{e}\cdot y)\left(1 + O(|d|^{-1+2\sigma})\right).$$

This yields the first relation in the lemma. The second one is obtained in a similar way. We skip the detail. $\square$

It is in general difficult to control the error terms on the right side of the relations in Lemma 5.1. It prevents us from deriving the asymptotic formula with further terms associated with the trajectories oscillating between centers of two solenoidal fields.

6. Additional remarks

We conclude the paper by making three comments to the main theorem.

Remark 6.1. We discuss the case in which $\omega_- = \pm\hat{e}$ or $\omega_+ = \pm\hat{e}$. We can derive the asymptotic formula with only the leading term even in this case. The formula takes a different form. We assume that $\omega_- \neq \omega_+$. If, for example, $\omega_- = \hat{e}$ and $\omega_+ \neq -\hat{e}$, then $f_h = f_h(\hat{e} \to \omega_+)$ behaves like

$$\begin{aligned} f_h &= \exp(i(\alpha_2/h)\tau(-\hat{e};\hat{e},\omega_+))f_h(\hat{e}\to\omega_+;\alpha_1,e_1) \\ &+ \cos(\alpha_1/h)\pi\exp(i(\alpha_1/h)\tau(\hat{e};\hat{e},\omega_+))f_h(\hat{e}\to\omega_+;\alpha_2,e_2) + o(h^{1/2}) \end{aligned}$$

and the backward scattering amplitude $f_h = f_h(\hat{e} \to -\hat{e})$ obeys

$$f_h = f_h(\hat{e}\to-\hat{e};\alpha_1,e_1) + \cos^2(\alpha_1/h)\pi\, f_h(\hat{e}\to-\hat{e};\alpha_2,e_2) + o(h^{1/2}).$$

Here we note that $\gamma(\omega;\omega)$ is still interpreted as $\gamma(\omega;\omega) = \pi$.

Remark 6.2. The theorem can be extended to the case of scattering by many solenoidal magnetic fields. We consider the operator

$$H_h = (-ih\nabla - A)^2, \quad A(x) = \sum_{j=1}^{N} \alpha_j \Lambda(x - e_j)$$

with $e_j \neq e_k$. This operator is self–adjoint under the conditions $\lim_{|x-e_j|\to 0} |u(x)| < \infty$ for $1 \le j \le N$. We set $e_{jk} = e_k - e_j$ and $\hat{e}_{jk} = e_{jk}/|e_{jk}|$ for pair (j,k) with $j \neq k$. We assume that

$$\text{there is no other center on the line joining } e_j \text{ and } e_k. \tag{6.1}$$

If $\omega_- \neq \hat{e}_{jk}$ and $\omega_+ \neq \hat{e}_{jk}$, then $f_h = f_h(\omega_- \to \omega_+)$ is shown to obey

$$\begin{aligned} f_h &= \sum_{j=1}^{N} \tau_j(h) f_h(\omega_-\to\omega_+;\alpha_j,e_j) \\ &+ \sum_{1\le j,k\le N,\ j\neq k} \tau_{jk}(h)|e_{jk}|^{-1/2} f_h(\omega_-\to\hat{e}_{jk};\alpha_j,e_j) f_h(\hat{e}_{jk}\to\omega_+;\alpha_k,e_k) \\ &+ o(h), \qquad h \to 0, \end{aligned}$$

for $\omega_- \neq \omega_+$, where

$$\tau_j(h) = \prod_{k=1,\ k\neq j}^{N} \exp\left(i(\alpha_k/h)\tau(\hat{e}_{kj};\omega_-,\omega_+)\right)$$

and

$$\tau_{jk}(h) = \left(\prod_{1\le m\le N,\ m\neq j} e^{i(\alpha_m/h)\tau(\hat{e}_{mj};\omega_-,\hat{e}_{jk})}\right)\left(\prod_{1\le m\le N,\ m\neq k} e^{i(\alpha_m/h)\tau(\hat{e}_{mk};\hat{e}_{jk},\omega_+)}\right).$$

We can also compute the third term, but it takes too lengthy a form to write out.

Remark 6.3. The Aharonov–Bohm effect is much more strongly reflected in the case when assumption (6.1) is violated. For example, this is the case when all the centers of fields are on an even line. For brevity, we consider the case $N = 3$. We assume that $\hat{e} = \hat{e}_{12} = \hat{e}_{23} = (1, 0)$. The asymptotic formula depends on the ratio of distances between adjacent centers as well as on the fluxes of fields. We define $\theta_\pm \pi$ as the angles between vectors $(0, \pm 1)$ and $(1, -\delta_0)$, where $\delta_0 = (|e_{23}|/|e_{12}|)^{1/2}$. By definition, $0 < \theta_\pm < 1$ and $\theta_+ + \theta_- = 1$. If $\omega_+ \neq \pm\hat{e}$, then we can prove that $f_h = f_h(\hat{e} \to \omega_+)$ behaves like

$$\begin{aligned} f_h &= \tau_1(h) f_h(\hat{e} \to \omega_+; \alpha_1, e_1) + \tau_2(h) \cos(\alpha_1/h)\pi\, f_h(\hat{e} \to \omega_+; \alpha_2, e_2) \\ &+ \tau_3(h)\left(\theta_+ \cos(\alpha_1/h + \alpha_2/h)\pi + \theta_- \cos(\alpha_1/h - \alpha_2/h)\pi\right) f_h(\hat{e} \to \omega_+; \alpha_3, e_3) \\ &+ o(h^{1/2}), \qquad h \to 0, \end{aligned}$$

where

$$\begin{aligned} \tau_1(h) &= \exp(i(\alpha_2/h)\tau(-\hat{e}; \hat{e}, \omega_+)) \exp(i(\alpha_3/h)\tau(-\hat{e}; \hat{e}, \omega_+)) \\ \tau_2(h) &= \exp(i(\alpha_1/h)\tau(\hat{e}; \hat{e}, \omega_+)) \exp(i(\alpha_3/h)\tau(-\hat{e}; \hat{e}, \omega_+)) \\ \tau_3(h) &= \exp(i(\alpha_1/h)\tau(\hat{e}; \hat{e}, \omega_+)) \exp(i(\alpha_2/h)\tau(\hat{e}; \hat{e}, \omega_+)). \end{aligned}$$

We can also show that the backward amplitude $f_h = f_h(\hat{e} \to -\hat{e})$ obeys

$$f_h \sim f_1 + (\cos(\alpha_1/h)\pi)^2 f_2 + \left(\theta_+ \cos((\alpha_1 + \alpha_2)\pi/h) + \theta_- \cos((\alpha_1 - \alpha_2)\pi/h)\right)^2 f_3$$

as $h \to 0$, where $f_j = f_h(\hat{e} \to -\hat{e}; \alpha_j, e_j)$.

The results in Remarks 6.1 and 6.3 are obtained as immediate consequences of the previous works [8, Theorem 1.1] and [9, Theorem 1.2, Remark 1.1] respectively, where we have studied the magnetic scattering by two or three point–like fields at large separation.

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