On the Riemann-Hilbert problem and stable vector bundles on the Riemann sphere*

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À la mémoire d'Andrey Bolibrukh

Abstract

In this note we give a brief survey of recent results on the classical Riemann-Hilbert problem for differential equations on the Riemann sphere. We emphasize geometrical aspects of the problem involving the notion of stability of vector bundles with connections.

Keywords: Riemann-Hilbert problem, vector bundles, connections, irreducibility, stability, monodromy representation, fuchsian differential systems. 2000 MSC: 35C10, 35C20.

1 Introduction

Let us briefly recall what is meant by the Riemann-Hilbert problem for differential equations on the Riemann sphere. This problem was included by D. Hilbert in his famous list under the number twenty one and can be reformulated as follows:

Given a representation of the fundamental group of the punctured Riemann sphere,

$$\chi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S; z_0) \to \mathrm{GL}(p, \mathbb{C})$$

where $S = \{a_1, \ldots, a_2\}$ *is a set of points in* $\mathbb{C}$*, does there exist a fuchsian differential system on* $\mathbb{P}^1(\mathbb{C})$*,*

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i}{z - a_i} \right) y,$$

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*This note is an extended version of the talk given by the author in Luminy *where* B_i*,* $1 \leq i \leq n$*, are* $p \times p$*-matrices with entries in* $\mathbb{C}$ *satisfying* $\sum_{i=1}^{n} B_i = 0$*, (so that* ∞ *is not a singular point), for which* χ *is a monodromy representation ?*

This problem has a long story. For more than seventy years, it was commonly believed that it had a positive solution and had been completely solved by J. Plemelj in 1908. But at the beginning of the eighties an error was discovered in the proof, see [11]. It turned out that J. Plemelj could only obtain a positive answer in the case of differential systems with regular singularities, see [19]. Later on, W. Dekkers solved the problem positively in dimension 2, see [5].

Then, in 1989, A. Bolibrukh gave a final and surprising answer to the problem. *It turns out that this problem has in general a negative answer.* A. Bolibrukh published an important counterexample for a representation of dimension three with four singular points on the Riemann sphere. He also classified all the representations in dimension three that can be realized as monodromy representations of fuchsian systems, see [1],[3]. This classification in dimension three has been established recently using tools from complex algebraic geometry, see [9]. In 2000, a classification for the representations in dimension four was given by A. Gladyshev, see [10].

In 1992, A. Bolibrukh showed that for irreducible representations, the problem has a positive solution, see [1],[3], a result also obtained independently by V. Kostov at the same time, see [12]. More recently, the subject has been revisited in a more algebraic setting, see [21], [23], and generalizations have been obtained when $\mathbb{P}^1(\mathbb{C})$ is replaced by a Riemann surface of positive genus, see [7].

2 The geometrical approach

The methods introduced by A. Bolibrukh use to a large extend the geometry of vector bundles on the Riemann sphere. To understand his approach, we will state the Riemann-Hilbert problem in a more geometrical setting.

Let us first recall the method of attack of P. Deligne to handle the problem in the case of regular singularities, see [6].

It is a classical fact that starting from the representation χ, one can construct a vector bundle $\hat{E}$ on the open manifold $\mathbb{P}^1(\mathbb{C}) \setminus S$, endowed with a flat holomorphic connection $\hat{\nabla}$ with χ as its holonomy or monodromy representation, see [6]. By a classical theorem of Stein, we know that $\hat{E}$ is in fact holomorphically trivial on $\mathbb{P}^1(\mathbb{C}) \setminus S$. In terms of differential equations, one gets a differential system

$$(D) \qquad dy = \omega y$$

with χ as monodromy representation and S as the singular divisor of the differential form ω, and with regular singularities at the points a_i, $1 \leq i \leq n$, see also [20].

Now consider, for all $1 \leq i \leq n$, the matrices

$$E_i = \frac{1}{2i\pi} \log \chi(\sigma_i),$$

for a given determination of the logarithm, where σ_i denotes the homotopy class of a simple loop around a_i with base point z_0 enclosing no other a_j. Modulo conjugation with a matrix S_i, we may assume that the matrix E_i is upper triangular, for $1 \leq i \leq n$.

We also consider local differential systems $dy = \omega_i y$ defined on a neighborhood of a_i by

$$\omega_i(z) = \frac{E_i}{(z-a_i)}dz.$$

By construction, each such local system is fuchsian at a_i and has the requested local monodromy. The idea of P. Deligne was to glue together these local systems with the help of the vector bundle $(\hat{E}, \hat{\nabla})$ in order to get a vector bundle E on $\mathbb{P}^1(\mathbb{C})$ endowed with a connection ∇ which has logarithmic singularities at the points a_i, $1 \leq i \leq n$. This construction of E provides what P. Deligne calls the *canonical extension* of $\hat{E}$ on $\mathbb{P}^1(\mathbb{C})$.

Instead canonical extensions, A. Bolibrukh considered extensions of $\hat{E}$ on $\mathbb{P}^1(\mathbb{C})$ by means of local fuchsian systems of the form $dy = \omega_i^{\Lambda_i} y$ where

$$\omega_i^{\Lambda_i}(z) = (\Lambda_i + (z-a_i)^{\Lambda_i} E_i (z-a_i)^{-\Lambda_i})\frac{dz}{z-a_i},$$

where Λ_i is a diagonal matrix with integer entries such that the matrix

$$(z-a_i)^{\Lambda_i} E_i (z-a_i)^{-\Lambda_i}$$

is holomorphic at a_i, for all $1 \leq i \leq n$. This idea came from what are called *Levelt decompositions* of fundamental matrices of differential systems with regular singularities, see [1],[3],[8],[13].

This construction provides an infinite family $\mathcal{E}$ of vector bundles $(E^\Lambda, \nabla^\Lambda)$ on $\mathbb{P}^1(\mathbb{C})$, where the connections ∇^Λ have logarithmic singularities on S, parametrized by $n-$tuples $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$.

A. Bolibrukh has moreover shown that any extension of $\hat{E}$ on $\mathbb{P}^1(\mathbb{C})$ with a connection ∇ having logarithmic singularities, can be obtained in this manner [1],[3]. As a result, the Riemann-Hilbert problem can be stated as follows:

A representation

$$\chi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S; z_0) \to \mathrm{GL}(p, \mathbb{C})$$

is given. Does there exist n diagonal matrices Λ_i, $1 \leq i \leq n$, with integer entries such that $(E^\Lambda, \nabla^\Lambda) \in \mathcal{E}$ is holomorphically trivial on $\mathbb{P}^1(\mathbb{C})$ for $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$.

The result obtained in 1992 by A. Bolibrukh can be reformulated in the following way:

Let

$$\chi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S; z_0) \to \mathrm{GL}(p, \mathbb{C})$$

be an irreducible representation. Then, there exist n diagonal matrices Λ_i, $1 \leq i \leq n$, with integer entries such that $(E^\Lambda, \nabla^\Lambda) \in \mathcal{E}$ is holomorphically trivial on $\mathbb{P}^1(\mathbb{C})$ for $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$.

One of the main geometrical ingredients in the proof was to observe that if the representation χ is irreducible then the splitting type of the bundle E^Λ on $\mathbb{P}^1(\mathbb{C})$,

$$(\mathcal{O}) \qquad E^\Lambda \cong \mathcal{O}(c_1) \oplus \cdots \oplus \mathcal{O}(c_p)$$

satisfies the important property that

$$|c_i - c_j| \leq (n-2)p, \tag{B}$$

for all $1 \leq i, j \leq p$.

3 The Riemann-Hilbert problem and stability assumptions

In this section, we mainly restate recent results of A. Bolibrukh that give new sufficient conditions to solve positively the Riemann-Hilbert problem on Riemann surfaces of genus $g \geq 0$. But, for simplicity, we will focus here on the case of the Riemann sphere only and we will explain the results obtained in [14] in a more geometrical language.

It is known, from the work of C. Simpson, that the notion of irreducibility is actually related to the concept of *stability* of vector bundles with connections, see [22]. Let us first recall the definition of it.

Definition 1 *A pair (F, ∇) of a vector bundle F and a connection ∇ is called* stable *if for any proper subbundle F', $0 \subsetneq F' \subsetneq F$, that is stabilized by the connection ∇,*

$$\nabla(F') \subset F' \otimes \Omega^1(\log S),$$

the slope $\mu(F') = \deg(F')/\mathrm{rank}(F')$ of F' is smaller than the slope $\mu(F)$ of F,

$$\mu(F') < \mu(F). \tag{$*$}$$

This notion of stability has to be distinguished from the classical one, where the inequality $(*)$ has to be satisfied for *all* proper subbundles F' of F, see for instance [18]. In particular, one easily sees that there exists no stable (in the classical sense) vector bundle F of degree zero on $\mathbb{P}^1(\mathbb{C})$. Indeed, one should have the relations $c_1 + \cdots + c_p = 0$ and $c_i < 0$, $1 \leq i \leq p$, for the splitting $(\mathcal{O})$ of the bundle F on $\mathbb{P}^1(\mathbb{C})$, which is impossible.

We are now able to state the main result of this note, see [4].

Theorem 1 *Let*

$$\chi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S; z_0) \to \mathrm{GL}(p, \mathbb{C})$$

be a representation. Assume that among the constructed pairs $(E^\Lambda, \nabla^\Lambda) \in \mathcal{E}$, there is a stable pair $(E^{\Lambda^0}, \nabla^{\Lambda^0})$. Then, one can construct another pair $(E^{\tilde{\Lambda}^0}, \nabla^{\tilde{\Lambda}^0}) \in \mathcal{E}$ that is stable, of degree zero and holomorphically trivial on $\mathbb{P}^1(\mathbb{C})$. The Riemann-Hilbert problem has therefore a positive solution for the representation χ.

Sketch of Proof The first part of the proof involves much calculation. Starting from the initial stable pair $(E^{\Lambda^0}, \nabla^{\Lambda^0}) \in \mathcal{E}$, one constructs a new stable pair $(E^{\Lambda^1}, \nabla^{\Lambda^1}) \in \mathcal{E}$ of *degree zero*, with large differences (in fact larger than the integer $(n-2)p^2$) between the entries $\lambda_i^{j,1}$, $1 \leq j \leq p$ of each Λ_i^1. This property turns out to be crucial as we will see later. The details of this construction are explained in [4].

The vector bundle $\hat{E}$ on $\mathbb{P}^1(\mathbb{C}) \setminus S$ (see section 2) is described by means of a locally constant cocycle $\{g_{\alpha,\beta}\}_{\alpha,\beta\in\mathcal{L}}$ corresponding to a covering $\{U_\alpha\}_{\alpha\in\mathcal{L}}$ of $\mathbb{P}^1(\mathbb{C}) \setminus S$. By construction, the vector bundle E^{Λ^1} is described by a cocycle $\{g_{\alpha,\beta}(z), g_{i,\alpha}(z)\}_{1\le i\le n,\alpha,\beta\in\mathcal{L}}$, for a covering $\{O_i, U_\alpha\}_{1\le i\le n,\alpha\in\mathcal{L}}$ of $\mathbb{P}^1(\mathbb{C})$ which is defined as follows. For a small neighborhood O_i of a_i, $1 \le i \le n$, the function $g_{i,\alpha}(z)$ defined on $O_i \cap U_\alpha$ is of the form

$$g_{i,\alpha}(z) = (z - a_i)^{\Lambda_i^1}(z - a_i)^{E_i}.$$

From now on, to simplify the notation, we assume that $a_1 = 0$. Again, by the result of Stein, the vector bundle E^{Λ^1} is holomorphic trivial on $\mathbb{P}^1(\mathbb{C}) \setminus \{0\}$, and without loss of generality, we may assume that all the functions $g_{i,\alpha}(z)$, for $i \neq 1$, $\alpha \in \mathcal{L}$, split as products $g_{i,\alpha}(z) = \Gamma_i^{-1}(z)\Gamma_\alpha(z)$ where $\Gamma_i^{-1}(z)$ is holomorphic invertible on O_i and $\Gamma_\alpha(z)$ is holomorphic invertible on U_α. By the holomorphic triviality of $\hat{E}$ on $\mathbb{P}^1(\mathbb{C}) \setminus S$, the functions $g_{\alpha,\beta}$ also split.

From the decomposition $(\mathcal{O})$ for the vector bundle E^{Λ^1}, we get in particular that there exist holomorphic invertible matrices $\Gamma_1(z)$ (resp. $\Gamma_\alpha(z)$) on a neighborhood of 0 (resp. on a neighborhood of ∞) such that

$$(C) \qquad \Gamma_1(z)z^K\Gamma_\alpha(z) = g_{1,\alpha}(z) = z^{\Lambda_1^1}z^{E_1},$$

on $O_1 \cap U_\alpha$, where

$$K = \operatorname{diag}(c_1, \ldots, c_p)$$

and $c_1 \ge \cdots \ge c_p$ with $c_1 + \cdots + c_p = 0$.

Now, the geometrical key-ingredient of the proof is that the boundedness property of the splitting type is preserved when one replaces the notion of irreducibility by the notion of stability. More precisely, when a pair $(E^\Lambda, \nabla^\Lambda)$ is stable, then we get the estimates

$$(B) \qquad |c_i - c_j| \le (n - 2)p,$$

for all $1 \le i, j \le p$, in its decomposition $(\mathcal{O})$ on $\mathbb{P}^1(\mathbb{C})$. For an analytical proof, see [14] and for a more geometrical proof based on Harder-Narasimhan filtrations, see [4].

On the other hand, due to a lemma of A. Bolibrukh, see [3], there exists a matrix $\tilde{\Gamma}_\alpha(z)$, holomorphic invertible on a neighborhood of infinity such that

$$\tilde{\Gamma}_\alpha(z)z^{-K}\Gamma_1^{-1}(z) = \tilde{\Gamma}_1(z)z^{K^\sigma},$$

where $\tilde{\Gamma}_1(z)$ is holomorphic invertible on a neighborhood of 0, and

$$K^\sigma = \operatorname{diag}(c_{\sigma(1)}, \ldots, c_{\sigma(p)})$$

with σ a permutation of $\{1, \ldots, p\}$. From (C) and the latter formula, we deduce that

$$(S) \qquad z^{K^\sigma}g_{1,\alpha}(z) = (\tilde{\Gamma}_1(z))^{-1}\tilde{\Gamma}_\alpha(z)\Gamma_\alpha(z),$$

on $O_1 \cap U_\alpha$, where $(\tilde{\Gamma}_1(z))^{-1}$ is holomorphic invertible on a neighborhood of 0 and $\tilde{\Gamma}_\alpha(z)\Gamma_\alpha(z)$ is holomorphic invertible on a neighborhood of ∞.

Let

$$\tilde{g}_{1,\alpha}(z) = z^{\Lambda_1^1+K^\sigma}z^{E_1}$$

and $\tilde{g}_{i,\alpha}(z) = g_{i,\alpha}(z)$, for $i \neq 1$.

We notice the crucial fact that the matrix

$$z^{(\Lambda_1^1 + K^\sigma)} E_1 z^{-(\Lambda_1^1 + K^\sigma)},$$

is holomorphic invertible at 0, since by construction the diagonal matrix Λ_1^1 has entries with differences larger than $(n-2)p^2$ and the matrix K^σ has entries whose differences are bounded by $(n-2)p$.

By construction, the cocycle $\{\tilde{g}_{i,\alpha}, g_{\alpha,\beta}\}$ related to the covering $\{O_i, U_\alpha\}_{1\le i\le n, \alpha\in\mathcal{L}}$ describes the vector bundle $(E^{\tilde{\Lambda}^0}, \nabla^{\tilde{\Lambda}^0})$ where $\tilde{\Lambda}^0 = \{\Lambda_1^1 + K^\sigma, \Lambda_2^1, \ldots, \Lambda_n^1\}$. From the relation (S), we finally get that the vector bundle $E^{\tilde{\Lambda}^0}$ is holomorphically trivial on $\mathbb{P}^1(\mathbb{C})$, which proves the result. □

Remark: One observes that if the representation χ is irreducible, then by definition the pairs $(E^\Lambda, \nabla^\Lambda)$ are stable for all Λ. From Theorem 1, we recover the fact that the Riemann-Hilbert problem has a positive solution for irreducible representations.

In the case of a reducible representation χ, we get a more precise result. The following theorem restates in a geometrical setting the main result obtained in [14].

Theorem 2 *Let*

$$\chi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S; z_0) \to \mathrm{GL}(p, \mathbb{C})$$

be a reducible representation. Assume that among the constructed pairs $(E^\Lambda, \nabla^\Lambda)$, there is a stable pair $(E^{\Lambda^0}, \nabla^{\Lambda^0})$. Then, one can construct a pair $(E^{\tilde{\Lambda}^0}, \nabla^{\tilde{\Lambda}^0})$ which is holomorphically trivial on $\mathbb{P}^1(\mathbb{C})$ and which in addition has a holomorphically trivial proper subbundle $0 \subsetneq \tilde{F} \subsetneq E^{\tilde{\Lambda}^0}$ that is stabilized by the connection $\nabla^{\tilde{\Lambda}^0}$.

In terms of differential equations, the representation χ can be realized as the monodromy representation of a fuchsian system

$$\frac{dy}{dz} = \left(\sum_{i=1}^{n} \frac{B_i}{z - a_i} \right) y,$$

with $\sum_{i=1}^n B_i = 0$, where the coefficient matrices B_i are reducible,

$$B_i = \begin{pmatrix} B_i^1 & * \\ 0 & B_i^2 \end{pmatrix},$$

for all $1 \le i \le n$.

Sketch of Proof The idea in the first step of the proof is the following. From $(E^{\Lambda^0}, \nabla^{\Lambda^0})$, one construct another a pair $(E^{\Lambda^1}, \nabla^{\Lambda^1}) \in \mathcal{E}$ which has a subbundle F_1 stabilized by the connection ∇^{Λ^1}. This pair is constructed in such a way that the pairs $(F_1, \nabla^{\Lambda^1}_{|F_1})$ and $(E^{\Lambda^1}/F_1, \nabla_q^{\Lambda^1})$, where $\nabla_q^{\Lambda^1}$ is the connection constructed from ∇^{Λ^1} on the quotient bundle E^{Λ^1}/F_1, are *stable of degree zero*, and as in the proof of Theorem 1, with large differences between the entries $\lambda_i^{j,1}$, $1 \le j \le p$ of each Λ_i^1. The details of this construction are explained in [14]. The rest of the proof follows the same lines as the proof of Theorem 1 and will not be reproduced here, see [14]. □

Several applications of these results have been investigated.

- A new method of constructing counterexamples to the Riemann-Hilbert problem by means of direct sums of representations, has been obtained in special cases, see [15].

- Later on, the reduction procedure introduced in Theorem 2 has been applied in the framework of isomonodromic Schlesinger deformations. A constructive method to get solutions of non-linear systems of partial differential equations called *Schlesinger equations* has been obtained. These equations are written as follows,

  $$(S) \qquad dA_i(a) = - \sum_{j=1, j\neq i}^{n} \frac{[A_i(a), A_j(a)]}{a_i - a_j} d(a_i - a_j) \quad , \quad i = 1, \dots, n,$$

  and are obtained as isomonodromic deformations of a family of fuchsian systems

  $$(F_a) \qquad \frac{dT}{dx} = \left(\sum_{i=1}^{n} \frac{A_i(a)}{x - a_i} \right) T \quad , \quad \sum_{i=1}^{n} A_i(a) = 0$$

  depending holomorphically on the parameter $a = (a_1, \dots, a_n) \in D(a^0)$, where $D(a^0)$ is a small disck with center $a^0 = (a_1^0, \dots, a_n^0)$ in the space $\mathbb{C}^n \setminus \cup_{i\neq j}\{(a_i - a_j) = 0\}$.

  The method consists in *rational* reductions of these equations to blocked upper triangular forms. More precisely, under the hypothesis of reducibility of the monodromy representation of the family (F_a), we have shown that there exist *rational* transformations Γ_i in their arguments such that the matrices

  $$\Gamma_i(a_1, \dots, a_n, \{(A_i)_{k,l} \ / \ 1 \leq i \leq n, k \in I_i, l \in J_i\}) = B_i = \begin{pmatrix} B_i^1 & B_i^3 \\ 0 & B_i^2 \end{pmatrix}$$

  where I_i, J_i are subsets of $\{1, \dots, p\}$, for $1 \leq i \leq n$, satisfy again Schlesinger equations. It is easy to see that the functions $(B_i^j)_{1\leq i\leq n}$ satisfy Schlesinger equations for $j = 1, 2$ and that the funtions $(B_i^3)_{1\leq i\leq n}$ satisfy systems of linear partial differential equations with coefficients involving rational functions in $a_1, \dots, a_n$ and functions B_i^j, $1 \leq i \leq n$, $j = 1, 2$. In that way, we have *reduced* the study of the initial Schlesinger equations (S) to the study of two Schlesinger equations of smaller dimensions, see [17].

- An other application concerns the famous Birkhoff reduction problem for linear differential equations, see [16].

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