A SOLIDIFICATION PHENOMENON IN RANDOM PACKINGS

L. BOWEN, R. LYONS, C. RADIN, AND P. WINKLER

Abstract. We prove that uniformly random packings of copies of a certain simply-connected figure in the plane exhibit global connectedness at all sufficiently high densities, but not at low densities.

1. Introduction

The densest way to cover a large area with non-overlapping unit disks is as in Figure 1.

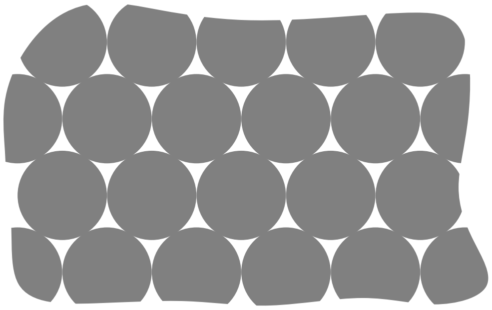

Figure 1. The densest packing of unit disks in the plane

A **packing** is a collection of congruent copies of a subset with pairwise disjoint interiors. See [Fej] for a proof that the above packing is indeed densest possible for unit disks.

It is an old unsolved problem to understand whether densest packings of spheres, simplices or other shapes, in a Euclidean or hyperbolic space of any dimension, exhibit crystallographic symmetry. For instance, this is the spirit of Hilbert's eighteenth problem; see [Fej, Rad] for background.

Using physics models of two- and three-dimensional matter as a guide, we are tempted to try to gain insight about densest packings by considering packings at densities below the maximum. (For an example concerning spheres in $\mathbb{R}^3$, see [KRS].) In effect, we are emphasizing not so much that densest packings and sparse packings differ by their *symmetry*, as that they differ in some fundamental geometric fashion. Indeed, it is commonly suggested in the physics literature (see for instance [AH]) that 2-dimensional models of matter do

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not exhibit crystallographic symmetry, and it is sometimes said by mathematicians that in high dimensional Euclidean space, densest packings of spheres may not have crystallographic symmetry. So perhaps it is appropriate to reexamine the precise manner in which densest packings differ fundamentally from sparse packings, and to use packings at less than optimum density as a guide.

The density of the unit disk packing of Figure 1 is $\widetilde{d} := \pi/\sqrt{12} \approx 0.91$; packings of density 0.89 (say) can be obtained by shrinking the disks slightly, shaking the whole collection, and then expanding the whole picture to recover unit-size disks. One might ask whether by doing this one preserves long-range correlations. Experiments and discrete models indicate that such long-range correlations are indeed preserved, but this has never been proved. See [BLRW] for background.

In this work we prove the conjectured behavior, not for disks but for deformed disks, copies of a "zipper" tile designed specifically for the purpose; see Figure 2. This tile can cover the plane completely, in which case the packing has density 1, and is completely connected in any sense. What we show is that even at somewhat lower densities, the uniform random packing still has rich structure; in particular it has a form of connectedness associated with site percolation [Gri]. What this means for packing large but finite boxes (with torus boundary conditions) is that the necessary gross irregularities of most packings at such high densities occur, but not in a way to disconnect the packings. Although we define "uniform random packing" of the plane by limits of measures on packings of finite boxes, the key to our proof is to examine isometry-invariant probability measures on packings of the whole plane and to show that the ones that maximize "degrees of freedom per tile" are unique for high densities.

In physics one is interested in change of behavior as density changes. We show that at high density there is a nonzero probability of an infinite linked component, and that this probability is zero at low density. Thus, there are different "phases" of the packings [BLRW]. (This is the same phase transition seen in continuum percolation, where one looks at overlapping disks with random independent centers, but our methods are quite different. Indeed, no such result is known for packings of disks.)

Although we believe such a result also holds for packings of disks or of spheres — pairs of which would be called "linked" if sufficiently close — we are able to prove the result only for our tiles, which are shaped to allow three well-defined levels of pairwise separation. It is generally understood that crystalline behavior is not seen in two dimensions, so the form of connectedness we use may be useful in understanding the role of geometry in Hilbert's problem.

2. DESCRIPTION OF THE TILE

We consider packings by a deformed disk denoted by t, referred to as "the tile" and depicted in Figure 2. In this section, we define it precisely.

Let H be a regular hexagon of area 1. Let r be the radius of the in-circle of H. Let D be a disk concentric with the in-circle and of radius $r + \rho$, where $0 < \rho \ll 1$ is a number we shall choose more precisely later. We shall construct the shape t by modifying H as follows; D will be called the **shadow disk** of t.

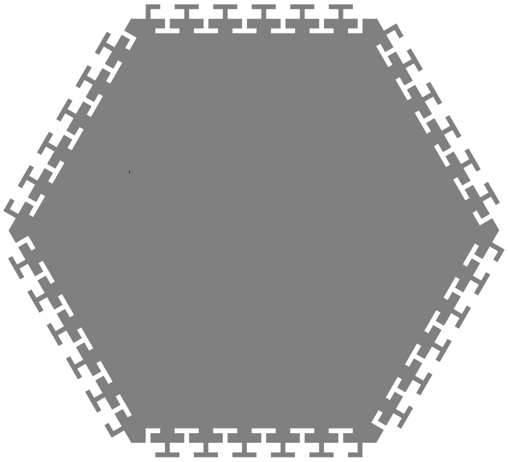

Figure 2. The zipper tile

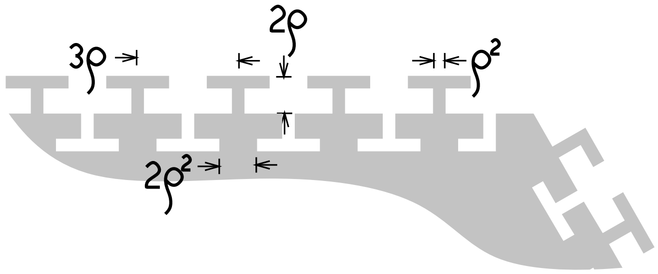

Figure 3. Close-up view of the fringe

As shown in Figure 2, the tile t equals H with each side modified by a "fringe" and each corner modified by a hook and inlet, where a hook is about half an element of the fringe. As shown in Figure 3, the fringe height is 2ρ. The elements of the fringe have two different size "necks", one of size ρ^2 and one of size $2\rho^2$, allowing neighboring tiles to be linked in either of two well-defined modes, "tight linked" and "loose linked", the former illustrated in Figure 4 and the latter illustrated in Figure 5. We say that two tiles t are **linked** (tightly or loosely) if when one is held fixed, the other can be moved continuously only by a bounded amount (without overlapping the first). A **tight** link is one that permits no movement of one tile while fixing the other, while a link that is not tight is called **loose**. A key feature of our model is that when two tiles are tightly linked, any motion of one would necessitate a corresponding motion of the other. As we shall explain, the uniform probability distribution on packings of the plane at given density is a limit of such a distribution on packings of larger and larger tori. In our model, these distributions on packings of finite tori are concentrated on packings with the maximal number of degrees of freedom, and therefore intuitively the fewest possible number of tiles bound by tight links. This gives us useful control on the packings in the support of our distributions.

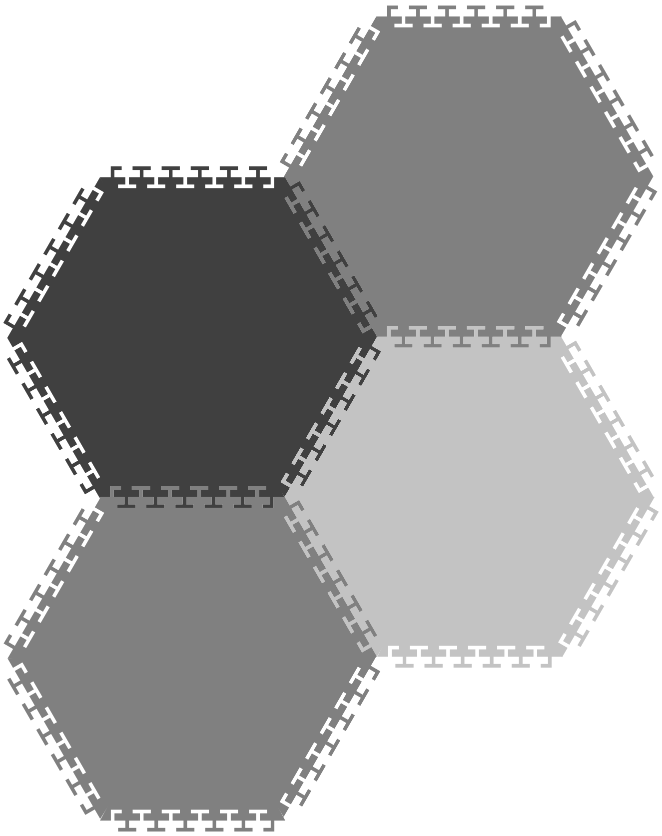

Figure 4. Tight-linked tiles

A tile is called **fully linked on one side** if it is linked with another tile on that side in such a way that either they are tight linked and the line joining their centers goes through the midpoint of the sides of the corresponding hexagons; or the tiles are not tight linked but can be moved continuously so that their shadow disks touch each other. A tile is **fully linked** if it is fully linked on all sides. We note that the fully tight-linked packing (Figure 4) corresponds to a tiling by the original hexagon and has density 1, and that the tile has area 1 by construction.

3. Statement of Results

To state our results we need some notation. Let X be the space of all packings of the plane by the tile t. Given a compact subset K of the plane and two packings of the plane, we consider the distance between the two packings with respect to K to be the Hausdorff distance between the unions of the tiles in the respective packings intersected with K. Then

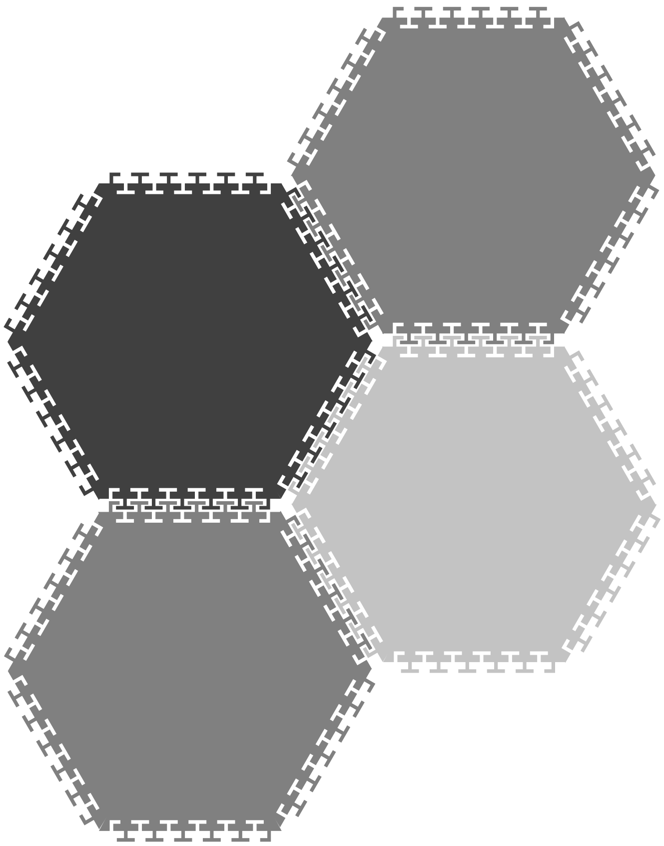

FIGURE 5. Loose-linked tiles

X is endowed with the topology of Hausdorff convergence on compact subsets; X is compact. Intuitively, two packings are close in X if they are close in the Hausdorff sense in a large ball centered at the origin. We shall define a probability measure on X that is “uniform” on the set of all packings of a fixed density. For this, we shall need the space X_n of all packings by the tile of the $n \times n$ torus $\mathbb{R}^2/(n\mathbb{Z})^2$.

For any integer m, let $X_{n,m} \subset X_n$ consist of those packings which contain exactly m tiles ($X_{n,m}$ is empty if m is large enough). To each tile, we assign the set of 6 unit vectors based on its center and pointing to the center of each of its edges. Through this assignment, we can view $X_{n,m}$ as a subset of T_n^m/Σ_m, where T_n is the unit tangent bundle of the $n \times n$ torus modulo a $2\pi/6$ rotation and the symmetric group Σ_m acts by permuting the factors.

When m/n^2 is small, $X_{n,m}$ is $(3m)$-dimensional. However, when m/n^2 is sufficiently large, the dimension of $X_{n,m}$ inside T_n^m/Σ_m is less than $3m$. This is because at least two tiles in any packing of $X_{n,m}$ will have to be tightly linked, so that it is impossible to move one continuously without moving the other. Thus it is useful to decompose $X_{n,m}$ into a (finite) disjoint union of sets $X_{n,m,k}$ of packings containing exactly k tight links. Generically, the dimension of $X_{n,m,k}$ is $3(m-k)$. The dimension can be strictly less than this if the packings are jammed in the sense of [DTSC], although this fact will not be important for us. The **top dimension** of $X_{n,m}$ means the maximum dimension of all $X_{n,m,k}$. Let $\mu_{n,m}$ be the probability measure on $X_{n,m}$ obtained by normalizing the Hausdorff measure on $X_{n,m}$ in the top dimension of $X_{n,m}$ with respect to the natural metric inherited from T_n^m/Σ_m. We interpret $\mu_{n,m}$ as being a uniform measure. The fact that $\mu_{n,m}$ is supported on those packings with the fewest number of tight links will be crucial in the analysis to follow.

Let $\widetilde{X}_n$ be the space of all $(n\times n)$-periodic packings of the plane. In other words, $\widetilde{X}_n$ consists of those packings that are preserved under translations by $n\mathbb{Z}\times n\mathbb{Z}$. Under the quotient map, this space is naturally identified with X_n. Therefore, we can view the measures $\mu_{n,m}$ as living on $\widetilde{X}_n \subset X$.

For a fixed density $d \in [0, 1]$, let $\mu^{(d)}$ be any measure obtained as the weak* limit of measures of the form $\mu_{n,m}$ such that $n \to \infty$ and $m/n^2 \to d$. (Note that m/n^2 is the density of every packing in the support of $\mu_{n,m}$ and d is the average density of a packing chosen with respect to $\mu^{(d)}$: see Lemma 5.1.) A priori, $\mu^{(d)}$ may not be unique, although we shall prove that it is for large enough d.

As we have seen, two tiles t can be linked. Therefore it makes sense to speak of a **linked component** of a packing; it is a maximal sub-packing such that for every two tiles t, t' in it, there is a sequence $t = t_1, t_2, \ldots, t_n = t'$ such that t_i is linked to t_{i+1} $(i = 1, \ldots, n-1)$. A **tight-linked component** is defined similarly, but we require t_i to be tightly-linked to t_{i+1}.

We say that a measure on the space X of packings is invariant if it preserved under the full isometry group of the plane. All the measures we consider are probability measures unless stated otherwise.

Let λ_0 be the unique invariant measure on tilings (packings that cover $\mathbb{R}^2$) by our tile. Let λ_1 be the unique invariant measure on packings by t such that all tiles are fully loose linked, are as close as possible to each other, and the packing has hexagonal symmetry. Write $\lambda_s := s\lambda_1 + (1-s)\lambda_0$.

Our main results are the following:

Theorem 3.1. *There exists* $0 < d_1 < 1$ *such that if* $d \geq d_1$, $\mu^{(d)}$ *is unique and equals* λ_s, *where* $s := (1-d)/(1-d_1)$.

Corollary 3.2. *The* $\mu^{(d)}$*-probability that the origin is inside a tile belonging to an infinite linked component is nonzero for* $d \geq d_1$.

Proposition 3.3. *For some* $d_2 > 0$, *the probability (with respect to any* $\mu^{(d)}$ *for any* $d < d_2$*) that the origin is inside an infinite linked component is zero.*

4. Tile properties

Lemma 4.1. *For small* ρ, *if tiles* t_1 *and* t_2 *are not tightly linked and do not overlap, then the distance between their centers is at least* $2r + 2\rho$.

Proof. Consider the line segment from the center of t_1 to the center of t_2. If this segment traverses near a corner of t_1 or t_2, then it must be longer than $2r + 2\rho$ for small enough ρ. Supposes it crosses a fringe of t_1 and of t_2. If the tiles are not linked, then the claim is obvious. If they are linked, then to minimize the distance, it must be that their fringes match up (so they are fully linked on one side). Thus the closest they can come is if the two are pushed flat up against each other so that their shadow disks touch. In this case, the distance between the centers is exactly $2r + \rho$. □

We shall say that two tiles are **densely loose linked** if they are loose linked and their shadow disks touch. There is a unique invariant measure on maximally dense packings by congruent disks [BHRS]. Hence the probability measure λ_1 that we defined earlier is the unique invariant measure on packings by t such that all tiles are fully and densely loose linked. Let d_1 be the density of such a packing.

Given a tile t in a packing P, we denote by $V(t)$ the Voronoi cell of the center of t with respect to the centers of the other tiles; that is, $V(t)$ is the open set of points closer to the center of t than to the center of any other tile. We denote the area of a region A of the plane by $|A|$.

Lemma 4.2. *The following holds for small enough $\rho > 0$. For any packing P, if $t \in P$ is a tile that has no tight links, then the area of $V(t)$ is least $1/d_1$. Moreover, equality holds iff the configuration of tiles determining $V(t)$ is congruent to a corresponding configuration of a packing in the support of λ_1.*

Proof. For a tile t, let $H(t)$ denote the hexagon from which t is created. For $x > 0$, let $H_x(t)$ denote the homothetic copy $\frac{r+x}{r}H(t)$ about the center of $H(t)$.

Suppose t is a tile of P without any tight links. Consider the rays $R_1, \ldots, R_6$ from the center of the hexagon $H(t)$ through each of its 6 vertices. These rays divide the plane into 6 sectors, $S_1, \ldots, S_6$.

By construction, if t and t_1 are loose linked, then $|H_\rho(t) \cap H_\rho(t_1)| = \mathrm{O}(\rho^2)$: The hexagon interiors do not intersect if they are parallel, while if they are not parallel, they can intersect only very slightly at a corner. The openings at which a corner can enter have area $\mathrm{O}(\rho^2)$ as $\rho \to 0$.

Thus, we have proved that whenever t is loose linked in the sector S_i, then $|V(t) \cap S_i| \geq |H_\rho(t)|/6 - \delta_1$, with $\delta_1 = \mathrm{O}(\rho^2)$ as $\rho \to 0$.

Similarly, if t and t_2 are not linked at all, then $|H_{2\rho}(t) \cap H_{2\rho}(t_2)| = \mathrm{O}(\rho^2)$: Again, their interiors do not intersect if they are parallel, while if they are not parallel, they can intersect only very slightly at a corner. So there exists $\delta_2 > 0$ such that whenever t is not linked in the sector S_i, we have $|V(t) \cap S_i| \geq |H_{2\rho}(t)|/6 - \delta_2$, with $\delta_2 = \mathrm{O}(\rho^2)$ as $\rho \to 0$.

Therefore, if t has no tight links but is not fully linked, then

$$|V(t)| \geq j\Big(\frac{|H_\rho(t)|}{6} - \delta_1\Big) + (6-j)\Big(\frac{|H_{2\rho}(t)|}{6} - \delta_2\Big)$$

for some j with $0 \leq j \leq 5$. Given that δ_1, δ_2 are of order ρ^2 while $|H_{2\rho}(t)| - |H_\rho(t)|$ is of order ρ, for ρ small enough we may conclude that $|V(t)| > |H_\rho(t)|$ in this case.

On the other hand, the geometry of a tile is such that for small ρ, if t_1 and t_2 are two tiles loose linked to t, then t_1 cannot be tight linked to t_2. Now suppose that t is fully loose linked. Then the Voronoi cell of the center of t is determined by six tiles $t_1, \ldots, t_6$ all loose linked to t and all with the property that their shadow disks $D, D_1, \ldots, D_6$ do not overlap (by the previous lemma). It follows [Fej2] that $|V(t)| \geq |H_\rho(t)|$, with equality iff each of the disks $D_1, \ldots, D_6$ touches D. But there is only one way in which this can occur (up to isometry). So $V(t) = H_\rho(t)$ in this case. This implies that the configuration $t, t_1, \ldots, t_6$ is congruent to a corresponding configuration of a packing in the support of λ_1. ☐

It is easy to see that given $\rho > 0$, there exists $\varepsilon > 0$ such that for any finite component c of tight-linked tiles in any packing, the union V_c of the Voronoi cells of the centers of the tiles of c has area at least $j_c + \varepsilon \mathrm{Per}_c$. Here j_c is the number of tiles in c and Per_c is the perimeter of the union of hexagons corresponding to c. Let ε be the largest such constant. Let $\delta > 0$ be such that the area of the Voronoi cell in the fully densely loose-linked packing equals $1 + \varepsilon \mathrm{Per}_1 + \delta$, where Per_1 is the perimeter of the hexagon of a single tile. Since $\varepsilon = \rho + \mathrm{O}(\rho^2)$ and $\delta = \mathrm{O}(\rho^2)$, we have:

Lemma 4.3. *For sufficiently small ρ, there are $\varepsilon, \delta > 0$ such that for any finite tight-linked component c,*

$$\begin{aligned} d_1 &= \frac{1}{1 + \mathrm{Per}_1 \varepsilon + \delta}, \\ |V_c| &\geq j_c + \varepsilon \mathrm{Per}_c, \end{aligned}$$

and

$$\delta \leq \varepsilon / 100 .$$

5. High Density

Recall that X is the compact space of all packings of the plane by the tile (with the topology of Hausdorff-metric convergence on compact subsets). Let $\widetilde{M}$ be the space of isometry-invariant Borel probability measures on X. For any $\mu \in \widetilde{M}$, we denote by $|\mu| := \mu(A_0)$ the **density** of μ, where A_0 is the set of all packings $P \in X$, one of whose tiles contains the origin. Since a tile is the closure of its interior, A_0 is a closed set.

Lemma 5.1. *If $\mu_i \in \widetilde{M}$ converges to μ in the weak* topology, then $|\mu_i|$ converges to $|\mu|$.*

Proof. Let $\widehat{P}$ denote the union of tiles in a packing, P. For any invariant probability measure ν and any $z \in \mathbb{R}^2$, we have

$$|\nu| = \int \mathbf{1}_{\{0 \in \widehat{P}\}} \, d\nu(P) = \int \mathbf{1}_{\{z \in \widehat{P}\}} \, d\nu(P) .$$

Integrating over z in a unit-area disk, D, with respect to Lebesgue measure and using Fubini's theorem gives the identity $|\nu| = \int |\widehat{P} \cap D| \, d\nu(P)$. Since the function $P \mapsto |\widehat{P} \cap D|$ is continuous on X, the lemma follows. ☐

Recall that λ_0 is the unique invariant measure on tilings by our tile, so that $|\lambda_0| = 1$. Recalling that d_1 is the density of a fully densely loose-linked tiling, fix a density d with $d_1 \leq d \leq 1$. Let μ_N be the uniform measure on configuration of tiles at density d_N in an $N \times N$ torus, where $d_N \to d$ as $N \to \infty$. To prove theorem 3.1, we shall show that the weak* limit of μ_N exists and equals λ_s, where $s := (1 - d)/(1 - d_1)$.

We shall use several lemmas that depend on the following notation. Given a packing $P \in X$, let

- t_P be the tile of P such that the origin belongs to $V(t_P)$ (this exists as long as the origin is not on the boundary of a Voronoi cell),
- K_P be the tight-linked component containing t_P,
- j_P be the number of tiles in K_P, and
- $f(P) := 3/j_P$ if j_P is finite and t_P contains the origin, and 0 otherwise.

Thus $f(P)$, in a sense, measures the number of degrees of freedom per tile near the origin.

Lemma 5.2. *If ν is any measure in $\widetilde{M}$, then $\int f\, d\nu(f) \leq 3|\nu|$, with equality iff t_P has no tight links for ν-almost every packing P.*

The proof is immediate.

Lemma 5.3. *If a sequence $\langle \nu_n \rangle \subset \widetilde{M}$ converges to ν in the weak* topology, then $\int f\, d\nu_n$ converges to $\int f\, d\nu$.*

Lemma 5.3 is proven in a manner similar to Lemma 5.1.

Given a finite tight-linked component c, let the congruence class of c be C and let $X_C \subset X$ be the space of all packings P for which t_P exists and K_P is in C. Let X'' be the space of all packings P with density 1, where "density" refers to the usual concept of the limit of the proportion of the area of P inside a large disk centered at the origin as the radius tends to infinity. Let $X' \subset X$ be the space of all packings P such that K_P is infinite and either the density of P is less than 1, the density is not defined, or t_P does not exist. Thus, X is the disjoint union of X', X'' and the collection of X_C for all C.

Let ν be any invariant probability measure with density d. Let ν_C be ν conditioned on X_C, ν' be ν conditioned on X' and ν'' be ν conditioned on X''. Since λ_0 is the only invariant probability measure with support in X'', we have $\nu'' = \lambda_0$. Thus,

$$\nu = \nu(X')\nu' + \nu(X'')\lambda_0 + \sum_C \nu(X_C)\nu_C\,.$$

Define the density $|\omega| := \omega(A_0)$ as before, but for any (invariant or non-invariant) probability measure ω on X. We have

$$d = |\nu| = \nu(X')\,|\nu'| + \nu(X'') + \sum_C \nu(X_C)\,|\nu_C|$$

and

$$\int f\, d\nu = \sum_C \nu(X_C) \int f\, d\nu_C = \sum_C \nu(X_C) \frac{3}{j_C} |\nu_C|\,,$$

where j_C is the number of tiles in C.

Lemma 5.4. *Let $\nu \in \widetilde{M}$ and C be a finite-component class. Suppose that $0 \leq s \leq 1$ is such that $|\lambda_s| = |\nu_C|$. Then $\int f\, d\lambda_s \geq \int f\, d\nu_C$. Moreover, equality holds only if $j_C = 1$.*

Proof. As in the proof of Lemma 5.1, we have that $|\nu_C| = \int j_C/|V(t_P)|\,d\nu_C(P)$.

First suppose that $j_C = 1$. Then $|\nu_C| = \int 1/|V(t_P)|\,d\nu_C(P) \le d_1$ by Lemma 4.2. This means that $s = 1$ and $\int f\,d\nu_C = 3\,|\nu_C| = 3\,|\lambda_s| = \int f\,d\lambda_s$.

Now assume that $j = j_C > 1$ and put $p := \mathrm{Per}_C$. By definition,

$$\int f\,d\lambda_s = s\int f\,d\lambda_1 + (1-s)\int f\,d\lambda_0 = s\int f\,d\lambda_1 = 3sd_1\,.$$

Since $\nu_C(f) = 3|\nu_C|/j_C = 3|\lambda_s|/j_C = 3(sd_1 + 1 - s)/j$, it suffices to show that

$$sd_1 > \frac{sd_1 + (1-s)}{j}\,,$$

which is equivalent to

$$s(jd_1 - d_1 + 1) > 1\,.$$

Now $sd_1 + (1-s) = |\nu_C| \le \frac{j}{j+\varepsilon p}$, where ε is from Lemma 4.3. Solving for s gives

$$s \ge \frac{1 - \frac{j}{j+\varepsilon p}}{1 - d_1}\,,$$

whence it is enough to show that

$$(jd_1 - d_1 + 1)\frac{1 - \frac{j}{j+\varepsilon p}}{1 - d_1} > 1\,.$$

This boils down to

$$d_1(p\varepsilon + 1) > 1\,.$$

Now, $j > 1$ implies that $p \ge (7/6)\mathrm{Per}_1$, where Per_1 is the perimeter of a single tile. Since $\varepsilon/100 > \delta$ (by Lemma 4.3), this implies that $p\varepsilon + 1 > 1 + \varepsilon\mathrm{Per}_1 + \delta = 1/d_1$, proving the last inequality. ☐

Lemma 5.5. *We have $\int f\,d\nu \le \int f\,d\lambda_s$ for all $\nu \in \widetilde{M}$ with $|\nu| = |\lambda_s|$. Equality holds only if*

- *$\nu(X_C) = 0$ for every component class C with $j_C > 1$ and*
- *whenever $\nu(X_C) > 0$ and $j_C = 1$, we have $|\nu_C| = d_1$.*

Proof. Recall that

$$\int f\,d\nu = \sum_C \nu(X_C)\int f\,d\nu_C\,.$$

For each component class C, let s_C be defined as follows:

- if there exists $s \in [0,1]$ such that $|\nu_C| = sd_1 + (1-s)$, then set $s_C := s$;
- otherwise, set $s_C := 1$.

Let $\omega_C := s_C\lambda_1 + (1 - s_C)\lambda_0$ and

$$\sigma := \big(\nu(X') + \nu(X'')\big)\lambda_0 + \sum_C \nu(X_C)\omega_C\,.$$

From the previous lemma, if $|\nu_C| \geq d_1$, then $\int f\,d\nu_C \leq \int f\,d\omega_C$, with equality only if $j_C = 1$. If $|\nu_C| < d_1$, then $s_C = 1$ and

$$\int f\,d\nu_C = \frac{3\,|\nu_C|}{j_C} < 3d_1 = \int f\,d\omega_C\,.$$

Summing up, we obtain

$$\begin{aligned}\int f\,d\sigma &= \sum_C \nu(X_C) \int f\,d\omega_C \\ &\geq \sum_C \nu(X_C) \int f\,d\nu_C = \int f\,d\nu\,.\end{aligned}$$

Moreover, equality holds only if $\nu(X_C) = 0$ for every component C with $j_C > 1$ and $|\nu_C| = d_1$ whenever $j_C = 1$. Since $|\omega_C| \geq |\nu_C|$, we have

$$\begin{aligned}|\sigma| &= \nu(X') + \nu(X'') + \sum_C \nu(X_C)\,|\omega_C| \\ &\geq \nu(X')\,|\nu'| + \nu(X'') + \sum_C \nu(X_C)\,|\nu_C| \\ &= |\nu| \\ &= |\lambda_s|\,.\end{aligned}$$

Since σ and λ_s are both convex combinations of λ_0 and λ_1, this implies that $\int f\,d\sigma \leq \int f\,d\lambda_s$ with equality iff $\sigma = \lambda_s$. Thus, $\int f\,d\nu \leq \int f\,d\lambda_s$. In the equality case we must have $\int f\,d\nu = \int f\,d\sigma = \int f\,d\lambda_s$ and $\sigma = \lambda_s$. This implies that $\nu(X_C) = 0$ if $j_C > 1$ and $|\nu_C| = d_1$ if $j_C = 1$. ☐

Lemma 5.6. *Let* $\nu \in \widetilde{M}$. *If* $|\nu| = |\lambda_s|$, *then* $\int f\,d\nu \leq \int f\,d\lambda_s$. *Equality holds iff* $\nu = \lambda_s$.

(Informally, λ_s uniquely maximizes the number of degrees of freedom per tile for invariant measures of a fixed density.)

Proof. The previous lemma implies $\int f\,d\nu \leq \int f\,d\lambda_s$. Assume $\int f\,d\nu = \int f\,d\lambda_s$; then

$$\nu = \nu(X')\nu' + \nu(X'')\lambda_0 + \nu(X_C)\nu_C$$

where C is the component of size 1 and $|\nu_C| = d_1$. This gives $\int f\,d\nu = \nu(X_C)3d_1 = \int f\,d\lambda_s = 3sd_1$. Hence $\nu(X_C) = s$. Since ν' has density strictly less than $1 = |\lambda_0|$ but $|\nu| = |\lambda_s|$, we must have $\nu(X') = 0$. That is,

$$\nu = \nu(X'')\lambda_0 + \nu(X_C)\nu_C\,.$$

Since ν and λ_0 are isometry invariant, ν_C must also be isometry invariant. By Lemma 4.2, λ_1 is the unique isometry-invariant measure with support in X_C and with density d_1. Hence $\nu_C = \lambda_1$. This implies $\nu = \lambda_s$ and the proof is finished. ☐

Proof of Theorem 3.1: It is easy to see that one can pack the $N \times N$ torus in such a way that there is a large region of tight-linked tiles and a large region of densely loose-linked tiles, in such a way that the interface between the two regions has a density which approaches zero as N tends to infinity, and the density d_N of the packing P_N tends to d. Let ω_N be the invariant measure supported on isometric copies of $\widetilde{P}_N$ (a pull-back of P_N to the plane). Then ω_N tends to λ_s in the weak* topology. By Lemma 5.3, this implies that $\int f\,d\omega_N \to \int f\,d\lambda_s$.

Now μ_N, the uniform measure of density d_N on the $N \times N$ torus, satisfies $\int f\,d\mu_N \geq \int f\,d\omega_N$. This is because μ_N is by definition supported on packings with the maximal number of degrees of freedom for the given density d_N. Hence $\liminf_N \int f\,d\mu_N \geq \liminf_N \int f\,d\omega_N = \int f\,d\lambda_s$.

Therefore, if μ_∞ is any weak* subsequential limit of $\langle \mu_N \rangle_N$, then $\int f\,d\mu_\infty \geq \int f\,d\lambda_s$. But $d_N \to d$, so $|\mu_\infty| = |\lambda_s|$ by Lemma 5.1. The previous lemma now implies that $\mu_\infty = \lambda_s$. $\square$

Returning to the discussion of the introduction, we note that from simulations of hard disks, one would expect the corollary to hold even for a range of densities below d_1, but we do not know how to prove this.

Remark on Higher Dimensions

The basic features of our argument can be generalized to dimension 3 or higher, except for our use in Lemma 4.2 of [Fej2] on the minimal Voronoi region in disk packings in the plane. It would be of interest if this part of our proof could be replaced by an argument insensitive to dimension.

6. LOW DENSITY

In this final section, we confirm the intuition that at low densities, there will be no infinite loose-linked component. It is obvious that there is no infinite *tight*-linked component at densities smaller than d_1.

We begin with a lemma that holds for any tile shape (in fact, for any *collection* of shapes and sizes, as long as each can be fit into a disk of some fixed radius s, and "density" is interpreted as number of tiles per unit area).

Lemma 6.1. *For small enough density d, if a packing P is drawn from $\mu^{(d)}$, then the probability that the disk B_R of radius R about the origin contains more than $9R^2d$ tile centers goes to zero as $R \to \infty$.*

Proof. Let s be the radius of the smallest disk containing the tile (in our case, s is about $2^{1/2} \cdot 3^{-3/4} \cdot (1+2\rho)$) and choose

$$0 < d < \frac{.05}{13\pi s^2}\ ;$$

for our zipper tiles with small enough ρ, $d \leq .003$ suffices. Let T be the set of tiles whose centers fall in B_R, $k := \lceil \pi R^2 d \rceil$ and $\ell > 9R^2 d$. Letting $\mu^{(d)}(\cdot)$ denote the probability of an event with respect to the measure $\mu^{(d)}$, we shall show that

$$\frac{\mu^{(d)}(|T| = \ell)}{\mu^{(d)}(|T| = k)} \leq \gamma^{\ell - k}$$

for some constant $\gamma < 1$. It then follows that

$$\mu^{(d)}(|T| > 9R^2d) \le \mu^{(d)}(|T|{=}k) \sum_{\ell=\lceil 9R^2d\rceil}^{\infty} \gamma^{\ell-k} \le \mu^{(d)}(|T|{=}k) \frac{\gamma^{\lfloor(9-\pi)R^2d\rfloor}}{1-\gamma} \to 0$$

as $R \to \infty$, as desired.

The measure $\mu^{(d)}$ is the limit of uniform distributions of configurations on the $N \times N$ torus $\mathbf{T}_N$, in turn obtainable by choosing a sequence of $n = \lfloor N^2d\rfloor$ points from the Lebesgue distribution λ on $\mathbf{T}_N^n$ as the centers of the tiles, orienting each tile independently and uniformly at random, and finally conditioning on no overlap. We denote by $\lambda(|T| = j)$ the *a priori* probability that exactly j points fall inside B_R (which we take to be some fixed disk in the torus).

Let Φ be the event that there is no overlap among the tiles whose centers lie in B_R, and Ψ the event that there is no overlap involving any tile whose center falls *outside* B_R. Then

$$\frac{\mu^{(d)}(|T|{=}\ell)}{\mu^{(d)}(|T|{=}k)} = \frac{\lambda(|T|{=}\ell)}{\lambda(|T|{=}k)} \cdot \frac{\lambda(\Phi \mid |T|{=}\ell)}{\lambda(\Phi \mid |T|{=}k)} \cdot \frac{\lambda(\Psi \mid |T|{=}\ell \wedge \Phi)}{\lambda(\Psi \mid |T|{=}k \wedge \Phi)} .$$

and our job is to bound the three fractions on the right.

For the first, we note that $|T|$ is binomially distributed in the measure λ, hence

$$\begin{aligned}\frac{\lambda(|T|{=}\ell)}{\lambda(|T|{=}k)} &= \frac{\binom{n}{\ell}\left(\frac{\pi R^2}{N^2}\right)^{\ell}\left(1-\frac{\pi R^2}{N^2}\right)^{n-\ell}}{\binom{n}{k}\left(\frac{\pi R^2}{N^2}\right)^{k}\left(1-\frac{\pi R^2}{N^2}\right)^{n-k}} \le \frac{(n{-}k)!/(n{-}\ell)!}{\ell!/k!}\left(\frac{\frac{k}{n}}{1-\frac{k}{n}}\right)^{\ell-k} \\ &< \frac{(n{-}k)^{\ell-k}}{(\ell/e)^{\ell-k}} \cdot \frac{k^{\ell-k}}{(n{-}k)^{\ell-k}} = \left(\frac{ek}{\ell}\right)^{\ell-k} \le 0.95^{\ell-k}\end{aligned}$$

for large R.

The next fraction is easy: since we may throw the first k centers into B_R, then the remaining $\ell{-}k$, we have

$$\frac{\lambda(\Phi \mid |T|{=}\ell)}{\lambda(\Phi \mid |T|{=}k)}$$

is the probability that the additional $\ell{-}k$ centers do not cause a collision, which is at most 1.

For the (inverse of) the third fraction, we throw $n{-}\ell$ centers into the region outside B_R, then the remaining $\ell{-}k$. A new point, if it lands at distance greater than $2s$ from any previous point or from the disk B_R, causes no new overlap; and at each stage there are fewer than $n{-}k$ points already placed. Hence

$$\begin{aligned}\frac{\lambda(\Psi \mid |T|{=}k \wedge \Phi)}{\lambda(\Psi \mid |T|{=}\ell \wedge \Phi)} &> \left(\frac{N^2 - \pi(R+2s)^2 - (n{-}k)4\pi s^2}{N^2 - \pi R^2}\right)^{\ell-k} \\ &\ge \left(1 - 4\pi s^2 d - \frac{4\pi s R + 4\pi s^2}{N^2 - \pi R^2}\right)^{\ell-k} > \left(1 - 13 s^2 d\right)^{\ell-k}\end{aligned}$$

for $N \gg R$.

Putting the inequalities together, we have

$$\frac{\mu^{(d)}(|T|=\ell)}{\mu^{(d)}(|T|=k)} \le \left(\frac{.95}{1-13s^2d}\right)^{\ell-k} = \gamma^{\ell-k}$$

where $\gamma := .95/(1-13s^2d) < 1$ by choice of d. □

Proposition 6.2. *For some $d_2 > 0$, the $\mu^{(d)}$-probability that the origin is inside an infinite connected component of loosely-linked tiles is zero for $d < d_2$.*

Proof. Let $d \in (0, .003)$ be a density to be chosen later. Let P be a packing drawn from $\mu^{(d)}$; we aim to show that the probability that the origin is connected by a loose-linked chain of tiles of P to some point at distance R approaches zero as $R \to \infty$.

We again choose some large radius R and let T be the set of tiles of P whose centers fall inside the disk B_R.

Fix the positions of the tiles of $P \setminus T$ (the black tiles of Figure 6) and consider the space of packings having these tiles plus n tiles whose centers fall in B_R. We think of this space as being a subset of $T_1(B_R)^n/\Sigma_n$, where $T_1(B_R)$ is the unit tangent bundle of B_R (modulo a $2\pi/6$ rotation to take into account the symmetries of the tile) and the symmetric group acts by permuting the factors.

If α_n is the volume (in $T_1(B_R)^n/\Sigma_n$-space) of this space and $m < n$, then by packing $n-m$ tiles into B_R and then the remaining m in the left-over space, we have

$$\alpha_n \ge \frac{1}{\binom{n}{n-m}} \alpha_{n-m} \frac{1}{m!} \left[\pi(R-2s)^2 - n\pi(2s)^2\right]^m ,$$

where s is, as before, the radius of the circle circumscribing a tile. This takes into account possible intrusion of tiles in $P - T$ into B_R, and the fact that a tile center at point x can exclude nearby centers but only within distance $2s$ of x.

Let β denote the "wiggle room" of a tile t loose linked to a stationary tile t', that is, the 3-dimensional volume of the space of positions of t; then $\beta = \mathrm{O}(\rho^3)$ (but we use only that β is bounded by a constant). If a packing "percolates", that is, contains a chain of loose-linked tiles connecting the center to the boundary of B_R, let $t_1, \dots, t_m$ be a shortest such chain (the dark grey tiles of Figure 6). Note that $m \ge R/(2s)$. For each $i > 2$, the tile t_i is linked to one of the three sides of t_{i-1} farthest from the side of t_{i-1} linked to t_{i-2}, and has wiggle room at most β with respect to t_{i-1}. Accounting for the orientation of t_1 and allowing the remaining $n-m$ tile centers to fall anywhere in B_R, we have that the $3n$-dimensional volume of the set of percolating packings is bounded by $(\pi/3) \cdot 6 \cdot 3^{m-2} \cdot \beta^{m-1} \cdot \alpha_{n-m} < 3^m \beta^{m-1} \alpha_{n-m}$.

Comparing with the lower bound for α_n, we find that given $|T| = n \le 9dR^2$, the probability of percolation is less than

$$\frac{3^m \beta^{m-1} \alpha_{n-m}}{\alpha_n} \le \frac{3^m \beta^{m-1}\, n!/(n-m)!}{\left[\pi(R-2s)^2 - n\pi(2s)^2\right]^m} < \left(\frac{27\beta d R^2}{\pi\left[(R-2s)^2 - 36dR^2s^2\right]}\right)^m \Big/ \beta ,$$

which goes to zero as R (thus also m) increases, for suitably chosen d. Since we know from Lemma 6.1 that $\mu^{(d)}(|T| \le 9dR^2)$ approaches 1 as $R \to \infty$, the proposition follows. □

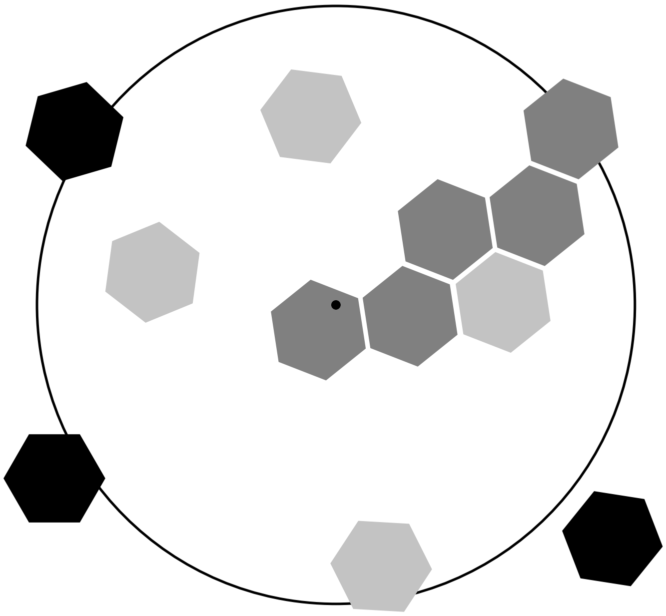

Figure 6. An unlikely configuration of tiles in and around B_R

A more careful argument would prove Proposition 6.2 for any density below $1/\big(4\pi(2/3\sqrt{3})\big) = .2067^+$ for sufficiently small ρ, but clearly the probability of percolation will remain 0 for much higher densities than that.

7. A Conjecture

We have shown that high-density random packings of zipper tiles in the plane contain an infinite loose-linked component with positive probability, while low-density random packings do not. What happens in the case of ordinary disks, where there is no apparent linking mechanism? We believe, but cannot prove, the following

Conjecture. *Suppose $\mu^{(d)}$ is defined as above for geometric disks of radius 1. Join two centers by an edge if their distance is at most $2+\varepsilon$ for some fixed $\varepsilon \ll 1$. Then for sufficiently high density d below the maximum, the graph resulting from a configuration drawn from $\mu^{(d)}$ will contain an infinite connected component a.s.*

This statement can be shown by a standard Peierls-type argument for large ε; this may be known already. In general, there is some parameter set of $(\varepsilon, d) \subset (0,\infty) \times (0,1)$ for which there is an infinite component. For small d or for large ε, the problem is quite similar to continuum percolation, where one connects by an edge two points of a Poisson point process if their distance is at most r. Because of homotheties, one may fix the intensity of the point process to be 1. Then there is a phase transition in r. Our problem is quite different in really having two parameters, but our conjecture is that there is a phase transition in d for every ε nevertheless.

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Department of Mathematics, Indiana University, Bloomington, IN 47405
E-mail address: `lpbowen@indiana.edu`

Department of Mathematics, Indiana University, Bloomington, IN 47405
E-mail address: `rdlyons@indiana.edu`

Department of Mathematics, University of Texas, Austin, TX 78712
E-mail address: `radin@math.utexas.edu`

Department of Mathematics, Dartmouth College, Hanover, NH 03755
E-mail address: `peter.winkler@dartmouth.edu`