THE PARAMETERIZATION METHOD FOR ONE-DIMENSIONAL INVARIANT MANIFOLDS OF HIGHER DIMENSIONAL PARABOLIC FIXED POINTS

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Abstract. We use the parameterization method to prove the existence and properties of one-dimensional submanifolds of the center manifold associated to the fixed point of C^r maps with linear part equal to the identity. These maps appear as Poincaré maps in Celestial Mechanics.

1. Introduction

We consider C^r maps of $\mathbb{R}^{1+n}$ having a parabolic fixed point and study the existence of one-dimensional invariant manifolds passing through this fixed point.

We assume that the fixed point is the origin and that the linear part of the map at the fixed point is the identity. Then a whole neighborhood of the origin is a center manifold. However there may exist invariant submanifolds of points which go to the origin by the iteration of the map. In this setting we refer to such submanifolds as stable manifolds. In the same way we can speak of unstable manifolds.

These problems appear naturally in Celestial Mechanics. In this application, often the fixed point is image of infinity under a suitable transformation and the invariant manifolds are the separation from bounded and unbounded motions. See for example, [Slo58, McG73, Rob84, Eas84] for studies of parabolic invariant manifolds in $\mathbb{R}^2$ and applications to Celestial Mechanics.

Another situation where this problem appears is in complex analytical dynamics in the neighborhood of a fixed point whose linearization is a root of unity. Studies of one-dimensional complex dynamics in neighborhoods of fixed points in complex dimension one go back to Fatou. A survey including more recent developments is [Mil91]. Since the parameterization method includes a description of the dynamics in the manifold, these studies will be relevant for us. See also [EW85, EW86, Éca85] for other studies.

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Proofs of existence of (complex) one-dimensional invariant manifolds near parabolic points in analytic maps of $\mathbb{C}^p$ appear in [Hak98].

In renormalization group theory, eigenvalues of modulus 1 are called marginal eigenvalues. The behavior of the renormalization group associated to these eigenvalues has been studied in [GKT85]. Indeed, the manifolds tangent to the marginal eigenvalues seem to be closely related to the renormalons.

In fluid mechanics, parabolic manifolds play a role in the separation of fluids from a boundary [Pra04]. Important studies of parabolic manifolds in this case are [KHN05, SGH05].

A very systematic study of analytic and continuous invariant manifolds (of any dimension) for parabolic points of maps of arbitrary dimensions can be found in [BF04].

See [Fon93, CFN92, Fon99] for studies of the dynamics in maps with parabolic points with nilpotent linear part. The previous papers consider either analytic mappings or use topological methods for differentiable maps and obtain analytical or Lipschitz results respectively.

In this paper, we consider finitely differentiable mappings and obtain finite differentiable results. See Theorem 2.1 for a precise statement of the main result.

The method we use in this paper is the parameterization method introduced recently in [CFdlL03a, CFdlL03b, CFdlL05]. In this method, one tries to look at the same time for a parameterization of the invariant manifold and for a version of the dynamics on it. We give an overview of the method in Section 2.

The solution of the resulting functional equation for the parameterization and the dynamics, has two parts. In the first part, carried out in Section 3, we obtain a polynomial approximation of the solution just matching powers, and in the second part, we study the equation for the parameterization minus the polynomial approximation.

The resulting functional equation has several remarkable functional analysis properties and we show it can be solved using the contraction mapping principle in an appropriate Banach space. This has the consequence that, given some approximate solutions, there is a true solution nearby, and hence the proof can be used to validate numerical computations.

See Section 4 for the treatment of the functional equation. We anticipate that the main technique is the introduction of appropriate Banach spaces that incorporate the fact that the functions we are interested in vanish at the origin with their derivatives (see Section 4.1). In these spaces, the equation of the parameterization method can be treated as a fixed point equation.

We also point out that the parameterization method is very well suited for numerical implementations. When carried out with the help of a computer, the power matching methods in the first part of the proof yield a very accurate solution. Of course, given the "a-posteriori" form of our functional analysis treatment, we know that the true solution will be close to the computed one. We present the results of a computer implementation in Section 5. Even if preliminary, this implementation suggests several conjectures. We also present some comparisons with other algorithms. Besides issues of accuracy and speed, we point out that the parameterization can follow turns of the manifold which prevent to treat it as a graph.

2. Statement of the main result

In this paper we look for one-dimensional invariant manifolds of maps $F : U \subset \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$. We will write $(x, y) \in \mathbb{R} \times \mathbb{R}^n$. Without any loss of generality we will assume that the fixed point is the origin and that, at the fixed point, the manifold, if it exists, is tangent to the first axis.

The parameterization method consists in looking simultaneously for an embedding $K : I_0 \subset \mathbb{R} \to \mathbb{R}^{1+n}$ and a map $R : I_0 \subset \mathbb{R} \to \mathbb{R}$ such that

$$F \circ K = K \circ R. \tag{2.1}$$

K is a parameterization of a curve and R is a representation of the dynamics on the curve. Condition (2.1) ensures that the range of K is invariant by F. The fact that the curve passes through the origin and is tangent to the first axis is ensured by the supplementary conditions

$$K(0) = 0, \qquad DK(0) = (1, 0).$$

It is known that invariant manifolds associated to eigenvalues equal to one are, in general, not smooth at the fixed point. We will take I_0 as an interval of the form $[0, t_0)$ and we will obtain certain regularity on $[0, t_0)$ and the same regularity as the map on $(0, t_0)$.

The map R is one-dimensional with the origin as a parabolic fixed point. These maps are considered in [Tak73] where it is proved that if $f(x) = x + \dots$ is C^∞ there is a C^∞ change of variables φ such that

$$\varphi^{-1} \circ f \circ \varphi(x) = x \pm x^k + cx^{2k-1},$$

where the sign ± 1 and c are completely determined by the $2k - 1$ jet of f.

The main result of this paper is

Theorem 2.1. *Let $F = (F^1, F^2) : U \subset \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ be a C^r map, $r \geq 2$ or $r = \infty$, such that $F(0,0) = 0$, $DF(0,0) = \mathrm{Id}$*

$$D^j F^1(0,0) = 0, \qquad \text{for} \quad 2 \leq j \leq N - 1 \tag{2.2}$$

$$D^j F^2(0,0) = 0, \qquad \text{for} \quad 2 \leq j \leq M - 1 \tag{2.3}$$

and

$$\frac{\partial^N F^1}{\partial x^N}(0,0) < 0, \qquad \frac{\partial^M F^2}{\partial x^M}(0,0) = 0 \tag{2.4}$$

for some $2 \le N, M \le r$. *In the case that* $M \le N$ *assume furthermore*

$$\operatorname{Spec} \frac{\partial^M F^2}{\partial x^{M-1}\partial y}(0,0) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}. \tag{2.5}$$

Let $L = \min(N, M)$ *and* $\eta = 1 + N - L$. *We assume that* $r > 2N - 1$.

Then there exist a C^p *map* $K : [0, t_0) \subset \mathbb{R} \to \mathbb{R}^{1+n}$, *with* $p = [(r-N+1)/\eta]-1$, *of class* C^r *in* $(0, t_0)$ *and a polynomial* $R : \mathbb{R} \to \mathbb{R}$ *such that*

$$F \circ K = K \circ R. \tag{2.6}$$

Moreover $K(t) = (t, 0) + O(t^2)$ *and* $R(t) = t + d_N t^N + O(t^{2N-1})$ *with* $d_N = \frac{\partial^N F^1}{\partial x^N}(0,0) < 0$.

Remark 2.2. The fact that $d_N < 0$ implies that the origin is an attractor for the map R. Let $[0, t_*)$ be contained in the basin of attraction of R. Using the formula $F^n \circ K = K \circ R^n$, which is directly obtained from (2.6), we deduce that $K([0, t_*))$ is attracted to the origin by F, and hence is a stable manifold.

The parameterization given in Theorem 2.1 provides a stable manifold in the half space $\{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid x > 0\}$. As usual, considering the inverse map we can deal with an unstable manifold.

We can also use Theorem 2.1 to study the invariant manifolds in the half space $\{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid x < 0\}$ making the change of coordinates $(x, y) \mapsto (-x, y)$. For instance, if the hypotheses hold and N, M are odd, there is stable manifold in $\{(x, y) \in \mathbb{R} \times \mathbb{R}^n \mid x < 0\}$. If N, M are even, there is an unstable manifold in the same domain.

Remark 2.3. In [BF04] it is proved that if we are under the hypotheses of Theorem 2.1 and Condition (2.5) holds, there is a stable manifold which is locally unique among the Lipschitz ones, in the sense that within a sector whose vertex is the origin, there is no other Lipschitz stable manifold. However if Condition (2.5) does not hold the map may have an open stable set. For instance, for the map

$$F(x, y) = (x - x^3, y - x^4 y)$$

we have $N = 3$, $M = 5$, F satisfies the hypotheses of Theorem 2.1 and the origin attracts an open set of full measure.

The scheme of the proof consists of first looking for polynomials $K^{\le}$ and R of sufficiently large degree —actually of degree $2N - L$ for $K^{\le}$ and degree $2N - 1$ for R— such that $K^{\le}$ so that the invariance equation (2.6) is satisfied up to a high order error. Then we set $K = K^{\le} + K^{>}$ and we seek $K^{>}$ satisfying $F \circ (K^{\le} + K^{>}) = (K^{\le} + K^{>}) \circ R$.

The computation of $K^{\leq}$ and R is done by matching powers of t in (2.6). One obtains a hierarchy of equations that can be solved recursively. The proof is very explicit and is the basis of a practical algorithm which we implement in Section 5

The high order part of the parameterization $K^{>}$ will be found as the fixed point of the contraction operator $\mathcal{N} = \mathcal{S}^0 \circ \mathcal{F}$ (see (4.53) below). It is well known that if $K_*^{>}$ verifies $\|K_*^{>} - \mathcal{N}(K_*^{>})\| \leq \varepsilon$ then the distance of $K_*^{>}$ to the true solution $K^{>}$ is $\|K^{>} - K_*^{>}\| \leq \frac{\ell\varepsilon}{1-\ell}$, where $\ell = \mathrm{Lip}\,(\mathcal{N})$. This is "a posteriori" estimate which provides an explicit bound of the error in the computed approximation, $K^{\leq}$.

3. Polynomial Approximation

We shall denote by E_1 the one-dimensional subspace generated by the first variable and by E_2 the n-dimensional space generated by the last n variables. We shall denote π_i, $i = 1, 2$, the corresponding projectors onto E_1 and E_2 respectively, and $\pi_{2,l} = \tilde{\pi}_l \pi_2 : \mathbb{R}^{1+n} \to \mathbb{R}$, $1 \leq l \leq n$, where $\tilde{\pi}_l$ is the projector from E_2 onto its l coordinate. Also $E_{2,l} = \pi_{2,l} E_2$.

We write $F = (F^1, F^2) = \mathrm{Id} + \sum_{j=L}^{r} F_j + \tilde{F}_{r+1}$ in the form

$$F^1(x, y) \quad = \quad x + \sum_{j=N}^{r} F_j^1(x, y) + o(|(x, y)|^r), \tag{3.1}$$

$$F^2(x, y) \quad = \quad y + \sum_{j=M}^{r} F_j^2(x, y) + o(|(x, y)|^r), \tag{3.2}$$

where F_j^1, F_j^2 are homogeneous polynomials of degree j taking values in E_1 and E_2 respectively and $\tilde{F}_{r+1}(x, y) = o(|(x, y)|^r)$.

Moreover we will write $F_j^i = \pi_i F_j$ and $F_j^{2,l} = \pi_{2,l} F_j$, etc. We also introduce the notation

$$F_k^1(x, y) = \sum_{j+|m|=k} a_{j,m} x^j y^m, \qquad F_k^{2,l}(x, y) = \sum_{j+|m|=k} b_{j,m}^l x^j y^m,$$

where $y^m = y_1^{m_1} \cdots y_n^{m_n}$, $a_{N,0} = a_{N,0,\ldots,0}$, the vector $b_{M,0} = (b^1_{M,0,\ldots,0}, \ldots, b^n_{M,0,\ldots,0})^{\mathrm{T}}$ and the matrix

$$B_{M-1,1} = \begin{pmatrix} b^1_{M-1,1,0,\ldots,0} & \cdots & b^1_{M-1,0,0,\ldots,1} \\ & \cdots & \\ b^n_{M-1,1,0,\ldots,0} & \cdots & b^n_{M-1,0,0,\ldots,1} \end{pmatrix}.$$

Condition (2.4) on F implies that $a_{N,0} < 0$, $b_{M,0} = 0$, and Condition (2.5) implies that the eigenvalues of $B_{M-1,1}$ have positive real part.

Lemma 3.1. *Let $F = (F^1, F^2) : U \subset \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ be a C^r map, such that $F(0, 0) = 0$, $DF(0, 0) = \mathrm{Id}$ and satisfies (2.2)–(2.5). Given $2 \leq m \leq r$, there exist polynomials $K = K_1 + \sum_{j=2}^{m-L+1} K_j$ and $R = R_1 + \sum_{j=N}^{m} R_j$, where $L =$*

$\min(N, M)$ and K_j and R_j are homogeneous polynomials of degree j, with $K_1(t) = (t, 0)$, $R_1(t) = t$, such that

$$F \circ K(t) - K \circ R(t) = o(t^m).$$

Moreover, we can choose R and K of the form

$$R = \begin{cases} R_1 + R_N & \text{if} \quad N \le m < 2N - 1 \\ R_1 + R_N + R_{2N-1} & \text{if} \quad m \ge 2N - 1 \end{cases}$$

$$K = \begin{cases} \sum_{j=1}^{m-N+1} K_j & \text{if} \quad N \le M \\ (\sum_{j=1}^{m-N+1} K_j^1, \ \sum_{j=1}^{m-M+1} K_j^2) & \text{if} \quad M < N. \end{cases}$$

Remark 3.2. *If $N < M$ we do not need to use Condition (2.5) to determine K and R.*

Proof. Since we want that $K(0) = 0$ and $R(0) = 0$ we look for K and R in the form

$$K(t) = \sum_{j=1}^{m} K_j(t), \qquad R(t) = \sum_{j=1}^{m} R_j(t). \tag{3.3}$$

We introduce the notation

$$K^1(t) = \sum_{i=1}^{m} c_i^1 t^i, \qquad K^{2,l}(t) = \sum_{i=2}^{m} c_i^{2,l} t^i, \qquad R(t) = \sum_{i=1}^{m} d_i t^i.$$

We substitute (3.3) into $F \circ K - K \circ R = 0$ to determine K_j and R_j order by order. At first order we obtain

$$K_1 - K_1 \circ R_1 = o(t).$$

Here we have several possible choices, however we take

$$K_1(t) = (K_1^1(t), K_1^2(t)) = (t, 0), \qquad R_1(t) = t.$$

Case $N \le M$. If $2 \le j < N$ (of course this case is void if $N = 2$) from $F \circ K - K \circ R = 0$ we can write

$$K_1 + K_2 + \cdots + K_j - K_1 \circ R - K_2 \circ R - \cdots - K_j \circ R = o(t^j).$$

Comparing the terms of order two we get $K_2 - K_1 \circ R_2 - K_2 \circ R_1 = 0$ and hence $R_2 = 0$ and K_2 is free. Assuming inductively that $R_p = 0$ for $2 \le p < l \le j$ we have

$$K_1 + \cdots + K_l - K_1 \circ (R_1 + R_l) - \cdots - K_l \circ (R_1 + R_l) = o(t^l)$$

and comparing the terms of order l we get the condition $K_l - K_1 \circ R_l - K_l \circ R_1 = 0$. Projecting onto the first component we obtain $R_l = 0$. Moreover K_l is free. Therefore $R_2 = \cdots = R_{N-1} = 0$.

For $j \ge N$ we write

$$\sum_{i=1}^{j} K_i + \sum_{i=N}^{j} F_i \circ K - \sum_{i=1}^{j} K_i \circ (R_1 + R_N + \cdots + R_j) = o(t^j).$$

When $j = N$, comparing terms of order N we have

$$K_N + F_N \circ K_1 - K_1 \circ R_N - K_N \circ R_1 = 0. \tag{3.4}$$

Since $R_1(t) = t$ we get that K_N is free and $F_N^1 \circ K_1 = R_N$. Thus $d_N = a_{N,0}$. We note that the projection onto E_2 of the left-hand side of (3.4) vanishes. Indeed, if $M > N$, by the form of F, $F_N^2 = 0$. If $M = N$ we have $F_N^2 \circ K_1(t) = F_M^2(t,0) = 0$ by Condition (2.5).

When $j > N$, the terms of order j can be obtained from

$$\begin{aligned} & K_1 + \cdots + K_j + F_N \circ (K_1 + \cdots + K_{j-N+1}) + \cdots + F_j \circ K_1 \\ = \quad & K_1 \circ (R_1 + R_N + \cdots + R_j) + \cdots + K_j \circ R_1 + o(t^j). \end{aligned} \tag{3.5}$$

Actually from (3.5) we will obtain K_{j-N+1} and R_j assuming we already know K_p for $p < j - N + 1$ and R_p for $p < j$. Indeed, projecting (3.5) onto $E_{2,l}$ by $\pi_{2,l}$, and equating the terms of order j we get

$$\begin{aligned} & c_j^{2,l} + b^l_{N-1,1,0,\dots,0} c^{2,1}_{j-N+1} + b^l_{N-1,0,1,\dots,0} c^{2,2}_{j-N+1} + \cdots + b^l_{N-1,0,0,\dots,1} c^{2,n}_{j-N+1} \\ & \quad = (j - N + 1) c^{2,l}_{j-N+1} d_N + c_j^{2,l} + \Gamma_j^{2,l}, \end{aligned}$$

where $\Gamma_j^2 = (\Gamma_j^{2,1}, \dots, \Gamma_j^{2,n})$ depends on $F, K_2, \dots, K_{j-N}, R_N, \dots, R_{j-1}$. Hence, in matrix notation

$$(B_{N-1,1} - (j - N + 1) a_{N,0} \mathrm{Id}\,) c^2_{j-N+1} = \Gamma_j^2. \tag{3.6}$$

If $N = M$ the matrix $B_{N-1,1} - (j - N + 1)a_{N,0}\mathrm{Id}$ is invertible because, by the property on the eigenvalues of $B_{M-1,1}$ and the fact that $a_{N,0} < 0$, all its eigenvalues have positive real part, and hence they are non-zero. If $N < M$ then $B_{N-1,1} = 0$ and $(j - N + 1)a_{N,0}\mathrm{Id}$ is invertible. Hence in both cases we can solve (3.6) and obtain K^2_{j-N+1}.

Projecting (3.5) onto E^1 we have

$$\begin{aligned} & c_j^1 + N a_{N,0,0,\dots,0} c^1_{j-N+1} + a_{N-1,1,0,\dots,0} c^{2,1}_{j-N+1} + a_{N-1,0,1,\dots,0} c^{2,2}_{j-N+1} + \dots \\ & \quad = d_j + (j - N + 1) c^1_{j-N+1} d_N + c_j^1 + \Gamma_j^1, \end{aligned} \tag{3.7}$$

where Γ_j^1 depends on $F, K_2, \dots, K_{j-N}, R_N, \dots, R_{j-1}$.

Condition (3.7) becomes

$$(2N - j - 1) a_{N,0} c^1_{j-N+1} - d_j = \text{ known terms.} \tag{3.8}$$

Therefore, if $j \neq 2N - 1$ we can take $d_j = 0$ and determine c^1_{j-N+1} from (3.8). If $j = 2N - 1$ we have c_N^1 free and we must determine d_{2N-1} from (3.8).

Case $M < N$**.** If $2 \le j < M$ (this case is void if $M = 2$) we have, as in the previous case, $R_j = 0$ and K_j is free.

If $M \le j < N$ we claim that, having fixed K_1 and R_1, $R_j = 0$, K_j^1 is free and K_j^2 is uniquely determined. Indeed, we check the claim by induction. We write

$$\sum_{i=1}^{j} K_i + \sum_{i=M}^{j} F_i \circ K - \sum_{i=1}^{j} K_i \circ (R_1 + R_j) = o(t^j). \tag{3.9}$$

When $j = M$, projecting onto E_1 we get $K_M^1 - K_1^1 \circ R_M - K_M^1 \circ R_1 = 0$, which implies that $R_M = 0$ and K_M^1 remains free.

Projecting (3.9) onto $E_{2,l}$ by $\pi_{2,l}$ and considering the terms of order M we obtain

$$K_M^2 + F_M^2 \circ K_1 - K_M^2 \circ R_1 = 0.$$

Since $b_{M,0} = 0$, the terms of order M agree.

We assume the claim is true for $M \le l < j$. Projecting (3.9) onto E_1, we get $K_j^1 - K_1^1 \circ R_j - K_j^1 \circ R_1 = 0$ from which we obtain $R_j = 0$ and K_j^1 is free. Projecting (3.9) onto E_2

$$K_j^2 + \sum_{i=M}^{j} [F_i^2 \circ (K_1 + \cdots + K_{j-M+1})]_j = K_j^2 \circ R_1,$$

where $[\cdot]_k$ stands for the terms of order k of the expression contained in brackets. From this we obtain

$$B_{M-1,1} c_{j-M+1}^2 = \text{ known terms}$$

and, since $B_{M-1,1}$ is invertible, we can determine uniquely K_{j-M+1}^2 from the previous condition.

Next, when $j = N$, from (3.9) with $j = N$, using that $R_j = 0$ for $2 \le j \le N-1$, projecting onto E_1 and considering the terms of order N we have $K_N^1 + F_N^1 \circ K_1 - K_1^1 \circ R_N - K_N^1 \circ R_1 = 0$ which implies $R_N(t) = F_N^1(t, 0)$ and hence $d_N = a_{N,0}$. K_N^1 remains free.

Projecting onto $E_{2,l}$ and considering the terms of order N we have

$$K_N^2 + [F_M^2 \circ (K_1 + \cdots + K_N)]_N + \cdots + [F_N^2 \circ (K_1 + \cdots + K_N)]_N - K_1^2 R_N - K_N^2 \circ R_1 = 0.$$

As in the previous case we deduce that $B_{M-1,1} c_{N-M+1}^2 =$ known terms, which permits to obtain K_{N-M+1}^2.

Finally, when $j > N$, projecting $F \circ K - K \circ R = 0$ onto E_1 and considering the terms of order j we have

$$\begin{aligned} K_j^1 &+ [F_N^1 \circ (K_1 + \cdots + K_j)]_j + \cdots + [F_j^1 \circ (K_1 + \cdots + K_j)]_j \\ &- K_1^1 R_j - [K_2^1 \circ (R_1 + R_N + \cdots + R_{j-1})]_j - \cdots - K_j^1 \circ R_1 = 0. \end{aligned}$$

Then

$$N a_{N,0} c_{j-N+1}^1 + \text{ known terms } - d_j - [K_{j-N+1}^1 (R_1 + R_N)^{j-N+1}]_j = 0.$$

This gives $(2N-j-1)a_{N,0}c^1_{j-N+1} - d_j =$ known terms. This equation coincides with (3.8) and we deal with it as we did there.

Projecting onto $E_{2,l}$ and considering the terms of order j we have

$$K^2_j + \sum_{i=M}^{j} [F^2_i \circ (K_1 + \cdots + K_j)]_j - \sum_{i=1}^{j} [K^2_i \circ (R_1 + R_N + \cdots + R_j)]_j = 0. \quad (3.10)$$

This gives as before $B_{M-1,1}c^2_{j-M+1} =$ known terms, and hence we can obtain K^2_{j-M+1} uniquely from it. □

If we try to follow the same scheme for two dimensional invariant manifolds, we get into trouble because matching coefficients in $t \in \mathbb{R}^2$ in the equation $F \circ K - K \circ R = 0$ we obtain more conditions than the number of coefficients we have to determine and, in general, these conditions are not compatible.

4. Invariant manifold

Let k be an integer such that $2N-1 \le k \le r$. We decompose $F = P + Q_k$, where P is the Taylor polynomial of degree $k-1$ of F, and by (2.2)–(2.5) has the form (using T to denote transport)

$$P(x,y) = \begin{pmatrix} x + a_{N,0}x^N + y^{\mathrm{T}} f_{N-1}(x,y) + f_{N+1}(x,y) \\ y + B_{M-1,1}x^{M-1}y + y^{\mathrm{T}} g_{M-2}(x,y)y + g_{M+1}(x,y) \end{pmatrix}$$

with $y^{\mathrm{T}} g_{M-2}(x,y)y = (y^{\mathrm{T}} g^1_{M-2}y, \dots, y^{\mathrm{T}} g^n_{M-2}y)^{\mathrm{T}}$ and $g^j_{M-2} = O(|(x,y)|^{M-2})$ is a $n \times n$ matrix for $j \in \{1,\dots,n\}$ and $Q_k = O(|(x,y)|^k)$.

Let $K^{\le} : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^n$ and $R : \mathbb{R} \to \mathbb{R}$ be polynomials obtained applying Lemma 3.1 with $F = P$ and $m = k-1$. We have $K^{\le}(t) = (t, K^2_2 t^2) + (O(t^2), O(t^3))$ and

$$P \circ K^{\le} - K^{\le} \circ R = T_k, \quad (4.1)$$

where T_k is a polynomial such that $T_k = o(t^{k-1})$ and hence $D^l T_k = O(t^{k-l})$ for all $0 \le l \le r$.

Our goal is to find $K^{>}$ such that

$$F \circ (K^{\le} + K^{>}) - (K^{\le} + K^{>}) \circ R = 0. \quad (4.2)$$

For that we will transform (4.2) into a fixed point equation for $K^{>}$ and we will look for $K^{>}$ in a space of differentiable functions of order $O(t^k)$.

4.1. **The Banach spaces $\mathcal{X}^k_r$.** We fix $\eta = 1 + N - L$ as in Theorem 2.1.

Given E a Banach space, $t_0 \in (0,1)$, $r \ge 0$ and $k \in \mathbb{R}$, we introduce the Banach space

$$\mathcal{X}^k_r = \{f : (0,t_0) \to E \mid \ f \in C^r, \ \max_{0 \le j \le r} \sup_{t \in (0,t_0)} t^{-k+j\eta} |D^j f(t)| < \infty\},$$

with the norm

$$\|f\|_{r,k} := \max_{0\le j\le r} \sup_{t\in(0,t_0)} t^{-k+j\eta}|D^j f(t)|.$$

The following proposition is an elementary consequence of the definition of the Banach spaces.

Proposition 4.1. *The following three conditions are equivalent:*

$$f \in \mathcal{X}_r^k. \tag{4.3}$$

$$D^l f \in \mathcal{X}_{r-l}^{k-l\eta}, \quad 0 \le l \le r. \tag{4.4}$$

$$f \in C^r(0,t_0) \quad \text{and} \quad D^l f \in \mathcal{X}_0^{k-l\eta} \quad 0\le l\le r. \tag{4.5}$$

If $f \in \mathcal{X}_r^k$, $\|f\|_{r,k} = \max_{0\le l\le r}\|D^l f\|_{r-l,k-l\eta} = \max_{0\le l\le r}\|D^l f\|_{0,k-l\eta}$.

$$k_1 \ge k_2 \implies \mathcal{X}_r^{k_1}\subset \mathcal{X}_r^{k_2}, \quad \|f\|_{r,k_2}\le \|f\|_{r,k_1}, \quad \text{for } f\in \mathcal{X}_r^{k_1}.$$

$$r_1 \ge r_2 \implies \mathcal{X}_{r_1}^{k}\subset \mathcal{X}_{r_2}^{k}, \quad \|f\|_{r_2,k}\le \|f\|_{r_1,k}, \quad \text{for } f\in \mathcal{X}_{r_1}^{k}.$$

Proposition 4.2. *If* $f(t)\in L(\mathbb{R}^p,\mathbb{R}^q)$ *with* $f\in\mathcal{X}_r^k$ *and* $g(t)\in\mathbb{R}^p$ *with* $g\in\mathcal{X}_r^l$ *then* $fg\in\mathcal{X}_r^{k+l}$ *and*

$$\|fg\|_{r,k+l}\le 2^r\|f\|_{r,k}\|g\|_{r,l}. \tag{4.6}$$

More generally, if $f(t)\in L^p(X_2,X_3)$ *with* $f\in\mathcal{X}_r^k$ *and* $h_i(t)\in L^{q_i}(X_1,X_2)$*, with* $h_i\in\mathcal{X}_r^{m_i}$, $1\le i\le p$*, where* X_1*,* X_2 *and* X_3 *are Banach spaces, we have* $f\,h_1\cdots h_p\in\mathcal{X}_r^{k+m_1+\cdots+m_p}$ *and*

$$\|f\,h_1\cdots h_p\|_{r,k+m_1+\cdots+m_p}\le (1+p)^r\|f\|_{r,k}\|h_1\|_{r,m_1}\cdots\|h_p\|_{r,m_p}. \tag{4.7}$$

Proof. This follows easily from

$$\begin{aligned} t^{-k-l+j\eta}|D^j(fg)(t)| &= t^{-k-l+j\eta}\Big|\sum_{m=0}^{j}\binom{j}{m}D^{j-m}f(t)D^m g(t)\Big| \\ &\le \sum_{m=0}^{j}\binom{j}{m}t^{-k-l+j\eta}t^{k-(j-m)\eta}\|f\|_{r,k}t^{l-m\eta}\|g\|_{r,l} \\ &= 2^j\|f\|_{r,k}\|g\|_{r,l}. \end{aligned}$$

□

The following proposition deals with the composition in $\mathcal{X}_k^r$ spaces. It will be used in Section 4.10.

Proposition 4.3. *Let* $G: U\subset\mathbb{R}^{1+n}\to E$ *be a* C^r *map, where* E *is a Banach space, and* $m\in\mathbb{R}$ *be such that* $|D^pG(x,y)|\le M_p|(x,y)|^{m-p}$ *for all* $0\le p\le r$ *and* $(x,y)\in U\setminus\{(0,0)\}$*. Then,*

a) *If* $g\in\mathcal{X}_j^1$ *with* $0\le j\le r$ *and* $g((0,t_0))\subset U$*, then* $G\circ g\in\mathcal{X}_j^m$*.*

b) *If* $g \in \mathcal{X}_j^1$ *with* $0 \le j \le r$, $g(0,t_0)) \subset U$ *and* $h_i \in \mathcal{X}_j^{m_i}$ *for some* $m_i \in \mathbb{R}$, $i = 1, \cdots, l$, *and* $0 \le l \le r$, *then* $(D^l G \circ g) h_1 \cdots h_l \in \mathcal{X}_s^{m-l+m_1+\cdots+m_l}$, *where* $s = \min\{r-l, j\}$.

Proof. It is clear that $G \circ g \in C^j$ and since for all $t \in (0, t_0)$,

$$|G \circ g(t)| \le M_0 |g(t)|^m \le M_0 \|g\|_{j,1}^m t^m,$$

$G \circ g \in \mathcal{X}_0^m$. Moreover, for $0 \le l \le r$, by Faa-di-Bruno's formula,

$$D^l(G \circ g)(t) = \sum_{i=1}^{l} \sum_{\substack{1 \le l_1, \cdots, l_i \le l \\ l_1 + \cdots + l_i = l}} \sigma D^i G \circ g \, D^{l_1} g \cdots D^{l_i} g,$$

(where σ is a combinatorial coefficient) and the fact

$$\begin{aligned} & t^{-m+l\eta} |D^i G(g(t))| \, |D^{l_1} g(t)| \cdots |D^{l_i} g(t)| \\ & \qquad \le t^{-m+i} |D^i G(g(t))| \, t^{1-l_1\eta} |D^{l_1} g(t)| \cdots t^{1-l_i\eta} |D^{l_i} g(t)| \le M_i \|g\|_{r,1}^i \end{aligned}$$

we obtain that $D^l(G \circ g) \in \mathcal{X}_0^{m-l\eta}$ and therefore we conclude that $G \circ g \in \mathcal{X}_r^m$.

To prove b) we use that, by a) applied to $D^l G$ instead of G, $D^l G \circ g \in \mathcal{X}_{r-l}^{m-l}$. Then applying (4.7) we obtain $(D^l G \circ g) h_1 \cdots h_l \in \mathcal{X}_{r-l}^{m-l+m_1+\cdots+m_l}$. □

4.2. **A motivating example.** To motivate the choice of the Banach spaces $\mathcal{X}_r^k$, we consider the following example to emphasize the fact that we may lose more of one order in the scale of spaces when taking derivatives. Let $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x, y) = (x - x^N + f(x, y), \, y + x^{M-1} y + g(x, y)),$$

where $f, g \in C^r$, $r > 2N - 1$, and $f, g = O(|(x, y)|^r)$. Taking $K^{\le}(t) = (t, 0)$ and $R(t) = t - t^N$ and have that

$$T_r := F \circ K^{\le} - K^{\le} \circ R = (f(t, 0), g(t, 0)) = O(t^r).$$

Now we look for $K^{>}$ satisfying (4.2). Is it easy to see that (4.2) is equivalent to

$$(DF \circ K^{\le}) K^{>} - K^{>} \circ R = -T_r - \mathcal{N}(K^{>}) \tag{4.8}$$

with $\mathcal{N}(K^{>}) = F \circ (K^{\le} + K^{>}) - F \circ K^{\le} - (DF \circ K^{\le}) K^{>}$.

In view of Lemma 3.1 we consider $K^{\le}$ as a polynomial of degree $r - N$. To solve this equation we will work in a space of differentiable functions $K^{>}$ of order $O(t^{r-N+1})$. In such case, using Taylor's theorem, we get $\mathcal{N}(K^{>})(t) = O(t^{r+1})$. Indeed, $\mathcal{N}(K^{>}) \approx (1/2)(D^2 F \circ K^{\le})(K^{>})^2$ and hence $\mathcal{N}(K^{>})(t) = O(t^{L-2}) O(t^{2(r-N+1)})$. We have $L - 2 \ge 0$ and since $r \ge 2N - 1$ then $2(r - N + 1) \ge r + 1$.

Using

$$K^>(t) - K^>(t - t^N) = -\int_0^1 DK^>(t - st^N)(-t^N)\, ds$$
$$DF(K^\leq)K^> - K^> \circ R = [DF(K^\leq) - \mathrm{Id}]K^> + K^> - K^> \circ R,$$

we rewrite equation (4.8) more explicitly as:

$$\begin{pmatrix} -Nt^{N-1}\pi_1 K^> \\ t^{M-1}\pi_2 K^> \end{pmatrix} + t^N \int_0^1 DK^>(t - st^N)\, ds = -T_r - \mathcal{N}(K^>) + O(t^{2r-N}). \quad (4.9)$$

Our goal is to determine η such that $DK^>(t) = O(t^{r-N+1-\eta})$. We observe that, if the second component of equation (4.9) is satisfied, we have to match terms of orders $O(t^{r+M-N})$, $O(t^{r+1-\eta})$ and $O(t^r)$. Hence, if $M < N$, we must have $r + M - N = r + 1 - \eta$, which implies $\eta = 1 + N - M$, and if $N \leq M$, we must have $\eta = 1$. To deal simultaneously with the cases $M < N$ and $N \leq M$ we take $\eta = 1 + N - L$, as in the statement of Theorem 2.1.

4.3. **Decomposition of the interval.** To obtain bounds of several objects we will use the decomposition of $(0, t_0)$, associated to he map R, given by the following lemma.

Lemma 4.4. *Let R be an analytic map in a neighborhood of the origin of the form $R(z) = z + d_N z^N + O(z^{2N-1})$ with $d_N < 0$ and let $\alpha = 1/(N-1)$. If t_0 is small enough there exist s and a collection of intervals $I_k = [\frac{c_{k+1}}{(s+k+1)^\alpha}, \frac{c_k}{(s+k)^\alpha}]$ such that*

1) $(0, t_0] = \cup_{k\geq 0} I_k$.
2) $R(I_k) = I_{k+1}$.

Moreover $c_k = c_0 + O(\frac{1}{k^\beta})$, where $c_0 = (-\alpha/d_N)^\alpha$ and β is any number in $(0, 1)$.

Proof. We will use the Fatou coordinates in the attracting petal which intersects the positive real axis (see for instance [Mil91]). First we conjugate R by $\varphi_1(w) = c_0 w^{-1/(N-1)}$, where $c_0 = \big(\frac{-1}{(N-1)d_N}\big)^{1/(N-1)}$. This gives

$$\begin{aligned} G_1(w) &:= \varphi_1^{-1} \circ R \circ \varphi_1(w) \\ &= \varphi_1^{-1}\Big(\frac{c_0}{w^{1/(N-1)}} + d_N \frac{c_0^N}{w^{N/(N-1)}} + O\big(\frac{1}{w^{(2N-1)/(N-1)}}\big)\Big) \\ &= w\Big[1 + d_N \frac{c_0^{N-1}}{w} + O\big(\frac{1}{w^{(2N-1)/(N-1)}}\big)\Big]^{-1} \\ &= w + 1 + O\Big(\frac{1}{w}\Big). \end{aligned}$$

Then by Lemma 7.8 in [Mil91] there exists an analytic φ_2 which conjugates G_1 to the translation $w \mapsto w + 1$. In that lemma the asymptotic expression for φ_2 is not made explicit, but working out some more details of the proof we obtain that $\varphi_2(w) = w + O(w^{1-\beta})$ with $\beta \in (0,1)$. Therefore

$$\varphi(w) := \varphi_1 \circ \varphi_2(w) = \frac{c_0}{w^\alpha} + O\big(\frac{1}{w^{\alpha+\beta}}\big)$$

conjugates R to $z \mapsto z+1$, i.e. $R(\varphi(z)) = \varphi(z+1)$. Let s such that $\varphi(s) = t_0$. If t_0 is small enough, s is big and φ is monotonically decreasing in (s,∞). Then $(\varphi(s+k))_{k\geq 0}$ converges monotonically to zero and since $R(\varphi(s+k)) = \varphi(s+k+1)$ then $R([\varphi(s+k+1), \varphi(s+k)]) = [\varphi(s+k+2), \varphi(s+k+1)]$. We define $c_k = (s+k)^\alpha \varphi(s+k) = c_0 + O(1/k^\beta)$. Then the proof is complete. □

4.4. **Scaling and preliminary lemmas.** As a first adjustment, we scale the y-variable through $E_\delta(x,y) = (x, \delta y)$. After the scaling, equations (4.1) and (4.2) become

$$\widetilde{P} \circ \widetilde{K}^{\leq} - \widetilde{K}^{\leq} \circ R = \widetilde{T}_k \tag{4.10}$$

and

$$\widetilde{F} \circ (\widetilde{K}^{\leq} + K^{>}) - (\widetilde{K}^{\leq} + K^{>}) \circ R = 0, \tag{4.11}$$

where $\widetilde{F} = E_\delta^{-1} \circ F \circ E_\delta$, $\widetilde{P} = E_\delta^{-1} \circ P \circ E_\delta$, $\widetilde{Q}_k = E_\delta^{-1} \circ Q_k \circ E_\delta$, $\widetilde{K}^{\leq} = E_\delta^{-1} \circ K^{\leq}$ and $\widetilde{T}_k = E_\delta^{-1} \circ T_k$.

We have:

$$\widetilde{P}(u,v) = \begin{pmatrix} u + a_{N,0}u^N + \delta v^{\mathrm{T}} f_{N-1}(u,\delta v) + f_{N+1}(u,\delta v) \\ v + B_{M-1,1}u^{M-1}v + \delta v^T g_{M-2}(u,\delta v)v + \delta^{-1} g_{M+1}(u,\delta v) \end{pmatrix} \tag{4.12}$$

and

$$\widetilde{K}^{\leq}(t) = (t, \delta^{-1}K_2^2 t^2) + (O(t^2), \delta^{-1}O(t^3)). \tag{4.13}$$

From now on, we drop the tilde in (4.10), (4.11) (4.12) and (4.13) and we assume that δ is small. Let

$$\sigma := \delta\alpha |a_{N,0}|^{-1} \sup_{t\in(0,t_0)} |f_{N-1}(t,0)t^{-N+1}|. \tag{4.14}$$

As a second adjustment, in the case that $M \leq N$, we choose a norm in E_2 such that

$$\|\mathrm{Id} - B_{M-1,1}t^{M-1}\| \leq 1 - \mu t^{M-1} \tag{4.15}$$

for some $\mu > 0$. This is possible by Condition (2.5). Indeed, in a basis where $B_{M-1,1}$ is in Jordan form, with small non-diagonal terms, $\mathrm{Id} - B_{M-1,1}t^{M-1}$ is also in Jordan form. Then we can take the max norm in this basis.

Finally in $\mathbb{R}^{1+n}$ we take the norm $|(x,y)| = \max(|x|,|y|)$, $(x,y) \in \mathbb{R}\times\mathbb{R}^n$.

Lemma 4.5. *If t_0 is small enough, for all $t \in (0,t_0)$,*

$$\|(DP)^{-1}(K^{\leq}(t))\| \leq 1 - a_{N,0}t^{N-1}(N + \sigma\alpha^{-1}) + Ct^N. \tag{4.16}$$

Proof. Taking into account (4.12) and (4.13) we can write $DP(K^{\leq}(t))$ as

$$\begin{pmatrix} 1+Na_{N,0}t^{N-1}+O(t^N) & \delta f_{N-1,0}(t)+O(t^N) \\ \delta^{-1}O(t^M)+(M-1)B_{M-1,1}K_2^2t^M+O(t^{M+2}) & \mathrm{Id}+B_{M-1,1}t^{M-1}+O(t^M) \end{pmatrix}. \tag{4.17}$$

Let $t_0=t_0(\delta)>0$ be such that $t_0^{1/2}\delta^{-1}<1$. Then since $DP\circ K^{\leq}$ is close to the identity,

$$(DP)^{-1}(K^{\leq}(t))=\begin{pmatrix} 1-Na_{N,0}t^{N-1}+O(t^N) & -\delta f_{N-1,0}(t)+O(t^N) \\ -\delta^{-1}O(t^M) & \mathrm{Id}-B_{M-1,1}t^{M-1}+O(t^M) \end{pmatrix}. \tag{4.18}$$

Now we compute the matrix norm of $(DP\circ K^{\leq})^{-1}$. Since we use the max norm in $\mathbb{R}\times\mathbb{R}^n$, the matrix norm of $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}$ is less or equal than $\max\{|c_{11}|+|c_{12}|, |c_{21}|+\|c_{22}\|\}$. For the first row, using that $a_{N,0}<0$ we have

$$\begin{aligned} &|1-Na_{N,0}t^{N-1}+O(t^N)|+|\delta f_{N-1}(t,0)+O(t^N)| \\ &\leq\ 1-Na_{N,0}t^{N-1}+\delta t^{N-1}\sup_{t\in(0,t_0)}|f_{N-1}(t,0)t^{-N+1}|+O(t^N) \\ &\leq\ 1-a_{N,0}t^{N-1}(N+\sigma\alpha^{-1})+O(t^N). \end{aligned} \tag{4.19}$$

For the block of the n remaining rows, using that $t_0^{1/2}\delta^{-1}<1$ and the hypotheses we have that, if $M\leq N$,

$$\begin{aligned} &\|\mathrm{Id}-B_{M-1,1}t^{M-1}+O(t^M)\|+\delta^{-1}O(t^M) \\ &\qquad\leq\|\mathrm{Id}-B_{M-1,1}t^{M-1}\|+O(t^{M-1/2})<1 \end{aligned} \tag{4.20}$$

if t_0 is small enough.

When $N<M$ we bound the right hand side of (4.20) by $1+t^{M-1}\|B_{M-1,1}\|+O(t^{M-1/2})$. Recalling that $a_{N,0}<0$, (4.19) implies (4.16) in this case. When $N\geq M$, we have (4.16) because of the second adjustment (4.15) and (4.19). □

4.5. **Weak contraction generated by the nonlinear terms.** From now on C will be a generic constant depending only on t_0, N and k, that can take different values at different places.

Although the origin is not hyperbolic we get some contraction from the nonlinear terms. The next result gives some quantitative estimates which are consequence of the weak hyperbolicity provided by the standing hypotheses.

Lemma 4.6. *Let $\{I_n\}_{n\geq 0}\subset(0,t_0)$ and $s>0$ be as in Lemma 4.4. There exists a constant C depending only on t_0, N and k, such that for any $n\geq 0$ such that $I_n\subset(0,t_0)$, $t\in I_n$ and $j\geq 0$,*

$$\prod_{l=0}^{j}\|(DP)^{-1}(K^{\leq}(R^l(t)))\|\leq C\left(\frac{s+n+j}{s+n}\right)^{N\alpha+\sigma}, \tag{4.21}$$

$$\|D[(DP)^{-1}\circ K^{\leq}](t)\| \leq C(s+n)^{-\alpha(L-2)}, \tag{4.22}$$

$$|DR^j(t)| \leq C\left(\frac{s+n}{s+n+j}\right)^{N\alpha}, \quad |D^2R^j(t)| \leq C(s+n)^{N\alpha-1}|DR^j(t)|. \tag{4.23}$$

Proof. Let $t \in I_n$. First we observe that, since $R^l(t) \in I_{n+l}$ by bound (4.16) we have that:

$$\begin{aligned}
&\|(DP)^{-1}(K^{\leq}(R^l(t)))\| \\
&\qquad \leq 1 - a_N\left(\frac{c_{n+l}}{(s+n+l)^{\alpha}}\right)^{N-1}(N+\sigma\alpha^{-1}) + C\left(\frac{c_{n+l}}{(s+n+l)^{\alpha}}\right)^{N} \\
&\qquad \leq 1 + (N\alpha+\sigma)\frac{1}{s+n+l} + C\frac{1}{(s+n+l)^{1+\gamma}},
\end{aligned}$$

where $\gamma = \min\{\beta, \alpha\}$. Recall that, if $N = 2$, $\alpha = 1$ and $\beta < 1$, otherwise we can take $\beta > \alpha$.

Therefore

$$\begin{aligned}
&\prod_{l=0}^{j}\|(DP)^{-1}(K^{\leq}(R^l(t)))\| \\
&\qquad \leq \exp\Big(\sum_{l=0}^{j}\log\big(1 + (N\alpha+\sigma)\frac{1}{s+n+l} + C\frac{1}{(s+n+l)^{1+\gamma}}\big)\Big) \\
&\qquad \leq \exp\Big(\sum_{l=0}^{j}(N\alpha+\sigma)\frac{1}{s+n+l} + C\frac{1}{(s+n+l)^{1+\gamma}}\Big) \\
&\qquad \leq \exp\Big(\log\Big(\frac{s+n+j}{s+n-1}\Big)^{N\alpha+\sigma} + \frac{C}{\gamma}\Big(\frac{1}{(s+n-1)^{\gamma}} - \frac{1}{(s+n+j)^{\gamma}}\Big)\Big).
\end{aligned}$$

This proves the first bound of Lemma 4.6. The second one follows differentiating (4.17) and using that $t \in I_n$.

Now we deal with the bounds involving the derivatives of R. Since $R(t) = t + a_{N,0}t^N + d_{2N-1}t^{2N-1}$, we have that for $t \in I_n$,

$$|DR(t)| \leq 1 - N\alpha\frac{1}{s+n+1} + C\frac{1}{(s+n+1)^{1+\beta}}. \tag{4.24}$$

Using

$$DR^j = \prod_{l=0}^{j-1} DR\circ R^l, \qquad j \geq 1 \tag{4.25}$$

together with (4.24) we have that

$$\begin{aligned}|DR^j(t)| &\leq \exp\Big(\sum_{l=0}^{j-1}\log\big(1-\frac{N\alpha}{s+n+l+1}+\frac{C}{(s+n+l+1)^{1+\beta}}\big)\Big)\\ &\leq \exp\Big(-\sum_{l=0}^{j-1}\frac{N\alpha}{s+n+l+1}+\frac{C}{(s+n+l+1)^{1+\beta}}\Big)\\ &\leq \exp\Big(\log\Big(\frac{s+n+j}{s+n}\Big)^{-N\alpha}+\frac{C}{\beta}\Big(\frac{1}{(s+n)^\beta}-\frac{1}{(s+n+j)^\beta}\Big)\Big).\end{aligned}$$

The first bound of (4.23) follows easily from the above inequality. Now we deal with the bound of D^2R^j. Differentiating formula (4.25), we have that

$$D^2R^j(t)=\sum_{l=0}^{j-1}D^2R\circ R^l\cdot DR^l\frac{\prod_{i=0}^{j-1}DR\circ R^i}{DR\circ R^l}=DR^j\sum_{l=0}^{j-1}D^2R\circ R^l\frac{DR^l}{DR\circ R^l}. \quad (4.26)$$

We note that $|DR(R^l(t))|\geq 1/2$ and $|DR^2(R^l(t))|\leq 2N(N-1)|a_{N,0}||R^l(t)|^{N-2}$ if t_0 is small enough. Hence using the first bound in (4.23), for $t\in I_n\subset(0,t_0)$, we have that

$$\begin{aligned}\sum_{l=0}^{j-1}|D^2R\circ R^l(t)|\frac{|DR^l(t)|}{|DR\circ R^l(t)|} &\leq C\sum_{l=0}^{j-1}\frac{(s+n)^{N\alpha}}{(s+n+l)^{\alpha(N-2)+N\alpha}}\\ &= C(s+n)^{N\alpha}\sum_{l=0}^{j-1}\frac{1}{(s+n+l)^2}\leq C(s+n)^{N\alpha}\frac{1}{s+n-1}.\end{aligned}$$

Therefore, from (4.26), we obtain the second bound of (4.23) with some constant C independent of n and j. □

4.6. **The operators $\mathcal{L}^j$.** Our goal is to find a solution of equation (4.2) after scaling, that is a solution of equation (4.11). Assuming that $K^{\leq}$ satisfies equation (4.1), then $K^{>}$ is a solution of equation (4.2) (we recall that $F=P+Q_k$) if and only if

$$\begin{aligned}&(DP\circ K^{\leq})K^{>}-K^{>}\circ R \qquad (4.27)\\ &= -T_k-Q_k\circ(K^{\leq}+K^{>})-P\circ(K^{\leq}+K^{>})+P\circ K^{\leq}+(DP\circ K^{\leq})K^{>}.\end{aligned}$$

This motivates the definition of the linear operator

$$\mathcal{L}^0(S)=(DP\circ K^{\leq})S-S\circ R. \quad (4.28)$$

When dealing with the derivatives of $K^{>}$ we will need the operators

$$\mathcal{L}^j(H)=(DP\circ K^{\leq})H-H\circ R\,(DR)^j, \qquad j\geq 1. \quad (4.29)$$

We note that if S is a C^r solution of $\mathcal{L}^0(S)=T$, with $T\in C^r$, then for $0\leq j\leq r$, $H=D^jS$ is a solution of equation

$$\mathcal{L}^j(H)=T^j, \quad (4.30)$$

where T^j is defined by the recurrence

$$\begin{aligned} T^0 &= T \\ T^{j+1} &= DT^j - D(DP\circ K^{\leq})D^jS + jD^jS\circ R\,(DR)^{j-1}D^2R. \end{aligned} \tag{4.31}$$

We recall that $\eta = 1 + N - L$ and σ is defined in (4.14).

Lemma 4.7. *If $k > 2N-1$ and $\sigma < \alpha(k-2N+1)$, the operators $\mathcal{L}^j : \mathcal{X}_0^{k-N+1-j\eta} \to C^0$, $j \geq 0$, defined by (4.28) and (4.29) are one to one.*

Remark 4.8. *The conditions for injectivity become weaker when j grows. For instance, if $j = 1$ it is enough $k > 2N - L$ and $\sigma < \alpha(k - 2N + L)$.*

Here we could take $k \geq 2N - 1$ and $\sigma < \alpha j(N - \eta)$.

Proof. Let $j \geq 0$. We look for the kernel of $\mathcal{L}^j$. Let $S \in \mathcal{X}_0^{k-N+1-j\eta}$ be such that $\mathcal{L}^j(S) = 0$. This is equivalent to $S = [(DP)^{-1}\circ K^{\leq}]\, S\circ R(DR)^j$. Using iteratively this condition and (4.25) we obtain,

$$S = \Big[\prod_{m=0}^{i}(DP)^{-1}\circ K^{\leq}\circ R^m\Big] S\circ R^{i+1}(DR^{i+1})^j, \qquad i \geq 0. \tag{4.32}$$

Let $t \in I_n$. We can bound the norms of the terms in the right-hand side of (4.32) by using Lemma 4.6 and the fact that $R^{i+1}(t) \in I_{n+i+1}$. We obtain

$$\begin{aligned} |S(t)| &\leq C\left(\frac{s+n+i}{s+n}\right)^{N\alpha+\sigma}\left(\frac{s+n}{s+n+i+1}\right)^{N\alpha j}|S\circ R^{i+1}(t)| \\ &\leq C\|S\|_{0,k-N+1-j\eta}\frac{(s+n)^{N\alpha j-N\alpha-\sigma}}{(s+n+i)^{-N\alpha-\sigma+N\alpha j+(k-N+1-j\eta)\alpha}}. \end{aligned}$$

Hence, since $\alpha(k-2N+1)-\sigma > 0$ and $N-\eta \geq 1$,

$$|S(t)| \leq \lim_{i\to\infty} C\|S\|_{0,k-N+1-j\eta}\frac{(s+n)^{N\alpha(j-1)-\sigma}}{(s+n+i)^{\alpha(k-2N+1)-\sigma+\alpha j(N-\eta)}} = 0.$$

Thus $S = 0$ and therefore $\operatorname{Ker}\mathcal{L}^j = 0$. □

4.7. **The operators $\mathcal{S}^j$.** To obtain a formal solution of $\mathcal{L}^j(H) = T$ we rewrite it as the fixed point equation

$$H = [(DP)^{-1}\circ K^{\leq}]H\circ R\,(DR)^j + [(DP)^{-1}\circ K^{\leq}]\,T. \tag{4.33}$$

Iterating (4.33), assuming that $\Big[\prod_{m=0}^{i}(DP)^{-1}\circ K^{\leq}\circ R^m\Big]H\circ R^{i+1}\,(DR^{i+1})^j$ goes to 0 as i tends to ∞, we obtain $H = \mathcal{S}^j(T)$ with

$$\mathcal{S}^j(T) = \sum_{i\geq 0}\Big[\prod_{m=0}^{i}(DP)^{-1}\circ K^{\leq}\circ R^m\Big]T\circ R^i\cdot(DR^i)^j. \tag{4.34}$$

The following two sections are devoted to study the operators defined by (4.34) in different spaces. In particular in Lemma 4.9 we will prove that if $T \in \mathcal{X}_0^{k-j\eta}$

the right-hand side of (4.34) is absolutely convergent. Then if $T \in \mathcal{X}_0^{k-j\eta}$ we can compute $\mathcal{L}^j(\mathcal{S}^j(T))$ rearranging terms and we obtain

$$\mathcal{L}^j(\mathcal{S}^j(T)) = \lim_{i\to\infty} [T - \prod_{m=1}^{i+1} (DP)^{-1} \circ K^{\leq} \circ R^m \, T \circ R^{i+1}(DR^{i+1})^j] = T. \tag{4.35}$$

4.8. **The operators $\mathcal{S}^j$ on spaces of low regularity.** In this section, given $j \in \mathbb{Z}$, $j \geq 0$, we consider the operator $\mathcal{S}^j$ defined by (4.34) on the spaces $\mathcal{X}_0^{k-j\eta}$ and $\mathcal{X}_1^{k-j\eta}$.

Lemma 4.9. *If $k > 2N-1$ and $\sigma < \alpha(k-2N+1)$ then, (4.34) defines a bounded linear operator*

$$\mathcal{S}^j : \mathcal{X}_0^{k-j\eta} \longrightarrow \mathcal{X}_0^{k-N+1-j\eta}.$$

We have

$$\mathcal{L}^j \circ \mathcal{S}^j = \mathrm{Id} \qquad on \quad \mathcal{X}_0^{k-j\eta}.$$

Moreover (4.34) also defines a bounded linear operator

$$\mathcal{S}^j : \mathcal{X}_1^{k-j\eta} \longrightarrow \mathcal{X}_1^{k-N+1-j\eta}$$

and if $T \in \mathcal{X}_1^{k-j\eta}$

$$D[\mathcal{S}^j(T)] = \mathcal{S}^{j+1}(\widetilde{T}),$$

where

$$\widetilde{T} = DT - D(DP \circ K^{\leq})\, \mathcal{S}^j(T) + j\mathcal{S}^j(T) \circ R\, (DR)^{j-1} D^2 R. \tag{4.36}$$

Proof. Let $t \in I_n \subset (0, t_0)$ and $T \in \mathcal{X}_0^{k-j\eta}$. We denote $S = \mathcal{S}^j(T)$. Bounding the right-hand side of (4.34) using Lemma 4.6 we obtain

$$\begin{aligned}
|S(t)| &\leq \sum_{i\geq 0} \Big[\prod_{m=0}^{i} \|(DP)^{-1}(K^{\leq}(R^m(t)))\| \Big] |T(R^i(t))|\, |DR^i(t)|^j \\
&\leq \sum_{i\geq 0} C \left(\frac{s+n+i}{s+n}\right)^{N\alpha+\sigma} \|T\|_{0,k-j\eta} |R^i(t)|^{k-j\eta} |DR^i(t)|^j \\
&\leq \sum_{i\geq 0} C \left(\frac{s+n+i}{s+n}\right)^{N\alpha+\sigma} \|T\|_{0,k-j\eta} \left(\frac{c_{n+i}}{(s+n+i)^\alpha}\right)^{k-j\eta} \left(\frac{s+n}{s+n+i}\right)^{N\alpha j},
\end{aligned}$$

if $k - j\eta \geq 0$. In the case that $k - j\eta < 0$, we use that $R^i(t) > \frac{c_{n+i+1}}{(s+n+i+1)^\alpha}$ and hence

$$|S(t)| \leq \sum_{i\geq 0} C \left(\frac{s+n+i}{s+n}\right)^{N\alpha+\sigma} \|T\|_{0,k-j\eta} \left(\frac{c_{n+i+1}}{(s+n+i+1)^\alpha}\right)^{k-j\eta} \left(\frac{s+n}{s+n+i}\right)^{N\alpha j}.$$

Hence in both cases we have that

$$|S(t)| \leq C\|T\|_{0,k-j\eta}(s+n)^{N\alpha(j-1)-\sigma} \sum_{i\geq 0} (s+n+i)^{-\alpha(k-N+j(N-\eta))+\sigma}. \tag{4.37}$$

We have $N-\eta \geq 1$ and by hypothesis $\alpha(k-N)-\sigma > \alpha(N-1) = 1$, thus the series in the right-hand side of (4.37) is convergent. Moreover

$$\begin{aligned} |S(t)| &\leq C\|T\|_{0,k-j\eta}(s+n)^{N\alpha(j-1)-\sigma} \int_{s+n-1}^{\infty} \frac{dx}{x^{\alpha(k-N+j(N-\eta))-\sigma}} \\ &\leq C\|T\|_{0,k-j\eta}(s+n)^{1-\alpha(k-j\eta)} \end{aligned}$$

and therefore, since $t \in I_n$, $|t^{-k+N-1+j\eta}S(t)| \leq C\|T\|_{0,k-j\eta}$ which implies that

$$\mathcal{S}^j(T) \in \mathcal{X}_0^{k-N+1-j\eta}, \qquad \|\mathcal{S}^j(T)\|_{0,k-N+1-j\eta} \leq C\|T\|_{0,k-j\eta}. \tag{4.38}$$

This also proves the uniform convergence of the right-hand side of (4.34). Hence substituting (4.34) into (4.30) we can reorder the terms and check that $\mathcal{S}^j(T)$ indeed solves (4.30). See formula (4.35). This ends the proof of the first part of the lemma.

We claim that, if $T \in \mathcal{X}_1^{k-j\eta}$, then $\widetilde{T} \in \mathcal{X}_0^{k-(j+1)\eta}$. Indeed, it follows from $DT \in \mathcal{X}_0^{k-(j+1)\eta}$, (4.38), the fact that $D(DP \circ K^{\leq}) \in \mathcal{X}_0^{L-2}$, $D^2R \in \mathcal{X}_0^{N-2}$ and the definition of η.

Next we prove that $D\mathcal{S}^j(T) = \mathcal{S}^{j+1}(\widetilde{T})$. Let $T \in \mathcal{X}_1^{k-j\eta}$. We observe that, if $\mathcal{S}^j(T)$ is differentiable, then differentiating equation (4.30), we obtain that $D[S^j(T)]$ must be a solution of

$$\mathcal{L}^{j+1}(H) = DP \circ K^{\leq} \circ H - H \circ R\,(DR)^{j+1} = \widetilde{T}. \tag{4.39}$$

Thus, if $\mathcal{S}^j(T)$ is differentiable and its derivative belongs to $\mathcal{X}_0^{k-N+1-(j+1)\eta}$, the uniqueness result of Lemma 4.7 applied to $\mathcal{L}^{j+1}$ implies that $D[\mathcal{S}^j(T)] = \mathcal{S}^{j+1}(\widetilde{T})$, since both $D[\mathcal{S}^j(T)]$ and $\mathcal{S}^{j+1}(\widetilde{T})$ are solutions of equation (4.39) belonging to $\mathcal{X}_0^{k-N+1-(j+1)\eta}$.

Therefore it remains to prove that $\mathcal{S}^j(T)$ is differentiable and its derivative belongs to $\mathcal{X}_0^{k-N+1-(j+1)\eta}$.

Differentiating formally (4.34) we obtain $D[\mathcal{S}^j(T)] = \Delta S_1 + \Delta S_2 + \Delta S_3$, where

$$\begin{aligned} \Delta S_1 &= \sum_{i\geq 0} \Big[\prod_{m=0}^{i} (DP)^{-1} \circ K^{\leq} \circ R^m\Big] DT \circ R^i\,(DR^i)^{j+1}, \\ \Delta S_2 &= \sum_{i\geq 0} \Big[\prod_{m=0}^{i} (DP)^{-1} \circ K^{\leq} \circ R^m\Big] T \circ R^i\, j(DR^i)^{j-1}\, D^2R^i, \\ \Delta S_3 &= \sum_{i\geq 0}\sum_{m=0}^{i} \Big[\prod_{\nu=0}^{m-1} (DP)^{-1} \circ K^{\leq} \circ R^\nu\Big] D\big((DP)^{-1} \circ K^{\leq} \circ R^m\big) \times \\ &\quad \times \Big[\prod_{\nu=m+1}^{i} (DP)^{-1} \circ K^{\leq} \circ R^\nu\Big] T \circ R^i\,(DR^i)^j. \end{aligned}$$

Next we prove that $\Delta S_1, \Delta S_2$ and ΔS_3 are absolutely convergent and hence $\mathcal{S}^j(T)$ is differentiable.

First we deal with ΔS_1. Since $T \in \mathcal{X}_1^{k-j\eta}$, $DT \in \mathcal{X}_0^{k-(j+1)\eta}$ and therefore, by (4.38) $\Delta S_1 = \mathcal{S}^{j+1}(DT) \in \mathcal{X}_0^{k-N+1-(j+1)\eta}$ and

$$\|\Delta S_1\|_{0,k-N+1-(j+1)\eta} \le C\|DT\|_{0,k-(j+1)\eta} \le C\|T\|_{1,k-j\eta} \tag{4.40}$$

Next we consider ΔS_2. Let $t \in I_n \subset (0, t_0)$. By Lemma 4.6,

$$\begin{aligned}
&\Big\|\Big[\prod_{m=0}^{i}(DP)^{-1}(K^{\le}\circ R^m(t))\Big]T\circ R^i(t) j(DR^i(t))^{j-1}\, D^2R^i(t)\Big\| \\
&\quad\le jC(s+n)^{N\alpha-1}\Big[\prod_{m=0}^{i}\|(DP)^{-1}(K^{\le}\circ R^m(t))\|\Big]|T\circ R^i(t)|\,|DR^i(t)|^j \\
&\quad\le jC(s+n)^{N\alpha-1}\left(\frac{s+n+i}{s+n}\right)^{N\alpha+\sigma}\frac{\|T\|_{0,k-j\eta}}{(s+n+i)^{\alpha(k-j\eta)}}\left(\frac{s+n}{s+n+i}\right)^{N\alpha j} \\
&\quad= jC\|T\|_{0,k-j\eta}\frac{(s+n)^{N\alpha j-1-\sigma}}{(s+n+i)^{\alpha(k-N+j(N-\eta))-\sigma}}.
\end{aligned}$$

Therefore, since $\alpha(k-N)-\sigma > 1$ and $N-\eta \ge 1$ we have that

$$\begin{aligned}
|\Delta S_2(t)| &\le C\|T\|_{0,k-j\eta}(s+n)^{N\alpha j-1-\sigma}\int_{s+n-1}^{\infty}\frac{dx}{x^{\alpha(k-N+j(N-\eta))-\sigma}} \\
&\le C(s+n)^{-\alpha(k-N-j\eta)}\|T\|_{0,k-j\eta} \qquad (4.41)
\end{aligned}$$

which implies that the series defining ΔS_2 converges uniformly, $\Delta S_2 \in \mathcal{X}_0^{k-N-j\eta} \subset \mathcal{X}_0^{k-N+1-(j+1)\eta}$ and

$$\|\Delta S_2\|_{0,k-N+1-(j+1)\eta} \le \|\Delta S_2\|_{0,k-N+1-j\eta} \le C\|T\|_{0,k-j\eta}.$$

Finally we deal with ΔS_3. Let $t \in I_n \subset (0, t_0)$. Applying Lemma 4.6 and the chain rule

$$\begin{aligned}
\|D\big((DP)^{-1}(K^{\le}\circ R^m(t))\big)\| &\le C(s+n+m)^{-\alpha(L-2)}|DR^m(t)| \\
&\le C(s+n+m)^{-2+\alpha(\eta-1)}(s+n)^{N\alpha}.
\end{aligned}$$

Since $2-\alpha(\eta-1) > 1$ we also have $\sum_{m=0}^{i}(s+n+m)^{-2+\alpha(\eta-1)} \le C(s+n)^{-1+\alpha(\eta-1)}$.

Proceeding as in (4.41) and using again Lemma 4.6 and the fact that $\|(DP)^{-1}(K^{\leq}\circ R^{\nu}(t))\| \geq 1$, we get

$$
\begin{aligned}
|\Delta S_3(t)| \leq C \sum_{i\geq 0}\sum_{m=0}^{i} \Big[\prod_{\nu=0}^{i} \|(DP)^{-1}(K^{\leq}\circ R^{\nu}(t))\|\Big] \times \\
\times \frac{(s+n)^{N\alpha}}{(s+n+m)^{2-\alpha(\eta-1)}} \frac{\|T\|_{0,k-j\eta}}{(s+n+i)^{\alpha(k-j\eta)}} \left(\frac{s+n}{s+n+i}\right)^{N\alpha j} \\
\leq C\|T\|_{0,k-j\eta}(s+n)^{N\alpha j-1+\alpha(\eta-1)-\sigma} \sum_{i\geq 0} \frac{1}{(s+n+i)^{\alpha(k-N+j(N-\eta))-\sigma}} \\
\leq C(s+n)^{-\alpha(k-N+1-(j+1)\eta)}\|T\|_{0,k-j\eta}.
\end{aligned}
$$

This implies that the series defining ΔS_3 is uniformly convergent, $\Delta S_3 \in \mathcal{X}_0^{k-N+1-(j+1)\eta}$ and

$$\|\Delta S_3\|_{0,k-N+1-(j+1)\eta} \leq C\|T\|_{1,k-j\eta}.$$

Collecting the previous estimates we deduce that $D[\mathcal{S}^j(T)] \in \mathcal{X}_0^{k-N+1-(j+1)\eta}$, $\mathcal{S}^j(T) \in \mathcal{X}_1^{k-N+1-j\eta}$ and $\mathcal{S}^j : \mathcal{X}_1^{k-j\eta} \to \mathcal{X}_1^{k-N+1-j\eta}$ is a bounded operator. □

4.9. **The operators $\mathcal{S}^0$ and $\mathcal{S}^1$ in spaces of higher regularity.** To deal with the r-derivative of $K^{>}$ we will need to work with the operators $\mathcal{S}^0$ and S^1 defined on the space $\mathcal{X}_s^{k-N+1}$ with $s \leq r$.

Proposition 4.10. *Let $r > 0$, $k > 2N-1$ and $\sigma < \alpha(k-2N+1)$. Then, if $0 \leq s \leq r$*

$$\mathcal{S}^0 : \mathcal{X}_s^{k} \to \mathcal{X}_s^{k-N+1} \qquad \text{and} \qquad \mathcal{S}^1 : \mathcal{X}_s^{k-\eta} \to \mathcal{X}_s^{k-N+1-\eta}$$

are bounded linear operators.

Proof. Given $T \in \mathcal{X}_r^k \subset \mathcal{X}_0^k$ we introduce the sequence $(T^j)_{0\leq j\leq r}$ defined inductively by

$$
\begin{aligned}
T^0 &= T \\
T^{j+1} &= DT^j - D(DP\circ K^{\leq})\,\mathcal{S}^j(T^j) + j\mathcal{S}^j(T^j)\circ R\,(DR)^{j-1}D^2R, \qquad (4.42)
\end{aligned}
$$

for $0 \leq j \leq r-1$, where the operators $\mathcal{S}^i$ are defined by (4.34).

By Lemma 4.9, $\mathcal{S}^0(T^0) \in \mathcal{X}_1^{k-N+1}$ and $\|\mathcal{S}^0(T^0)\|_{0,k-N+1} \leq C\|T^0\|_{0,k}$.

We claim that for $1 \leq j \leq r$ the following three properties hold

a) T^j has the form

$$T^j = D^jT + \sum_{l=0}^{j-1}\Big(A_l^j\mathcal{S}^l(T^l) + p_l^j\mathcal{S}^l(T^l)\circ R\Big), \qquad (4.43)$$

where

i) A_l^j are matrices whose coefficients are polynomials in the variable t and belong to $\mathcal{X}_r^{L-1+l-j}$.

ii) p_l^j are polynomials in t and belong to $\mathcal{X}_r^{L-1+l-j}$.

Moreover $T^j \in \mathcal{X}_1^{k-j\eta}$ if $j \le r-1$. For $j = r$ we have that $T^r \in \mathcal{X}_0^{k-r\eta}$.

b) $\mathcal{S}^j(T^j) \in \mathcal{X}_1^{k-N+1-j\eta}$ and $\|\mathcal{S}^j(T^j)\|_{0,k-N+1-j\eta} \le c_j \|T^j\|_{0,k-j\eta}$ if $j \le r-1$. If $j = r$ we have that $\mathcal{S}^r(T^r) \in \mathcal{X}_0^{k-N+1-r\eta}$ and $\|\mathcal{S}^r(T^r)\|_{0,k-N+1-r\eta} \le c_r \|T^r\|_{0,k-r\eta}$.

c)
$$D[\mathcal{S}^{j-1}(T^{j-1})] = \mathcal{S}^j(T^j). \tag{4.44}$$

To prove the claim we proceed by induction. When $j = 1$ we have

$$T^1 = DT + A_0^1\, \mathcal{S}^0(T^0) + p_0^1\, \mathcal{S}^0(T^0) \circ R$$

with $A_0^1(t) = -D(DP \circ K^{\le})(t) = O(t^{L-2})$ and $p_0^1(t) = 0$. Since $DT \in \mathcal{X}_{r-1}^{k-\eta}$, $A_0^1 \in \mathcal{X}_r^{L-2}$ and $\mathcal{S}^0(T) \in X_1^{k-N+1}$, using (4.6) we deduce that $T^1 \in X_1^{k-\eta}$.

Then, by Lemma 4.9 and taking into account formula (4.17), $\mathcal{S}^1(T^1) \in \mathcal{X}_1^{k-N+1-\eta}$, $\|\mathcal{S}^1(T^1)\|_{0,k-N+1-\eta} \le c_1 \|T^1\|_{0,k-\eta}$ and $D[\mathcal{S}^0(T^0)] = \mathcal{S}^1(T^1)$.

We assume that a), b) and c) hold true for l with $1 \le l \le j \le r-1$. First we check that T^{j+1} has the form (4.43).

Differentiating (4.43) with respect to t we have that

$$\begin{aligned} DT^j =& D^{j+1}T + \sum_{l=0}^{j-1} \Big(DA_l^j\, \mathcal{S}^l(T^l) + A_l^j \mathcal{S}^{l+1}(T^{l+1}) \Big) \\ &+ \sum_{l=0}^{j-1} \Big(Dp_l^j\, \mathcal{S}^l(T^l) \circ R + p_l^j \mathcal{S}^{l+1}(T^{l+1}) \circ R\, DR \Big). \end{aligned} \tag{4.45}$$

Substituting formula (4.45) into (4.42) we get that T^{j+1} is of the form (4.43) with A_l^{j+1} and p_l^{j+1} given by

$$\begin{aligned} A_0^{j+1} &= DA_0^j, & p_0^{j+1} &= Dp_0^j, \\ A_l^{j+1} &= DA_l^j + A_{l-1}^j, & p_l^{j+1} &= Dp_l^j + p_{l-1}^j\, DR, \qquad 1 \le l \le j-1, \\ A_j^{j+1} &= A_{j-1}^j - D(DP \circ K^{\le}), & p_j^{j+1} &= p_{j-1}^j\, DR + j(DR)^{j-1} D^2 R. \end{aligned} \tag{4.46}$$

One immediately checks that A_l^{j+1} and p_l^{j+1} satisfy i) and ii) respectively.

From (4.43) we deduce that $T^{j+1} \in \mathcal{X}_1^{k-(j+1)\eta}$ if $j < r-1$, and $T^r \in \mathcal{X}_0^{k-r\eta}$.

Now by Lemma 4.9, $D[\mathcal{S}^j(T^j)] = \mathcal{S}^{j+1}(T^{j+1})$ and there exists $c_{j+1} > 0$ such that $\|\mathcal{S}^{j+1}(T^{j+1})\|_{0,k-N+1-(j+1)\eta} \le c_{j+1} \|T^{j+1}\|_{0,k-(j+1)\eta}$. This proves a), b) and c) for $j+1$.

Applying iteratively (4.44) we have that

$$D^j[\mathcal{S}^0(T)] = \mathcal{S}^j(T^j) \in \mathcal{X}_0^{k-N+1-j\eta}, \qquad j \le r,$$

and hence $\mathcal{S}^0(T) \in \mathcal{X}_r^{k-N+1}$.

Finally we prove that $\mathcal{S}^0$ is a bounded operator from $\mathcal{X}_s^k$ to $\mathcal{X}_s^{k-N+1}$. First we notice that there exists a constant $b > 0$ such that for all $0 \le l \le j-1$, $j \le r$, $\|A_l^j\|_{r,L-1+l-j}, \|p_l^j\|_{r,L-1+l-j} \le b$. Moreover, since $L-1+l-j \ge N-1-(j-l)\eta$ if $0 \le l \le j-1$, we also have that $A_l^j,\, p_l^j \in \mathcal{X}_r^{N-1-(j-l)\eta}$ and $\|A_l^j\|_{r,N-1-(j-l)\eta}, \|p_l^j\|_{r,N-1-(j-l)\eta} \le b$. Since, if $0 \le j \le r$ we can express T^j in form (4.43), by Lemma 4.9, Proposition 4.1 and (4.6) we get that

$$\begin{aligned}
\|T^j\|_{0,k-j\eta} \le & \|D^jT\|_{0,k-j\eta} + \sum_{l=0}^{j-1} \|A_l^j\|_{0,N-1-(j-l)\eta} \|\mathcal{S}^l(T^l)\|_{0,k-N+1-l\eta} \\
& + \sum_{l=0}^{j-1} \|p_l^j\|_{0,N-1-(j-l)\eta} \|\mathcal{S}^l(T^l)\circ R\|_{0,k-N+1-l\eta} \\
\le & \|T\|_{j,k} + 2b \sum_{l=0}^{j-1} c_l \|T^l\|_{0,k-l\eta},
\end{aligned} \tag{4.47}$$

where we have used that

$$\begin{aligned}
\|\mathcal{S}^l(T^l)\circ R\|_{0,m} &= \sup \frac{|\mathcal{S}^l(T^l)(R(t))|}{t^m} \le \sup \frac{|\mathcal{S}^l(T^l)(R(t))|}{|R(t)|^m} \sup \frac{|R(t)|^m}{t^m} \\
&\le \|\mathcal{S}^l(T^l)\|_{0,m}.
\end{aligned}$$

We claim that (4.47) implies that there exist constants $d_j > 0$ such that

$$\|T^j\|_{0,k-j\eta} \le d_j \|T\|_{j,k}, \qquad 0 \le j \le r. \tag{4.48}$$

Indeed, we prove inequality (4.48) by induction. If $j = 0$, (4.48) is satisfied taking $d_0 = 1$. We assume that (4.48) is true for $l \le j$. Then, from (4.47) we get

$$\|T^{j+1}\|_{0,k-(j+1)\eta} \le \|T\|_{j+1,k} + 2b \sum_{l=0}^{j} c_l d_l \|T\|_{l,k} \le \Big(1 + 2b\sum_{l=0}^{j} c_l d_l\Big) \|T\|_{j,k}$$

which satisfies (4.48) if we take $d_{j+1} = 1 + 2b\sum_{l=0}^{j} c_l d_l$.

Finally, since $\mathcal{S}^0(T) \in \mathcal{X}_r^{k-N+1} \subset \mathcal{X}_s^{k-N+1}$ and $D^j[\mathcal{S}^0(T)] = \mathcal{S}^j(T^j)$, $0 \le j \le r$, we have

$$\begin{aligned}
\|\mathcal{S}^0(T)\|_{s,k-N+1} &= \max_{0\le j\le s} \|\mathcal{S}^j(T^j)\|_{0,k-N+1-j\eta} \le \max_{0\le j\le s} c_j \|T^j\|_{0,k-j\eta} \\
&\le \max_{0\le j\le s} c_j d_j \|T\|_{j,k} \le (\max_{0\le j\le s} c_j d_j) \|T\|_{s,k}.
\end{aligned}$$

This ends the proof that $\mathcal{S}^0 : \mathcal{X}_s^k \to \mathcal{X}_s^{k-N+1}$ is a bounded operator.

The proof of the statement for $\mathcal{S}^1$ is quite similar to the one of $\mathcal{S}^0$. Here, given $T \in \mathcal{X}_r^{k-\eta} \subset \mathcal{X}_0^{k-\eta}$ we define $(T^j)_{0\le j\le r}$ by

$$T^0 = T$$

$$T^{j+1} = DT^j - D(DP\circ K^{\le})\,\mathcal{S}^{j+1}(T^j) + (j+1)\mathcal{S}^{j+1}(T^j)\circ R\,(DR)^j D^2R, \quad (4.49)$$

for $0 \le j \le r-1$.

By Lemma 4.9, $\mathcal{S}^1(T^0) \in \mathcal{X}_1^{k-N+1-\eta}$ and $\|\mathcal{S}^1(T^0)\|_{0,k-N+1-\eta} \le C\|T^0\|_{0,k-\eta}$.

Also we can prove by induction that for $1 \le j \le r$ we have

a) T^j has the form

$$T^j = D^jT + \sum_{l=0}^{j-1}\Big(B_l^j\mathcal{S}^{l+1}(T^l) + q_l^j\mathcal{S}^{l+1}(T^l)\circ R\Big), \quad (4.50)$$

where

i) B_l^j are matrices whose coefficients are polynomials in the variable t and belong to $\mathcal{X}_r^{L-1+l-j}$.

ii) q_l^j are polynomials in t and belong to $\mathcal{X}_r^{L-1+l-j}$.

Moreover $T^j \in \mathcal{X}_1^{k-(j+1)\eta}$ if $j \le r-1$, and $T^r \in \mathcal{X}_0^{k-(r+1)\eta}$.

b) $\mathcal{S}^{j+1}(T^j) \in \mathcal{X}_1^{k-N+1-(j+1)\eta}$ and $\|\mathcal{S}^{j+1}(T^j)\|_{0,k-N+1-(j+1)\eta} \le c_j\|T^j\|_{0,k-(j+1)\eta}$ if $j \le r-1$. For $j=r$, $\mathcal{S}^{r+1}(T^r) \in \mathcal{X}_0^{k-N+1-(r+1)\eta}$ and $\|\mathcal{S}^{r+1}(T^r)\|_{0,k-N+1-(r+1)\eta} \le c_r\|T^r\|_{0,k-(r+1)\eta}$.

c)

$$D[\mathcal{S}^j(T^{j-1})] = \mathcal{S}^{j+1}(T^j). \quad (4.51)$$

Next the proof proceeds in a completely analogous way as for $\mathcal{S}^0$. □

4.10. **Fixed point equation.** Using the definition of the operator $\mathcal{L}^0$, we can rewrite equation (4.27) as

$$\mathcal{L}^0(K^>) = \mathcal{F}(K^>),$$

where

$$\mathcal{F}(K^>) = -T_k - Q_k\circ(K^{\le}+K^>) - P\circ(K^{\le}+K^>) + P\circ K^{\le} + (DP\circ K^{\le})\,K^>. \quad (4.52)$$

Assuming formally that $\mathcal{L}^0\circ\mathcal{S}^0 = \mathrm{Id}$ (the fact we can use this property in appropriate spaces will be justified later on in this section), it is sufficient to solve

$$K^> = \mathcal{S}^0\circ\mathcal{F}(K^>). \quad (4.53)$$

Note that, since T_k and $Q_k\circ K^{\le}$ belong to $\mathcal{X}_r^k$, Proposition 4.10 implies that $\mathcal{S}^0\circ\mathcal{F}(0) = \mathcal{S}^0(-T_k - Q_k\circ K^{\le})$ belongs to $\mathcal{X}_r^{k-N+1}$. By this reason, we will look for the solution of equation (4.53) in $\mathcal{X}_r^{k-N+1}$. However, we will first obtain a solution of class C^{r-1}.

Since

$$\begin{aligned}\|\mathcal{S}^0 \circ \mathcal{F}(0)\|_{r,k-N+1} &= \|\mathcal{S}^0(-T_k - Q_k \circ K^{\le})\|_{r,k-N+1} \\ &\le \|\mathcal{S}^0\|\big(\|T_k\|_{r,k} + \|Q_k \circ K^{\le}\|_{r,k}\big),\end{aligned}$$

we will find the solution in the ball $\mathcal{B}^{k-N+1}_{r-1,\rho} \subset \mathcal{X}^{k-N+1}_{r-1}$ of radius $\rho = 2\|\mathcal{S}^0\|\big(\|T_k\|_{r,k}+ \|Q_k \circ K^{\le}\|_{r,k}\big)$ with t_0 so small that $K^{\le}(t) + K^{>}(t)$ belongs to the domain of F. In the next section we will prove that the solution obtained is indeed of class C^r.

Proposition 4.11. *Under the hypotheses of Theorem 2.1, if t_0 is small enough, equation* (4.53) *has a unique fixed point $K^{>} : [0, t_0) \to \mathbb{R}^{1+n}$ in $\mathcal{B}^{k-N+1}_{r-1,\rho}$.*

We postpone the proof of Proposition 4.11 to the end of this section after we have developed some preliminary lemmas.

We write

$$\mathcal{F}(K^{>})(t) = -T_k(t) - H(t, K^{>}(t))$$

with

$$H(t, z) = Q_k(K^{\le}(t) + z) + P(K^{\le}(t) + z) - P(K^{\le}(t)) - DP(K^{\le}(t))z, \tag{4.54}$$

$t \in \mathbb{R}$, $z \in \mathbb{R}^{1+n}$. We observe that H is C^r.

In the following lemma we collect some properties of H that we will use hereafter.

Lemma 4.12. *Assume F is C^r and $2N - 1 < k \le r$. Then, if $K^{>} \in \mathcal{B}^{k-N+1}_{j,\rho}$, $0 \le j \le r - 1$, and t_0 is small*

(1) $H \circ (\mathrm{Id}\,, K^{>}) \in \mathcal{X}^k_j$.
(2) $\frac{\partial H}{\partial t} \circ (\mathrm{Id}\,, K^{>}) \in \mathcal{X}^{k-\eta}_j$.
(3) $\frac{\partial H}{\partial z} \circ (\mathrm{Id}\,, K^{>}) \in \mathcal{X}^{k-\eta}_j$.

Proof. We start the proof of (1). By Taylor's theorem,

$$H(t, K^{>}(t)) = Q_k(K^{\le}(t) + K^{>}(t)) + \int_0^1 (1-\theta) D^2P(K^{\le}(t) + \theta K^{>}(t))\big(K^{>}(t)\big)^2 \, d\theta.$$

We note that $K^{\le} + K^{>} \in \mathcal{X}^1_j$, $0 \le j \le r-1$, and $|D^l Q_k(x, y)| \le C|(x, y)|^{k-l}$, hence by a) of Proposition 4.3 we have $Q_k \circ (K^{\le} + K^{>}) \in \mathcal{X}^k_j$ and using b) of Proposition 4.3 with $h_1 = h_2 = K^{>}$ we get that, since P is a polynomial, $D^2P(K^{\le}(t) + \theta K^{>}(t))\big(K^{>}(t)\big)^2 \in \mathcal{X}^{L-2+2(k-N+1)}_j \subset \mathcal{X}^k_j$.

To establish (2), differentiating H with respect to t, applying Taylor's theorem and substituting $z = K^{>}(t)$, we can write

$$\begin{aligned}\frac{\partial H}{\partial t}(t, K^{>}(t)) =& DQ_k(K^{\le}(t) + K^{>}(t))DK^{\le}(t) \\ &+ \int_0^1 (1-\theta) D^3P(K^{\le}(t) + \theta K^{>}(t))K^{>}(t)DK^{\le}(t)\, d\theta.\end{aligned}$$

Using property b) of Proposition 4.3 with $G = Q_k$, $g = K^{\le} + K^{>}$ and $h_1 = DK^{\le} \in \mathcal{X}_r^0$ we get that $DQ_k(K^{\le} + K^{>})DK^{\le} \in \mathcal{X}_j^{k-1} \subset \mathcal{X}_j^{k-\eta}$ and using b) of Proposition 4.3 with $G = P$, $g = K^{\le} + K^{>}$ and $h_1 = K^{>}$, $h_2 = DK^{\le}$, we obtain $D^3P(K^{\le} + \theta K^{>})K^{>}DK^{\le} \in \mathcal{X}_j^{L-3+k-N+1} \subset \mathcal{X}_j^{k-\eta}$.

Finally, we check (3). Differentiating H with respect to z and substituting $z = K^{>}(t)$ we can write

$$\frac{\partial H}{\partial z}(t, K^{\le}(t)) = DQ_k(K^{\le}(t) + K^{>}(t)) + \int_0^1 D^2P(K^{\le}(t) + \theta K^{>}(t))K^{>}(t)\, d\theta.$$

Applying property b) of Proposition 4.3 as before we obtain (3). □

Lemma 4.13. *Assume F is C^r and $k - 2N + L > 0$. Let $\mathcal{F}$ be the operator defined in (4.52). Let $\rho_r = 2\|\mathcal{S}^0\|(\|T_k\|_{r,k} + \|Q_k \circ K^{\le}\|_{r,k})$ and $\rho \ge \rho_r$. Then, there exist $t^* \in (0,1)$ such that, for any $0 < t_0 < t^*$,*

(1) *$\mathcal{F}$ is well defined on $\mathcal{B}_{j,\rho}^{k-N+1}$ and $\mathcal{F}(\mathcal{B}_{j,\rho}^{k-N+1}) \subset \mathcal{X}_j^k$ for $0 \le j \le r$.*
(2) *$\mathcal{F} : \mathcal{B}_{r-1,\rho}^{k-N+1} \to \mathcal{X}_{r-1}^k$ is Lipschitz, with Lipschitz constant bounded by*

$$\operatorname{Lip} \mathcal{F} \le Ct_0^{k-2N+L}. \tag{4.55}$$

Remark 4.14. Even though in this lemma the condition on k is to be greater than $2N - L$, in other previous results, namely Proposition 4.10, we have to require the stronger condition $k > 2N - 1$.

Proof. We use the expression $\mathcal{F}(K^{>})(t) = -T_k(t) - H(t, K^{>}(t))$ with H defined by (4.54).

Let $K^{>} \in \mathcal{B}_{j,\rho}^{k-N+1}$. By Lemma 4.12 the fact that $\mathcal{F}(\mathcal{B}_{j,\rho}^{k-N+1}) \subset \mathcal{X}_j^k$ follows from

$$\|\mathcal{F}(K^{>})\|_{j,k} \le \|T_k\|_{j,k} + \|H \circ (\mathrm{Id}\,, K^{>})\|_{j,k}.$$

To establish the second statement we take $K_1^{>}, K_2^{>} \in \mathcal{B}_{r-1,\rho}^{k-N+1}$ and $0 \le l \le r-1$. Then, using (1) of Lemma 4.12,

$$\begin{aligned}
|D^l(\mathcal{F}(K_1^{>}) - \mathcal{F}(K_2^{>}))(t)| &= |D^l[H(t, K_1^{>}(t)) - H(t, K_2^{>}(t))]| \\
&\le \sup_{\theta \in (0,1)} |D^{l+1}H(t, \theta K_1^{>}(t) + (1-\theta)K_2^{>}(t))|\, |K_1^{>}(t) - K_2^{>}(t)| \\
&\le Ct^{k-(l+1)\eta}t^{k-N+1}\|K_1^{>} - K_2^{>}\|_{r-1,k-N+1}.
\end{aligned}$$

Hence, since $\eta = 1 + N - L$

$$\begin{aligned}
\|\mathcal{F}(K_1^{>}) - \mathcal{F}(K_2^{>})\|_{r-1,k} &= \sup_{0 \le l \le r-1} \|D^l(\mathcal{F}(K_1^{>}) - \mathcal{F}(K_2^{>}))\|_{0,k-l\eta} \\
&\le \sup_{0 \le l \le r-1} \sup_{t \in (0,t_0)} Ct^{-\eta}t^{k-N+1}\|K_1^{>} - K_2^{>}\|_{r-1,k-N+1} \\
&\le Ct_0^{k-2N+L}\|K_1^{>} - K_2^{>}\|_{r-1,k-N+1}.
\end{aligned}$$

□

Proof of Proposition 4.11. Taking ρ as in Lemma 4.13 and using Proposition 4.10 we have

$$\mathcal{S}^0 \circ \mathcal{F} : \mathcal{B}^{k-N+1}_{r-1,\rho} \to \mathcal{X}^{k-N+1}_{r-1}.$$

Moreover, for any $K^> \in \mathcal{B}^{k-N+1}_{r-1,\rho}$, adding and subtracting $\mathcal{F}(0) = -T_k - Q_k \circ K^{\leq}$ we can write

$$\begin{aligned}\|\mathcal{S}^0 \circ \mathcal{F}(K^>)\|_{r-1,k-N+1} &\leq \|\mathcal{S}^0\| \big(\|\mathcal{F}(0)\|_{r-1,k} + \|\mathcal{F}(K^>) - \mathcal{F}(0)\|_{r-1,k}\big) \\ &\leq \|\mathcal{S}^0\| \big(\|T_k\|_{r,k} + \|Q_k \circ K^{\leq}\|_{r,k}\big) + C t_0^{k-2N+L} \|\mathcal{S}^0\| \, \|K^>\|_{r-1,k-N+1}.\end{aligned}$$

Then, we can choose t_0 such that $\|\mathcal{S}^0 \circ \mathcal{F}(K^>)\|_{r-1,k-N+1} < \rho$, and

$$\mathrm{Lip}\,(\mathcal{S}^0 \circ \mathcal{F})_{|\mathcal{B}^{k-N+1}_{r-1,\rho}} \leq C t_0^{k-2N+L} \|\mathcal{S}^0\| < 1.$$

We conclude that equation (4.53) has a unique fixed point in $\mathcal{B}^{k-N+1}_{r-1,\rho}$. □

By Lemma 4.12, if $K^> \in \mathcal{B}^{k-N+1}_{r-1,\rho}$, $\mathcal{F}(K^>) \in \mathcal{X}^k_j \subset \mathcal{X}^k_0$ and hence, by (4.35), $\mathcal{L}^0(\mathcal{S}^0(\mathcal{F}(K^>)) = \mathcal{F}(K^>)$ and then the solution of the fixed point equation (4.53) is also a solution of (4.27). At this point we have already proved that $K = K^{\leq} + K^>$ and R are a solution of (2.6) with $K \in C^{r-1}$.

4.11. **The C^∞ case.** Up to this point, we have established that if F is C^r, with $r > 2N-1$, there exists a C^{r-1} invariant manifold. More concretely, in the previous sections we have proved that, once $K^{\leq}$ and R are fixed, there exists ρ_r such that for any $\rho > \rho_r$ there exist t_r and a unique $K^>_r : [0, t_r) \to \mathbb{R}^{1+n}$, $K^>_r \in \mathcal{B}^{k-N+1}_{r-1,\rho} \subset \mathcal{X}^{k-N+1}_{r-1}$, such that $K = K^{\leq} + K^>_r$ is a solution of $F \circ K = K \circ R$.

Now we assume that F is also $C^{r'}$, $r' > r$. Taking the same $K^{\leq}$ and R, the preceding procedure yields the existence of a unique function $K^>_{r'} : [0, t_{r'}) \to \mathbb{R}^{1+n}$, $K^>_{r'} \in \mathcal{B}^{k-N+1}_{r'-1,\rho'} \subset \mathcal{X}^{k-N+1}_{r-1}$. We claim that both $K^>_r$ and $K^>_{r'}$ coincide in their common domain $[0, t_0)$. Indeed, it is enough to take $\rho^* = \max\{\rho, \rho'\}$, then both $\mathcal{B}^{k-N+1}_{r-1,\rho}$ and $\mathcal{B}^{k-N+1}_{r'-1,\rho'}$ are contained in $\mathcal{B}^{k-N+1}_{r-1,\rho^*}$, and, since the solution is unique in $\mathcal{B}^{k-N+1}_{r-1,\rho^*}$, the claim follows.

We finally prove that if F is C^∞ — and it is a diffeomorphism —, the K thus obtained actually is C^∞. Indeed, since F is C^∞, it is C^r, for any $r > 0$. We fix some $r_0 > 2N - 1$, and we obtain $K = K^{\leq} + K^>_{r_0}$ as a C^{r_0-1} parameterization of the invariant manifold, defined in some interval $[0, t_0)$. Let $r > r_0$. By the previous comments, there exists $K^>_r$ defined in $[0, t_r)$, which coincides with $K^>_{r_0}$. This establishes that $K^>_{r_0}$ is C^{r-1} in $[0, t_r)$. Now we use the invariance equation $F \circ K = K \circ R$ to extend the differentiability to $[0, t_r)$. Indeed, there exists $k \geq 0$ such that $R^k([t_r, t_0)) \subset (0, t_r)$. Now, the relation $K = F^{-k} \circ K \circ R^k$ proves that K is C^{r-1} in $[0, t_0)$.

4.12. **Sharp regularity.** In Section 4.10 we have proved the existence of a solution $K^> \in \mathcal{X}_{r-1}^{k-N+1}$ of the equation

$$\mathcal{L}^0(K^>) = \mathcal{F}(K^>).$$

Since $K^>$ is $(r-1)$-times differentiable and $r \geq 2$, we can differentiate both sides of this last equality to obtain

$$D[\mathcal{L}^0(K^>)] = D[\mathcal{F}(K^>)].$$

From the definitions of $\mathcal{L}^0$ and $\mathcal{L}^1$ we can write

$$D[\mathcal{L}^0(K^>)] = \mathcal{L}^1(DK^>) + D(DP \circ K^{\leq})K^>,$$

where $\mathcal{L}^1$ is defined in (4.29).

On the other hand,

$$D\mathcal{F}(K^>)(t) = -DT_k(t) - \frac{\partial H}{\partial t}(t, K^>(t)) - \frac{\partial H}{\partial z}(t, K^>(t))DK^>(t).$$

It is clear that $DK^>$ is a solution of

$$\mathcal{L}^1(DK^>) = \mathcal{A}(DK^>) + B, \tag{4.56}$$

where

$$\mathcal{A}(DK^>)(t) = -\frac{\partial H}{\partial z}(t, K^>(t))DK^>(t) \tag{4.57}$$

and

$$B(t) = -D(DP \circ K^{\leq})(t)K^>(t) - DT_k(t) - \frac{\partial H}{\partial t}(t, K^>(t)). \tag{4.58}$$

Lemma 4.15. *The operator $\mathcal{A}$, defined by* (4.57)*, is a bounded linear operator from $\mathcal{X}_s^{k-N+1-\eta}$ to $\mathcal{X}_s^{k-\eta}$ for $0 \leq s \leq r-1$. In every case*

$$\|\mathcal{A}\| \leq C t_0^{k-2N+L}. \tag{4.59}$$

Moreover, $B \in \mathcal{X}_{r-1}^{k-\eta}$.

Before addressing the proof of Lemma 4.15, we finish the proof of Theorem 2.1.

End of the proof of Theorem 2.1. We have already justified that $DK^> \in \mathcal{X}_{r-2}^{k-N+1-\eta}$ is a solution of equation (4.56). Then, by Lemma 4.15 the right-hand side of (4.56) belongs to $\mathcal{X}_{r-2}^{k-\eta}$. We would like to apply $\mathcal{S}^1$ to both sides of (4.56). We note that

$$\mathcal{S}^1(\mathcal{L}^1(DK^>)) = \lim_{i\to\infty}[DK^> - \prod_{m=0}^{i} DP^{-1} \circ K^{\leq} \circ R^m\; DK^> \circ R^{i+1}(DR^{i+1})^j] = DK^>$$

because $DK^> \in \mathcal{X}_0^{k-N+1-\eta}$. Then we can write

$$DK^> = \mathcal{S}^1 \circ \mathcal{A}(DK^>) + \mathcal{S}^1 \circ B.$$

By Proposition 4.10, $\mathcal{S}^1 : \mathcal{X}_s^{k-\eta} \to \mathcal{X}_s^{k-N+1-\eta}$, $s \leq r-1$, is a bounded linear operator and by Lemma 4.15 the norm of $\mathcal{A}$ can be made small by taking t_0 small and hence we can have $\|\mathcal{S}^1\circ\mathcal{A}\|_{L(\mathcal{X}_s^{k-N+1-\eta},\mathcal{X}_s^{k-N+1-\eta})}$ less than one for $s = r-2, r-1$. Therefore

$$DK^> = (\mathrm{Id} - \mathcal{S}^1 \circ \mathcal{A})^{-1}\mathcal{S}^1 B.$$

Moreover, since $B \in \mathcal{X}_{r-1}^{k-\eta}$, $DK^> \in \mathcal{X}_{r-1}^{k-N+1-\eta}$ and therefore $K^> \in \mathcal{X}_r^{k-N+1}$. □

Proof of Lemma 4.15. To prove the statement for $\mathcal{A}$, let $\psi \in \mathcal{X}_s^{k-N+1-\eta}$, with $0 \le s \le r-1$. Using Lemma 4.12 we calculate

$$\begin{aligned} |D^m[\mathcal{A}\psi](t)| &\le C\sum_{j=0}^{m}\left|D^j\frac{\partial H}{\partial z}(t, K^>(t))\right||D^{m-j}\psi(t)| \\ &\le C\sum_{j=0}^{m} t^{k-\eta-j\eta}t^{k-N+1-\eta-(m-j)\eta}\|\psi\|_{s,k-N+1-\eta} \\ &\le Ct^{k-(m+1)\eta+k-2N+L}\|\psi\|_{s,k-N+1-\eta}, \end{aligned} \tag{4.60}$$

where $0 \le m \le s$. Finally, from (4.60) we obtain

$$\begin{aligned} \|\mathcal{A}\psi\|_{s,k-\eta} &= \sup_{0\le m\le s}\sup_{0<t<t_0}|t^{-k+(m+1)\eta}D^m[\mathcal{A}\psi](t)| \\ &\le \sup_{0\le m\le s}\sup_{0<t<t_0} C(1+\rho)t^{k-2N+L}\|\psi\|_{s,k-N+1-\eta} \\ &\le Ct_0^{k-2N+L}\|\psi\|_{s,k-N+1-\eta}. \end{aligned}$$

Now we check that $B \in \mathcal{X}_{r-1}^{k-\eta}$. We claim that the three terms in the right-hand side of (4.58) belong to $\mathcal{X}_{r-1}^{k-\eta}$. Indeed, since $D(DP\circ K^\le)$ is a polynomial matrix such that $|D(DP\circ K^\le)(t)| \le Ct^{L-2}$, then $D(DP\circ K^\le) \in \mathcal{X}_{r-1}^{L-2}$. Therefore, since $K^> \in \mathcal{X}_{r-1}^{k-N+1}$ we have that $D(DP\circ K^\le)K^> \in \mathcal{X}_{r-1}^{L-2+k-N+1} = \mathcal{X}_{r-1}^{k-\eta}$. Since $T_k \in \mathcal{X}_r^k$ we have $DT_k \in \mathcal{X}_{r-1}^{k-\eta}$. Finally, Lemma 4.12 asserts that $\frac{\partial H}{\partial t}\circ(\mathrm{Id}, K^>) \in \mathcal{X}_{r-1}^{k-\eta}$. This finishes the proof. □

5. Numerical implementation and examples

In this section we implement the algorithm given in Section 3 to compute a stable invariant manifold tangent to the x axis at the origin and we describe some features of it on two examples.

Let F be a map satisfying the hypotheses of Theorem 2.1. The parameterization method gives polynomials

$$K^\le(t) = \left(t + \sum_{i=2}^{k} c_i^1 t^i, \sum_{i=2}^{k} c_i^2 t^i\right) \quad \text{and} \quad R(t) = t + d_N t^N + d_{2N-1}t^{2N-1}$$

satisfying $F\circ K^\le - K^\le\circ R = o(t^{k+L-1})$ if r, the differentiability of F, is bigger than $k+L-1$.

We want to stress that, as we pointed out in Section 3, the parameterization method lefts free the coefficients c_i^1, c_i^2 for $i = 2, N-1$ and c_N^1. We will discuss how the choice of these free coefficients may be used to increase the domain where $K^{\leq}$ gives a good approximation of the invariant manifold. We also emphasize that a suitable choice of these free coefficients can stabilize numerically the method.

In the examples we admit that $K^{\leq}$ and R as accurate approximations of the invariant manifold and the dynamics on it respectively in $[0, t_0]$ if they satisfy

$$\max_{i=0,\cdots,I} |F \circ K^{\leq}(t_i) - K^{\leq} \circ R(t_i)| \leq 10^{-17}, \qquad I = 1000,\ t_i = t_0 \frac{i}{I}. \tag{5.1}$$

We have performed the numerical computations with long double precision.

5.1. **Example 1.** The first example is given by:

$$F(x, y) = \begin{pmatrix} x - a^2x^2 + ay^2 + a^2x^3 \\ y + xy - ax^3 + ax^4 \end{pmatrix} \tag{5.2}$$

with $a = 0.1$. This map satisfies the hypotheses of Theorem 2.1, hence it has a stable manifold which is tangent at x axis at the origin. In this example $N = M = 2$ and the coefficient c_2^1 is free.

We have computed a polynomial approximation $K^{\leq}$ of degree 200 of the invariant manifold and also the polynomial R, the dynamics on the invariant manifold. The computations take a few seconds. As we pointed out before, the fact that $K^{\leq}$ is an accurate approximation of an invariant manifold on the domain $[0, t_0]$, in the sense of condition (5.1), may depend on the choice of c_2^1. The following table illustrates this phenomenon. It shows the maximum value of t_0 for which (5.1) holds for different values of c_2^1.

c_2^1	t_0	$\pi^1 K^{\leq}(t_0)$
-0.5000	0.6975	0.6148036
-0.4375	0.7605	0.6876998
-0.3750	0.8340	0.7773399
-0.3125	0.9255	0.8960931
-0.2500	1.0110	1.0157500
-0.1875	1.0605	1.0848160
-0.1250	1.0620	1.0808590
-0.0625	1.0650	1.0789480

c_2^1	t_0	$\pi^1 K^{\leq}(t_0)$
0.0000	1.0635	1.0716890
0.0625	1.0050	1.0051940
0.1250	0.9360	0.9377373
1.8750	0.8775	0.8865733
0.2500	0.8265	0.8454586
0.3125	0.7725	0.8030672
0.3750	0.7320	0.7736656
0.4375	0.6945	0.7470789

The biggest value of t_0 as a function of c_2^1 is $t_0 = 1.0695$ and is obtained for $c_2^1 = -0.12249$. The computations give $R(t) = t - a^2t^2 + a^2t^3$. The image of $K^{\leq}$ is predicted in Figure.1:

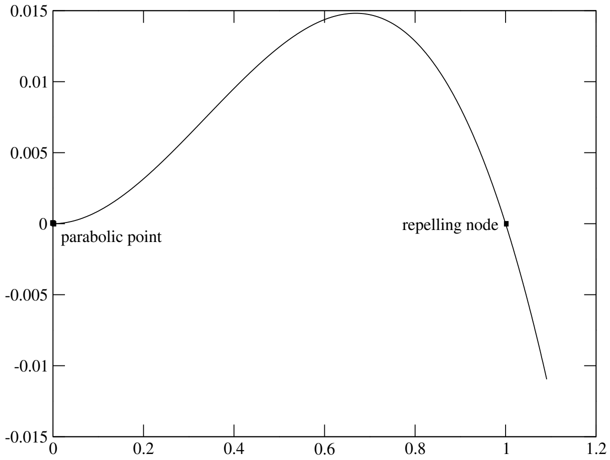

Figure 1

It is worth to note that $(1,0) = K^{\leq}(1)$ and that it is a repelling node of F (in fact F only has two fixed points). Hence there is numerical evidence that $K^{\leq}$ parameterizes a connection between both fixed points.

We note that since, $R(t) = t - a^2t^2 + a^2t^3$, $t = 1$ is a fixed point of R and the stable manifold is given by $K^{\leq}$ restricted to $[0,1)$.

An important feature of this method is that $K^{\leq}$ parameterizes a curve which contains, and goes beyond, the fixed point $(1,0)$. Moreover the dynamics on the manifold given by R catches this fixed point and the dynamics around it (restricted to the invariant manifold). In the neighborhood of $(1,0)$, the image of $K^{\leq}$ is the slow manifold of $(1,0)$, that is, the invariant manifold tangent to the spectral subspace associated to the smallest eigenvalue (for slow manifolds see [CFdlL03a]).

Note that this piece of the curve can not be obtained by globalizing the local stable manifold of $(0,0)$.

We have performed another experiment: we have plotted in clear grey the points of the rectangle $D = [0,1.1] \times [-0.02, 0.02]$ (where the invariant manifold is contained) such that its iterate by F which is not in D has the y component less than -0.02. Analogously we have plotted in dark grey the points of D escaping from above from D. We have also plotted in black the invariant manifold computed before. The results are showed in Figure 2 where we clearly see that the invariant manifold separates different dynamical behaviors.

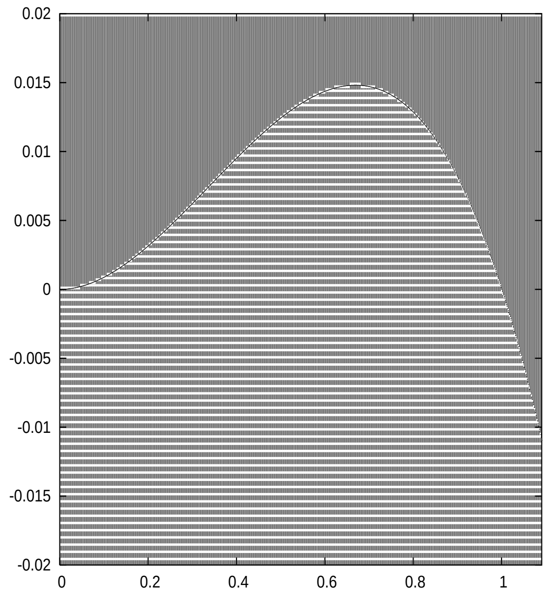

Figure 2

This is consequence of the fact that $\frac{\partial^M F_2}{\partial x^{M-1}\partial y} = 1 > 0$ which provides a weak expansion in the y direction and is the reason of the uniqueness of the manifold.

¿From now on we restrict ourselves to the interval $(0,1)$, since the stable manifold is the given by $K((0,1))$.

In order to check that $K^{\leq}$ is a good approximation of an invariant manifold of F in $[0,t_0]$, we have computed the distance between the curves $F^m(K^{\leq})$ and $K^{\leq}$ for $m = 1,5,10,15,20$ by the formula

$$\max_{i=0,\cdots,I} \operatorname{dist}(F^m(K^{\leq}(t_i)), K^{\leq}) \ \text{ with } \ I = 1000, \ \text{ and } \ t_i = t_0 \frac{i}{I}. \tag{5.3}$$

The results are displayed in the next table.

m	$\operatorname{dist}(F^m(K^{\leq}), K^{\leq})$
20	$5.801640 \cdot 10^{-12}$
15	$1.389271 \cdot 10^{-13}$
10	$3.369653 \cdot 10^{-15}$
5	$7.892456 \cdot 10^{-17}$
1	$1.944933 \cdot 10^{-18}$

Finally we have compared the parameterization method with the graph transform method. To compare both methods, we have also computed an approximation of the stable manifold of F as the graph of a polynomial of degree 200 following the graph transform method, that is we have looked for φ of the form $\varphi(x) = \varphi_2 x^2 + \cdots + \varphi_{200} x^{200}$ such that $F^2(x,\varphi(x)) - \varphi(F^1(x,\varphi(x))) = O(x^{202})$.

We have obtained that the curve $y = \varphi(x)$ approximates the stable manifold of F for values of x in $[0, 0.015]$ with an error of order 10^{-10}, that is:

$$\max_{i=0,\cdots,I} |\pi^2 F(x_i, \varphi(x_i)) - \varphi(\pi^1 F(x_i, \varphi(x_i)))| \leq 10^{-10}, \qquad I = 1000,\ x_i = 0.015\frac{i}{I}.$$

It is clear that the parameterization method is, at least in this example, much better than the graph transform method.

5.2. **Example 2.** The second example we consider is given by:

$$F_\alpha(x, y) = \begin{pmatrix} x - 10x^2 + (x-y)^4 - 0.01y^2 \\ y + 20xy + 400(x-y)^3 - 2000(x-y)^4 + \alpha x^{10}(1+2y)^{10}\sin(1/x) \end{pmatrix}.$$

Note that $F_\alpha \in C^4$ but $F_\alpha \notin C^5$. The map F satisfies the hypotheses of Theorem 2.1 for any $\alpha \in \mathbb{R}$ with $N = M = 2$ and $r = 4$.

Let us consider first the case $\alpha = 0$. In this case $F_0 \in C^\infty$and the polynomials $K^\leq(t) = (t - 10t^2, -10t^2)$ and $R(t) = t - 10t^2$ satisfy the invariance condition $F_0 \circ K^\leq - K^\leq \circ R = 0$. To check the parameterization method we have computed numerically the coefficients of R and $K^\leq$ up to order 50 with $c_2^1 = -10$ using the algorithm and we have obtained the expressions for $K^\leq$ and R given above, that is, the coefficients of the terms of order bigger or equal than 3 are zero.

Next we consider $\alpha = 0.2$. We have computed numerically the coefficients of R and $K^\leq$ up to order 9 for different choices of c_2^1 (which is free) and we have obtained that $K^\leq$ is an accurate approximation of the invariant manifold at $[0, t_0]$ with:

c_2^1	t_0	$\pi^2 K^\leq(t_0)$
-18	0.003600	$-1.221890 \cdot 10^{-4}$
-17	0.003600	$-1.230796 \cdot 10^{-4}$
-16	0.003400	$-1.108668 \cdot 10^{-4}$
-15	0.003400	$-1.116350 \cdot 10^{-4}$
-14	0.004000	$-1.547944 \cdot 10^{-4}$
-13	0.003400	$-1.131959 \cdot 10^{-4}$
-12	0.003200	$-1.010587 \cdot 10^{-4}$
-11	0.003000	$-8.944563 \cdot 10^{-5}$

c_2^1	t_0	$\pi^2 K^\leq(t_0)$
-9	0.002800	$-7.885386 \cdot 10^{-5}$
-8	0.002400	$-5.817098 \cdot 10^{-5}$
-7	0.002200	$-4.906015 \cdot 10^{-5}$
-6	0.002000	$-4.066134 \cdot 10^{-5}$
-5	0.002000	$-4.082926 \cdot 10^{-5}$
-4	0.001800	$-3.312480 \cdot 10^{-5}$
-3	0.001800	$-3.324798 \cdot 10^{-5}$
-2	0.001800	$-3.337185 \cdot 10^{-5}$
0	0.001600	$-2.645351 \cdot 10^{-5}$

Again for $\alpha = 0$ we take the value $c_2^1 = -10$ and obtain the functions $K^\leq(t) = (t - 10t^2, -10t^2)$ and $R(t) = t - 10t^2$. The plot of the image of $K^\leq$ is displayed in Fig. 3. We restrict the manifold to values of the parameter t in $[0, 0.1]$ because we are looking for invariant manifolds in the right-hand side plane. We remark that the image of $K^\leq$ provides a global invariant manifold where the global dynamics of the restriction of F is not invertible.

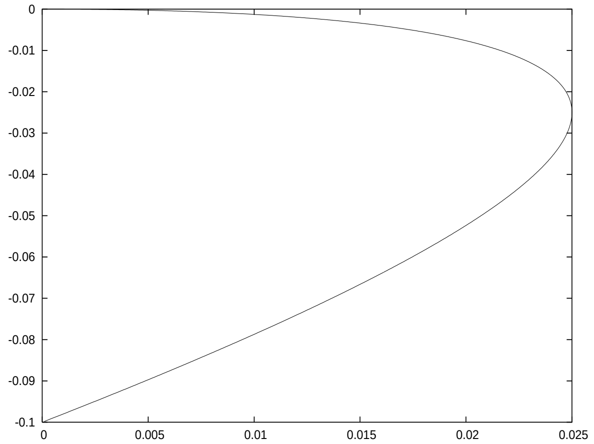

Figure 3

In the same way as in (5.3) we have computed the distance between the curves $F^m(K^{\leq})$ and $K^{\leq}$ for some values of m. The results are shown in the following table

m	$\mathrm{dist}(F^m(K^{\leq}), K^{\leq})$
1	$4.235369 \cdot 10^{-15}$
50	$3.114468 \cdot 10^{-15}$
100	$8.152378 \cdot 10^{-15}$
500	$7.720018 \cdot 10^{-15}$
1000	$1.962527 \cdot 10^{-14}$
5000	$4.593030 \cdot 10^{-13}$
10000	$1.829293 \cdot 10^{-12}$

We see that in this example the parameterization is extremely accurate.

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