DESTRUCTION OF INVARIANT CIRCLES OF NONTWIST MAPS

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Abstract. The aim of this paper is two–fold: first, to prove that the main cause of the destruction of invariant circles of the quadratic nontwist standard map is its folding property, and second, to illustrate numerically and give arguments that support a conjecture concerning the regularity of motion in the region where the map changes its sign of twist.

In order to locate regions in the infinite annulus, where a quadratic nontwist map exhibits different types of motion one associates to the map some non—invariant curves, such that their definition and relative position in the phase space give information on the dynamics of the orbits crossing them.

1. Introduction

Area preserving diffeomorphisms (APMs) of the infinite annulus $\mathbb{A} := \mathbb{S}^1 \times \mathbb{R}$ are local models for a Poincaré map associated to a cross section in an energy manifold of a two–degree of freedom Hamiltonian flow. An APM for which the twist condition is violated is called nontwist map. Invariant circles in both classes of maps (twist and nontwist) act as transports barriers in the phase space. Their destruction leads to loss of stability and confinement.

Twist property is a basic assumption of the KAM theorem for APMs and Aubry–Mather theory (for a survey of twist maps see [16]). During the last twenty five years much work was done on the study of the twist APMs, while the investigation of the nontwist maps started only in the last decade. Iso-energetically degenerate two–degree of freedom Hamiltonian systems, and nontwist APMs are good models for phenomena in a variety of fields. Such systems can describe the behaviour of magnetic field lines in reversed shear tokamaks [4], chaotic advection in fluids [5, 24], chaotic ionization in atomic physics [8], oscillations in astrophysics [17] (additional references can be found in [6],[25]).

The need to understand and explain these physical phenomena led to the study of nontwist APMs. There exist only few theoretical results concerning dynamical behaviour of these maps. In [9] and [23] are proved KAM–type results, that is the persistence to perturbations of the twistless circle and nearby invariant circles of diophantine rotation number. Regions of instability for nontwist APMs are studied in [11]. Recently it was proved that the vanishing of twist can also occur in a one–parameter family of APMs of the plane near the 1:3 resonance [10], that is whenever the elliptic fixed point passes through a triplication, a twistless bifurcation occurs.

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Having KAM–type results, a natural question is how do the persisting invariant circles of nontwist maps behave at a larger perturbation? Which of them break up first, and in which regions of the phase space are they more robust? While for the twist standard map the breakup hierarchy of the invariant circles is conjectured to be connected with the number theoretical properties of their rotation number, computer experiments with quadratic nontwist maps reveal that the breakup process occurs successively from the regions of monotone twist property towards the twistless circle.

The breakup of the twistless circle having rotation number $1/\gamma$, respectively $1/\gamma^2$, where $\gamma = (1+\sqrt{5})/2$ is the golden mean, was studied in [7, 1, 2] and the results interpreted within the renormalization group framework. Regularity of a parameterization of the twistless circle of noble rotation number, at the threshold of breakup, is investigated in [3]. Renormalization of iso–energetically degenerate two degree of freedom Hamiltonian systems is also reported in [12].

Numerical and theoretical studies [6], [25], [19], [18], [23] of the behaviour of nontwist APMs revealed a sequence of local and global bifurcations in the region where the twist property is violated, as well as the presence of invariant sets having special topological type (Figs. 12b, 5b, 5c). Moreover in this region the invariant circles appear as being much more robust than those lying in the regions of positive or negative twist of the map.

The goal of this paper is two–fold: to give arguments for the regularity of motion in the region of the phase space, where a quadratic nontwist map changes its sign of twist, and to prove that the main cause of the destruction of invariant circles of such a map is its folding property.

First we associate to a general quadratic nontwist map some non–invariant curves (critical twist circles, nontwist circle, folding curve) whose definition and position in the phase space give information on the dynamical behaviour of the orbits crossing them.

We conjecture that for some parameter pairs there exists a bounded invariant annulus containing the nontwist circle, on which the quadratic nontwist standard map is the time–one map of a one degree of freedom Hamiltonian system. The arguments to support this conjecture derive from considering the restriction of the map to such an annulus as a perturbation of a zero–twist map.

In order to decipher and explain the scenario of destruction of invariant circles of nontwist APMs, we exploit the folding property of the map. It is shown that for points in some region of the parameter plane, the corresponding nontwist map exhibits a bounded invariant annulus, including the critical twist circles, where the motion is quasi–integrable. Outside this annulus no invariant circle can exist, and moreover there are restrictions on the position and distribution of points of a regular hyperbolic periodic orbit.

Since the quadratic nontwist maps exhibits a lot of new phenomena we provide many phase portraits which illustrate them, and eventually can suggest how to prove or disprove the conjectured properties.

The paper is organized as follows: Section 2 presents properties of twist and nontwist APMs which are used in the paper. In Section 3 we study geometric properties of the quadratic nontwist maps, that is, how they act on the horizontal and vertical foliation of the annulus $\mathbb{S}^1 \times \mathbb{R}$. These properties are characterized by a sequence of cone fields (pairs of vector fields) defined on the phase space, more precisely by the relative position of the associated vectors at each point. We show that a quadratic nontwist map and its inverse

have distinct lines along which the twist condition is violated. Any invariant sub-annulus including one of them must include the second, too.

We also show that the folding property of an exact twist APM, allows the 1–cone crossing along an entire curve (the 1–cone crossing at a point in the phase space of a twist map is the strongest condition of non–existence of an invariant circle through that point).

Section 4 is devoted to the characterization of the dynamics of the quadratic nontwist standard map in the presence of foldings, as well as to the numerical illustration of the dynamics within the conjectured bounded invariant annulus, and of different scenarios of destruction of the last invariant circles.

2. Preliminaries

In the sequel we refer to exact area and orientation preserving C^r–diffeomorphisms, $r \geq 1$, defined on the infinite annulus $\mathbb{A} := \mathbb{S}^1 \times \mathbb{R}$, and called simply, area preserving maps (APMs).

Denote by $z = (\theta, y)$ an arbitrary point of the annulus. If for some $q \geq 1$, $f^q(z) = z$ then z is called periodic point of period q. The linear stability of a periodic point of an APM is characterized by its *residue*:

$$R(z) = \frac{2 - \mathrm{tr}(D_z f^q)}{4}, \tag{1}$$

where tr denote the *trace* of $D_z f^q$, the derivative at z of the q^{th}–iterate of the map. If $R(z) < 0$, then the periodic orbit is regular hyperbolic. For $0 < R(z) < 1$ the orbit is elliptic, while for $R(z) > 1$ the orbit is inverse hyperbolic. $R(z) = 0, 1$ corresponds to parabolic periodic points.

A closed curve of the annulus $\mathbb{A}$ which cannot be stretched to a point is called *rotational circle* or *homotopically nontrivial circle*. A rotational circle C with the property that $f(C) = C$ is called *rotational invariant circle*. In the sequel we refer only to such invariant circles and we omit the term rotational.

A lift of an APM $f : \mathbb{A} \to \mathbb{A}$ to the covering space $\mathbb{R}^2$ of the annulus $\mathbb{A}$ is a map $F : \mathbb{R}^2 \to \mathbb{R}^2$, with the property that $\pi \circ F = f \circ \pi$, where $\pi : \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{R}$ is the projection $\pi(x, y) = (x \pmod 1, y)$.

In the sequel ∂_θ, ∂_y denote the partial derivative with respect to θ, respectively to y.

An area preserving diffeomorphism $f : \mathbb{A} \to \mathbb{A}$, $f(\theta, y) = (\overline{\theta}, \overline{y})$, is *a positive* (*negative*) *twist map* if it has the property:

$$\partial_y \overline{\theta} \geq K > 0, \quad \text{respectively} \quad \partial_y \overline{\theta} \leq K' < 0 \tag{2}$$

We note that any lift F, of a twist map f, $F(x, y) = (\overline{x}, \overline{y})$, satisfies similar conditions. Geometrically, the twist condition ensures that the F–image of every vertical line $x = x_0$ in $\mathbb{R}^2$ is a graph over x.

Remark 1. *The F–image of a vertical is also a graph over y iff $\partial_y \overline{y} \neq 0$.*

Definition 2.1. *A twist APM whose associated function $\partial_y \overline{y}$ has not a constant sign is said to have a degenerate twist.*

An APM for which the twist condition is violated is called *nontwist map*. Obviously, the twist can be violated in many ways. We study the APMs with the property that on

each vertical line $\theta = \theta_0 \in \mathbb{S}^1$, the twist is violated at a unique point $y(\theta_0)$, which is a non–degenerate critical point of the map $y \mapsto \overline{\theta}(\theta_0, y)$:

$$\frac{\partial \overline{\theta}(\theta_0, y(\theta_0))}{\partial y} = 0, \quad \text{but} \quad \frac{\partial^2 \overline{\theta}(\theta_0, y(\theta_0))}{\partial y^2} \neq 0. \tag{3}$$

The rotational circle of equation $y = y(\theta)$ is called the *critical twist circle*, and we denote it by C_1.

Let $f : (\theta, y) \mapsto (\overline{\theta}, \overline{y})$ be an APM and $D_z f$ the derivative of f at $z = (\theta, y)$:

$$D_z f \equiv \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = \begin{pmatrix} \partial_\theta \overline{\theta} & \partial_y \overline{\theta} \\ \partial_\theta \overline{y} & \partial_y \overline{y} \end{pmatrix} (\theta, y) \tag{4}$$

Because of the area preservation property and the relation $D_{f(z)} f^{-1} = (D_z f)^{-1}$ we have:

$$D_z f^{-1} \equiv \begin{pmatrix} d(f^{-1}(z)) & -b(f^{-1}(z)) \\ -c(f^{-1}(z)) & a(f^{-1}(z)) \end{pmatrix} \tag{5}$$

Hence if f is a positive twist map, i.e. $b(z) > 0$ for any z, then f^{-1} has a negative twist.

Definition 2.2. *An APM having $d(z) > 0$ for every $z \in \mathbb{S}^1 \times \mathbb{R}$ is called $(+d)$–APM.*

The prototype of APM defined on the annulus $\mathbb{A}$, is the standard map defined by:

$$\begin{aligned} \overline{\theta} &= \theta + \Omega(\overline{y}) \ (\text{mod } 1) \\ \overline{y} &= y - \frac{k}{2\pi} \sin 2\pi\theta, \end{aligned} \tag{6}$$

where k is the perturbation parameter, and Ω is a differentiable function, called rotation number function. For $\Omega(y) = y$ we get the positive twist standard map, while for Ω a quadratic function, $\Omega(y) = \omega \pm y^2$, $\Omega(y) = \omega(1 \pm y^2)$, $\omega \in \mathbb{R}$, we have a nontwist standard map.

A survey of the properties of twist APMs can be found in [16]. In the sequel, we summarize some of them.

The average rotation of orbits around the annulus is measured by their rotation number. In order to define this number, let us fix a lift F of the map f, and take a point $(x, y) \in \mathbb{R}^2$ such that $\pi(x, y) = (\theta, y)$. Denote by $(x_n, y_n) = F^n(x, y)$, $n \in \mathbb{N}$ its orbit. If the limit

$$\rho = \lim_{n \to \infty} \frac{x_n - x}{n} \tag{7}$$

exists, then it is called *the rotation number* of the orbit of $z = \pi(x, y)$.

If (θ, y) is a periodic point of least period $q > 0$, and the orbit of its lift (x, y) has the property that:

$$x_q = x_0 + p, \text{ for some } p \in \mathbb{Z}, \tag{8}$$

then (θ, y) is called (p, q)–type periodic orbit. The rotation number of such an orbit is p/q.

A Poincaré–Birkhoff chain containing periodic orbits of type (p, q) will be called in the sequel, (p, q)–type Poincaré–Birkhoff chain.

The quasi-periodic orbits are those having an irrational rotation number. The closure of such an orbit can be an invariant circle or a Cantor set, called *cantorus*.

By Birkhoff theorem [15] a rotational invariant circle γ of a twist APM is the graph of a function $h : \mathbb{S}^1 \to \mathbb{R}$, and this function, regarded as $h : [0,1] \to \mathbb{R}$, with $h(0) = h(1)$ is a Lipschitz function [14], that is, there exist two constants A, B, such that

$$A \leq \frac{h(x) - h(x')}{x - x'} \leq B, \ \forall \ x, x' \in [0,1] \tag{9}$$

This property has been exploited in [14] to derive criteria ensuring that there is no invariant circle through a point in the phase space of a twist APM. In order to present the so called *cone crossing criterion*, recall that usually, a cone in the tangent space $T_z\mathbb{A}$, generated by two vectors $v^-(z), v^+(z) \neq 0$, is the set:

$$V(z) = \{w \in \mathrm{T}_z\mathbb{A} \mid w = \mu_1 v^-(z) + \mu_2 v^+(z), \mu_1, \mu_2 \geq 0\}. \tag{10}$$

Thus the cone $V(z)$ is represented by the pair of vectors $(v^-(z), v^+(z))$. Let $\beta^+(z), \beta^-(z) \in \mathbb{S}^1$ be such that $u^-(z) = (\cos 2\pi\beta^-(z), \sin 2\pi\beta^-(z))$, $u^+(z) = (\cos 2\pi\beta^+(z), \sin 2\pi\beta^+(z))$ are the unit vectors associated to $v^-(z)$, respectively to $v^+(z)$, i.e. $\|u^\pm\| = 1$ and $v^\pm(z) = \lambda^\pm(z)u^\pm(z)$, $\lambda^\pm(z) > 0$.

To an arbitrary (twist or nontwist) APM one associates a sequence of vector pairs $(v_n^-(z), v_n^+(z))$, $z \in \mathbb{A}$, defined as follows:

$$v_n^-(z) = D_{f^n(z)}f^{-n}(-e_2), \quad v_n^+(z) = D_{f^{-n}(z)}f^n(e_2), \text{where } e_2 = (0,1). \tag{11}$$

The directions of the vectors $v_n^-(z)$, $v_n^+(z)$ illustrate how the linear maps $D_{f^n(z)}f^{-n}$, $D_{f^{-n}(z)}f^n$ deviate the downward vertical through $f^n(z)$, respectively the upward vertical through $f^{-n}(z)$. $v_0^-(z) = -e_2$, $v_0^+(z) = e_2$. Denote by $\beta_n^-(z)$, $\beta_n^+(z) \in \mathbb{S}^1$, $n \geq 1$, the corresponding values associated as above. At an elliptic q–periodic point, z, the derivatives $D_{f^{nq}(z)}f^{-nq}$, $D_{f^{-nq}(z)}f^{nq}$ rotate the vertical vector e_2 more than 2π, for $n \geq N(z)$. That is why for each $z \in \mathbb{A}$ we represent the vectors $v_n^-(z)$, $v_n^+(z)$ by $\beta_n^-(z)$, $\beta_n^+(z)$ in $\mathbb{R}$, the covering space of $\mathbb{S}^1$. The pair $(\beta_n^-(z), \beta_n^+(z)) \in \mathbb{R}^2$ is called the n–cone associated to the point z, and is denoted $V_n(z)$, $n \in \mathbb{N}$.

We associate to the field of n–cones, V_n, the function $\alpha_n : \mathbb{A} \to \mathbb{R}$ defined by:

$$\alpha_n(z) = \beta_n^+(z) - \beta_n^-(z), n \in \mathbb{N}. \tag{12}$$

We call $\alpha_n(z)$ *the extended angle of the n–cone $V_n(z)$*.

Let us characterize the field of n–cones of a positive twist APM. The 0–cone $V_0(z)$ at the point z is represented by the pair $(\beta_0^-(z), \beta_0^+(z)) \in \mathbb{R}^2$, with $\beta_0^-(z) = -1/4$, and $\beta_0^+(z) = 1/4$. By the positive twist property, $\beta_1^+(z) < 1/4$, $\beta_1^-(z) > -1/4$. The derivative, D_zf, of an APM acts increasingly on slopes, that is, if two vectors $v, w \in T_z\mathbb{A}\backslash\{0\}$ have the slopes s_v, s_w such that $-\infty \leq s_v < s_w \leq \infty$, then the order relation between the slopes of their images, $D_zf(v)$, $D_zf(w)$, is preserved. As a consequence the representatives $\beta_n^-(z), \beta_n^+(z)$ of the n–cones $V_n(z)$ of a positive twist map are related by:

$$\begin{aligned} -\frac{1}{4} = \beta_0^-(z) < \beta_1^-(z) < \cdots < \beta_n^-(z) < \cdots \\ \frac{1}{4} = \beta_0^+(z) > \beta_1^+(z) > \cdots > \beta_n^+(z) > \cdots \end{aligned} \tag{13}$$

For a negative twist map the sequences $(\beta_n^-(z))$, $(\beta_n^+(z))$, $n \in \mathbb{N}$, have the property:

$$\begin{aligned} -\frac{1}{4} = \beta_0^-(z) > \beta_1^-(z) > \cdots > \beta_n^-(z) > \cdots \\ \frac{1}{4} = \beta_0^+(z) < \beta_1^+(z) < \cdots < \beta_n^+(z) < \cdots \end{aligned} \tag{14}$$

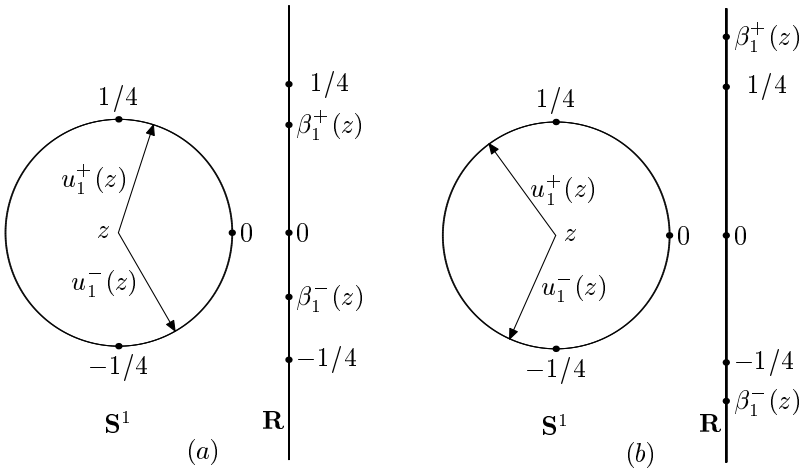

FIG. 1. 1–cone represented by two unit vectors $u_1^+(z)$, $u_1^-(z)$ and the corresponding real values $\beta_1^+(z)$, $\beta_1^-(z)$ associated to a positive twist APM (a), respectively to a negative twist APM (b).

REMARK 2. *For a positive twist APM the function $\alpha_n < 1/2$, while for a negative twist APM, $\alpha_n > 1/2$, for any $n \geq 1$.*

The typical relative position of the vectors defining the 1–cone at a point z of a positive twist, respectively negative twist APM is illustrated in Fig.1, as well as the position of the associated values $\beta_1^+(z)$, $\beta_1^-(z) \in \mathbb{R}$.

DEFINITION 2.3. *Let f be a nontwist APM. A n–cone $V_n(z)$ whose associated value $\alpha_n(z)$ is equal to $1/2$ is called indefinite cone.*

If $\gamma = \text{graph}(h)$ is an invariant graph of a positive twist APM, then h satisfies the cone conditions [14]:

$$\beta_n^-(x, h(x)) < \beta_n^+(x, h(x)), \ \forall \ n \in \mathbb{N} \tag{15}$$

This property led to the following *converse KAM* criterion [14]:

CONE CROSSING CRITERION: *If for some $n \geq 1$, $\beta_n^-(z) > \beta_n^+(z)$, then no invariant circle can pass through the point z, and one says that the n–cone crossing has occurred at the point z or that the cone $V_n(z)$ has crossed.*

In order to detect the mechanism of destruction of invariant circles in nontwist APMs we consider maps which are perturbations of integrable nontwist APMs, having as a rotation number function Ω, a concave quadratic function, depending on the parameter ω, the maximum value of Ω. The general form of such a two–parameter APM, $f_{\omega,k} : (\theta, y) \mapsto (\overline{\theta}, \overline{y})$, is:

$$\begin{array}{rcl} \overline{\theta} & = & \theta + \Omega(\overline{y}, \omega) - k\varphi_1(\theta, \overline{y}) \ (\text{mod } 1) \\ \overline{y} & = & y - k\varphi_2(\theta, \overline{y}) \end{array} \tag{16}$$

where, by area preservation condition the two perturbing functions are related by: $\partial_\theta \varphi_1 = -\partial_{\overline{y}} \varphi_2$.

The prototype of quadratic nontwist map is the nontwist standard map (NSM), studied numerically in [6, 7, 1, 2, 25], i.e. the map (6) with $\Omega(y, \omega) = \omega(1 - y^2)$. By the change of

coordinates and parameter, $\varphi(\theta, y, k) = (\theta, \sqrt{\omega}y, \sqrt{\omega}k)$, NSM is equivalent to the nontwist standard–like map corresponding to the rotation number function expressed in normal form $\Omega(y, \omega) = \omega - y^2$. We chose to investigate the former map, because it puts into evidence the dependence of some dynamical properties (the width of the regular region (transport barrier), the folding properties) not only on the perturbation parameter k, but also on the parameter ω.

NSM is a time–reversible map, f, with respect to the involution

$$I_0(\theta, y) = (-\theta \pmod 1, y - \frac{k}{2\pi}\sin 2\pi\theta), \tag{17}$$

i.e. $I_0^2 = \mathrm{id}$, and $f^{-1} = I_0 \circ f \circ I_0$. f decomposes as a product of involutions $f = I_1 \circ I_0$, and it also has a spatial symmetry $S(\theta, y) = (\theta + 1/2 \pmod 1, -y)$, that is $f \circ S = S \circ f$. The composition of the two symmetries, $S \circ I_0$, is also a reversing symmetry [18], that is:

$$f^{-1} = (S \circ I_0) \circ f \circ (S \circ I_0) \tag{18}$$

If the rotation number function Ω, of a quadratic nontwist map is a concave function, then the map can have an invariant circle of maximum rotation number, ρ_{max}, called *twistless* or *shearless circle*. For low values of the perturbation parameter and rational numbers p/q, such that $p/q << \rho_{max}$ there can exist two Poincaré–Birkhoff chains of type (p, q), called *twin chains*, one included in the region where the map has a positive twist, and another in the region of negative twist. As the perturbation parameter increases the two chains approach each other and their periodic points, and invariant manifolds undergo local and global bifurcations [6, 23, 18, 19].

A quadratic nontwist map can exhibit not only invariant graphs, but also meanders (Fig. 10b), that is invariant circles along which momenta y are not univalent functions of the angle θ.

In order to motivate our study, let us consider the integrable quadratic nontwist map:

$$\begin{aligned} \overline{\theta} &= \theta + \Omega(y, \omega) \\ \overline{y} &= y \end{aligned} \tag{19}$$

Denote by y_t the point of maximum of the rotation number function Ω. The motion occurs on the circles $\mathbb{S}^1 \times \{y\}$, and the rotation number of the orbit of a point (θ, y) is $\Omega(y)$. Hence $\omega = \Omega(y_t)$ is the maximum rotation number, and the circle, $y = y_t$, is both the twistless circle and the critical twist circle ($\partial\overline{\theta}_y(\theta, y_t) = 0$). For $y < y_t$ the map (19) has a positive twist, while for $y > y_t$ a negative one. After a slight perturbation, the orbits of maximum rotation number fills either a circle, if this number is irrational or an invariant set of special topological type (see Figs. 5b, 12b), when the maximum rotation number is a rational number. The invariant twistless circle or set is generically, distinct from the critical twist circle. In case of the NSM, for example, the critical twist circle C_1 is by (3), the circle $y = \frac{k}{2\pi}\sin 2\pi\theta$, which obviously is not an invariant circle. The twistless circle of the NSM, if it exists, is generated by the orbits of the two fixed points of the reversing symmetry $S \circ I_0$ [18], i.e. the points $p_0 = (1/4, k/4\pi)$, $p_0' = (-1/4, -k/4\pi)$, called *indicator points* [22].

This distinction between the two sets associated to a non–integrable quadratic nontwist APM, one invariant (the twistless circle) and another non–invariant (the critical twist circle), suggests that none of them divides the phase space into two semi–infinite annuli,

where the map has opposite signs of twist, and the main results on twist maps can be applied. We will show in the sequel that indeed this is the case.

3. Non–invariant curves associated to a quadratic nontwist map

In order to characterize the dynamical behaviour of a quadratic nontwist map, in different regions of the phase space, we study the geometric properties of such a map, that is, how it acts on the horizontal, respectively the vertical foliation of the annulus. Consider an exact quadratic nontwist area preserving diffeomorphism f of the form (16), which preserves the orientation and is $(+d)$, i.e. $d = \partial_y \overline{y} > 0$. The inverse map, f^{-1}, is also a quadratic nontwist APM. Let us denote by C_1, C_{-1}, the critical twist circles associated to f, respectively to f^{-1}.

Lemma 3.1. *The critical twist circles C_1, C_{-1} associated to a quadratic nontwist APM are related by $C_{-1} = f(C_1)$, and if the map is non–integrable they intersect each other at an even number of points.*

Proof.

By (4)-(5), we get that C_1 has the equation $b(z) = 0$, and C_{-1} the equation $b(f^{-1}(z)) = 0$. Hence $f(C_1) = C_{-1}$.

The two critical twist circles associated to the perturbed quadratic nontwist maps f, f^{-1}, are rotational non-invariant circles, and they are graphs of real functions defined on the unit circle $\mathbb{S}^1$. The map being exact it has the intersection property. That is any rotational circle has to have an intersection with its image. Thus we have $C_1 \cap f(C_1) \neq \emptyset$, i.e $C_1 \cap C_{-1} \neq \emptyset$. C_1, C_{-1} being graphs of periodic functions, which intersect each other, the number of points of intersection is even. □

Remark 3. *A bounded invariant annulus containing one of the two critical twist circles contains also the second one.*

The critical twist circles of NSM:

$$C_1 : y - \frac{k}{2\pi} \sin 2\pi\theta = 0, \quad C_{-1} : y = 0 \tag{20}$$

intersect each other at exactly two points (Fig. 2).

The sign of twist of an APM gives information on how the map acts on the vertical foliation of the annulus. A natural question is whether we can extract information on the dynamics of a map, investigating the way of acting on the horizontal foliation. To give an answer, we consider a circle $\mathbb{S}^1 \times \{y_0\}$, $y_0 \in \mathbb{R}$. Its f–image is a curve γ parameterized by $\theta \in \mathbb{S}^1$:

$$\begin{aligned} \Theta(\theta) &= \overline{\theta}(\theta, y_0) \\ Y(\theta) &= \overline{y}(\theta, y_0) \end{aligned} \tag{21}$$

If $\dot{\Theta}(\theta) \neq 0$, for all $\theta \in \mathbb{S}^1$, then γ is the graph of a function of θ. Otherwise, it is not a graph.

Definition 3.1. *A rotational circle $\mathbb{S}^1 \times \{y\}$ is said to be folded by an APM f, if its f–image is not a graph of a function of $\theta \in \mathbb{S}^1$.*

In order to point out the regions of the phase space, where the map has folding effect, we associate to it *the folding curve*, denoted Γ_f, i.e. the curve the equation:

$$a(\theta, y) = 0 \iff \partial_\theta \overline{\theta} = 0. \tag{22}$$

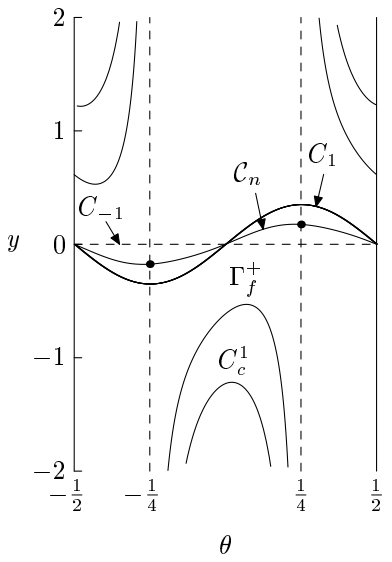

FIG. 2. The non–invariant curves associated to NSM, corresponding to $\omega = 0.37$, $k = 2.2$. C_1, C_{-1} are the critical twist circles, $\mathcal{C}_n$ is the nontwist circle, Γ_f^+ a component of the folding curve, and C_c^1 a component of the 1–cone crossing curve. The second component of each of the last two curves, is also illustrated by symmetry with respect to S.

REMARK 4. *The folding curve cannot be a rotational circle. Hence except for a finite number of points,* Γ_f *is transversal to the horizontal foliation of the annulus.*

Any circle $\mathbb{S}^1 \times \{y\}$ of the horizontal foliation, which crosses the curve Γ_f is folded by the APM (Fig.6).

The NSM has a folding curve with two connected components, denoted Γ_f^+, Γ_f^- (see Fig. 2):

$$\begin{aligned} &\Gamma_f^+ : \ y = \frac{k}{2\pi}\sin 2\pi\theta - \frac{1}{2k\omega\cos 2\pi\theta}, \quad \theta \in (-\frac{1}{4}, \frac{1}{4}) \\ &\Gamma_f^- = S(\Gamma_f^+), \end{aligned} \tag{23}$$

where S is the symmetry defined in Section 2. Both have the lines $\theta = \pm 1/4$ as asymptotic lines.

REMARK 5. *The segment of a circle* $\mathbb{S}^1 \times \{y\}$ *intercepted by the folding curve is mapped to a graph of function over* y.

REMARK 6. *The folding curve of a quadratic nontwist APM cannot cross the critical twist circle* C_1 *of that map, because at a point of intersection the area preservation property would be violated. The critical twist circle* C_{-1} *associated to the inverse map can be crossed by the folding curve* (*see Fig.16*).

Because NSM can be treated analytically, we use it to illustrate the consequences of the folding properties. Numerical investigation of a more general quadratic nontwist APM of the form (16), as a global model of Poincaré map associated to a one and a half degree

of freedom Hamiltonian system, which moreover is a symmetry free APM is reported in [20].

Analyzing the equation of the folding curve of NSM, one observes that keeping ω fixed, and increasing k, or conversely, its components approach the critical twist circle C_1, but by Remark 6 do not cross it. Due to the symmetry S of the map, we restrict our attention to the component Γ_f^+.

DEFINITION 3.2. *A pair of parameters (ω, k) for which there is an invariant annulus $\mathbb{A}' = \mathbb{S}^1 \times (-\infty, u(\omega, k)]$ including the folding curve Γ_f^+ and such that $f_{|\mathbb{A}'}$ has a positive twist is called twist–folding pair.*

We denote by:

$$\mathcal{F}^+ = \{(\theta, y) \mid b(\theta, y) > 0, b(f^{-1}(\theta, y)) > 0, a(\theta, y) \leq 0\}, \tag{24}$$

the region bounded by Γ_f^+, and included in the annulus $\mathbb{A}'$. In the sequel we call the set $\mathcal{F}^+$ *the folding region.*

LEMMA 3.2. *If the NSM corresponding to a twist–folding pair has an invariant circle crossing the folding curve Γ_f^+, and $z_1 = (\theta_1, y_1), z_2 = (\theta_2, y_2)$, with $\theta_1 < \theta_2$ (cyclic order), are the points of intersection with Γ_f^+, then the function $\varphi : \mathbb{S}^1 \to \mathbb{R}$ whose graph is the invariant circle, is a strictly increasing function on the interval (θ_1, θ_2).*

Proof. By the corollary of the Birkhoff theorem [14] the function φ is a Lipschitz function and at each point $z = (\theta, \varphi(\theta))$ of its graph the slopes associated to the cone $V_1(z)$, $s_1^-(z) = -a(z)/b(z)$, $s_1^+(z) = d(f^{-1}(z))/b(f^{-1}(z))$, are local Lipschitz constants for φ. For $\theta \in (\theta_1, \theta_2)$, these slopes are both positive, and thus the function $\varphi_{|(\theta_1, \theta_2)}$ is strictly increasing. □

Now we interpret the folding properties of the map f from the point of view of how the inverse map $f_{|\mathbb{A}'}^{-1} : (\theta, y) \mapsto (\tilde{\theta}, \tilde{y})$ tilts the verticals to the left.

By (5) we have that at each point $u \in f(\Gamma_f^+)$,

$$\frac{\partial \tilde{y}(u)}{\partial y} = \frac{\partial \overline{\theta}(f^{-1}(u))}{\partial \theta} = 0 \tag{25}$$

This means that a vertical $V : x = x_0$, crossing the set $F(\mathcal{F}^+)$ is mapped by the inverse F^{-1} of a lift F, onto the graph of a function over x, but not over y, too (see the Remark 1). Such a vertical V is tilted to the left, but it is folded over $\mathcal{F}^+$ (Fig. 3). Since each invariant circle of a twist APM intersects only once each vertical, as well as its image, the folding of the curve $F^{-1}(V)$ imposes a restriction on the invariant circles which are to intersect the set $\mathcal{F}^+$, namely the crossing of this set must occur at sufficiently high slopes in order to avoid a double intersection with $F^{-1}(V)$.

REMARK 7. *The folding property of a twist APM is equivalent to the degenerate twist property of its inverse.*

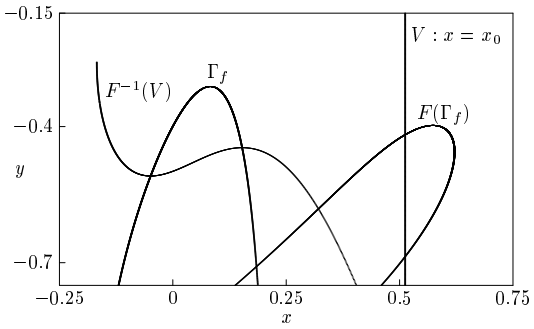

FIG. 3. The folding curve Γ_f^+ and its image under a lift F of NSM, and the pre-image $F^{-1}(V)$ of a vertical V, folding over the folding region $\mathcal{F}^+$; $\omega = 0.68, k = 1.85$

4. Location of the region of nontwist property

If an APM has a positive twist on $\mathbb{S}^1 \times \mathbb{R}$, then its inverse has a negative twist, and conversely. In the case of a quadratic nontwist APM, such as NSM, the two critical twist circles C_1, C_{-1} intersect each other at an even number of points. Hence there exist subsets in the phase space, bounded by arcs of the circles C_1, C_{-1}, where the two maps f and f^{-1} have the same sign of twist. This suggests that for NSM corresponding to a twist–folding pair the transition from the maximal semi–infinite invariant annulus $\mathbb{A}^+$, where the map has a positive twist, to $\mathbb{A}^- = S(\mathbb{A}^+)$, where the map has a negative twist, does not occur sharply. In order to locate exactly the region of the phase phase where this transition occurs, we exploit the properties of the field of cones V_1 defined in Section 2, more precisely the properties of the associated function $\alpha_1 : \mathbb{A} \to \mathbb{R}$. By Remark 2 the function α_1 is greater than $1/2$ in the semi–infinite annulus $\mathbb{A}^-$ and it is lower than $1/2$ in $\mathbb{A}^+$. Being a continuous function it follows that on each vertical line $\theta = \theta_0$ there is a unique point $y_0(\theta_0)$ such that at $z_0 = (\theta_0, y_0)$ the function α_1 crosses the value $1/2$, that is the 1–cone $V_1(z_0)$ is indefinite. In the following we deduce the position of such a point z_0 for a general quadratic nontwist map.

PROPOSITION 4.1. *Let f be a $(+d)$–quadratic nontwist APM with $a = \partial\overline{\theta}/\partial\theta > 0$ in a neighbourhood $\mathcal{V}$ of the critical twist circle C_1. Then there is a rotational non–invariant graph $\mathcal{C}_n \subset \mathcal{V}$ such that at its points the 1–cones are indefinite cones.*

Proof. To make a choice, suppose that our map is a perturbation of an integrable nontwist map with a concave rotation number function Ω. Similar arguments lead to the same result for a convex rotation number function.

Let $F : (x, y) \mapsto (\overline{x}, \overline{y})$ be a fixed lift of the map f. We denote also by C_1, C_{-1} the graphs of periodic real functions which project onto the critical twist circles of the nontwist map, and by $\mathcal{W} \subset \mathbb{R}^2$ a neighbourhood of the circle C_1, where $a = \partial\overline{x}/\partial x > 0$.

By Lemma 3.1 the two critical twist sets C_1, C_{-1} intersect each other. If u is a point of intersection then $b(u) = 0$ and $b(f^{-1}(u)) = 0$, and as a consequence $v_1^+(u) = (0, d(f^{-1}(u))$, and $v_1^-(u) = (0, -a(u))$, i.e. the cone $V_1(u)$ is indefinite.

Let u_1, u_2 be two consecutive points of intersection of the circles $C_1, C_{-1} \subset \mathbb{R}^2$. Between these points either C_1 is above C_{-1} or conversely. Let us consider the first case, and the segment of line $x = x_0$ crossing the two circles at $z_1 = (x_0, y_1)$, respectively at

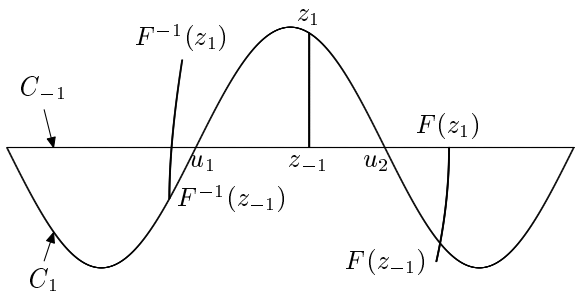

FIG. 4. The action of a quadratic nontwist map on a segment intercepted by critical twist circles C_1, C_{-1}.

$z_{-1} = (x_0, y_{-1})$ (Fig.4). Removing the end points we get an open segment $\mathbf{l}$ included in a region where both F, and F^{-1} have positive twist property. Hence $\mathbf{l}$ is tilted to the right by both maps. Let us consider first that $\mathcal{W}$ is a common neighborhood of C_1 and C_{-1}. Taking into account that $d(x,y), a(x,y) > 0$ for any $(x,y) \in \mathcal{W}$, we get by Eq. (5) and Remark 1 that $F(l)$ and $F^{-1}(l)$ are graphs of functions over y. By Lemma 3.1, $F^{-1}(z_{-1}) \in C_1$ and $F(z_1) \in C_{-1}$, and thus $F^{-1}(l)$ is included in the region of negative twist sign of F, while $F(l)$ is in the region of negative twist sign of F^{-1} (Fig. 4). As a consequence the 1–cone at $z \in l$ is $V_1(z) = (-1/4 + A(z), 1/4 + B(z))$ with $0 < A, B < 1/4$, and the associated extended angle is $\alpha_1(z) = 1/2 + B(z) - A(z)$. Since $\alpha_1(z_{-1}) = 1/2 - A(z_{-1})$ and $\alpha_1(z_1) = 1/2 + B(z_1)$, $0 < A(z_{-1}), B(z_1) < 1/4$, we get by the continuity of the function $\alpha_{1|l}$ that there is a unique point $z \in l$ where $\alpha_1(z) = 1/2$, i.e. the cone $V_1(z)$ is indefinite.

Now we consider the case when the neighbourhood $\mathcal{W}$ does not include C_{-1} completely, and we have $a(z') = 0$ at some $z' \in \mathbf{l}$. Because at every point $z \in \mathbf{l}$, lying between z_{-1} and z', we have $a(z) < 0$, and $v_1^-(z) = (b(z), -a(z))$, $v^+(z) = (b(F^{-1}(z), d(F^{-1}(z))$, it follows that $0 < \beta_1^-(z) < 1/4$ and $1/4 < \beta^+(z) < 1/2$, and thus $\alpha_1(z)$ cannot be $1/2$.

Using the same arguments as in the first case, we get that there is a unique point $z \in \mathbf{l}$, between z' and z_1, at which $\alpha_1(z) = 1/2$, because $V_1(z')$ is represented by the pair $(0, 1/4 + B(z')) \in \mathbb{R}^2$, with $0 < B(z') < 1/4$, and $V_1(z_1)$ is the same as in the first case.

The points z, on each segment intercepted by the critical twist circles, with $\alpha_1(z) = 1/2$, belong to the graph of a continuous periodic function, whose projection onto the annulus $\mathbb{S}^1 \times \mathbb{R}$ is the graph of a function $\psi : \mathbb{S}^1 \to \mathbb{R}$, which satisfies the stated property. □

DEFINITION 4.1. *The non–invariant graph $\mathcal{C}_n$ at whose points the 1–cones are indefinite cones is called nontwist circle of the quadratic nontwist map.*

Let $\varpi = dy \wedge d\theta$ be the area form of the annulus $\mathbb{S}^1 \times \mathbb{R}$. Then the nontwist circle can also be defined as the set $\{z \in \mathbb{S}^1 \times \mathbb{R} \mid \varpi(v_1^-(z), v_1^+(z)) = 0, a(z) > 0\}$. Since $D_{f^{-1}(z)}f(e_2) = (b(f^{-1}(z)), d(f^{-1}(z)))$, and $(D_z f)^{-1}(-e_2) = (b(z), -a(z))$, the equation $\varpi(v_1^-(z), v_1^+(z)) = 0$ becomes:

$$\mathcal{C}_n : \ a(z)b(f^{-1}(z)) + d(f^{-1}(z))b(z) = 0, a(z) > 0. \tag{26}$$

We note that the curve of equation $\varpi(v_1^-(z), v_1^+(z)) = 0$ can also have components in the region where $a < 0$. Such components are boundaries for sets where 1–cone–crossing occurs (see Section 5).

Since the folding curve cannot cross the critical twist circle C_1, the NSM exhibits for any pair of parameters, a neighbourhood $\mathcal{V}$ of C_1 where a is positive, hence a nontwist circle, as well. Its position in the phase space for a particular pair is illustrated in Fig.2.

We note that unlike the critical twist circles C_1, C_{-1} which are associated only to one of the maps f, f^{-1}, the nontwist circle is common for both maps. For the integrable nontwist map (19) all these three curves coincide with the circle $y = 0$. We expect that orbits starting on the nontwist circle have a particular dynamical behaviour.

REMARK 8. *The folding curve of a $(+d)$–APM cannot cross the nontwist circle.*

Actually, the equations $a(z) = 0$, Eq.26, of the two curves cannot have a common solution.

Now we are wondering about the position of the twistless circle, when it exists, with respect to the nontwist circle, and the critical twist circles C_1, C_{-1}. Due to the reversing–symmetry properties of NSM, the twistless circle, if it exists, contains the fixed points of the reversors $S \circ I_0$, $I_1 \circ S$ [18], that is the points $p_0 = (1/4, k/4\pi)$, $p_0' = (-1/4, -k/4\pi)$, respectively $p_1 = (\omega/2 - 1/4, 0)$, $p_1' = (\omega/2 + 1/4, 0)$ ($\mathbb{S}^1 \equiv [-1/2, 1/2)$). The points p_0, p_0' belong to the nontwist circle, while the points p_1, p_1' to the critical twist circle C_{-1}. Hence the twistless circle crosses the nontwist circle $\mathcal{C}_n$, as well as the critical twist circles $(C_1 = f^{-1}(C_{-1}))$.

Let (ω, k) be a twist–folding pair of parameters, and $\mathbb{A}^+$ the maximal semi–infinite invariant annulus, on which the corresponding NSM is a positive twist map. On $\mathbb{A}^+$ all the extended angles $\alpha_n(z)$ associated to the n–cones $V_n(z)$ are less than $1/2$, for $n \geq 1$, while $\alpha_n(z) > 1/2$ for every $z \in S(\mathbb{A}^+)$, and $n \geq 1$. Hence for each $\theta = \theta_0$, and $m \geq 1$, there is an $y_m(\theta_0)$, such that $\alpha_m(z_m) = 1/2$, with $z_m = (\theta_0, y_m(\theta_0))$, i.e. the m–cone $V_m(z_m)$ is an indefinite cone. As a consequence, there is for each twist–folding pair (ω, k) a rotational non–invariant circle, denoted $\mathcal{C}_n^m$ ($\mathcal{C}_n^1 \equiv \mathcal{C}_n$), which is the graph of the function $y_m(\theta)$ at whose points the cone V_m is indefinite. In order to deduce the relative position of the circles $\mathcal{C}_n^m$, $m > 1$, we prove:

PROPOSITION 4.2. *If z is a point in the fixed point set, Fix(SI_0), of the reversor SI_0 of the NSM, then for every $n \geq 1$ the cone $V_n(z)$ is indefinite.*

Proof. We show that for any $n \geq 1$ we have $v_n^+(z) = -v_n^-(z)$, i.e. $D_{f^{-n}(z)} f^n(e_2) = (D_z f^n)^{-1}(e_2)$. From (18) we get that $S \circ I_0$ is also a reversor for f^n, i.e.:

$$f^n = (S \circ I_0) \circ f^{-n} \circ (S \circ I_0). \tag{27}$$

Denoting $z_{-n} := f^{-n}(z)$, $z_n := f^n(z)$ and taking into account that $S \circ I_0(z) = z$, $D_u(S \circ I_0)(e_2) = -e_2$ for any u, and $D_z(S \circ I_0) = -\mathrm{id}$, we have by the chain rule:

$$\begin{aligned} &v_n^+(z) = D_{z_{-n}} f^n(e_2) = D_z(SI_0) \circ D_{z_n} f^{-n}(-e_2) = \\ &D_{z_n} f^{-n}(e_2) = -v_n^-(z). \end{aligned} \tag{28}$$

□

COROLLARY 4.1. *The non–invariant circles $\mathcal{C}_n^m$, $m \geq 1$, associated to the NSM corresponding to a twist–folding pair (ω, k) intersect each other at the indicator points $p_0, p_0' \in$ Fix(SI_0).*

Because a twist APM does not exhibits indefinite n–cones, $n \geq 1$, it is expected that all new phenomena exhibited by NSM to occur in an invariant annulus containing these circles, along which the cones are indefinite.

We recall that reconnection is a global bifurcation of the invariant manifolds of two hyperbolic periodic orbits belonging to twin (p, q)–type Poincaré–Birkhoff chains of a quadratic nontwist APM. At the reconnection threshold the two hyperbolic periodic orbits get connected by heteroclinic connections (Fig.5b). After reconnection the twin Poincaré–Birkhoff chains get dimerized island chains (Fig.5c), that is homoclinic loops to hyperbolic points of the downward moving chain surround elliptic points of the upward moving chain and conversely. For the upper dimerized island chain the extended angles α_{mq} associated to a q–periodic elliptic point e are $\alpha_{mq}(e) < 1/2$, $m \geq 1$, while those associated to the new corresponding hyperbolic point h are $\alpha_{mq}(h) > 1/2$, $m \geq 1$. Hence, on the segment connecting the points e and h, the functions α_{mq} must cross the value $1/2$, that is the circles $\mathcal{C}_n^{mq}$ cross that segment. The consequences of the existence of indefinite cones V_n along the orbits of NSM will be discussed in Section 5.3.

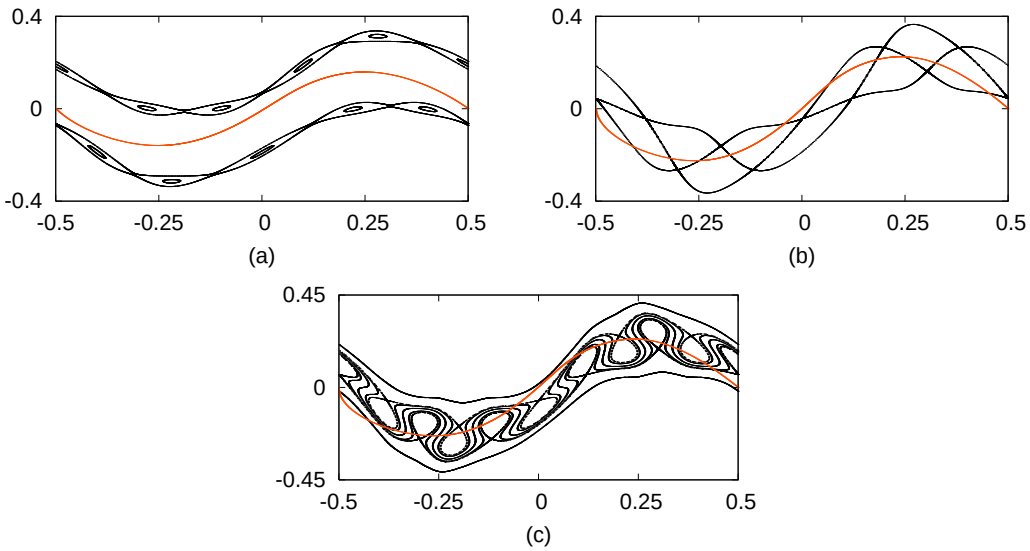

FIG. 5. Position of the nontwist circle (red) of NSM corresponding to $\omega = 0.621$, with respect to: a) twin Poincaré–Birkhoff chains of type $(3, 5)$; $k = 2$; b) the invariant manifolds of $(3, 5)$–periodic hyperbolic orbits, at the reconnection threshold; $k = 2.8245$; c) meanders and homoclinic connections created after reconnection; $k = 2.9$.

5. Dynamics of NSM in the regions of folding property

We investigate the behaviour of NSM within an invariant semi–infinite annulus $\mathbb{A}' = \mathbb{S}^1 \times (-\infty, u(\omega, k)]$ where the NSM is a positive twist map and the folding region $\mathcal{F}^+ \subset \mathbb{A}'$.

5.1. **Cone–crossing of positive twist maps.** Let $V_n(z) = (\beta_n^-(z), \beta_n^+(z))$ be the n–cone associated to a point z in the phase space of a positive twist map and $\alpha_n(z)$ the corresponding value, $\alpha_n(z) = \beta_n^+(z) - \beta_n^-(z) \in \mathbb{R}$. Because the map f is a C^r–diffeomorphism, $r \geq 1$, the vector fields v_n^-, v_n^+ are continuous for every n, as well as their extended oriented angle α_n. By (13), the sequence $(\alpha_n(z))$ is decreasing for any z. The sequence $(\alpha_n(z))_{n\in\mathbb{N}}$ associated to a point on a differentiable invariant circle decreases to 0, as $n \to \infty$.

If for some $m \geq 1$, we have $\alpha_m(z) < 0$, then by cone crossing criterion, no invariant circle can pass through the point z. Let us detect and characterize the subsets in the phase

space of an arbitrary positive twist APM, where the cone–crossing can occur, respectively the cone–crossing is impossible.

In the sequel, a curve of equation $\alpha_m(\theta, y) = 0$ is denoted C_c^m, and is called the *m–cone–crossing curve*. Taking into account the topology of the phase space (infinite annulus), a m–cone–crossing curve can have many components, and these components can be closed curves (contractible to a point or not) or unbounded curves.

LEMMA 5.1. *The functions α_n, $n \geq 1$, have positive values at any regular hyperbolic periodic point of a positive twist APM.*

Proof. At the points of a q– periodic regular hyperbolic orbit $z_0, z_1, \ldots, z_{q-1}$ the cone crossing cannot occur. It is obstructed by the stable and unstable one–dimensional invariant linear subspaces $E^s(z_i)$, $E^u(z_i)$, $i = 0, 1, \ldots, q-1$, because of the slope–order preserving property of the linear maps $D_z f$, $D_z f^{-1}$, and the fact that along such an orbit we have $D_{z_i} f(E^u_{z_i}) = E^u_{z_{i+1}}$, respectively $D_{z_{i+1}} f^{-1}(E^s_{z_{i+1}}) = E^s_{z_i}$, $i = \overline{0, q-1}$. Therefore $\alpha_n(z_i) > 0$, for any $i = \overline{0, q-1}$ and $n \geq 1$. □

From dynamically point of view a rotational m–cone–crossing closed curve, as a unique component of a curve of equation $\alpha_m(\theta, y) = 0$, $m \geq 1$, appears to be unlikely to exist, because it would separate the annulus into two disjoint components, one where the function α_m is positive and the other where it is negative. By cone crossing criterion, no invariant circle could pass through the latter component, and moreover by Lemma 5.1 no hyperbolic point could exist in that region, i.e. the dynamics would be very restrictive. As we shall see below, a multi–connected component m–cone–crossing curve of NSM corresponding to a twist–folding pair can have a rotational component crossing the folding region.

LEMMA 5.2. *Each point e of a periodic orbit of positive residue, included in a bounded invariant annulus of a twist APM, is surrounded by a sequence of closed n–cone–crossing curves, which are boundaries for closed sets $D_n \ni e$, $D_{n+1} \supset D_n$, $n \geq m(e)$, and convergent to a closed curve.*

Proof. Due to rotational motion about elliptic periodic points, the cone–crossing does occur at these points. If e is an elliptic periodic point, then there exists $m(e) > 0$ such that $\alpha_n(e) < 0$, for any $n > m(e)$. Obviously the same property is exhibited by inverse hyperbolic q–periodic points. Let $\mathbf{l}$ be the segment joining a periodic point of positive residue, e, with the nearest regular hyperbolic point u, within the same Poincaré–Birkhoff chain, situated, let us say at right of e. By Lemma 5.1 we have $\alpha_n(u) > 0$, for any $n \in \mathbb{N}$. From relations (13) it follows that at each point $z \in \mathbf{l}$ the sequence $(\alpha_n(z))$, $n \in \mathbb{N}$, is strictly decreasing. Thus there exists a monotone sequence of points $z_n \in \mathbf{l}$, such that $\alpha_n(z_n) = 0$, for any $n > m(e)$. This sequence "increases" on $\mathbf{l}$ in the direction from the elliptic (inverse hyperbolic) point towards the regular hyperbolic point. If $p \in \mathbf{l}$ is its limit, then $p \neq u$, and due to the continuity of the functions α_n, there is a sequence of n–cone crossing closed curves, $n > m(e)$, surrounding the point of positive residue and converging to a closed curve γ containing the point p. The inclusion of the Poincaré–Birkhoff chain in a bounded invariant annulus ensures that the n–cone–crossing curves cannot be rotational closed curves (see the above discussion, on the topological type of the components of a n–crossing curve). □

The breakup of an invariant circle separating two neighboring Poincaré–Birkhoff chains is probably due to the increase in diameter (as the perturbation increases) of the closed

sets D_n bounded by n–cone–crossing curves surrounding the periodic points of positive residues of the two chains and to their collision–merging. The study of the breakup of invariant circles of positive twist APMs based on this approach could give new insights into the process of destruction, as well as into that of the splitting of separatrices.

The strongest condition of non-existence of an invariant circle through a point z is the 1–cone–crossing condition at z, i.e. $\beta_1^-(z) > \beta_1^+(z)$ or equivalently, $\alpha_1(z) < 0$. In terms of the Jacobian matrix entries this condition becomes:

$$-\frac{a(z)}{b(z)} > \frac{d(f^{-1}(z))}{b(f^{-1}(z))} \tag{29}$$

PROPOSITION 5.1. *The 1–cone crossing can occur in the phase space of an exact $(+d)$ non–integrable positive twist APM only if the map exhibits the folding property.*

Proof. An exact APM of the annulus $\mathbb{S}^1 \times \mathbb{R}$ has intersection property, i.e. it maps each circle $\mathbb{S}^1 \times \{y\}$ to a rotational circle intersecting the first one. Hence if the map is non–integrable the function $c = \partial_\theta \overline{y}$ cannot have constant sign.

The inequality (29) can hold at a point z only if $a(z) < 0$. Since the functions b and d are positive, a cannot be negative everywhere in the phase space, because this would imply $c = (ad - 1)/b < 0$ everywhere. Hence the 1–cone–crossing can occur only if the set of equation $a(z) = 0$ is not empty, i.e. the map exhibits folding property. □

5.2. **Cone–crossing in the folding region of NSM.** In the sequel, we investigate the dynamical consequences of the folding property of NSM. Denote by $z^* = (0, y^*)$ the intersection of the folding curve Γ_f^+ to the line $\theta = 0$. First, we give heuristic arguments that the semi–line $l = \{(0, y) | y \leq y^*\} \subset \mathcal{F}^+$ and the point z^* have a particular connection with the folding property of the map, and as a consequence it is advisable to choose this semi–line among other vertical semi–lines included in the folding region $\mathcal{F}^+$ to investigate the possibility of being crossed or not by invariant circles.

Due to the asymmetry of Γ_f^+ with respect to $\theta = 0$, the danger of breaking up of invariant circles crossing the folding region seems to be more threatening for those which are to pass through the points $(0, y)$, $y < y^*$, than for those which are to pass through $(0, y)$, $y > y^*$. Let us support this assertion.

First we note that the points $(0, y)$ are mapped by NSM to points $(\overline{\theta}(0, y), y)$ at the same level y. A circle $\mathbb{S}^1 \times \{y\}$ divides the infinite annulus $\mathbb{S}^1 \times \mathbb{R}$ into two unbounded components: $W(y)$ above, and $W'(y)$ below it. Let us consider the subsets

$$U(y) = \{z \in W(y) \,|\, f^{-1}(z) \in W'(y)\},$$
$$D(y) = \{z \in W'(y) \,|\, f^{-1}(z) \in W(y)\}$$

called the upward moving set, respectively the downward moving set, associated to the circle $\mathbb{S}^1 \times \{y\}$ (these sets can be easily identified in Fig.6).

Identifying $\mathbb{S}^1 \equiv [-0.5, 0.5)$, NSM maps the segment $[-0.5, 0] \times \{y\}$ above the circle $\mathbb{S}^1 \times \{y\}$, and its complement below it. Suppose that an invariant circle $\gamma = \text{graph}(\varphi)$ could cross the folding curve Γ_f^+ at (θ_l, y_l), (θ_r, y_r). Let $(0, y')$ be its intersection with the line $\theta = 0$. By Lemma 3.2 the arc $\gamma' \subset \mathcal{F}^+$ of the invariant circle included in the folding region is the graph of an increasing function. Hence γ' should pass through the point $(0, y')$, from below the circle $\mathbb{S}^1 \times \{y'\}$ to above it, and as a consequence through $(\overline{\theta}(0, y'), y')$ from the upward moving set $U(y')$ to the downward moving set $D(y')$. If $y' < y^*$ then $\theta_l < 0 < \theta_r$, while for $y' > y^*$, $0 < \theta_l < \theta_r$. In the former case, the part of $f(\gamma')$ crossing the upward

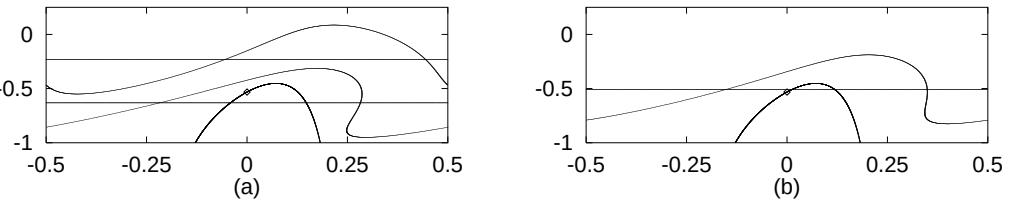

FIG. 6. The action of NSM on horizontal circles; a) the image of a circle above the folding curve and the image of a circle $\mathbb{S}^1 \times \{y\}$, $y < y^*$. b) the f–image of a circle crossing the folding curve above z^*.

moving set $U(y')$ is pushed to the right, while the part crossing the downward moving set $D(y')$, to the left, because all segments of horizontal circles intercepted by folding curve, and crossing γ' are mapped by Remark 5 to graphs of functions over y, which have a maximum within $U(y')$ and a minimum within $D(y')$. But such an action on an invariant graph could transform it to a non-graph, that is, could lead to the breakup of the invariant circle. In the latter case the horizontal segments intercepted by the folding curve, and crossing γ', are mapped only to the downward moving set $D(y')$, and as consequence the arc $f(\gamma')$ eventually could avoid the breakup. These remarks entails us to reduce the rigorous investigation of the existence/non–existence of invariant circles crossing the folding region to the points on the line $\theta = 0$.

First we note that the 1–cone–crossing does occur in NSM and the associated 1–cone–crossing curve, C_c^1, has the equation:

$$a(z)b(f^{-1}(z)) + b(z)d(f^{-1}(z)) = 0,\ a(z) < 0 \tag{30}$$

It has two unbounded components. The component included in $\mathcal{F}^+$ has the equation:

$$\overline{y} + y(1 + 2\omega k \overline{y} \cos 2\pi\theta) = 0,\ \ \theta \in (-1/4, 1/4). \tag{31}$$

It is the graph of a function $\phi : (-1/4, 1/4) \to \mathbb{R}$, having $\theta = \pm 1/4$ as asymptotic lines. In Fig.2 is illustrated its position in the phase space with respect to other non–invariant curves. Let $z_1 = (0, y_1)$ be the intersection between the line $\theta = 0$ and C_c^1. By the cone-crossing criterion no invariant circle can pass through a point $z = (0, y)$ with $y < y_1$. Hence within the semi–infinite annulus $\mathbb{S}^1 \times (-\infty, y_1)$, $\mathbb{S}^1 \times (-y_1, \infty)$, the orbits of NSM are not confined any more and the drifting of action occurs. In the sequel we show that there is a greater level $y^* > y_1$ below which no invariant circle can pass.

PROPOSITION 5.2. *Let $f_{\omega,k}$ be the NSM corresponding to a twist–folding pair (ω, k). If $z^* = (0, y^*)$ is the point of intersection of the folding curve to the line $\theta = 0$, then at each point $z = (0, y)$ with $y < y^*$, the cone crossing occurs.*

Proof. The vectors $v_m^+(z)$, $v_m^-(z)$, $m \geq 1$ associated to points of the 1–cone crossing curve are related as in the Fig. 7, that is their representative values are related by:

$$\begin{aligned} &\cdots > \beta_m^-(z) > \cdots > \beta_2^-(z) > \beta_1^-(z) = \\ &= \beta_1^+(z) > \beta_2^+(z) > \cdots > \beta_m^+(z) > \cdots, \quad z \in C_c^1 \end{aligned} \tag{32}$$

Since any invariant circle crossing the folding region has to intersect the vertical semi-lines included in $\mathcal{F}^+$, we restrict ourselves to the semiline $\theta = 0$, $y < y^*$.

Let us consider the sequence of functions $\xi_n : \mathbb{R} \to \mathbb{R}$, $\xi_n(y) = \alpha_n(0, y)$, $n \geq 1$. By Remark 2 in the region of positive twist property $\xi_n(y) < 1/2$, $n \geq 1$, while in the region

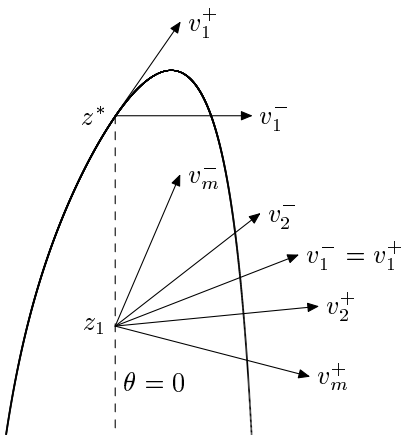

FIG. 7. The relative position of vectors $v_n^-(z_1)$, $v_n^+(z_1)$, where $z_1(0, y_1)$ belongs to the 1–cone crossing curve.

of negative twist, $\xi_n(y) > 1/2$, $n \geq 1$. By Propositions 4.1, 4.2, ξ_1 crosses the value $1/2$ at $y = 0$, while ξ_n, $n > 1$, reaches this value in a neighbourhood of zero. Because $\xi_n(y_1) \leq 0$, it follows that along the interval $(y_1, 0)$ these functions must be non–decreasing. Since the function ξ_1 increases strictly from $0 = \xi_1(y_1)$ to $\xi' = \xi_1(0) = 1/2$, as y increases from y_1 to 0, the functions ξ_n, $n > 1$ are forced to increase as well (due to continuity, and slope order preserving property of the maps $D_z f$, $D_z f^{-1}$), from a negative value at y_1, hence to vanish at some $y_n > y_1$. Thus $\{y_n\} = C_c^n \cap \{\theta = 0\}$ and $y_n \in [y_1, 0)$. Since (ξ_n) is a decreasing sequence of increasing functions, the bounded sequence (y_n) is increasing and converges to some $Y < 0$. In order to point out the position of the point $(0, Y)$ we recall that the line $\theta = 0$ is a symmetry line of the NSM as a reversible map. Hence on this line the symmetric periodic points lie. Since cone–crossing occurs at each point $(0, y)$, $y < Y$, no regular hyperbolic periodic point can lie on $\theta = 0$, below $(0, Y)$. If $(0, y)$, $y < Y$, is a periodic point of positive residue, the components of the n–cone–crossing curves C_c^n, with n greater than some $m(y) > 0$, associated to the periodic point $(0, y)$ cannot be contractible to a point closed curves, as in the case of Lemma 5.2, because such closed curves intersect at least two times the line $\theta = 0$, and this contradicts the fact that the sequence (y_n) is increasing. Thus the components of the n–cone–crossing curves, $n > m(y)$, associated to the elliptic or inverse hyperbolic periodic points on the symmetry line $\theta = 0$, below $(0, Y)$, can be rotational closed curves (see the component of C_c^3 in Fig. 8, associated to the elliptic periodic point of type $(1, 2)$) or eventually unbounded curves like the 1–cone crossing curve, with the periodic point below them.

Since the 1–cone–crossing which caused this behaviour can occur only in the region $\mathcal{F}^+$, where $a < 0$, the sequence (y_n), $n \geq 1$, must converge either to $y^* \in \Gamma_f^+$ or to a point y' slightly above y^*, in the case when z^* is very close to a periodic point of positive residue (Fig.10a). In both cases, on the line $\theta = 0$, below z^*, the n–cone–crossing occurs for some $n = n(z) \geq 1$. □

COROLLARY 5.1. *The NSM $f_{\omega,k}$, corresponding to a twist–folding pair (ω, k), cannot have an invariant circle through a point $(0, y)$ with $y < -1/(2\omega k)$, respectively through a point $(-0.5, y)$, with $y > 1/(2\omega k)$, and moreover there is no regular hyperbolic point on the symmetry line $\theta = 0$ below the point z^*, respectively on $\theta = -0.5$ above the point $S(z^*)$.*

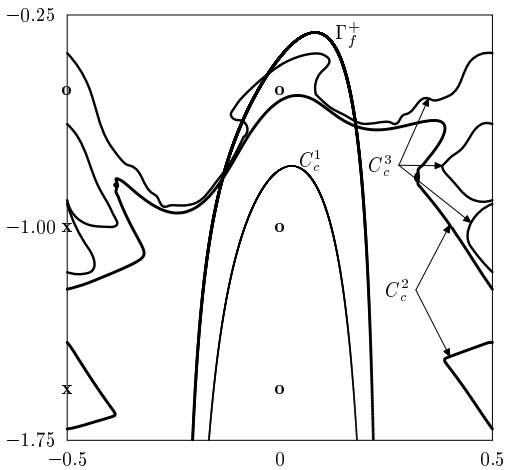

FIG. 8. Some cone–crossing curves in the phase space of the NSM corresponding to $\omega = 0.68, k = 1.85$. The points marked by o are elliptic periodic points, while those marked by x are regular hyperbolic periodic points. The phase portrait of this map is illustrated in Fig.10.

Any symmetric periodic orbit crossing the folding region of NSM and having a point on the symmetry line $\theta = 0$, respectively $\theta = -0.5$, is either elliptic or inverse hyperbolic.

More details concerning the dynamics of a twist map with folding property are presented in [21], where the twist standard map, corresponding to the perturbation parameter greater than 1, is also analyzed.

An invariant circle of NSM passing through a point $(0, y)$ above z^*, and crossing increasingly the folding region is illustrated in Figure 9. Hence the above hypothesis that such an invariant circle of NSM can avoid the breakup is confirmed (numerically).

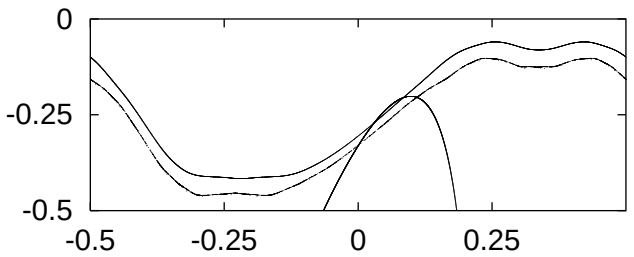

FIG. 9. The orbit of the point z^*, and an invariant circle through a point $(0, y)$ above z^*, crossing increasingly the folding curve; $\omega = 0.68$, $k = 2.22$.

From the definition of a twist–folding pair (ω, k) it follows that the corresponding map, $f_{\omega,k}$, has two boundary circles $\mathcal{C}_b \subset \mathbb{A}'$, $S(\mathcal{C}_b)$, separating a central invariant annulus, where the motion is quasi–integrable, from two irregular components, where the drifting of action occurs (Fig. 10). Moreover this invariant annulus includes the critical twist circles, and as a consequence twist and nontwist invariant sets can coexist within it. In Fig.10b are illustrated nontwist invariant sets, that is meanders and dimerized island chains, as well as invariant graphs, typical for twist maps.

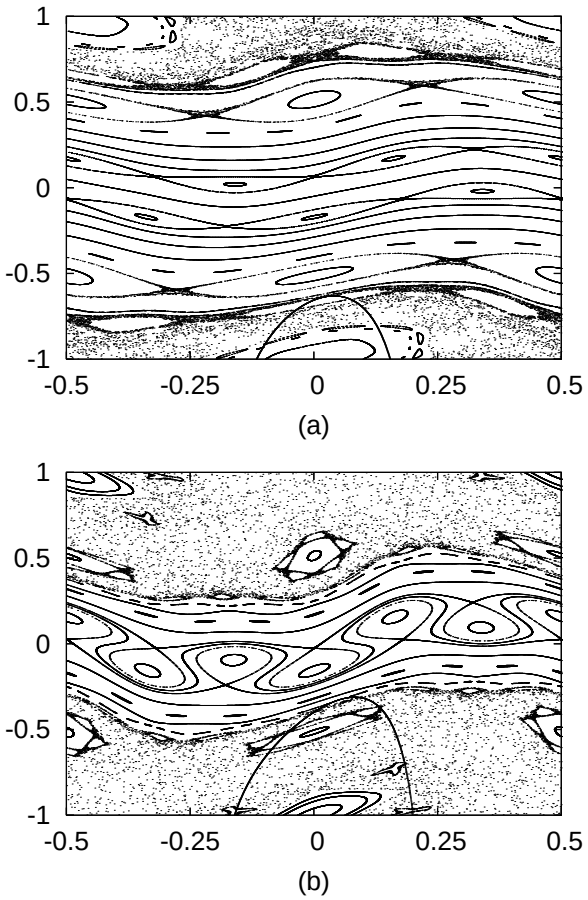

FIG. 10. Two phase portraits of NSM corresponding to two twist–folding pairs (ω, k), with $\omega = 0.68$ fixed, and distinct perturbation parameters. a) coexistence of regular and chaotic motion within the invariant annulus; $k = 1.12341$; b) meandering invariant circles, and homoclinic loops to two distinct $(2, 3)$–type hyperbolic periodic orbits within the invariant annulus; $k = 1.85$.

The boundary circle $\mathcal{C}_b$ passes through a point $(0, y)$, with $y \geq -1/(2\omega k)$. The approximate position of this circle, as well as its rotation number can be detected using, for example, a method presented in [13].

Having two twist–folding pairs (ω, k_1), (ω, k_2), with $k_1 < k_2$, keeping ω fixed and increasing k from k_1 to k_2, the point $z^* = (0, y^*)$, gets closer and closer to the critical twist circle C_1 and as a consequence the central invariant annulus shrinks (Fig.10) or equivalently, the destruction of boundary circles occurs progressively from the regions of monotone twist property toward the critical twist circle.

5.3. **Numerical observations on the destruction of invariant circles.** In order to discuss the scenario of destruction of invariant circles crossing the critical twist circles we first characterize the dynamics in the region of nontwist property of the map.

The NSM has the twist close to zero, around the critical twist circle C_1, i.e. $b = \partial_y \overline{\theta}$ is close to zero. Moreover it can be regarded as a perturbation of a zero–twist map.

DEFINITION 5.1. *An APM $Z : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$, $(\theta, y) \mapsto (\theta', y')$, with the property that $\partial_y \theta' = 0$, at each point of the annulus, is called zero–twist map.*

The APM $Z_{\omega,k} = (Z_1, Z_2)$ defined by:

$$\begin{aligned} \theta' &= \theta + \omega \ (\mathrm{mod}\ 1) \\ y' &= y - \frac{k}{2\pi} \sin 2\pi\theta \end{aligned} \tag{33}$$

is such a map. It is an integrable map, and all its orbits have the same rotation number ω. The NSM $f_{\omega,k} : (\theta, y) \mapsto (\overline{\theta}, \overline{y})$ can be expressed as:

$$\begin{aligned} \overline{\theta} &= Z_1(\theta, y) - \omega \overline{y}^2 \\ \overline{y} &= Z_2(\theta, y) \end{aligned} \tag{34}$$

The two maps coincide along the rotational non–invariant circle C_1 of equation $\overline{y}(\theta, y) = 0 \Leftrightarrow y - \dfrac{k}{2\pi} \sin 2\pi\theta = 0$, and NSM goes away from the integrable map $Z_{\omega,k}$ on both sides of C_1. Indeed, for a twist–folding pair (ω, k), the C^0–distance between these maps on the central invariant annulus K, bounded by the circles $\mathcal{C}_b$, $S(\mathcal{C}_b)$, is:

$$d(Z_{\omega,k}, f_{\omega,k}) = \sup_{(\theta,y)\in K} \omega |\overline{y}(\theta, y)|^2, \tag{35}$$

where $\overline{y}(\theta, y) = y - k/(2\pi) \sin 2\pi\theta$. By Remark 3 an invariant annulus containing the critical twist circle C_1 must also include C_{-1}, hence the nontwist circle $\mathcal{C}_n$, as well. Thus on the most narrow such an invariant sub–annulus, when it exists, $\omega |\overline{y}(\theta, y; k)|^2$ is sufficiently small, and thus the NSM is very close to an integrable map (zero–twist map). This property, as well as extensive computer experiments led us to state the following:

CONJECTURE 1. *Let $f_{\omega,k}$ be the NSM for which the folding region $\mathcal{F}^+$ is included in an invariant semi–infinite annulus $\mathbb{A}'$ where the map has positive twist. Denote by K' the closure of orbits starting on the nontwist circle $\mathcal{C}_n$, and by $T = \complement_{\mathbb{S}^1\times\mathbb{R}} K'$ its complement. Then $K'' = (\complement_{\mathbb{S}^1\times\mathbb{R}} T \setminus K') \cup K'$ is a bounded invariant set on which NSM is the time–one map of a Hamiltonian system defined by a function $H : K'' \to \mathbb{R}$.*

We note that the set $\complement_{\mathbb{S}^1\times\mathbb{R}} T \setminus K'$ usually consists in elliptic periodic points and invariant circles surrounding them, which do not cross the nontwist circle (Fig.11). Let us call K'' the *nontwist annulus*.

Let us give arguments to support this conjecture. First is the closeness to a zero twist map. The n–cones associated to such a map have constant extended angle, $\alpha_n = 1/2$. Along the orbits of NSM, within the nontwist annulus, the n–cones are almost indefinite, i.e. of extended angle close to $1/2$. This gives a large freedom to the motion unlike the twist maps where invariant circles are forced to flow through the interior of increasingly sharp cones. While for the twist maps defined on a bounded annulus the splitting of separatrices and the breakup of invariant circles seem to be due to the closed n–cone–crossing curves surrounding the elliptic points, which extend, colliding the invariant manifolds of the corresponding hyperbolic orbit, and breaking up nearby invariant circles, in the case of the nontwist annulus no danger was detected. As it was noted at the end of Section 4 the segment in a dimerized island, joining the elliptic point with the hyperbolic one is crossed by circles along which the cones are indefinite cones. The lack of an internal cause for the breakup of invariant circles and splitting of separatrices leads to admit the integrability.

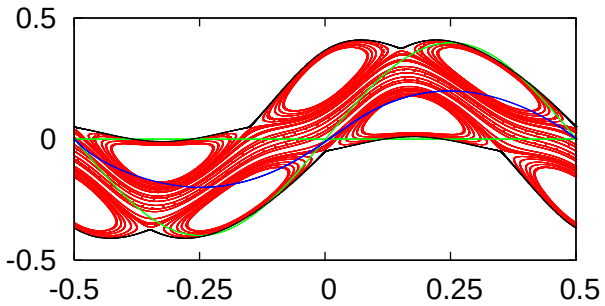

FIG. 11. Orbits in the invariant set K' (red) starting on the nontwist circle $\mathcal{C}_n$ (blue). The green lines are the critical twist circles, while the black orbits fill branches of invariant manifolds of the $(1,3)$–type hyperbolic orbit, which are boundaries of the the set K''; $\omega = 0.35$, $k = 2.5$

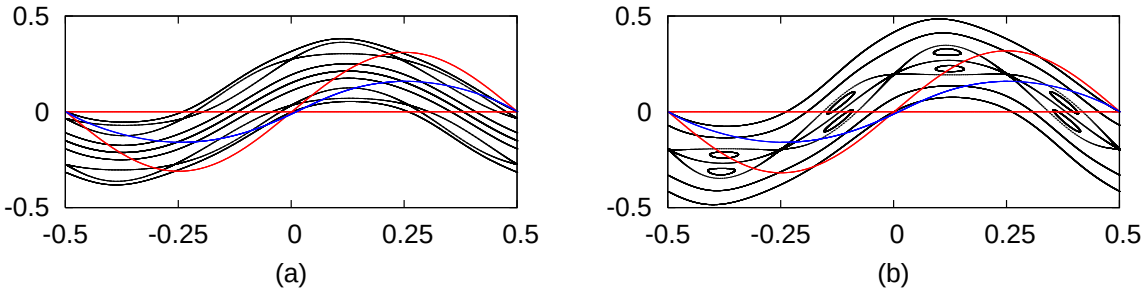

FIG. 12. Orbits within the invariant set K'' associated to NSM corresponding to two twist–folding pairs (ω, k). C_1, C_{-1} are drawn in red and $\mathcal{C}_n$ in blue. a) $\omega = 0.26$, $k = 2$; the set K'' contains the twin Poincaré–Birkhoff chains of type $(1,4)$. The invariant manifolds of each hyperbolic orbit coincide; b) $\omega = 0.26$, $k = 2.465$; the same chains at the threshold of collision.

The regular motion of NSM within the invariant set K'' is illustrated in Fig.12, where the invariant sets looks like the level sets of some function $H : K'' \to \mathbb{R}$. It can be also seen in Fig.10. The resemblance of the phase portrait of NSM, in the region where it changes its sign of twist, with that of a time–one map of a one degree of freedom Hamiltonian system is also evident in the first paper which analyzed in detail the new phenomena exhibited by the quadratic nontwist map (Figs. 4, 5, 7, 8 in [6]).

Within K'' one manifests nontwist property of the map. Actually, analyzing the equation (26) one can see that an orbit starting on the nontwist circle flips from the region of positive sign of twist for f to the region of negative sign, and conversely.

Numerical experiments with NSM corresponding to different twist–folding pairs (ω, k), with k varying within a closed interval $[k_r, k_c]$ (k_r the threshold of p/q–reconnection, $q > 0$ and odd, and k_c the threshold of saddle–center collision) reveal the integrable behaviour of the map within K'' (see Fig.4 in [6]). The invariant sets having special topological type which are born at the reconnection threshold or after reconnection belong to the invariant set K'' (Figs.5b,c).

The height of the bounded invariant set K'' depends on the twist–folding pair (ω, k).

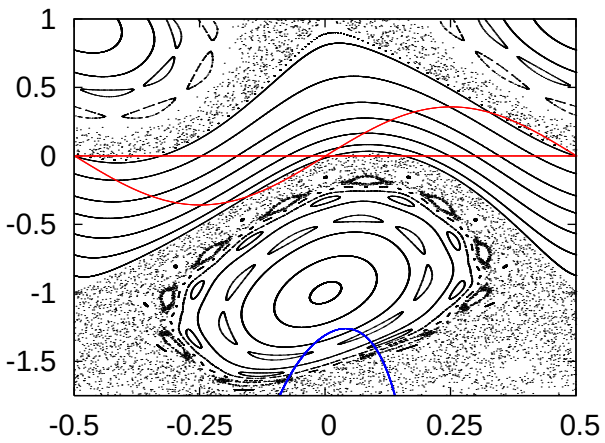

FIG. 13. The phase portrait of NSM corresponding to $\omega = 0.17$, $k = 2.25$. The critical twist circles are drawn in red, while the folding curve Γ_f^+ in blue. The point $z^* \in \Gamma_f^+$ lies below the elliptic fixed point $(0, -1)$, and the boundary circle $\mathcal{C}_b$ crosses the critical twist circles.

Keeping ω fixed and increasing the perturbation parameter k, the boundary circles $\mathcal{C}_b$, $S(\mathcal{C}_b)$ of the NSM corresponding to twist–folding pairs (ω, k) are destroyed progressively. Beyond a threshold $k'(\omega)$, the boundary circles cross the critical twist circles.

DEFINITION 5.2. *A pair of parameters (ω, k) is called nontwist–folding pair if there is a semi–infinite invariant annulus $\mathbb{A}'' = \mathbb{S}^1 \times (-\infty, l(\omega, k)]$ including the folding region $\mathcal{F}^+$, and such that there are points in $\mathcal{F}^+$ whose orbits cross the critical twist circles.*

If an orbit starting in the folding region $\mathcal{F}^+$ crosses the critical twist circle C_1, then along its points the relative position of the cones $V_n(z)$, V_{n+1}, $n \geq 1$ is variable, and the above arguments for the non–existence of invariant circles crossing the folding curve are not valid anymore. Computer experiments illustrate however that for a nontwist–folding pair no invariant circle can pass through a point $z(0, y)$ with $y < \tilde{y}$, where $\tilde{y}$ is slightly below y^*, except the case when (ω, k) is such that the point z^* lies below the elliptic fixed point $(0, -1)$. In this case the boundary circle $\mathcal{C}_b$ is far enough from z^* (Fig.13). The explanation could be the fact that the invariant manifolds of the hyperbolic fixed point pass below z^*, where cone crossing occurs. Thus their transversal intersection gives rise to a chaotic motion which extends up to the critical twist circle C_1.

The destruction of invariant circles crossing the critical twist circles seems to be also due to the foldings which affect this region. The folding curve and of the 1–cone crossing curve, as well, move towards the nontwist circle, as the parameter k increases. It remains as an open problem to decipher the mechanism of destruction in this case.

Fig.14a illustrates that some orbits starting on the nontwist circle, $\mathcal{C}_n$, can enter the folding region and move erratically, while other still fill a bounded invariant set on which the NSM appears to be the time–one map of a one degree of freedom Hamiltonian system (Fig.14a). The bounded invariant set shrinks as k increases (Fig.14b, c), and the destruction occurs from outside towards the twistless circle. In (Fig.14c) the rotational invariant circle is the twistless circle (the orbit of the indicator point $p_0(1/4, k/(4\pi))$.

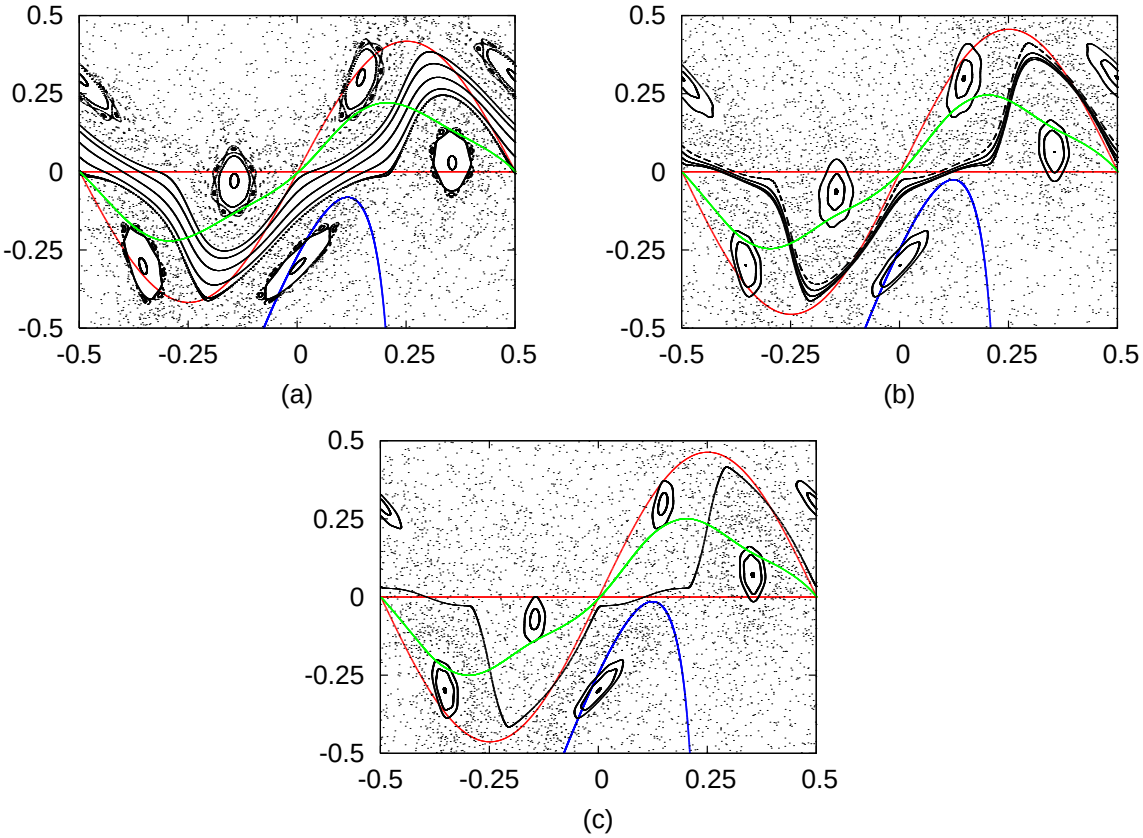

FIG. 14. The shrinking of the invariant annulus of regular dynamics of NSM corresponding to non twist–folding pairs (ω, k) with $\omega = 0.71$ fixed and k increasing. The critical twist circles are drawn in red, the nontwist circle in green, while the folding curve Γ_f^+ in blue; a) $k = 2.625$; b) $k = 2.86513$; c) $k = 2.90521$.

The nontwist circle acts as a frontier which stops the vertical motion of the folding curve, as the perturbation parameter increases, and ω is kept fixed. This property can be useful in the explanation of the breakup of last invariant circles of NSM.

In Fig.16 is illustrated the relative position of all non–invariant curves associated to NSM and the critical twistless circle of rotation number $1/\gamma$.

This is not the only scenario of destruction of the last invariant circles. We identified a different scenario for the nontwist–folding pair $(\omega = 0.48132, k)$, with k around 4.35. The orbit of the indicator point $p_0 \in \mathcal{C}_n$ of the map $f_{\omega,k}$, $\omega = 0.48132$, $k = 4.35$, fills a region of instability [11], which is included in an invariant annulus containing invariant circles which do not cross the nontwist circle (Fig.15). Hence the twistless circle is not the most robust in this case. We note that this pair (ω, k) corresponds to k slightly greater than the reconnection threshold of $(6, 13)$–type hyperbolic periodic orbits. Before reconnection, for $k = 4.3291$, the phase portrait (not included here) illustrates the same chaotic behaviour of orbits starting on the nontwist circle, except for the twistless circle which exists. After reconnection instead meanders, meandering cantori are born. Increasing the parameter k further, the central chaotic region extends and leads to the destruction of all circles.

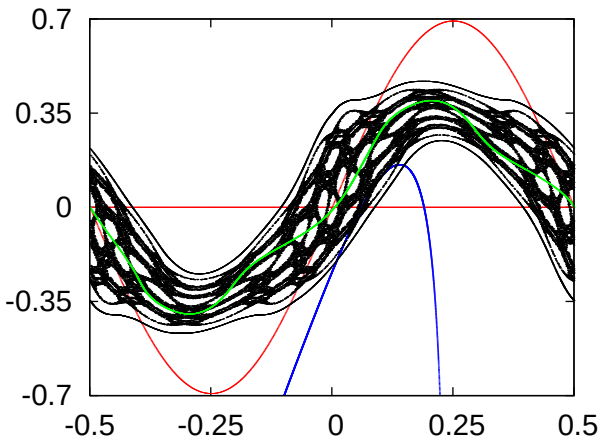

FIG. 15. $\omega = 0.48132$, $k = 4.35$; The region of instability generated by the orbit of the indicator point $p_0(1/4, k/(4\pi))$ and two invariant circles crossing the critical twist circles, but not the nontwist circle.

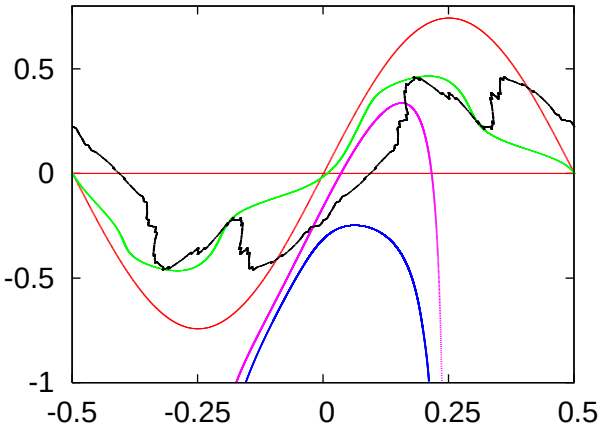

FIG. 16. The position of the critical twistless circle ($\omega = 0.686049$, $k = 4.66522193162601$ [6]) of rotation number $1/\gamma$, with respect to the circles C_1, C_{-1} (red), $\mathcal{C}_n$ (green) Γ_f^+ (magenta), and C_c^1 (blue).

Numerically computed breakup–diagram for NSM [22], [25], that is, the set of points in the parameter plane (ω, K), with $K = k/(2\pi)$, for which the twistless circle exists, reveals a fractal boundary for this set. Hence one expects that for some ω values, the twistless circle which has broken down at a threshold $K(\omega)$, to reappear at a higher perturbation value K.

6. Conclusions

In this paper we have investigated the scenario of destruction of invariant circles of quadratic nontwist maps. In order to locate the regions of different dynamical behaviour, we have associated to such a map four non–invariant curves: the critical twist circles C_1, C_{-1}, the nontwist circle $\mathcal{C}_n$ and the folding curve Γ_f. At the same time we have considered two subsets in the parameter plane ω, k of NSM: the subset of twist–folding

pairs, and the subset of nontwist–folding pairs (ω, k). For points in the first set we proved that the corresponding map exhibits a central invariant annulus where nontwist and twist behaviour can coexist, while outside it the motion is chaotic, due to the folding property of the map. We conjectured that within the "global" invariant annulus there is an invariant sub–annulus containing the nontwist circle, on which the NSM is the time–one map of a one degree of freedom Hamiltonian system. Theoretical arguments are suggested and numerical examples are given to support this conjecture.

Two scenarios of destruction of the last invariant circles of the NSM corresponding to a nontwist–folding pair are illustrated. The cause of destruction of the invariant circles following the first scenario appears to be the foldings which affects the horizontal circles crossing the critical twist circles, as the perturbation parameter increases, while for those following the second scenario this cause is not so evident.

It remains as an open problem to derive criteria which ensure the non–existence of invariant circles (graphs or meanders) crossing the critical twist circles.

7. Acknowledgements

The work has been supported by EURATOM-CEA Cadarache, France, and the Grant CNCSIS-MEC 957. The author thanks to BFR Group: J. Misguich, D. Constantinescu, B. Weyssov, I. Pavlenko, as well as to C. Chandre, R. Lima, M. Vittot, from CNRS Marseille-Luminy, for stimulating discussions.

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